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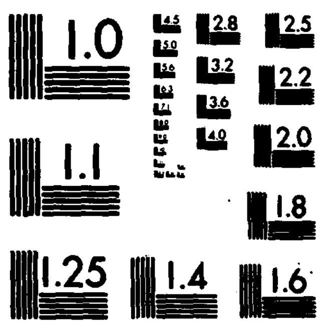

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SHOCK MODELS WITH PHASE TYPE
SURVIVAL AND SHOCK RESISTANCE

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ABSTRACT

New closure theorems for shock models in reliability theory are presented. If the number of shocks to failure and the times between the arrivals of shocks have probability distributions of phase type, then so has the time to failure. PH-distributions are highly versatile and may be used to model many qualitative features of practical interest. They are also well-suited for algorithmic implementation. The computational aspects of our results are discussed in some detail.

KEY WORDS

Reliability theory, shock models, distributions of phase type, computational probability.

1. INTRODUCTION

Shock models which relate the life distribution $H(\cdot)$ of a device, subject to failure by shocks occuring randomly in time, have received considerable attention in recent years. If $\bar{P}_k$ is the probability that the device survives $k \geq 0$, shocks and $N(t)$ is the random number of shocks in $(0,t]$, the survival probability $\bar{H}(\cdot) = 1 - H(\cdot)$, of such a device is given by

$$(1) \qquad \bar{H}(t) = E\bar{P}_{N(t)} = \sum_{k=0}^{\infty} \bar{P}_k \, P\{N(t)=k\} .$$

The most general shock models are those that correspond to (1), such that $\{N(t): t \geq 0\}$ is a general counting process and $1 \geq \bar{P}_0 \geq \bar{P}_1 \geq \bar{P}_2 \geq \cdots$. Interest in and published results for shock models center around proving that, subject to suitable assumptions on the point process $N(t)$ of shocks, various reliability characteristics of the shock resistance probabilities $\bar{P}_k$ are inherited by the survival probability $\bar{H}(\cdot)$ in continuous time.

The first systematic treatment of such shock models was given by Esary, Marshall and Proschan [5], when $N(t)$ is a homogeneous Poisson process. A-Hameed and Proschan considered the cases when $N(t)$ is a non-homogeneous Poisson Process [1] and a non-stationary pure birth process [2]. Block and Savits [4] treated the case when the interarrival time between shocks is NBUE (NBUE) or NBU(NWU) and Thall [8] derived interesting, but comparatively weaker, results when $N(t)$ is a clustered Poisson process.

In this paper, we obtain preservation theorems for the shock model (1) when $\bar{P}_k$ is of <u>phase-type</u> and so is the distribution of the interarrival time between shocks. $N(t)$ is then a phase type renewal process [7]. The relevance of phase type distributions (henceforth abbreviated as PH-distributions) to the algorithmic analysis of the time dependent behavior of of stochastic models has been discussed by Neuts in a series of papers starting with [6]. A comprehensive treatment may be found in Chapter 2 of [8]. PH-distributions provide an alternative point of departure in modelling real life distributions without the classic memoryless property and with possible proper unimodality or multimodality. PH-distributions include the exponential, Erlang and hyper-exponential distributions as very special cases. In addition, they have the desirable property of being closed under both finite convolutions and mixtures, a feature possessed by none of the well-known non-parametric classes of life distributions.

In Section 2, the basic properties of PH-distributions, needed in the sequel, are briefly reviewed. The main theoretical results are discussed in Section 3. Algorithmic considerations are presented in Section 4.

2. PH-DISTRIBUTIONS

A density $\{p_k\}$ on the nonnegative integers is <u>of phase type</u> if and only if there exists a finite Markov chain with transition probability matrix P of order $r + 1$ of the form

$$P = \begin{vmatrix} S & \underline{S}^{\circ} \\ \underline{0} & 1 \end{vmatrix} ,$$

and initial probability vector $[\underline{\beta}, \beta_{r+1}]$, such that $\{p_k\}$ is the density of the time till absorption in the state $r + 1$. The matrix $I - S$ is nonsingular and the stochastic matrix $S + (1-\beta_{r+1})^{-1}\underline{S}^{\circ}\cdot\underline{\beta}$ may be chosen to be irreducible.

The density $\{p_k\}$ is given by $p_0 = \beta_{r+1}$, and $p_k = \underline{\beta}\, S^{k-1}\, \underline{S}^{\circ}$, for $k \geq 1$. In this paper $\{p_k\}$ will be the density of the number of shocks to failure in a reliability shock model. We will assume throughout that

$\beta_{m+1} = 0$. We also clearly have that

$$\bar{P}_k = \sum_{\nu=k+1}^{\infty} p_k = \underline{\beta}\, S^k\, \underline{e}, \qquad \text{for } k \geq 0 .$$

The mean μ_1' of $\{p_k\}$ is given by $\underline{\beta}(I-S)^{-1}\underline{e}$.

A probability distribution $F(\cdot)$ on $[0,\infty)$ is <u>of phase type</u> if and only if there exists a finite Markov process with generator Q of the form

$$Q = \begin{vmatrix} T & \underline{T}^{\circ} \\ \underline{0} & 0 \end{vmatrix}$$

with initial probability vector $[\underline{\alpha}, \alpha_{m+1}]$, such that $F(\cdot)$ is the distribution of the time till absorption in the state $m+1$. The matrix T is nonsingular and the generator $T + (1-\alpha_{m+1})^{-1}\, \underline{T}^{\circ}\cdot\underline{\alpha}$ may be chosen to be irreducible. The distribution $F(\cdot)$ is given by

(2) $$F(x) = 1 - \underline{\alpha} \exp(Tx)\, \underline{e}\, , \qquad \text{for } x \geq 0 .$$

We shall denote $1 - F(x)$ by $\bar{F}(x)$. The mean λ_1' of $F(\cdot)$ is given by $\lambda_1' = -\underline{\alpha}\, T^{-1}\, \underline{e}$. The pairs $(\underline{\alpha},T)$ and $(\underline{\beta},S)$ are called <u>the representations</u> of $F(\cdot)$ and $\{p_k\}$ respectively. Renewal processes in which the underlying distribution $F(\cdot)$ is of phase type were discussed in [7].

Many derivatives related to PH-distributions involve the Kronecker product $L \otimes M$ of two matrices L and M. This is the matrix made up of the blocks $\{L_{ij}M\}$. Provided the matrix products are defined, we have that

(3) $$(L \otimes M)\,(K \otimes H) = LK \otimes MH .$$

This property is repeatedly used in the sequel.

3. CLOSURE THEOREMS

We first consider the Esary-Marshall-Proschan (E.M.P.) shock model [3,5] in which $\{N(t)\}$ is a Poisson counting process of rate λ.

Theorem 1

If the number of shocks to failure has a discrete PH-density $\{p_k, k \geq 0\}$ with representation $(\underline{\beta}, S)$, then the time to failure in the E.M.P. model has a continuous PH-distribution $H(\cdot)$ with representation $[\underline{\beta}, \lambda(S-I)]$.

Proof

Since $\overline{P}_k = \underline{\beta}\, S^k \underline{e}$, for $k \geq 0$, we obtain

$$\overline{H}(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \underline{\beta}\, S^k\, \underline{e} = \underline{\beta} \exp\, [\lambda(S-I)t]\, \underline{e}\,, \qquad \text{for} \quad t \geq 0\,.$$

This proves the stated result.

A number of interesting quantities may now be expressed in computationally convenient forms. The j-th noncentral moment of $H(\cdot)$ is given by

$$\mu_j' = j!\, \lambda^{-j}\, \underline{\beta}(I-S)^{-j}\, \underline{e}\,, \qquad \text{for} \quad j \geq 1\,. \tag{4}$$

The density $h(t) = H'(t)$, is given by

$$h(t) = \lambda\, \underline{\beta} \exp\, [\lambda(S-I)t]\, \underline{S}^{\circ}\,, \qquad \text{for} \quad t \geq 0\,, \tag{5}$$

and the failure rate $r(t) = h(t)\, \overline{H}^{-1}(t)$, equals

$$r(t) = \lambda \frac{\underline{\beta}\exp(\lambda tS)\underline{S}^{\circ}}{\underline{\beta}\exp(\lambda tS)\underline{e}}\,, \qquad \text{for} \quad t \geq 0 \tag{6}$$

Theorem 1 is a particular case of a more general result in which the arrivals of shocks occur according to a PH-renewal process [7]. This result is proved next.

Let the interarrival time distribution $F(\cdot)$ be of phase type with irreducible representation $(\underline{\alpha}, T)$ of order m. When $\alpha_{m+1} = 1 - \underline{\alpha}\, \underline{e}$, is positive, a geometrically distributed number of shocks occur simultaneously at each shock epoch. As in [7], we introduce the matrices $P(k,t)$, $k \geq 0$, $t \geq 0$, which satisfy the system of differential equations

$$
(7) \qquad \begin{aligned} P'(0,t) &= P(0,t)T , \\ P'(k,t) &= P(k,t)T + \sum_{\nu=1}^{k} \alpha_{m+1}^{\nu-1} P(k-\nu,t)\underline{T}^{\circ}\underline{\alpha} , \qquad k \geq 1 , \end{aligned}
$$

for $t \geq 0$, with initial conditions $P(k,0) = \delta_{0k} I$, for $k \geq 0$. The element $P_{ij}(k,t)$ is the conditional probability that the Markov process with generator $Q^* = T + (1-\alpha_{m+1})^{-1} \underline{T}^{\circ}\underline{\alpha}$, is in the state j at time t and that k shocks have occurred in $(0,t]$, given that it started in the state i at time 0.

The Markov process Q^* may be started according to any initial probability vector $\underline{\gamma}$. With $\underline{\gamma} = (1-\alpha_{m+1})^{-1} \underline{\alpha}$, the PH-renewal process is started immediately after a renewal epoch. With $\underline{\gamma} = -\lambda_1'^{-1}\underline{\alpha}T^{-1}$, where $\lambda_1' = -\underline{\alpha}\, T^{-1}\, \underline{e}$, is the mean time between shocks, we obtain the stationary version of the PH-renewal process.

Theorem 2

If the shocks occur according to a PH-renewal process with underlying representation $(\underline{\alpha},T)$ and the process Q^* *is started according to the probability vector* $\underline{\gamma}$ and if the probability density $\{p_k\}$ is of phase type with representation $(\underline{\beta},S)$ of order r, then the distribution $H(\cdot)$ is of phase type with the representation

$$
(8) \qquad \begin{aligned} \underline{\kappa} &= \underline{\gamma} \otimes \underline{\beta} , \\ K &= T \otimes I + \underline{T}^{\circ}\underline{\alpha} \otimes (I-\alpha_{m+1}S)^{-1} S , \end{aligned}
$$

of order rm.

Proof

By the law of total probability, we have

$$\bar{H}(t) = \underline{\gamma} \sum_{k=0}^{\infty} P(k,t)\underline{e} \cdot \underline{\beta}\, S^k\, \underline{e} \tag{9}$$

$$= (\underline{\gamma} \otimes \underline{\beta}) \sum_{k=0}^{\infty} P(k,t) \otimes S^k\, (\underline{e} \otimes \underline{e})$$

$$= (\underline{\gamma} \otimes \underline{\beta})\, Z(t)\, (\underline{e} \otimes \underline{e}), \qquad \text{for } t \geq 0 .$$

The matrix $Z(t)$ satisfies

$$Z'(t) = \sum_{k=0}^{\infty} P'(k,t) \otimes S^k = \sum_{k=0}^{\infty} P(k,t)\, T \otimes S^k$$

$$+ \sum_{k=1}^{\infty} \sum_{\nu=1}^{k} \alpha_{m+1}^{\nu-1} P(k-\nu,t)\, \underline{T}^{\circ}\underline{\alpha} \otimes S^k \quad \cdots\cdots$$

$$= Z(t)\, (T \otimes I) + \sum_{k=0}^{\infty} P(k,t)\, \underline{T}^{\circ}\underline{\alpha} \otimes S^{k+1}\, (I-\alpha_{m+1}S)^{-1}$$

$$= Z(t)\, [T \otimes I + \underline{T}^{\circ}\underline{\alpha} \otimes (I-\alpha_{m+1}S)^{-1}S] ,$$

and clearly $Z(0) = I \otimes I$.

This implies that $Z(t) = \exp(Kt)$, for $t \geq 0$. Upon substitution into (9), the proof is complete.

Particular Cases

1. If the number of shocks to failure is geometrically distributed, i.e. $\bar{P}_k = \theta^k$, for $k \geq 0$, then

$$\bar{H}(t) = \underline{\gamma} \sum_{k=0}^{\infty} P(k,t)\theta^k \underline{e} = \underline{\gamma} \exp\{[T+(1-\theta\alpha_{m+1})^{-1}\theta\, \underline{T}^{\circ}\underline{\alpha}]t\}\ \underline{e} , \tag{10}$$

for $t \geq 0$.

2. In the maximum shock model, failure occurs if and only if a shock occurs whose magnitude exceeds a critical randomized threshold Y with distribution

$G(\cdot)$. If the magnitudes of successive shocks are independent with common distribution $F(\cdot)$, then

$$\text{(11)} \qquad \bar{P}_k = \int_0^\infty F^k(x)\, dG(x) , \qquad \text{for } k \geq 0 .$$

It follows from (10) that

$$\text{(12)} \qquad \bar{H}(t) = \int_0^\infty \underline{\gamma} \exp \{[T + (1-\alpha_{m+1}F(x))^{-1}F(x)\underline{T}^\circ\underline{\alpha}]t\} \, \underline{e} \, dG(x) ,$$

for $t \geq 0$, so that $H(\cdot)$ is a mixture of PH-distributions. If $G(\cdot)$ is a discrete distribution with finite support, then $H(\cdot)$ itself is of phase type.

3. In the cumulative damage model, the damages are additive. With the same distributions $F(\cdot)$ and $G(\cdot)$ as in the preceding model, we obtain

$$\text{(13)} \qquad \bar{P}_k = \int_0^\infty F^{(k)}(x)\, dG(x) , \qquad \text{for } k \geq 0 .$$

If the distribution $G(\cdot)$ is of phase type with representation $(\underline{\delta},L)$, then

$$\bar{P}_k = \int_0^\infty \bar{G}(x)\, dF^{(k)}(x) = E\, \bar{G}(x_1 + \ldots + x_k)$$

$$= E\, \underline{\delta} \exp\, [L(x_1 + \ldots + x_k)]\underline{e} = \underline{\delta}\, A^k\, \underline{e} ,$$

where $A = \int_0^\infty \exp\,(Lx)\, dF(x)$. It is readily seen that A is a substochastic matrix of spectral radius less than one. The density $\{p_k\}$ is therefore of phase type. If the shocks occur according to a PH-renewal process, Theorem 2 may be applied to evaluate $\bar{H}(t)$. The matrix A is obtained by numerical integration for general distributions $F(\cdot)$. If $F(\cdot)$ itself is of phase type with representation $(\underline{\sigma},R)$, then

$$(14)\quad A = \int_0^\infty \exp(Lx)\underline{\sigma}\,\exp(Rx)\underline{R}^\circ dx = (I\otimes\underline{\sigma}) \int_0^\infty \exp(Lx)\otimes\exp(Rx)\,dx\,(I\otimes\underline{R}^\circ)$$

$$= -(I\otimes\underline{\sigma})\,[L\otimes I + I\otimes R]^{-1}\,(I\otimes\underline{R}^\circ)\quad .$$

The eigenvalues of L and R all lie in the open left half-plane. The same then holds true for the Kronecker sum $L\otimes I + I\otimes R$, so that the inverse exists.

The nonnegative rectangular matrix $V = -(L\otimes I + I\otimes R)^{-1}(I\otimes\underline{R}^\circ)$, may easily be computed by solving the system

$$(L\otimes I + I\otimes R)V = -\,I\otimes\underline{R}^\circ$$

by block Gauss-Seidel iteration.

4. ALGORITHMIC ASPECTS

We shall discuss the computation of the function $\overline{H}(t)$, which is given by Theorem 2. It readily follows from (1) that the mean h_1' of $H(\cdot)$ is given by $\lambda_1'\mu_1'$, where λ_1' and μ_1' are the means of $\{p_k\}$ and $F(\cdot)$ respectively, whenever the PH-renewal process of arrivals is started at a renewal epoch. With general initial conditions, the mean h_1' is given by $\lambda_1'\mu_1' + \tilde{\lambda}_1 - \lambda_1'$, where $\tilde{\lambda}_1 = -\,\underline{\gamma}\,T^{-1}\,\underline{e}$.

Knowledge of the mean h_1' of $H(\cdot)$ is useful in determining the interval over which we wish to evaluate $\overline{H}(t)$. We may e.g. wish to choose the mean as a convenient unit of time. This is accomplished by replacing K by $h_1'K$. A different rescaling may be chosen if the elements of $h_1'K$ are very large or if a different time scale is desirable for the practical problem at hand.

We now assume that the matrix K has been appropriately rescaled. The function $\overline{H}(t)$ is computed by numerical integration of the system of linear differential equations

$$(15)\qquad \begin{aligned} \underline{v}'(t) &= \underline{v}(t)\,K\,, \qquad \text{for} \quad t \geq 0\,,\\ \underline{v}(0) &= \underline{\gamma} \otimes \underline{\beta}\,, \end{aligned}$$

and setting $\overline{H}(t) = \underline{v}(t)\underline{e}$, for $t \geq 0$.

It is convenient to partition the vector $\underline{v}(t)$ as $[\underline{v}_1(t), \ldots, \underline{v}_m(t)]$, where the vectors $\underline{v}_i(t)$ are r-vectors. We also set $M = (I-\alpha_{m+1}S)^{-1}S$. The system (15) may then be rewritten as

$$(16)\qquad \underline{v}_j'(t) = \sum_{\nu=1}^{m} \underline{v}_\nu(t)\, T_{\nu j} + \alpha_j \left[\sum_{\nu=1}^{m} \underline{v}_\nu(t)\, T_\nu^\circ\right] M\,,$$

for $1 \leq j \leq m$. This system may be conveniently solved by a classical integration procedure, such as Runge-Kutta. We see that the vector $\left[\sum_{\nu=1}^{m} \underline{v}_\nu(t)\, T_\nu^\circ\right] M$ does not depend on j and needs to be evaluated only once in each computation of the right hand sides of (16).

In many PH-distributions of practical interest, such as e.g. finite mixtures of Erlang distributions, the order m of T may be large, but T, $\underline{T}^\circ$ and $\underline{\alpha}$ have very few nonzero entries. It is then advantageous to write a special purpose subroutine to evaluate the right hand side of (16). By so exploiting the sparsity of T, $\underline{T}^\circ$ and $\underline{\alpha}$, it is possible to reduce the computation time greatly. The mean h_1', or in general the scaling factor used in selecting the time unit, may also be utilized to choose the step size h in the numerical integration of the system (16). In similar problems, we have usually made two runs at least, one with 1/50 of the time unit and one with 1/100 of the time unit. If the results at corresponding time points are not sufficiently close, further runs with smaller steps are made. The computation times of such runs increase rapidly and efficient programming is desirable. Other methods with a variable step size and error control may also be implemented. These classical topics in the numerical integration of ordinary differential equations need not be belabored here. In all cases, the use of the particular structure of the matrix K is fully worthy of the additional programming effort.

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New closure theorems for shock models in reliability theory are presented. If the number of shocks to failure and the times between the arrivals of shocks have probability distributions of phase type, then so has the time to failure. PH-distributions are highly versatile and may be used to model many qualitative features of practical interest. They are also well-suited for algorithmic implementation. The computational aspects of our results are discussed in some detail.

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