AD-A090 787 TENNESSEE UNIV KNOXVILLE DEPT OF MATHEMATICS F/G 12/1
EQUIVALENT GAUSSIAN MEASURES WHOSE R-N DERIVATIVE IS THE EXPONE--ETC(U)
DEC 79 D M CHUNG, B S RAJPUT N00014-78-C-0468
UNCLASSIFIED TR-7906 NL

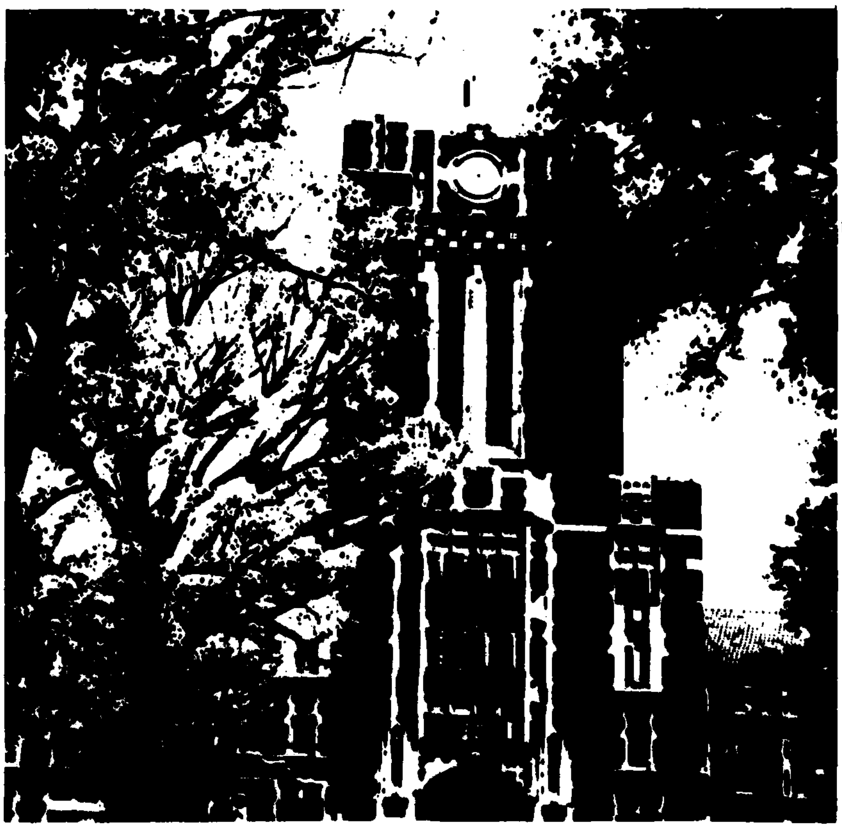
1 OF 1
AD A
090787

END
DATE
FILMED
11-80
DTIC

LEVEL II

(12)

DEPARTMENT OF MATHEMATICS
The University of Tennessee, Knoxville



(12)

EQUIVALENT GAUSSIAN MEASURES WHOSE R-N DERIVATIVE IS THE EXPONENTIAL OF A DIAGONAL FORM.

by

Dong M. Chung ~~and~~ Balram S. Rajput

University of Tennessee
Knoxville, Tennessee 37916

AMS 1970 subject classification. 60G15, 60G30.

Key words and phrases: Gaussian process, measurable process, Radon-Nikodým derivative.

(14) TR-7906

411894

JOB

ABSTRACT

A simple necessary and sufficient condition, on a trace-class kernel K , is given in order for the existence of a measurable (relative to the completed product σ-algebra) Gaussian process with covariance K . Using this result, sufficient conditions are given on the means and the covariances (relative to two equivalent (~) Gaussian measures P and P_λ) of a process X so that the Radon-Nikodým (R-N) derivative dp_λ/dP is the exponential of the diagonal form in X . Analogues of the last two results in the set up of Hilbert space are also proved.

1. INTRODUCTION

Let $(T,\mathcal{T},\nu)$ be an arbitrary σ-finite measure space and K a trace-class kernel on $T \times T$. We give a simple necessary and sufficient condition on K for the existence of a measurable (relative to the completed product σ-algebra) Gaussian process with covariance K (Theorem 1).

Assume that K satisfies the condition of the above theorem, so that there exists a measurable Gaussian process X on a probability space $(\Omega,\mathcal{F}, P)$ with covariance K. Assume, further, that the mean θ of X belongs to $L_2(\nu)$; then we, explicitly, evaluate $\int_\Omega \int_T \exp\{1/2\ \lambda|f(t)|X^2(t,\omega)\nu(dt)\}P(d\omega)$, where λ is a certain number and f is a certain measurable function (*Theorem 2, Corollary 2*). Assume the hypotheses and notation of the previous result and let, for each λ, a function θ_λ on T and a covariance function K_λ on $T \times T$ be given; then we give sufficient conditions on θ_λ and K_λ in order that (i) θ_λ and K_λ determine a probability measure P_λ on $(\Omega,\mathcal{F})$ with respect to which X is Gaussian, (ii) $P_\lambda \sim P$, and (iii) the R-N derivative dP_λ/dP is of the diagonal form in X; i.e., is expressible as $\int_T \exp\{1/2\ \lambda|f(t)|X^2(t,\omega)\}\nu(dt)$ (Theorem 3, Corollary 2).

The results of the previous paragraph are motivated by some of the work of D. E. Varberg [7] and L. A. Shepp [6], and they are generalizations of two results of the former author and are related to similar results of the latter. We may point out that these results are central and are best possible in the sense that they are proved under minimal hypotheses on the functions θ and K (see Remark 2). Analogues of Theorem 2 and 3, in the set up of separable Hibert spaces, are also proved (Theorem 4(i) and 4(ii)).

All results are stated and discussed in Section 2 and their proofs are given in Section 3.

2. STATEMENT AND DISCUSSION OF RESULTS

We begin by stating a few definitions, notation, and conventions that will be used throughout the paper.

(A.1) $(T, \mathcal{T}, \nu)$ denote an arbitrary σ-finite measure space; whenever we write T, it is implicitly assumed that $\mathcal{T}$ and ν are associated with it. If (Γ, A, γ) is a measure space, then $\bar{A}$ and $L_2(\gamma)$ denote, respectively, the completion of A relative to γ and the Hilbert space of real γ-square integrable functions.

(A.2) A real, nonnegative definite, symmetric and measurable function K on $T \times T$ is called a <u>kernel</u>; if, in addition, $\int_T K(t, t)\nu(dt) < \infty$, K is called a <u>trace-class kernel</u>. Let K be a trace-class kernel, and $\{\lambda_n\}$ and $\{\phi_n\}$ be, respectively, the positive eigenvalues (including multiplicities) and the corresponding (normalized) eigenfunctions of the integral equation

$$(2.1) \qquad \lambda\phi(s) = \int_T K(s, t)\phi(t)\nu(dt) ;$$

then K is called a <u>Mercer kernel</u> (<u>M-kernel</u>[1] for short), if K admits the representation

$$(2.2) \qquad K(s, t) = K_1(s, t) + K_2(s, t) , \quad s,t \in T ,$$

where K_1 and K_2 are trace-class kernels such that $K_2(t, t) = 0$ a.e. $[\nu]$, and

$$(2.3) \qquad K_1(s, t) = \sum_{n=1}^{\infty} \lambda_n \phi_n(s) \phi_n(t) , \quad s, t \in T ,$$

where the series converges absolutely, for all $s, t \in T$. We note that there exist numerous examples of such kernels.

(A.3) If K denotes an M-kernel on $T \times T$, then we denote, consistently, by $\{\lambda_n\}$ and $\{\phi_n\}$, respectively, the positive eigenvalues (including multiplicities) and the corresponding (normalized) eigenfunctions of the equation (2.1), and by K_1 and K_2 the kernels related to K as in (2.2). We will assume that the set $\{\lambda_n\}$ (and hence $\{\phi_n\}$) is not finite, since it is the only case of interest here.

(A.4) We consider here only real linear spaces and real stochastic processes.

Now, we are ready to state the first result of the paper.

THEOREM 1. Let K be a trace-class kernel on $T \times T$; then we have the following:

(a) If K is an M-kernel, then there exists a $\overline{T \times F}$ -measurable Gaussian process X on some probability space (Ω, F, P) such that K is the covariance of X ; further, if K_1, K_2, $\{\phi_n\}$ and $\{\lambda_n\}$ are realted to K as described in (A.3), then X can be so chosen that

$$X_t = Y_t + Z_t \quad , \quad t \in T ,$$

where Y and Z are independent Gaussian processes with covariances K_1 and K_2, respectively, and

$$(2.4) \qquad Y_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\ \phi_n(t)\ Y_n ,$$

where Y_n's are independent $N(0, 1)$ r.v.'s and the series converges in $L_2(P)$ and also a.s. $[P]$, for each fixed $t \in T$. Finally,

$$(2.5) \qquad Y(\cdot, \omega) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\ \phi_n(\cdot)\ Y_n(\omega) ,$$

where the series converges in $L_2(\nu)$ and also a.e. $[\nu]$, for every ω outside a P-null set.

(b) Conversely, if K is the covariance function of a $\overline{T \times F}$-measurable Gaussian process X on a probability space (Ω, F, P), then K is an M-kernel.

REMARK 1. It should be noted that, for a given M-kernel, Theorem 4.1(a) guarantees the existence of a Gaussian process which has the given kernel as its covariance and is measurable relative to the completed product σ-algebra. The question, whether for every M-kernel K there exists a Gaussian process which has covariance K and is measurable relative to the uncompleted product σ-algebra, has a negative answer. (see Remark 1, [2, p. 470]).

For the statements and the proofs of some of the following results, we need a few more notation and conventions which we record in the following:

(A.5) If K denotes an M-kernel on $T \times T$ (so that, in view of (A.3), $\{\lambda_n\}$ and $\{\phi_n\}$ are, respectively, the eigenvalues and corresponding eigenfunctions of (2.1)) and θ a ν-square integrable function on T, then we denote, consistently, by $\dot{\theta}$, the orthogonal projection of θ onto the space orthogonal to the $L_2(\nu)$-closure of the linear space of $\{\phi_n\}$, by λ, a real number such that $1 - \lambda\lambda_n > 0$, for all n, and by θ_λ, and K_λ the functions defined as follows

$$\theta_\lambda(t) = \theta(t) + \lambda \sum_{n=1}^{\infty} \lambda_n(1 - \lambda\lambda_n)^{-1} \langle\phi_n, \theta\rangle \phi_n(t), \quad t \in T, \tag{2.6}$$

$$K_\lambda(s,t) = \sum_{n=1}^{\infty} \lambda_n(1 - \lambda\lambda_n)^{-1} \phi_n(s)\, \phi_n(t) + K_2(s, t), \quad s, t \in T, \tag{2.7}$$

where $\langle , \rangle$ is the inner product in $L_2(\nu)$ and K_2 is related to K as in (2.2). Further, we consistently use the notation $D(\lambda)$ and $W(\lambda)$, respectively, for

$$\prod_{n=1}^{\infty} (1 - \lambda\lambda_n) \tag{2.8}$$

and

$$D(\lambda)^{1/2} \exp[-1/2\, \lambda\{||\dot{\theta}||^2 + \sum_{n=1}^{\infty} (1 - \lambda\lambda_n)^{-1} \langle\phi_n, \theta\rangle^2\}] \tag{2.9}$$

where $||\cdot||$ is the norm in $L_2(\nu)$. The series in (2.6) and (2.7) converge absolutely, respectively, for $t \in T$ and $s, t \in T$. This follows from the boundedness of the sequence $\{(1 - \lambda\lambda_n)^{-1}\}$ (recall that $\sum_{n=1}^{\infty} \lambda_n < \infty$), Cauchy inequality for sequences and (2.3). Since

$1 - \lambda\lambda_n > 0$, $\lambda_n > 0$ for all n , and $\sum_{n=1}^{\infty} \lambda_n < \infty$, we have that $0 < D(\lambda) < 1$. From this and the boundedness of the sequence $\{(1 - \lambda\lambda_n)^{-1}\}$, it follows that $W(\lambda)$ is a well defined positive real number.

In Theorem 2 and 3 and Corollaries 1 and 2, it will be assumed that the space $L_2(\nu)$ is separable.

We are now ready to state the following two results.

THEOREM 2. Let K be an M-kernel on $T \times T$ and $\theta \in L_2(\nu)$; then there exists a $\overline{\mathcal{T} \times \mathcal{F}}$ -measurable Gaussian process ξ on a probability space $(\Omega, \mathcal{F}, P)$ such that θ and K are , respectively, the mean and the covariance of ξ ; further, if λ and $W(\lambda)$ are related to θ and K as in (A.5), then

$$(2.10) \qquad \int_\Omega \exp\{1/2\ \lambda \int_T \xi^2(t,\omega)\nu(dt)\}\ P(d\omega) = W(\lambda)^{-1} < \infty.$$

THEOREM 3. Let K ,θ, ξ and $(\Omega, \mathcal{F}, P)$ be as in Theorem 2, and let λ ,θ_λ, K_λ and $W(\lambda)$ be related to θ and K as in (A.5). Then K_λ is a covariance function, and there exists a probability measure P_λ on $(\Omega, \mathcal{F})$ such that ξ is Gaussian on $(\Omega, \mathcal{F}, P_\lambda)$ with mean θ_λ and covariance K_λ , $P \sim P_\lambda$, and the R-N derivative $dP_\lambda | dP$ is given by

$$(2.11) \qquad dP_\lambda | dP(\omega) = W(\lambda) \exp\{1/2\ \lambda \int_T \xi^2(t,\omega)\nu(dt)\} \qquad \text{a.s } [P].$$

REMARK 2. It is clear, from (2.10) and (2.11), that in order to obtain results similar to Theorems 2 and 3 the functions θ and K appearing in these results must guarantee the existence of the process ξ which is measurable and whose almost all paths are ν-square integrable. Since, in view of Proposition 3.4 of [5] and Theorem 1, these conditions on ξ are equivalent to the facts that K is an M-kernel and that θ is ν-square integrable, it follows that Theorems 2 and 3 are best possible, i.e., they are proved under the weakest possible hypotheses on θ and K.

In order to point out the relation between the above two theorems and the corresponding results of Varberg (Theorems 1 and 2 of [7]) and Shepp [6, p. 352], we now state two corollaries. These corollaries are, essentially, the restatements of Theorems 1 and 2; nevertheless, their inclusion is necessary in order to compare our results with the corresponding results of Shepp and Varberg.

COROLLARY 1. Let r be a kernel on $T \times T$ (see (A.2)), and ρ and f be measurable with $|f(t)| > 0$ on T such that (i) $K(s,t) = r(s, t)|f(s)|^{1/2}|f(t)|^{1/2}$, $s, t \in T$, is an M-kernel, and (ii) $\theta(t) = \rho(t)|f(t)|^{1/2}$, $t \in T$, is ν-square integrable (both of these conditions are satisfied, for instance, when r is an M-kernel, $\rho \in L_2(\nu)$, and f is bounded, this follows from Theorem 1). Then there exists a $\overline{T \times F}$-measurable Gaussian process ζ on a probability space (Ω, F, P) such that ρ and r are, respectively, the mean and the covariance of ζ. Further, if λ and $W(\lambda)$ are related to θ and K as in (A.5), then

$$\int_{\Omega} \exp\{1/2\lambda\int_T |f(t)|\zeta^2(t,\omega)\nu(dt)\}P(d\omega) = W(\lambda)^{-1} . \tag{2.12}$$

COROLLARY 2. Let r, ρ, f, K and θ be as in Corollary 1 and let ζ and (Ω, F, P) be as obtained in Corollary 1 and let λ, θ_λ, K_λ and $W(\lambda)$ be related to θ and K as in (A.5). Then there exists a probability measure P_λ on (Ω, F) such that ζ is Gaussian on (Ω, F, P_λ) with mean $|f(t)|^{-1/2}\theta_\lambda(t)$, $t \in T$, and covariance $|f(s)|^{-1/2}|f(t)|^{-1/2} K_\lambda(s,t)$, $s,t \in T$, $P_\lambda \sim P$, and the R-N derivative $dP_\lambda|dP$ is given by

$$dP_\lambda/dP(\omega) = W(\lambda) \exp\{1/2\, \lambda\int_T |f(t)|\zeta^2(t,\omega)\nu(dt)\} \quad \text{a.s. } [P]. \tag{2.13}$$

REMARK 3. If $T = [a, b]$, $\mathcal{T}$ = the class of Borel subsets of T, ν = the Lebesgue measure, and if r is a continuous kernel on $T \times T$, then, by Mercer's theorem, r is an M-kernel on $T \times T$. Now if f is any bounded measurable function on T, then, as indicated in Corollary 1, $r(s, t)|f(s)|^{1/2}|f(t)|^{1/2}$, $s, t \in T$, is an M-kernel. From this it is now clear that Theorem 1 and Theorem 2 of Varberg [7] are special cases, respectively, of Corollary 1 and Corollary 2. These corollaries are also related to two results of Shepp that are given on pp. 350 and 352 of [6].

We shall now state two more results (Theorems 4(i) and 4(ii)). Theorem 4(i) is important in that it is needed for the proofs of Theorem 2 . Theorem 4 (iii) is included here to show that the analogue of Theorem 3 can be forumated for Gaussian measures defined on <u>abstract</u> separable Hilbert spaces.

We assume that the reader is familiar with the elementary properties of Gaussian measures in separable Hilbert spaces.

In the following theorem, H and $\mathcal{B}(H)$ denote, respectively, a separable Hilbert space and the σ-algebra generated by open sets of H ; and $<,>$ and $||\cdot||$ denote, respectively, the inner product and the norm in H .

THEOREM 4. Let μ be a Gaussian measure on $(H, \mathcal{B}(H))$ with mean m and covariance operator S . Denote by $\{\delta_n\}$ and $\{\psi_n\}$, the positive eigenvalues (including multiplicities) and the corresponding normalized eigenvectors of S , by δ , a real number such that $\delta\delta_n < 1$, for all n , and, by $\dot{m}$, the orthogonal projection of m onto the space orthogonal to the closed linear space generated by $\{\psi_n\}$. Define

$$S_\delta(x) = \sum_{n=1}^{\infty} \delta_n(1 - \delta\delta_n)^{-1} <\psi_n, x> \psi_n \quad , x \in H ,$$

$$m_\delta = m + \delta S_\delta(m) ,$$

and

$$U(\delta) = [\prod_{n=1}^{\infty} (1 - \delta\delta_n)]^{1/2} \exp[-1/2 \ \delta\{||\dot{m}||^2 + \sum_{n=1}^{\infty}(1 - \delta\delta_n)^{-1}<\psi_n, m>^2\}].$$

Then we have

(i) $\int_H \exp 1/2 \ \delta||x||^2\} \ \mu(dx)$

$$= [\prod_{n=1}^{\infty}(1 - \delta\delta_n)]^{-1/2}\exp[1/2 \ \delta\{||\dot{m}||^2 + \sum_{n=1}^{\infty}(1 - \delta\delta_n)^{-1}<\psi_n, m>^2\}] \tag{2.14}$$

$\equiv U(\delta)^{-1} < \infty$;

(ii) if μ_δ is the Gaussian measure[2] on $(H, \mathcal{B}(H))$ with mean m_δ and covariance operator S_δ, then $\mu_\delta \sim \mu$, and the R-N derivative $d\mu_\delta/d\mu$ is given by

$$d\mu_\delta/d\mu(x) = U(\delta)\exp\{1/2\ \delta\ ||x||^2\}, \qquad \text{a.s. } [\mu], \tag{2.15}$$

where $U(\delta)$ is as in (i).

3. PROOFS

<u>Proof of Theorem 1(a)</u>. For clarity, we devide our proof into three parts. In parts (i) and (ii), two auxilary processes Y^1 and Z^1 are defined; and, in part (iii), these are used to construct the required process X.

(i) There exists a $\overline{T \times F_1}$-measurable Gaussian process Y^1 with covariance K_1 defined on a probability space (Ω_1, F_1, P_1). Further, Y^1 can be so chosen that, for every fixed $t \in T$,

$$Y_t^1 = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\ \phi_n(t)\ Y_n^1,$$

where the series converges in $L_2(P_1)$ and also a.s. $[P_1]$, and Y_n^1's are independent $N(0, 1)$ r.v.'s on (Ω_1, F_1, P_1). Further

$$Y^1(\cdot,\omega) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\ \phi_n(\cdot)\ Y_n^1(\omega),$$

where the series converges in $L_2(\nu)$ and a.e. $[\nu]$, for every ω outside a P_1-null set.

<u>Proof of (i)</u>. Let $\{Y_n^1\}$ be a sequence of independent $N(0, 1)$ r.v.'s defined on a probability space (Ω_1, F_1, P_1). We now define two processes ξ and ζ in terms of λ_n's, ϕ_n's and Y_n^1's, and then define the required process Y^1 in terms of ξ and ζ.

We first define the process ξ. For each n, let

$$\psi_n(t,\omega) = \sqrt{\lambda_n}\ \phi_n(t)\ Y_n^1(\omega)\ , \quad (t,\omega) \in T \times \Omega_1\ .$$

Since $\sum_{j=1}^{\infty} \lambda_j < \infty$ and $\langle\psi_n, \psi_m\rangle_{L_2(\nu \times P_1)} = \sqrt{\lambda_n}\ \sqrt{\lambda_m}\ \delta_{n,m}$

($\delta_{n,m}$ is the Kronecker δ), it follows that

$$\left|\left| \sum_{j=n}^{m} \psi_j \right|\right|^2_{L_2(\nu \times P_1)} = \sum_{j=n}^{m} \lambda_j \to 0$$

as $n, m \to \infty$. Thus, $\{\sum_{j=1}^{n} \psi_j\}$ converges in $L_2(\nu \times P_1)$; and, so, there exists a subsequence $\{\sum_{j=1}^{n_k} \psi_j\}$ which converges pointwise off a $\nu \times P_1$-null set A. Define

$$\xi(t,\omega) = \begin{cases} \lim_k \sum_{j=1}^{n_k} \psi_j(t,\omega) & \text{off } A \\ 0 & \text{on } A \end{cases};$$

then, clearly, ξ is $\mathcal{T} \times \mathcal{F}_1$-measurable. Further, by Fubini's Theorem, there exists a ν-null set T_1 such that, for every fixed $t \notin T_1$, the set $A_t \equiv \{\omega: (t,\omega) \in A\}$ has P_1-measure zero, and, for every $\omega \notin A_t$,

$$\xi(t,\omega) = \lim_k \sum_{j=1}^{n_k} \psi_j(t,\omega) .$$

Now we define the process ζ. Since for every fixed $t \in T$, $\sum_{n=1}^{\infty} \lambda_n \phi_n^2(t) < \infty$ (see (2.3)) and Y_n^1's are independent mean 0 variance 1 r.v.'s, $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \phi_n(t)\, Y_n^1$ converges in $L_2(P_1)$ and also pointwise off a P_1-null set B_t, for each $t \in T$ [4, p. 147]. For each $t \in T$, define

$$\zeta_t(\omega) = \begin{cases} \lim_n \sum_{j=1}^{n} \lambda_j \phi_j(t)\, Y_j^1(\omega) , & \text{if } \omega \in B_t^c \\ \\ 0 , & \text{if } \omega \in B_t \end{cases},$$

where B_t^c denotes the complement of B_t.

Clearly, if $t \in T_1^c$, then $P_1(A_t^c \cap B_t^c) = 1$; further, if $\omega \in A_t^c \cap B_t^c$, then, since $\{\sum_{j=1}^{n_k} \psi_j(t,\omega)\}$ is a subsequence of $\{\sum_{j=1}^{n} \psi_j(t,\omega)\}$, $\xi(t,\omega) = \zeta(t,\omega)$. Thus, for every $t \in T_1^c$,

$$\text{(3.1)} \qquad \xi_t = \zeta_t \qquad \text{a.s. } [P_1] .$$

Finally, define

$$\text{(3.2)} \qquad Y^1(t,\omega) = \begin{cases} \xi(t,\omega) & \text{if } (t,\omega) \in T_1 \times \Omega_1 \\ \\ \zeta(t,\omega) & \text{if } (t,\omega) \in T_1^c \times \Omega_1 \end{cases} .$$

We now show that Y^1 is a required process.

Since, from (3.2), the set $\{(t, \omega): Y^1(t,\omega) \neq \xi(t,\omega)\}$ is contained in $\nu \times P_1$-null set $T_1 \times \Omega_1$, and since ξ is shown $\mathcal{T} \times \mathcal{F}_1$-measurable, it follows that Y^1 is $\overline{\mathcal{T} \times \mathcal{F}_1}$-measurable. Since, as shown above, the series $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\phi_n(t)\, Y_n^1$ converges to ζ_t in $L_2(P_1)$ and also a.s. $[P_1]$, for each fixed $t \in T$, and since, from (3.1) and (3.2), $Y_t^1 = \zeta_t$ a.s. $[P_1]$, for each $t \in T$, we have that $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\phi_n(t)\, Y_n^1$ converges to Y_t^1 in $L_2(P_1)$ and also a.s. $[P_1]$, for each $t \in T$. Also, from $L_2(P_1)$ convergence of the series to Y_t^1, $t \in T$, we have that Y^1 is Gaussian (recall that Y_n^1's are Gaussian) and that the covariance of Y^1 is $K_1(s,t) = \sum_{n=1}^{\infty}\lambda_n\phi_n(s)\phi_n(t)$, $s, t \in T$, where $\sum_{n=1}^{\infty} \lambda_n\phi_n(s)\phi_n(t)$ converges absolutely for $s, t \in T$.

To complete the proof of (i), it remains to prove that $\sum_{n=1}^{\infty}\sqrt{\lambda}\, \phi_n(\cdot)Y_n^1(\omega)$ converges to $Y^1(\cdot,\omega)$ in $L_2(\nu)$ and a.e. $[\nu]$, for almost all ω. Since, for $t \in T$, $\sum_{n=1}^{\infty}\sqrt{\lambda_n}\, \phi_n(t)\, Y_n^1$ is shown to converge to Y_t^1 a.s. $[P_1]$, we have, by an application of Fubini's theorem, that $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\phi_n(\cdot)\, Y_n^1(\omega)$ converges to $Y^1(\cdot, \omega)$ a.e. $[\nu]$, for almost all ω. Now we show the $L_2(\nu)$ convergence of $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \phi_n(\cdot)Y_n^1(\omega)$ to $Y^1(\cdot,\omega)$ a.e. $[P_1]$. Since

$$\int_\Omega\{\sum_{n=1}^{\infty} \lambda_n(Y_n^1)^2\}P_1(d\omega) = \sum_{n=1}^{\infty} \lambda_n < \infty \quad ,$$

we have that $\sum_{n=1}^{\infty} \lambda_n(Y_n^1(\omega))^2 < \infty$ a.s. $[P_1]$. Therefore,

$$\int_T\Big[\sum_{j=n}^{m}\sqrt{\lambda_j}\, \phi_j(t)\, Y_n^1(\omega)\Big]^2\nu(dt) = \sum_{j=n}^{m} \lambda_j(Y_j^1(\omega))^2 \to 0$$

a.s. $[P_1]$ as $n, m \to \infty$; consiquently, $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \phi_n\, Y_n^1(\omega)$ converges in $L_2(\nu)$, a.s. $[P_1]$. Now using the fact that $L_2(\nu)$ convergence implies the existence of a subsequence that converges to the same function a.e. $[\nu]$ and the fact that $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \phi_n\, Y_n^1(\omega)$ converges to $Y^1(\cdot,\omega)$ a.e. $[\nu]$, for almost all ω , we have that $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \phi_n\, Y_n^1(\omega)$ converges to $Y^1(\cdot,\omega)$ in $L_2(\nu)$, for almost all ω . The proof of (i) is now complete.

(ii) There exists a $\overline{T \times F}$-measurable Gaussian process Z^1 with covariance K_2 defined on some probability space (Ω_2, F_2, P_2) .

<u>Proof of (ii)</u>. By Kolmogorov's existence theorem, there exists a Gaussian process η with covariance K_2 defined on some probability space (Ω_2, F_2, P_2) . Let T_2 be the ν-null set of T such that $K_2(t,t) = 0$ off T_2 . Define

$$Z^1(t,\omega) = \eta(t,\omega)\, \chi_{T_2 \times \Omega_2}(t,\omega) ,$$

where $\chi_{T_2 \times \Omega_2}$ is the indicator of $T_2 \times \Omega_2$. Then, clearly, Z^1 is Gaussian with covariance K_2 ; further, since $Z^1(t,\omega) = 0$ a.e. $[\nu \times P_2]$, Z^1 is $\overline{T \times F_2}$-measurable.

(iii) Let $(\Omega_1, F_1, P_1), Y^1, Y_n^1$'s, and (Ω_2, F_2, P_2) , Z^1 be as in (i) and (ii), respectively. Let $(\Omega, F, P) = (\Omega_1 \times \Omega_2, F_1 \times F_2, P_1 \times P_2)$, and Π_j be the projection of $T \times \Omega$ onto $T \times \Omega_j$, $j = 1,2$. Let $Y_n = Y_n^1 \circ \Pi_1$, $Y = Y^1 \circ \Pi_1$, $Z = Z^1 \circ \Pi_2$ and

$$(3.3) \qquad X_t = Y_t + Z_t \ , \ t \in T ;$$

then the processes X, Y, Z and the r.v.'s Y_n's satisfy the required properties of Theorem 1(a).

<u>Proof of (iii)</u>. It is clear that Π_j is measurable from $(\Omega, \overline{T \times F})$ onto $(T \times \Omega_j, \overline{T \times F_j})$, $j = 1, 2$. Therefore, since by (i) and (ii) Y^1 and Z^1 are $\overline{T \times F_1}$ and $\overline{T \times F_2}$-measurable, respectively, Y and Z are $\overline{T \times F}$-measurable. The rest of the proof follows from (i) and (ii) and the observation that for any $t_1, \ldots, t_n, s_1, \ldots, s_m \in T$ and any $A \in \mathcal{B}(R^n)$, $B \in \mathcal{B}(R^m)$, $P\{(Y_{t_1}, \ldots, Y_{t_n}) \in A, (Z_{s_1}, \ldots, Z_{s_m}) \in B\} = P_1\{(Y^1_{t_1}, \ldots, Y^1_{t_n}) \in A\}.$ $P_2\{(Z^1_{s_1}, \ldots, Z^1_{s_m}) \in B\}$, where $\mathcal{B}(R^k)$ is the class of Borel subsets of the k-Euclidian space R^k. We omit the details.

<u>Proof of Theorem 1(b)</u>: This follows from Theorem 1 of [1] due to S. Cambanis.

<u>Proof of Theorem 2</u>. Let X be the Guassian process on (Ω, F, P) as constructed in Theorem 1(a) subject to the additional condition that $E(X_t) = 0$, $t \in T$. Note that, as follows from the proof of Theorem 1, this additional condition is satisfied by X if we choose the process Z^1 in the proof of Theorem 1(a) to have zero mean. Let

$$\xi_t = X_t + \theta(t), \quad t \in T ; \tag{3.4}$$

then, clearly, ξ is a $\overline{T \times F}$-measurable Gaussian process with mean θ and covariance K.

Since $\int_T K(t, t)\, \nu(dt) < \infty$ and $\theta \in L_2(\nu)$, $\xi(\cdot, \omega) \in L_2(\nu)$ a.s. [P], and since $L_2(\nu)$ is assumed separable, ξ induces a Gaussian measure μ on $L_2(\nu)$ via the map $\omega \mapsto \xi(\cdot,\omega)$ if $\xi(\cdot,\omega) \in L_2(\nu)$, $\omega \mapsto 0$ if $\xi(\cdot,\omega) \notin L_2(\nu)$ [5, Theorem 3.2]. For each $f \in L_2(\nu)$, define (pointwise)

$$S(f)(s) = \int_T K(s, t)f(t)\nu(dt) .$$

Then it follows from Lemma 3.2 and Proposition 3.5 of [5]; that θ and the operator S are, respectively, the mean and covariance operator of μ. Further, it is clear from the definition of S that its eigenvalues and corresponding eigenvectors, are respectively $\{\lambda_n\}$ and $\{\phi_n\}$ (see (A.3)). The proof of (2.10) now follows from Theorem 4(i), the above observations, and the following equation

$$\int_\Omega \exp\{1/2\ \lambda \int_T \xi^2(t,\omega)\nu(dt)\}P(d\omega) = \int_{L_2(\nu)} \exp\{1/2\ \lambda \int_T x^2(t)\nu(dt)\}\mu(dx),$$

which is a direct consequence of the change of variable formula [3, p. 163].

<u>Proof of Theorem 3</u>. Define, for every $B \in F$,

$$P_\lambda(B) = W(\lambda) \int_B \exp\{1/2\ \lambda \int_T \xi^2(t,\omega)\nu(dt)\}P(d\omega) , \tag{3.5}$$

then it is clear, from (2.10), that P_λ is a probability measure on (Ω,F), and, from (3.5), that $P_\lambda \sim P$ with the R-N derivative dP_λ/dP equal to the right side of (2.11) a.s. [P]. Thus, the proof of Theorem 3 will be complete, if we can show that ξ is Gaussian with mean θ_λ and covariance K_λ . We prove this in the following by showing that $E_\lambda[\exp\{i \sum_{j=1}^{n} s_j \xi_{t_j}\}]$ is the right n-dimensional characteristic function, where E_λ is the expectation relative to P_λ , and $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ are arbitrary elements of R and T , respectively.

Recall that $\xi_t = Y_t + Z_t + \theta(t)$, $t \in T$,(see (3.3) and (3.4)), and that $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \phi_n(\cdot)\, Y_n(\omega)$ converges to $Y(\cdot,\omega)$ in $L_2(\nu)$ a.s. [P] (see (2.5)). Using these, the independence of the families $\{Y_t: t \in T\}$, $\{Z_t: t \in T\}$ and the facts $E(Y_t) = E(Z_t) = 0$, $t \in T$, and $E(Z_t^2) = 0$ a.e. $[\nu]$, we have

$$(3.6) \quad \int_T \xi^2(t,\omega)\nu(dt) = \sum_{n=1}^{\infty} \lambda_n Y_n^2(\omega) + 2 \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle\phi_n,\theta\rangle Y_n(\omega) + \|\theta\|^2 \quad \text{a.s. } [P] .$$

Using (3.5) and (3.6),we have

$$E_\lambda[\exp\{i \sum_{j=1}^{m} s_j \xi_{t_j}\}]$$

$$= W(\lambda) E[\exp\{i \sum_{j=1}^{m} s_j \xi_{t_j} + 1/2\, \lambda \int_T \xi^2(t,\omega)\nu(dt)\}]$$

$$(3.7) \quad = W(\lambda)\, E[\exp\{i \sum_{j=1}^{m} s_j \xi_{t_j}\}$$

$$\times \exp\{1/2\, \lambda(\sum_{n=1}^{\infty} \lambda_n Y_n^2 + 2 \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle\phi_n,\theta\rangle Y_n + \|\theta\|^2)\}] .$$

Noting again that $\xi_t = Y_t + Z_t + \theta(t)$, $t \in T$, and that $\sum_{n=1}^{\infty} \sqrt{\lambda_n} \phi_n(t)\, Y_n$ converges to Y_t a.s. [P] (see(2.4)), the right side of (3.7) is

$$= W(\lambda)E\left[\lim_{k}\left[\exp\{i\sum_{j=1}^{m} s_j\left(\sum_{n=1}^{k}\sqrt{\lambda_n}\,\phi_n(t_j)\,Y_n + Z_{t_j} + \theta(t_j)\right)\}\right.\right.$$

$$\left.\left.\times \exp\{1/2\,\lambda\left(\sum_{n=1}^{k}\lambda_n Y_n^2 + 2\sum_{n=1}^{k}\sqrt{\lambda_n}\,\langle\phi_n,\theta\rangle\,Y_n + ||\theta||^2\right)\}\right]\right]$$

$$= W(\lambda)E\left[\lim_{k}\prod_{n=1}^{k}\left[\exp\{i\,Y_n\left(\sqrt{\lambda_n}\sum_{j=1}^{m} s_j\,\phi_n(t_j) - i\lambda\sqrt{\lambda_n}\,\langle\phi_n,\theta\rangle\right)+1/2\,\lambda\lambda_n Y_n^2\}\right]\right.$$

$$\left.\times \exp\{i\sum_{j=1}^{m} s_j\left(Z_{t_j} + \theta(t_j)\right)+ 1/2\,\lambda\,||\theta||^2\}\right] ,$$

which, by the dominated convergence theorem, is

$$(3.8) \qquad = W(\lambda)\lim_{k} E\left[\prod_{n=1}^{k}\left[\exp\{i\,Y_n\,B_n + 1/2\,\lambda\lambda_n\,Y_n^2\}\right]\right.$$

$$\left.\times \exp\,\{i\sum_{j=1}^{m} s_j\left(Z_{t_j} + \theta(t_j)\right) + 1/2\,\lambda\,||\theta||^2\}\right] ,$$

where

$$(3.9) \qquad B_n = \sqrt{\lambda_n}\sum_{j=1}^{n} s_j\phi_n(t_j) - i\lambda\sqrt{\lambda_n}\,\langle\phi_n,\theta\rangle .$$

Now using the independence of the r.v.'s Y_n's and the independence of the two families $\{Y_n: n = 1, 2, \ldots\}$, $\{Z_t: t \quad T\}$ and recalling that

$$E[\exp\{i\,Y_n\,B_n + 1/2\,\lambda\lambda_n\,Y_n^2\}] = (1 - \lambda\lambda_n)^{-1/2}\exp\{-1/2\,B_n^2(1 - \lambda\lambda_n)^{-1}\} ,$$

it follows that the expression in (3.8) is

$$(3.10) \quad = W(\lambda)\, D(\lambda)^{-1/2} \exp\{-1/2 \sum_{n=1}^{\infty}(1 - \lambda\lambda_n)^{-1}B_n^2\}$$

$$\times \exp\{i \sum_{j=1}^{m} s_j\theta(t_j) - 1/2 \sum_{j=1}^{m}\sum_{k=1}^{m} s_j s_k\, K_2(t_j, t_k)\}$$

$$\times \exp\{1/2\, \lambda\, ||\theta||^2\} .$$

Subsituting the value of B_n from (3.9) in (3.10) and observing that $||\theta||^2 = \sum_{n=1}^{\infty} <\phi_n,\theta>^2 + ||\dot{\theta}||^2$ (see (A.5)) , we see that the expression in (3.10) is

$$(3.11) \quad = W(\lambda)\, D(\lambda)^{-1/2} \exp[i \sum_{j=1}^{m} s_j\{\theta(t_j) + \sum_{n=1}^{\infty} \lambda\lambda_n(1 - \lambda\lambda_n)^{-1}\phi_n(t_j)<\phi_n,\theta>\}]$$

$$\times \exp[-1/2 \sum_{j=1}^{m}\sum_{k=1}^{m} s_j s_k\{K_2(t_j,t_k) + \sum_{n=1}^{\infty} \lambda_n(1 - \lambda\lambda_n)^{-1}\phi_n(t_j)\phi_n(t_k)\}]$$

$$\times \exp[1/2\, \lambda\{||\dot{\theta}||^2 + \sum_{n=1}^{\infty}(1 - \lambda\lambda_n)^{-1}<\phi_n,\theta>^2\}] ,$$

which, in view of (2.6) - (2.8), is

$$= W(\lambda)\, D(\lambda)^{-1/2} \exp\{i \sum_{j=1}^{m} s_j\theta_\lambda(t_j) - 1/2 \sum_{j=1}^{m}\sum_{k=1}^{m} s_j s_k\, K_\lambda(t_j, t_k)\}$$

$$\times \exp[1/2\, \lambda\{||\dot{\theta}||^2 + \sum_{n=1}^{\infty} (1 - \lambda\lambda_n)^{-1}<\phi_n,\theta>^2\}]$$

$$= W(\lambda)\, W(\lambda)^{-1} \exp\{i \sum_{j=1}^{m} s_j\theta_\lambda(t_j) - 1/2 \sum_{j=1}^{m}\sum_{k=1}^{m} s_j s_k\, K_\lambda(t_j, t_k)\} ,$$

by the definition of $W(\lambda)$ (see (2.9)). Thus,

$$\mathcal{E}_\lambda[\exp\{i \sum_{j=1}^{m} s_j \xi_{t_j}\}] = \exp\{i \sum_{j=1}^{m} s_j \theta_\lambda(t_j) - 1/2 \sum_{j=1}^{m} \sum_{k=1}^{m} s_j s_k K_\lambda(t_j, t_k)\},$$

as desired.

Proof of Corollary 1. Since K is an M-kernel and $\theta \in L_2(\nu)$, there exists, by Theorem 2, a $\overline{T \times F}$-measurable Gaussian process ξ on a probability space (Ω, F, P) with mean θ and covariance K. Let $\zeta_t = |f(t)|^{-1/2}\xi_t$, $t \in T$; then, clearly, ζ is $\overline{T \times F}$-measurable and Gaussian with mean ρ and covariance r; further, the proof of (2.12) follows immediately from (2.10).

Proof of Corollary 2. Define P_λ as in (3.5) replacing ξ_t by $|f(t)|^{1/2}\zeta_t$. Then by (2.12), P_λ is a probability measure, and, by the definition of P_λ, $P_\lambda \sim P$ with the R-N derivative dP_λ/dP equal to the right side of (2.13) a.s. [P]. Since the process ξ of Theorem 3 is related to ζ by $\xi_t = |f(t)|^{1/2}\zeta_t$ and since it is shown to be Gaussian on (Ω, F, P_λ) with mean θ_λ and covariance K_λ, it follows that ζ is Gaussian on (Ω, F, P_λ) with mean $|f(t)|^{-1/2}\theta_\lambda(t), t \in T$, and covariance $|f(s)|^{-1/2}|f(t)|^{-1/2} K_\lambda(s,t)$ $s, t \in T$.

Proof of Theorem 4(i): Choose an orthonormal set $\{\psi_k' : k = 1,2,\ldots,\ell\}$ of H so that $\{\psi_n\} \bigcup \{\psi_k'\}$ is a Hilbert basis of H, where ℓ is finite or $+\infty$. It follows that $\{\psi_n\} \bigcup \{\psi_k'\}$ is a family of independent r.v.'s on $(H, \mathcal{B}(H), \mu)$, that ψ_k''s are degenerate at $\langle\psi_k', m\rangle$ and that ψ_n's are Gaussian with mean $\langle\phi_n, m\rangle$ and variance δ_n. Using these facts, Parseval's relation and the monotone convergence theorem, we have

$$\int_H e^{1/2\delta||x||^2}\mu(dx) = \int_H \exp\{1/2\delta(\sum_{j=1}^{\infty}\langle\psi_j, x\rangle^2 + \sum_{j=1}^{\ell}\langle\psi_j', x\rangle^2)\}\mu(dx)$$

$$= \lim_n\{\int_H \exp\{1/2\ \delta\sum_{j=1}^{n}\langle\psi_j, x\rangle^2\}\mu(dx)\}$$

$$\times\ \{\int_H \exp\{1/2\ \delta\ \sum_{j=1}^{\ell}\langle\psi_j', x\rangle^2\}\mu(dx)\}$$

$$= \lim_n\ \{\prod_{j=1}^{n}\ \int_H \exp\ (1/2\ \delta\langle\psi_j, x\rangle^2)\mu(dx)\}$$

$$\times\ \{\prod_{j=1}^{\ell}\ \int_H \exp\ (1/2\ \delta\ \langle\psi_j', x\rangle^2)\mu(dx)\}$$

$$= \lim_n\ [\prod_{j=1}^{n}\ (1 - \delta\delta_j)^{-1/2}\ \exp\{1/2\ \delta\ \sum_{j=1}^{n}\ \langle\psi_j, m\rangle^2(1 - \delta\delta_j)^{-1}\}]$$

$$\times\ \exp\{1/2\ \delta\ \sum_{j=1}^{\ell}\ \langle\psi_j', m\rangle^2\}\ .$$

$$= [\prod_{j=1}^{\infty}(1 - \delta\delta_j)]^{-1/2}\ \exp\ [1/2\ \delta\{||\dot{m}||^2 + \sum_{j=1}^{\infty}\langle\psi_j, m\rangle^2(1 - \delta\delta_j)^{-1}\}]$$

$$= U(\delta)^{-1} < \infty\ .$$

Proof of Theorem 4(ii). Define, for every $B \in \mathcal{B}(H)$,

(3.12) $P_\delta(B) = U(\delta)\int_B \exp\{1/2\ \delta||x||^2\}\mu(dx)$,

then it is clear, from (2.14), that P_λ is a probability measure on $(H, \mathcal{B}(H))$, and, from (3.12), that $\mu \sim P_\delta$ with the R-N derivative $dP_\delta/d\mu$ equal to right side of (2.15) a.s. $[\mu]$. Let x be a fixed element of H , then, using arguments similar to the ones used in the proof of Theorem 3, it can be shown that

$$\int_H \exp\{i\langle x,y\rangle\} P_\delta(dy) = \exp\{i\langle x,\theta_\delta\rangle - 1/2 \langle x, S_\delta x\rangle\} \quad .$$

This shows that P_δ is Gaussian on H with mean θ_δ and covariance operator S_δ . Therefore, since in a separable Hilbert space the mean and the covariance operator determine the Gaussian measure uniquely (see, for example, [5, p. 399]), it follows that $P_\delta = \mu_\delta$. The proof is now complete.

REFERENCES

[1] Cambanis, S. (1973). Representation of stochastic processes of second order and linear operations, J. Math. Anal. App. 41, 603-620.

[2] Cambanis, S. (1975). The measurability of a stochastic process of second order and its linear space, Proc. Amer. Math. Soc., 47, 467-475.

[3] Halmos, P. R. (1950). Measure Theory. Van Nostrand, Princeton, N. J.

[4] Neveu, J. (1965). Mathematical Foundations of the Calculus of Probability. Holden-Day, San Francisco.

[5] Rajput, B. S. (1972). Gaussian measures on L_p spaces, $1 \leq p < \infty$, J. Multivariate Analysis 2, 382-403.

[6] Shepp, L. A. (1966). Radon-Nikodým derivatives of Gaussian measures. Ann. Math. Statist. 37, 321-354.

[7] Varberg, D. E. (1967). Equivalent Gaussian measures with a particularly simple Radon-Nikodým derivative. Ann. Math. Statist. 38, 1027-1030.

FOOTNOTES

1. This terminology is motivated by the classical theorem of Mercer, which asserts, in the present terminology, that every continuous (hence trace-class, relative to Lebesgue measure) kernel K on $[0, 1] \times [0, 1]$ admits expansion of the type given in (2.3).

2. Note that, since S_δ is a bounded, linear, nonnegative, self-adjoint and trace-class operator on H and $m_\delta \in H$, the measure μ_δ exists (see, for example, [5, p. 398]).

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: Technical Report #7906	2. GOVT ACCESSION NO.: AD90787
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Equivalent Gaussian measures whose R-N derivative is the exponential of a diagonal form.	5. TYPE OF REPORT & PERIOD COVERED: Interim
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): D.M. Chung and Balram S. Rajput	8. CONTRACT OR GRANT NUMBER(s): N00014-78-C-0468
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Department, University of Tennessee, Knoxville, TN 37916	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR 042-400
11. CONTROLLING OFFICE NAME AND ADDRESS: Statistics and Probability Program, Office of Naval Research, Arlington, VA 22217	12. REPORT DATE: December 1979
	13. NUMBER OF PAGES: 22
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: Distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Gaussian Processes, Radon-Nikodým derivative

Sigma

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A simple necessary and sufficient condition, on a trace-class kernel K, is given in order for the existence of a measurable (relative to the completed product σ-algebra) Gaussian process with covariance K. Using this result, sufficient conditions are given on the means and the covariances (relative to two equivalent (σ) Gaussian measures P and P_λ) of a process X so that the Radon-Nikodým (R-N) derivative dP_λ/dP is the exponential of the diagonal form in X. Analogues of the last two results in the set up of Hilbert space are also proved.

Sub Lambda

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)