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SET PARTITIONING BASED HEURISTICS
FOR INTERACTIVE ROUTING

by

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ABSTRACT

The set partitioning model is used as the basis for an interactive approach for solving a broad class of routing problems. A pricing mechanism is developed which can be used with a variety of methods in generating improving solutions. A version of the approach has been implemented via a colorgraphics display.

## 1. INTRODUCTION

We will consider here a set partitioning based approach for
solving a broad class of routing problems. The approach is designed
to take advantage of a high level of human interaction; the
current implementation is interactive via a colorgraphics display.
However, many of the concepts discussed here could be easily implemented
in an automatic system.

The routing problem which motivated much of this work is what
is called the static or subscriber dial-a-ride problem. This problem
will be discussed in detail in later sections. It is one of the more
complex members of the class of routing problems which are amenable to
the approach presented here. This class also includes many practical
delivery problems. In order to introduce the underlying methodology
which provides the basis for the approach, consider a very
simple delivery example. Assume that a depot is located at the
square box labeled D in Figure 1. From this depot a single delivery
is to be made to each of the points represented by numbered circles.
The numbers on arcs connecting the circles represents the travel dis-
tance between delivery points. Assume also that each vehicle (e.g.
truck) can deliver to a maximum of two points on a single trip. The
objective is to determine which vehicle should deliver to each point
and the routing for the vehicles which minimizes the total distance
travelled.

Each column in the matrix of Table 1 represents one possible vehicle
route. For example, column one represents a vehicle travelling from the
depot to delivery point (1) and returning. The $c_j$ row indicates the
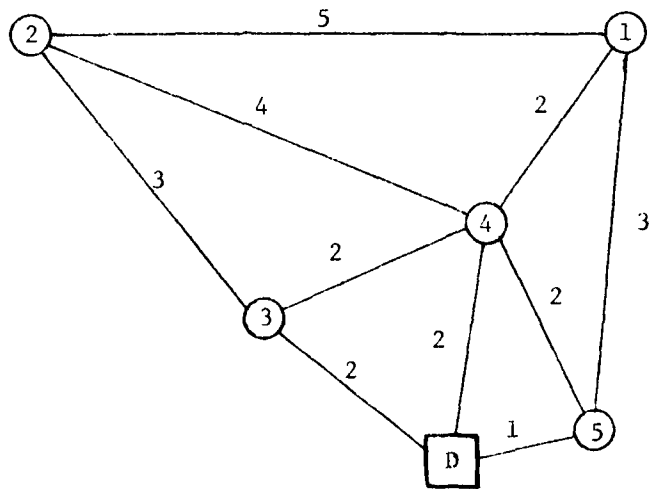
Figure 1.   Delivery Example Network

| route | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $c_j$ | 8 | 10 | 4 | 4 | 2 | 14 | 10 | 8 | 8 | 10 | 11 | 12 | 6 | 6 | 5 |
|  | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
|  | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
|  | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
|  | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
|  | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |

Table 1.   Matrix Corresponding to Routes in the Delivery

Example of Figure 1.

distance traveled for each trip. For example, column six represents a vehicle proceeding from the depot to delivery point (1), on to delivery point (2) and from there, back to the depot. The value of $c_6$ is 14, the total distance traveled for this trip. By enumerating each of the possibilities, as has been done for this matrix, the problem becomes one of selecting a set of columns such that every row is represented in exactly one column and the sum of the costs of the columns selected is the smallest possible. This integer program is called a "set partitioning model".

The set partitioning model was originally proposed for the delivery problem by Balinski and Quandt [2]. The model is very powerful in the sense that many realistic route constraints and cost functions can be handled easily in the column enumeration process. The obvious shortcoming of the model is that there are typically a very large number of columns to be enumerated and the resulting integer program is very large. The approach presented here is heuristic in the sense that we generate only a subset of the possible columns or routes and in general we do not solve the set partitioning model to optimality.

The set partitioning model has two very desirable features for interactive optimization. The first is that any route generated can be included as a column in the model. This allows the human interactor to utilize his/her intuition and spatial preception as well as a wide spectrum of mathematical techniques to generate new routes. The second feature is that, unlike more general integer programs, a feasible solution to the set partitioning model provides the basis for pricing

information which can be used to generate new candidate columns.

We will restrict the class of routing problems considered here to be those for which any subroute of a feasible route is also a feasible route with cost less than or equal to the cost of the original route. By imposing this restriction, as long as there is at least one 1 in every row of the partial set partitioning model that we have enumerated, we can easily generate feasible partitions. The only other restriction that we put on the class of routing problems is that we be able to pose them in a natural way as set partitioning problems. However, it should be noted that if there is not a nice spatial representation of the routing problem, the human interactor is much more restricted in his/her contribution.

## 2. SET PARTITIONING AND ROW PRICES

The set partitioning problem can be stated as

$$\text{minimize} \quad Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = 1 \quad \text{for } i = 1, 2, \ldots, m$$

$$x_j = 0 \text{ or } 1 \quad \text{for } j = 1, 2, \ldots, n.$$

For the delivery example in Table 1, the set partitioning model has the second row as the value of the $c_j$ and rows three through seven as the values of the $a_{ij}$. The variables which are set to one in a solution to the set partitioning problem will be called a "partition." We will denote a partition as $J^k = \{j \mid x_j^k = 1\}$. Balas and Padberg [1] provide a recent survey of results related to set partitioning problems.

A fundamental idea underlying much of the work presented here is the concept of "row prices."

Definition: $P^1 = (p_1^1, p_2^1, \ldots, p_m^1)$ is a set of feasible row prices corresponding to the partition $J^1$ if

$$\sum_{i=1}^{m} p_i^1 a_{ij} = c_j \quad \text{for } j \epsilon J^1.$$

It will be useful to interpret the price $p_i^1$ as an estimate of the cost to satisfy constraint i using solution $X^1$. For the delivery problem, $p_i^1$ is then an estimate of the cost of satisfying the requirement of delivery point i using the route corresponding to partition $J^1$.

Theorem 1: Given a set of feasible row prices $(p_1^1, p_2^1, \ldots, p_m^1)$ corresponding to partition $J^1$ with value $Z^1$, any other partition $J^2$ has value

$$z^2 = z^1 - \sum_{j \in J^2} \sum_{i=1}^{m} (p_i^1 a_{ij} - c_j)$$

Proof:

$$\sum_{j \in J^2} (\sum_{i=1}^{m} p_i^1 a_{ij} - c_j) = \sum_{i=1}^{m} p_i^1 \sum_{j \in J^2} a_{ij} - z^2$$

Since $J^2$ is a partition, $\sum_{j \in J^2} a_{ij} = 1$ for each $i = 1, 2, \ldots, m$. Also,

since $(p_1^1, p_2^1, \ldots, p_m^1)$ are feasible row prices corresponding to $X^1$ we have $\sum_{i=1}^{m} p_i^1 = z^1$. Hence the result follows.

Corrollary 1: For any set of feasible row prices $P^1$ corresponding to a partition $J^1$ if $\sum_{i=1}^{m} (p_i^1 a_{ij} - c_j) \leq 0$ for $j = 1, 2, \ldots, n$ then $X^1$ is optimum.

It can also be shown, using linear programming duality, that a set of feasible row prices $P^1$ satisfying Corrollary 1 exists if and only if $X^1$ is an optimum solution with the constraints $x_j = 0$ or 1 replaced by $x_j \geq 0$ for $j = 1, 2, \ldots, n$.

The quantity $\sum\limits_{i=1}^{m} p_i^1 a_{ij} - c_j$ will be interpreted as the "potential"

savings over the value of $Z^1$ which can result from constructing a partition that includes column j. Note from Theorem 1 that the potential savings can actually be achieved only if a partition can be constructed from columns with nonnegative potential savings.

## 3. POTENTIAL SAVINGS HEURISTIC

Given a partition $J^1$ a corresponding set of feasible row prices $P^1$, an attractive heuristic for attempting to generate a better partition is the following:

Step (0): Let $J^2 = \emptyset$ ($J^2$ will be the indices of columns in the new partition) and $N = \{1,2,\ldots,n\}$, (N will be the indices of columns which are candidate for inclusion in $J^2$)

Step (1): Calculate the potential savings $\sum_{i=1}^{m} p^1 a_{ij} - c_j$ for $j = 1,2,\ldots,n$.

Step (2): Determine $k \in N$ such that $\sum_{i=1}^{m} p_i^1 a_{ik} - c_k \geq \sum_{i=1}^{m} p_i^1 a_{ij} - c_j$ For all $j \in N$ (i.e., pick the column in N with the largest potential savings)

Step (3): If $\sum_{j \in J^2} a_{ij} = 1$ then set $a_{ik} = 0$. (Note from the assumption of section 1 that any subroute of a feasible route is also a feasible route the new column is legitimate)

Step (4): Let $J^2 = J^2 \cup \{k\}$ (i.e., put column k in the new partition)

Step (5): Delete from N all j for which $a_{ik} = 1$ and $a_{ij} = 1$ for some $i = 1,2,\ldots,m$.

Step (6): If $N = \emptyset$ stop. Otherwise go to step (2).

Note that under the assumption that any subset of a route is also a feasible route (discussed in section 1) this procedure will always terminate with $J^2$ as a partition although not necessarily a better partition than $J^1$. In addition, computation to date indicates that an optimum or near optimum solution to the set partitioning problem is determined very quickly by repeated application of the

potential savings heuristic. The heuristic is repeated until either optimality is proven (i.e., for some $X^k$ all $\sum_{i=1}^{m} p_i a_{ij} - c_j \leq 0$) or until some specified number of partitions has been generated.

To illustrate the potential savings hueristic, consider again the delivery example depicted in Figure 1 and Table 1. Suppose that we select $J^1 = \{1,2,\ldots,5\}$ as an initial partition. A set of feasible row prices $P^1$ is given in Table 2. (The question of how to generate "good" feasible row prices will be addressed in the next section.) The corresponding potential savings $\sum_{i=1}^{m} p_i^1 a_{ij} - c_j$ are also given in Table 2. Applying the potential savings heuristic and breaking ties by selecting the column with the lowest index yields the new partition $J^2 = \{6,13,5\}$. The new partition has a cost of $Z^2 = 22$ as compared to a cost $Z^1 = 28$ for the initial solution.

Using the row prices $P^2$ and potential savings $\sum_{i=1}^{m} p_i^2 a_{ij} - c_j$ of Table 2 and reapplying the potential savings heuristic yields the partition $J^3 = \{8,10,5\}$ which has a cost of $Z^3 = 20$. Again, from Table 2 we find that using the row prices $P^3$ given potential savings $\sum_{i=1}^{m} p_i^3 a_{ij} - c_j \leq 0$ for $j = 1,2,\ldots,n$. Hence, from Corrollary 1 the partition $J^3$ is optimum.

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | $p^1$ | $p^2$ | $p^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_j$ | 8 | 10 | 4 | 4 | 2 | 14 | 10 | 8 | 8 | 10 | 11 | 12 | 6 | 6 | 5 | | | |
| | 1 | | | | | 1 | 1 | 1 | 1 | | | | | | | 8.0 | 6.2 | 5.3 |
| | | 1 | | | | 1 | | | | 1 | 1 | 1 | | | | 10.0 | 7.8 | 7.1 |
| | | | 1 | | | | 1 | | | 1 | | | 1 | 1 | | 4.0 | 3.0 | 2.9 |
| | | | | 1 | | | | 1 | | | 1 | | 1 | | 1 | 4.0 | 3.0 | 2.7 |
| | | | | | 1 | | | | 1 | | | 1 | | 1 | 1 | 2.0 | 2.0 | 2.0 |
| $\sum_{i=1}^{m} p_i^1 a_{ij} - c_j$ | 0 | 0 | 0 | 0 | 0 | 4 | 2 | 4 | 2 | 4 | 3 | 0 | 2 | 0 | 1 | | | |
| $\sum_{i=1}^{m} p_i^2 a_{ij} - c_j$ | -1.8 | -2.2 | -1 | -1 | 0 | 0 | -.8 | 1.2 | .2 | .8 | -.2 | -2.2 | 0 | -1 | 0 | | | |
| $\sum_{i=1}^{m} p_i^3 a_{ij} - c_j$ | -2.7 | -2.9 | -1.1 | -1.3 | 0 | -1.6 | -1.8 | 0 | -.7 | 0 | -1.2 | -2.9 | -.4 | -1.1 | -.3 | | | |

Table 2. An Example Illustrating the Potential Saving Heuristic

with $J^1 = \{1,2,3,4,5,\}$, $J^2 = \{6,13,5\}$, and $J^3 = \{8,10,5\}$

## 4. ROW PRICING

For a given partition $X^k$ a set of feasible row prices is obtained by allocating the column cost $c_j$ for each $j \epsilon J^k$ among the rows having $a_{ij} = 1$. For the delivery example, this corresponds to allocating the trip cost among the delivery points of the trip. When a column $j \epsilon J^k$ contains only one $a_{ij} = 1$, the row price is $p_i^k = c_j$. However, when a column $j$ contains more than one $a_{ij} = 1$, there are an infinite number of possible sets of prices. As an example consider the partition $J^3 = \{8,10,5\}$ for the problem in Table 2. Since column 5 has only $a_{5,5} = 1$ the value $p_5^3 = 2$ is unique. Column 8 has both $a_{1,8} = 1$ and $a_{4,8} = 1$, hence the cost $c_8 = 8$ could be allocated between rows 1 and 4 in an infinite number of ways. Similiarly, column 10 has both $a_{2,10} = 1$ and $a_{3,10} = 1$, hence $c_{10} = 10$ could be allocated between rows 2 and 3 in an infinite number of ways. If we allocate $c_8$ as $p_1^3 = 4$ and $p_4^3 = 4$ and allocate $c_{10}$ as $p_2^3 = 5$ and $p_3^3 = 5$, the resulting $\sum_{i=1}^{m} p_i^3 a_{ij} - c_j$ do not indicate that $J^3$ is an optimum partition. Hence the set of prices $P^3$ given in Table 2 are clearly better since they do indicate that $J^3$ is on optimum partition.

Ideally, we would like a set of prices which would drive the potential savings heuristic toward an improving solution and would indicate optimality when no improving solution is possible (i.e., We would like the prices to be analogous to dual variables in linear programming). Unfortunately, it is easy to construct cases for which no such prices exist (i.e., any problem for which the integer and continuous solution differ). For the delivery problem and the more

complex dial-a-ride problem (to be discussed later), allocating column
cost in proportion to the cost of serving the delivery points one-at-
a-time is intuitively appealing and seems to work very well. As an
illustration consider again the partition $J^3$ in Table 2. Column 8
has $a_{1,8} = 1$ and $a_{4,8} = 1$. The cost of serving delivery point 1 if
it is the only point in a trip is $c_1 = 1$. The cost of serving delivery
point 4 if it is the only point in a trip is $c_4 = 4$. The prices for
rows 1 and 4 were determined as $p_1^3 = \dfrac{c_1 c_8}{c_1 + c_4} = 5.3$ and $p_4^3 = \dfrac{c_4 c_8}{c_1 + c_4}$
= 2.7. The other prices in Table 2 were determined similiarly.

## 5. COLUMN GENERATION

For large scheduling and routing problems it is generally not
practical to generate all columns of the corresponding set partitioning
model. The remainder of this paper will be concerned with using in-
formation gleaned from one solution via Theorem 1 to generate a new
and hopefully better solution. This is accomplished by either generating
new columns, adding them to the current set partitioning model, and
then resolving the model or by using the information from Theorem 1
directly to generate a new solution. In the latter case it is not
necessary to retain the columns of the set partitioning model. How-
ever, if the columns are retained, it is possible to further improve
the solution by periodically solving the set partitioning model.

We should note that for the class of scheduling and routing prob-
lems being considered here, it is very easy to generate an initial
solution. For our examples we use the identity solution (e.g., in
the delivery problem this is the solution which has each vehicle making
a single delivery) as our initial solution. However, any feasible
solution could be used as the initial solution.

Clearly there is a broad spectrum of possible approaches that
one might use to generate new columns and/or new solutions to the
set partitioning model. In fact, variations of many of the heuristics
which have been applied to delivery problems can be used very effectively
in conjunction with Theorem 1. In the next section we will discuss
the use of the Clarke and Wright [3] savings procedure in conjunction
with the delivery problem. In later sections we will discuss more
complex clustering and chaining heuristics as we have applied them
to the dial-a-ride problem.

## 6. CLARKE AND WRIGHT PROCEDURE WITH PRICING

The "savings" heuristic of Clarke and Wright [3] is the most widely known of the heuristics developed to date for delivery problems. The algorithm proceeds by calculating a savings for each pair of delivery points i and j defined as

$$s_0(i,j) = 2d_{0i} + 2d_{j0} - (d_{0i} + d_{ij} + d_{j0}) = d_{0i} + d_{j0} - d_{ij}$$

which is the savings in mileage of supplying delivery points i and j on the same route as opposed to supplying them individually directly from the depot ($d_{0i}$ is the distance from the depot to delivery point i). Routes are then constructed either one-at-a-time or in parallel by considering pairs of points in order of decreasing savings and including them in the same route if such a route is feasible.

Suppose that we consider once more the delivery example in Figure 1 and the covering solution information in Table 2. Note that the Clarke and Wright savings values are exactly the values of $\sum_{i=1}^{m} p_i^1 a_{ij} - c_j$ in Table 2 since $d_{0i} = p_i^1/2$ for $i = 1,2,\ldots,m$. Applying the C-W savings algorithm yields the same routing configuration as $J^2 = \{6,13,5\}$. At this point, the C-W algorithm would terminate. However, suppose that we set $d_{0i} = p_i^2/2$ for $i = 1,2,\ldots,m$. The new savings are then exactly the $\sum_{i=1}^{m} p_i^2 a_{ij} - c_j$ in Table 2. Applying C-W algorithm yields the same routing configuration as $J^3 = \{8,10,5\}$ which as noted previously is the optimum solution to this delivery example. Hence, for this example at least the C-W algorithm without pricing did not yield an optimum solution while the same algorithm when combined with pricing did yield the optimum solution.

Note that when routes are allowed to contain more than two trips, only a small subset of the set partitioning columns would be generated using this procedure. Also, note that the prices and savings can be calculated without ever generating the set partitioning matrix.

When the number of points allowed in a route exceeds two, some interesting questions arise as to exactly how the algorithm should be implemented. As an illustration, suppose that in the example we allow a vehicle to deliver to at most three points rather than two. Now suppose that we start with the solution $J^3$ in Table 2. We see that the two-at-a-time potential savings are all non-positive. However, if we ignore this fact and proceed with the algorithm, we would put points 1 and 4 in the same route since their potential savings of zero is maximum. If we then consider adding a third point to this route, we find that adding point 5 has a potential savings of 2 which is maximum. Finally, we combine points 2 and 3 into a single route since this has a potential savings of zero. The resulting routes (4,15) and (2,3) has a length of 10. Therefore, it appears that we can get better information by recalculating the potential savings after each augmentation of a route. This recalculation is not done in most implementations of the C-W algorithm.

Clearly, when constructing a route containing more than two points one must decide where in the route to put each additional point. The potential savings can be determined exactly only by solving a travelling salesmen problem over each new point which is a candidate to be added to the route. This is computationally expensive if a route can contain a large number of points. In most implementations of the C-W algorithm, new points are simply added on to the end of the route being constructed. When the

algorithm is implemented interactively using computer graphics, it appears that the human can perform an important role both in selecting candidate points and in inserting them logically into routes.

## 7. LOCATION - ALLOCATION WITH PRICING

Another procedure which Krolak and Nelson [5] have found effective in approaching the delivery problem utilizes the location - allocation model of Cooper [4]. This basic concept can also be used in conjunction with Theorem 1 to generate intuitively appealing columns to add to the set partitioning model.

To illustrate the procedure consider the delivery example illustrated in Figure 2. Again the circles represent delivery points and the square represents the depot. The basic idea is to use a surrogate distance rather than the actual distance in determining the points which are assigned to each vehicle. The surrogate distance is obtained by assuming that the vehicle travels from the depot to a specified cluster point, represented in Figure 2 by the dashed circles. It then makes the deliveries, returning after each delivery to the cluster point. After all deliveries have been made, the vehicle returns to the depot. Under this surrogate distance, the problem becomes one of locating the cluster points, one for each vehicle, and then assigning the delivery points to each vehicle.

If $(a_0, b_0)$ represents the coordinates of the depot, $(a_i, b_i)$ represents the coordinates of delivery point i, $(x_j, v_j)$ represents the coordinates of cluster point j, and assuming Euclidean distance, the problem can be modeled as follows:

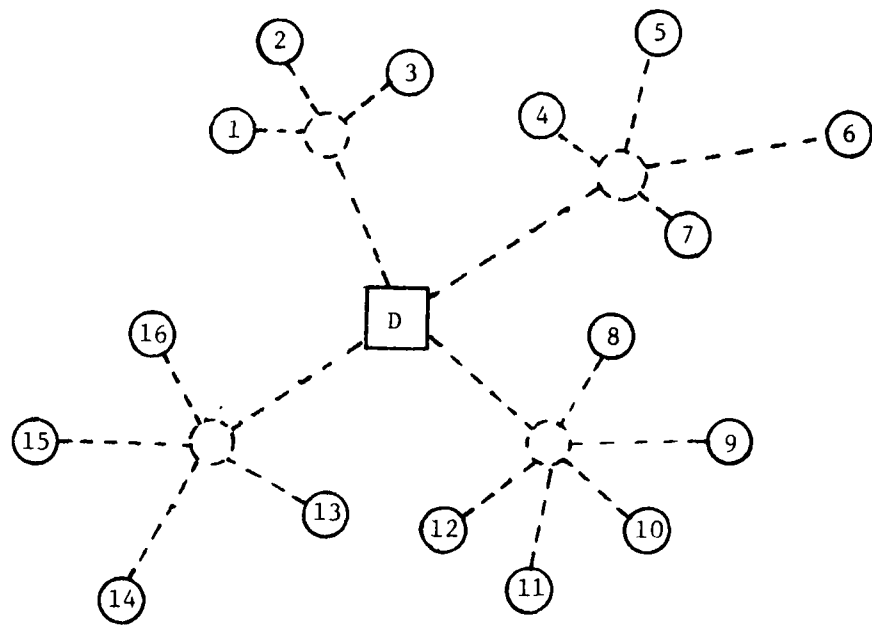$$\min_{x, v, z} \sum_{i=0}^{m} \sum_{j=1}^{n} 2[(x_j - a_i)^2 + (y_j - b_i)^2]^{1/2} z_{ij}$$

Figure 2.  Delivery Example to Illustrate the
Location – Allocation Model for Clustering.

$$\text{s.t.} \quad \sum_{i=1}^{m} z_{ij} \leq K \qquad j = 1,2,\ldots,m$$

$$\sum_{j=1}^{n} z_{ij} = 1 \qquad i = 1,2,\ldots,m$$

$$z_{ij} = 0 \text{ or } 1 \quad \text{all } i,j$$

where n is the number of vehicles and K is the vehicle capacity. When $z_{ij} = 1$, delivery point i is assigned to vehicle j and when $z_{ij} = 0$, delivery point i is not assigned to vehicle j.

While there is no method for efficiently solving this problem optimally, the following is an attractive heuristic. Pick a set of locations for the cluster points. With these coordinates fixed, solve the resulting assignment problem. With these values of $z_{ij}$ fixed, solve the resulting location problem. Continue alternating between the assignment and location problems for some specified number of iterations or until there is no further improvement in the objective. Once a cluster has been determined, the vehicle is then routed among the points of the cluster.

A slight modification of this model, together with Theorem 1, allows us to generate attractive new columns for the set partitioning problem. Suppose that we have a solution to the set partitioning problem and a set of row prices $p_1, p_2, \ldots, p_m$. Now consider the model

$$\min_{x,y,z} \sum_{i=0}^{m} \sum_{j=1}^{n} 2[(x_j - a_i)^2 + (y_j - b_i)^2]^{\frac{1}{2}} z_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{n} p_i z_{ij}$$

$$\text{s.t.} \sum_{i=1}^{m} z_{ij} \leq K \quad j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} z_{ij} \leq 1 \quad i = 1, 2, \ldots, m$$

$$z_{ij} = 0 \text{ or } 1.$$

Any cluster generated by this model will have a positive potential savings, with respect to the surrogate distance. Hence, it corresponds to an attractive column to add to the set partitioning problem (considering surrogate distances).

Since the second constraint has been changed to an inquality, not all delivery points will be assigned to cluster points. This simply means that the current row price for the delivery point is more attractive than the cost of serving the delivery point in alternative clusters considered by the model. The model can be solved using the same heuristic discussed for the earlier location - allocation model.

## 8. DIAL-A-RIDE PROBLEM

The dial-a-ride problem is a much more complex routing problem
than the delivery problem. In the dial-a-ride problem we are given
an origin-destination trip matrix and an underlying network on which
the trips are to be made. There is a single item (people, goods,
etc.) (demand for service) at each origin that needs to be transported
to its specified destination. The items are transported from origins
to destinations on vehicles each having capacity K. We wish to satisfy
the trip requirements while travelling the minimum distance.

This is a "static" version of the dial-a-ride problem since time
is not considered. There are a number of more complex versions of
this problem, but this version is sufficient to demonstrate the basic ideas
of our approach.

The set partitioning model for the dial-a-ride problem is analogus
to that of the delivery problem, but here rows represent trips rather
than simply delivery points. The vehicle capacity constraints are handled
by generating only routes which satisfy them.

## 9. DECOMPOSITION

In delivery problems, representative of the "one-ended" class of routing and scheduling problems, we need only be concerned with a single stop for a vehicle to satisfy a particular demand for service. In contrast, the "two-ended" class, which includes dial-a-ride problems, requires two stops in a specific order (sequence) to satisfy a specific demand for service. It is this requirement for sequencing of "pickup" and "dropoff" point pairs which adds greatly to the difficulty of handling the two-ended class of vehicle routing problems.

When one considers examples of two-ended problems, it becomes immediately apparent that the sequencing requirements greatly inhibit the complex pattern processing abilities of the human. Figure 3 illustrates an example of a 25 trip dial-a-ride problem. Rather than displaying order and structure, the problem resembles so much spaghetti. It is clear that for such problems the human interactor needs more help in generating good columns for the set partitioning model.

Unfortunately, it is also more difficult to apply straight forward methods such as the savings approach discussed earlier for the delivery example. In generating a dial-a-ride route one must be concerned with where both the origin and the destination occur in the sequence in order to calculate the potential savings. In addition, the capacity constraint may negate what otherwise appears to be good positions for the origin and destination in the sequence.

Because of this complexity, it is helpful to "decompose" the problem into two levels which we call "clustering" and "chaining". In essence, we consider a route to be made up of two components. Clusters
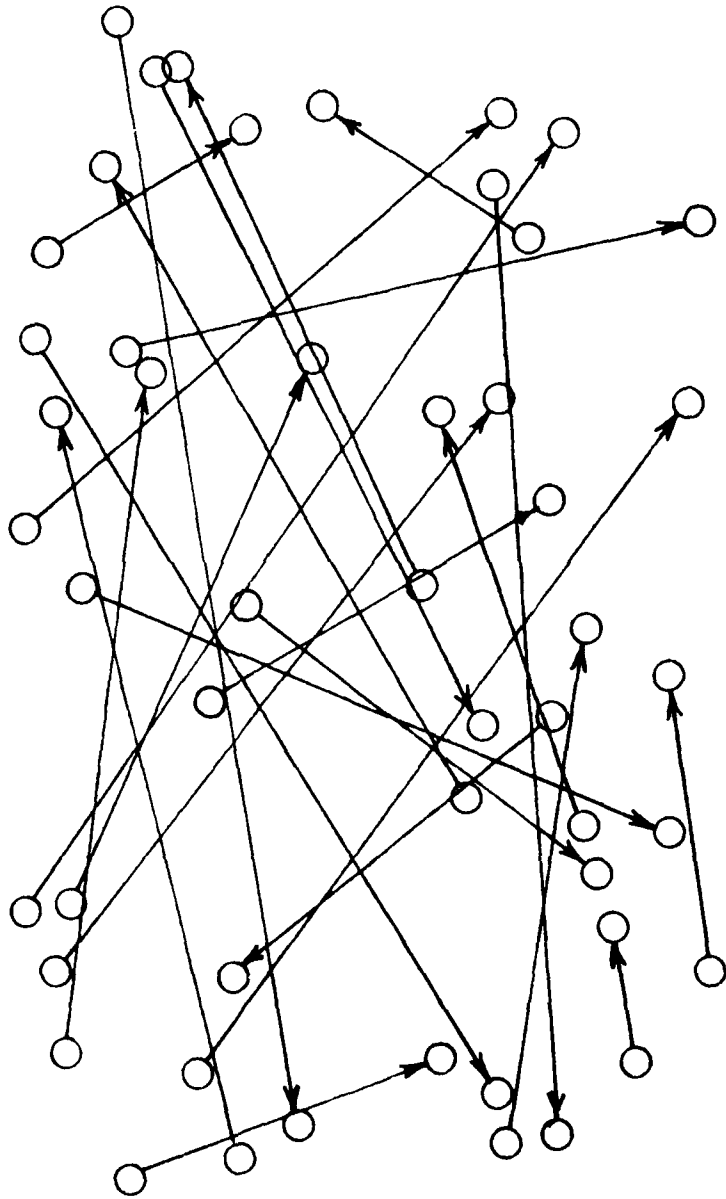
Figure 3. An Example of a 25 Trip Dial-a-Ride Problem

correspond to trips which can all be on a vehicle at one time, while chains correspond to movement from the end of one cluster to the beginning of the next. Figure 4 provides an example of five good clusters, while Figure 5 illustrates one way in which these five clusters might be linked into two chains.

The partitioning model and pricing concepts can be effectively exploited to aid in generating improving clusters and chains in a column generation approach to solving two-ended vehicle routing problems. One partitioning model can be utilized in generating improving clusters while another partitioning model helps identify better chains.

In the partitioning model for clustering, the columns represent clusters and the rows represent the individual trips (demand for service). We shall demonstrate, in the next section, just how the pricing information from the partitioning problem can be used to generate additional clusters.

In the partitioning model for chaining, the columns represent chains and the rows represent the individual trips. The function of the chaining partitioning model is to combine the clusters into good vehicle routes.

The overall solution procedure consists of linking the clustering models and chaining models together in an interactive manner. The clustering models, partitioning matrix and pricing information are used to generate good clusters. These clusters are then passed to the chaining phase where they are linked together via the chaining models, partitioning matrix and associated pricing information. After chaining, it is possible to return to the clustering phase to identify additional clusters.

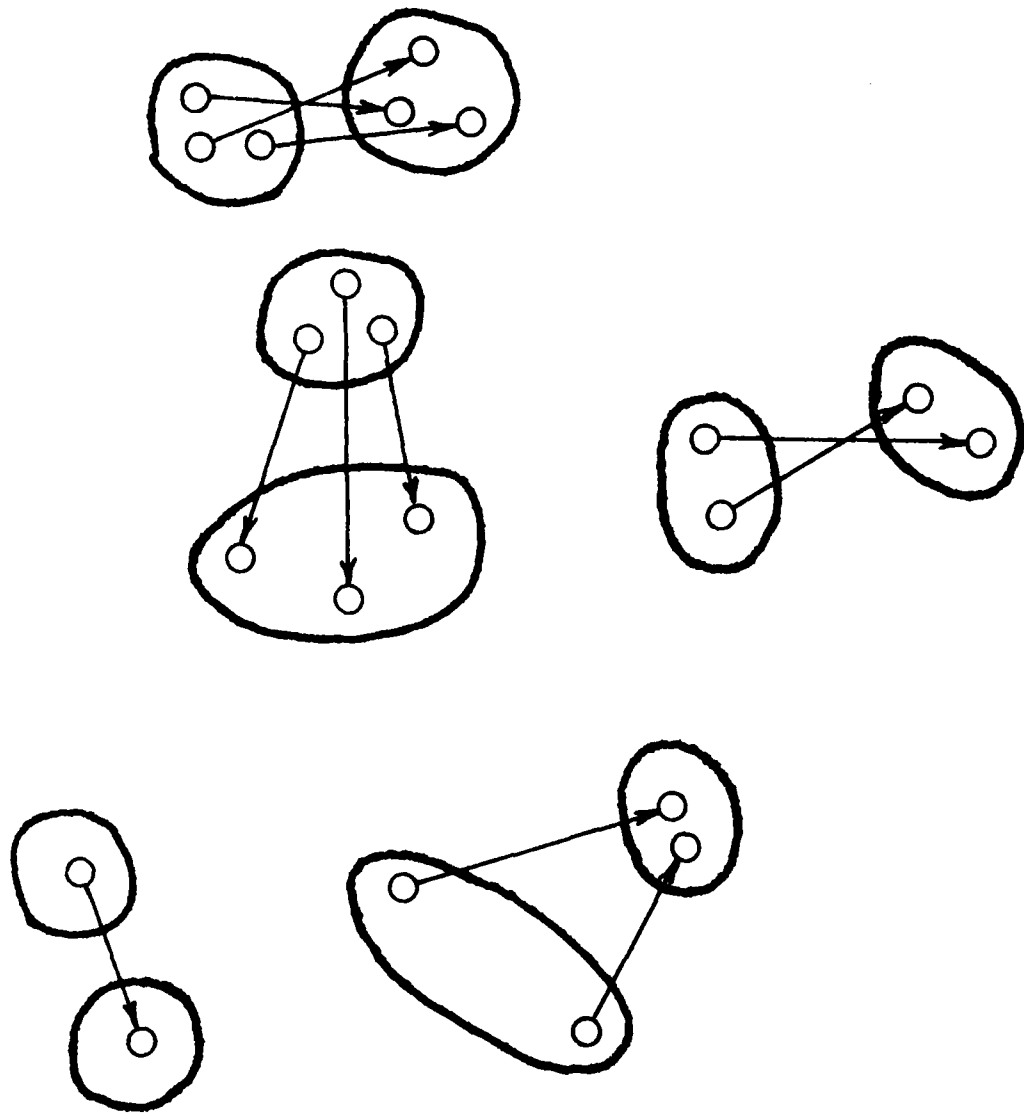The next two sections discuss the specifics of clustering and chaining
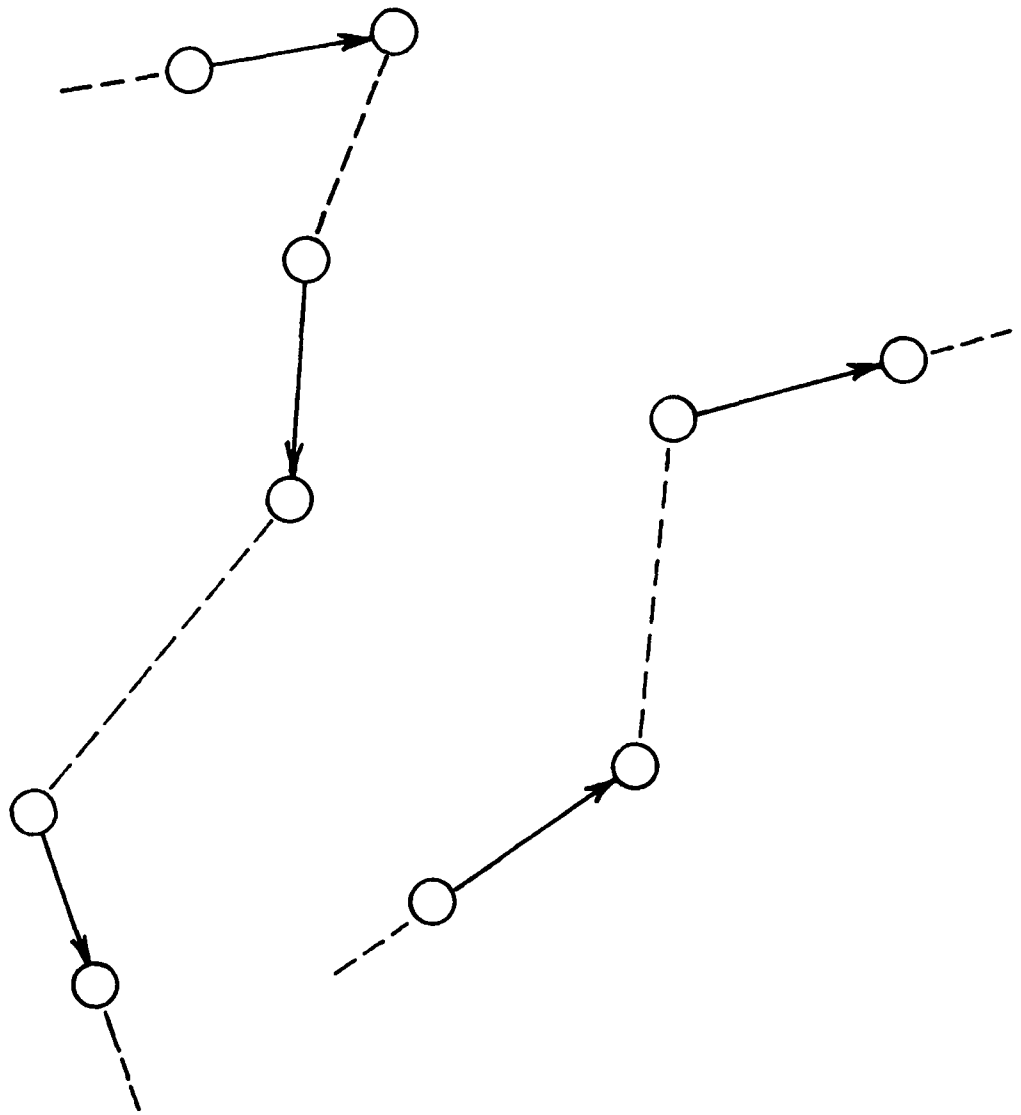
Figure 4. Example of Five Clusters

Figure 5. Example of Two Chains Linking the Five Clusters of Figure 4

## 10. CLUSTERING

In the previous section we introduced the clustering concept. In this section we shall provide more details of the concept as well as the structure and operation of various clustering models. Figure 6 depicts a typical cluster (in this case, three trips).

Clustering makes sense if the origins are reasonably close together and the destinations are also close together. One way to develop an evaluation of such circumstance is to locate the centroid of the origins, the centroid of the destinations and accumulate the resulting distances from the original trips. In Figure 6 we could evaluate the distances represented by (a+b+c) + (d+e+f). If this sum is small then it would make sense to cluster the trips.

By utilizing surrogate distances we lose the actual route distance evaluation; however, we gain the ability to evaluate large numbers of cluster possibilities conveniently and simultaneously. In Figure 6 we might employ Euclidean distances. In this case we could compare the sum of the row prices, $p_1 + p_2 + p_2$, generated in the covering model for clustering to the quantity $2(a+b+c) + 2(d+e+f) + g$ to determine whether clustering is appropriate. We can think of the latter quantity as a surrogate for the vehicle routing distance.

We can develop a straight forward extension of the Location - Allocation model discussed in Section 7 to identify good clusters for the dial-a-ride problem. In place of the quantity
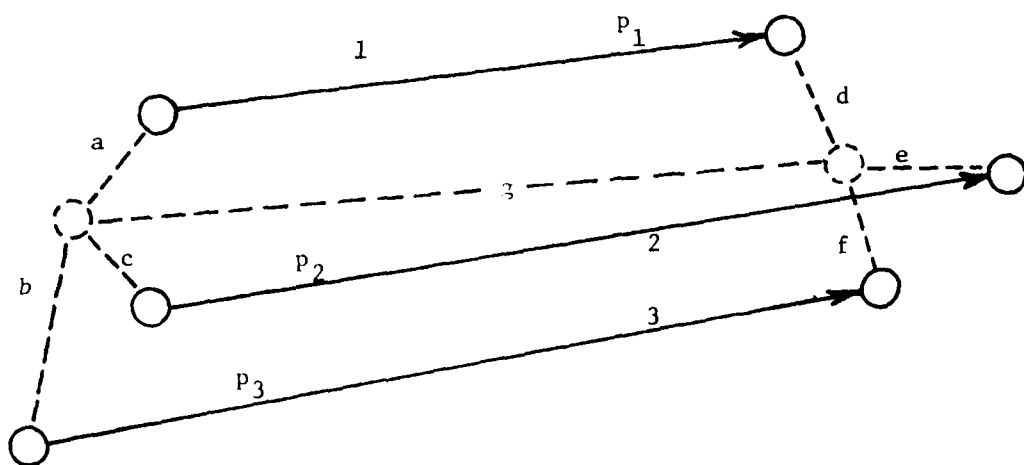
Figure 6.   Surrogate Distances for a Typical Cluster

$$\sum_i \sum_j 2[(x_j-a_i)^2 + (y_j-b_i)^2]^{\frac{1}{2}} z_{ij}$$

we would employ a quantity

$$\sum_i \sum_j 2[(\bar{x}_j-\bar{a}_i)^2 + (\bar{y}_j-\bar{b}_i)^2] z_{ij}$$

$$+ \sum_i \sum_j 2[(\hat{x}_j-\hat{a}_i)^2 + (\hat{y}_j-\hat{b}_i)^2]^{\frac{1}{2}} z_{ij}$$

where $\bar{x}_j$, $\bar{y}_j$, $\bar{a}_i$ and $\bar{b}_i$ correspond to origins and $\hat{x}_j$, $\hat{y}_j$, and $\hat{a}_i$ $\hat{b}_i$ correspond to destinations of trips.

It is also possible to develop a savings approach to clustering in a similiar manner to that described earlier for the delivery problem.

11.  CHAINING

Upon termination of the clustering process, a number of reasonably good clusters are available.  This set includes not only the "best" cluster sets as selected by the covering model for clustering, but also a number of other clusters which might be nearly as good, but which were not selected in the optimal solution to the covering model. Figure 7 illustrates a set of clusters which might result from the clustering process.  The clusters in the figure represent only the optimal solution to the clustering process.  In addition, we might have other clusters available, e.g., a cluster containing only trips 1 and 2.

The clusters obtained represent good possibilities for segments (legs) of a vehicle route.  The next step in the process is to link ("chain") these clusters into complete vehicles routes.  Figure 8 illustrates the chaining concept.  Trips 1 and 2 form one cluster, while trips 3, 4 and 5 form another cluster (see Figure 8a).  In Figure 8b, trips 1 and 2 are replaced by a single pseudo (cluster) trip, as is also the case for trips 3, 4 and 5.  These cluster trips are then chained together.

The interpretation of chaining is that a single vehicle will service the first set of trips (in Figure 8b these would be trips 1 and 2) and then proceed to service the next set of trips (i.e., trips 3, 4 and 5) in the chain.  Figure 9 illustrates the likely vehicle route to service the two clusters.

We shall demonstrate how mathematical models can be developed to aid the human in developing good chains (vehicle routes).
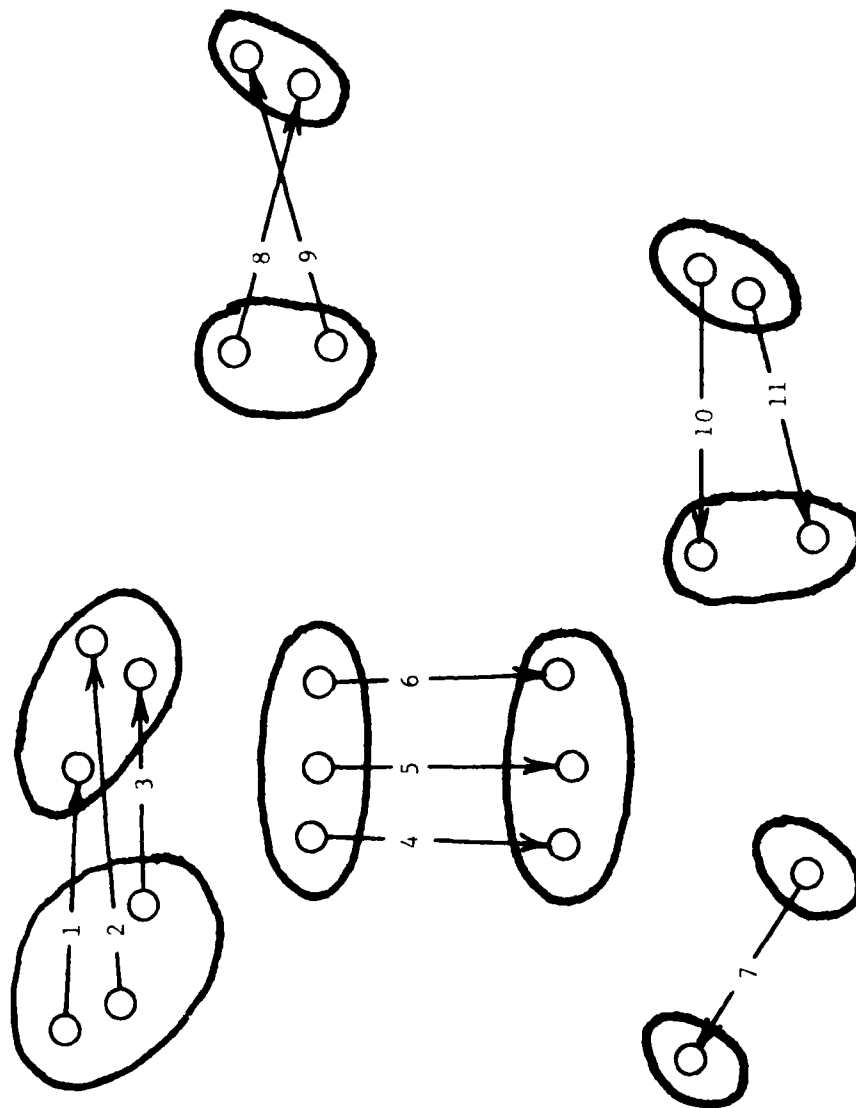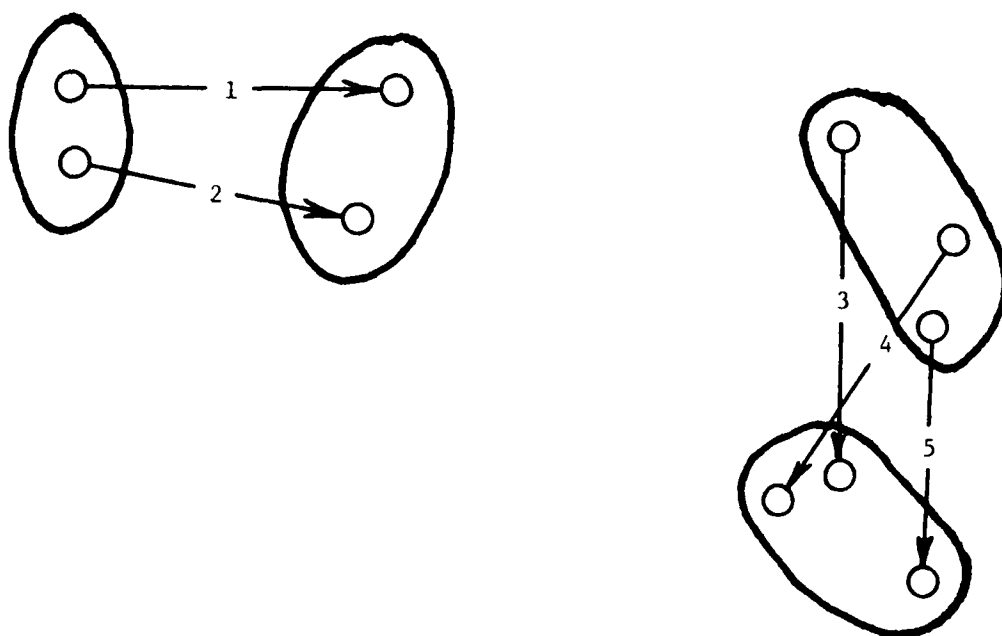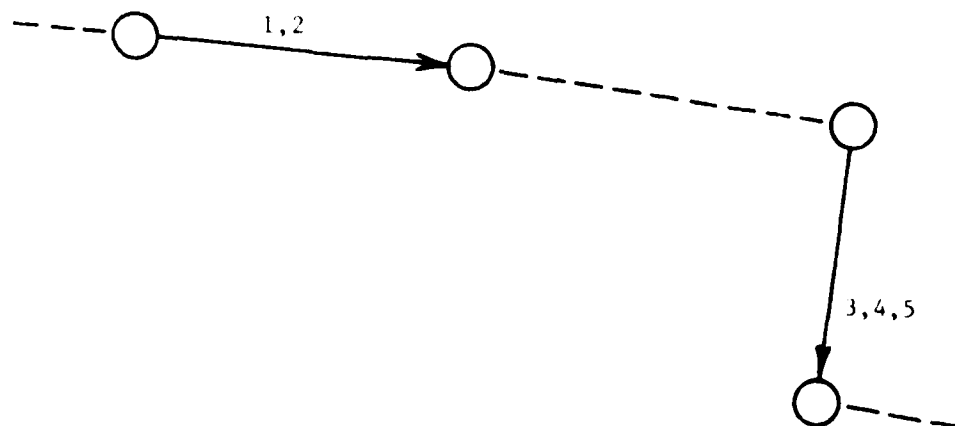
Figure 7. Example of Clusters Resulting from the Clustering Process

a.   Two Typical Clusters



b.   The Associated Cluster Arcs Chained Together

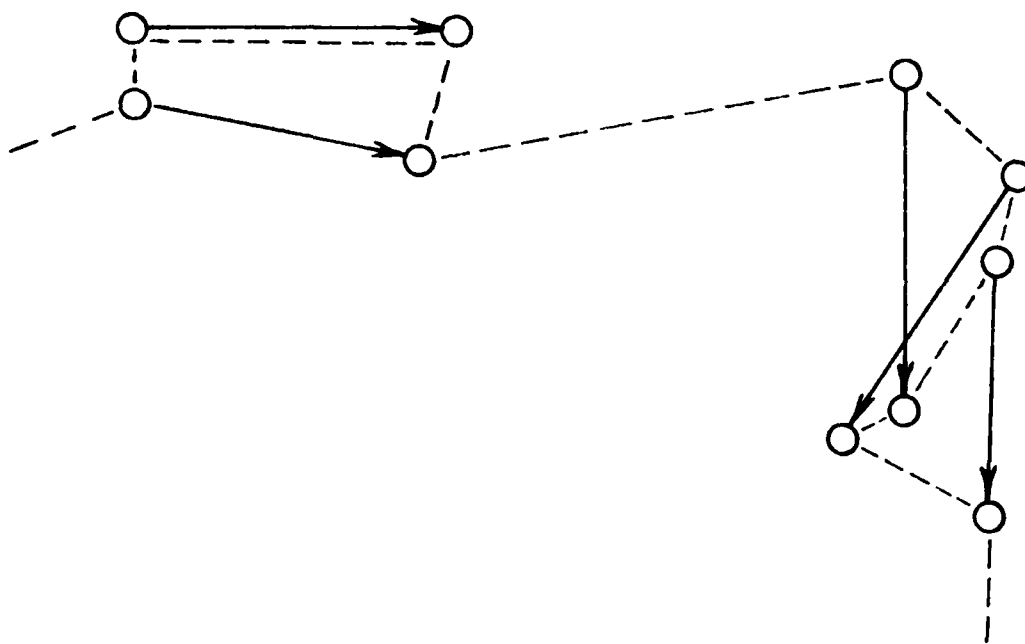Figure 8.   An Example of Chaining Clusters Together

Figure 9.    Probable Vehicle Route
for the Chain of Figure 8b

The most fundamental model of chaining is based on network flows.
In a network we represent each cluster trip by two nodes (an origin
and a destination node) and an arc. We then add additional arcs to the
network which link the destination node of one cluster arc to the origin
node of another cluster arc on the basis of proximity considerations.
In Figure 10, we indicate one network flow model for the clusters of
Figure 7.

To establish the flow variables for the network flow problem consider
the case where each trip appears in exactly one cluster. We assign every
arc an upper capacity of 1, the dashed arcs a lower capacity of 0 (zero)
and the solid cluster arcs a lower capacity of 1 indicating that these
trips must be serviced. Arc flow costs for the network flow problem
are the associated vehicle travel distances. If the "starting node"
and the "ending node" represents the vehicle storage depot, then we may
also assign additional costs to the arcs originating at the starting
node to reflect the fixed cost of using each vehicle (provided they
are all the same).

If the network flow problem for chaining contains no circuits and
if the same trip does not appear in two different clusters in the network,
then the resulting chains (vehicles routes) will be valid ones. If either
of these two conditions are explicitly included in the model, then the
underlying network flow problem becomes considerably more complicated.

We can circumvent some of the problems associated with the difficult
problems of chaining by utilizing a partitioning model of chaining. (Actually
we substitute one difficulty for another.) In a partitioning model of chaining,
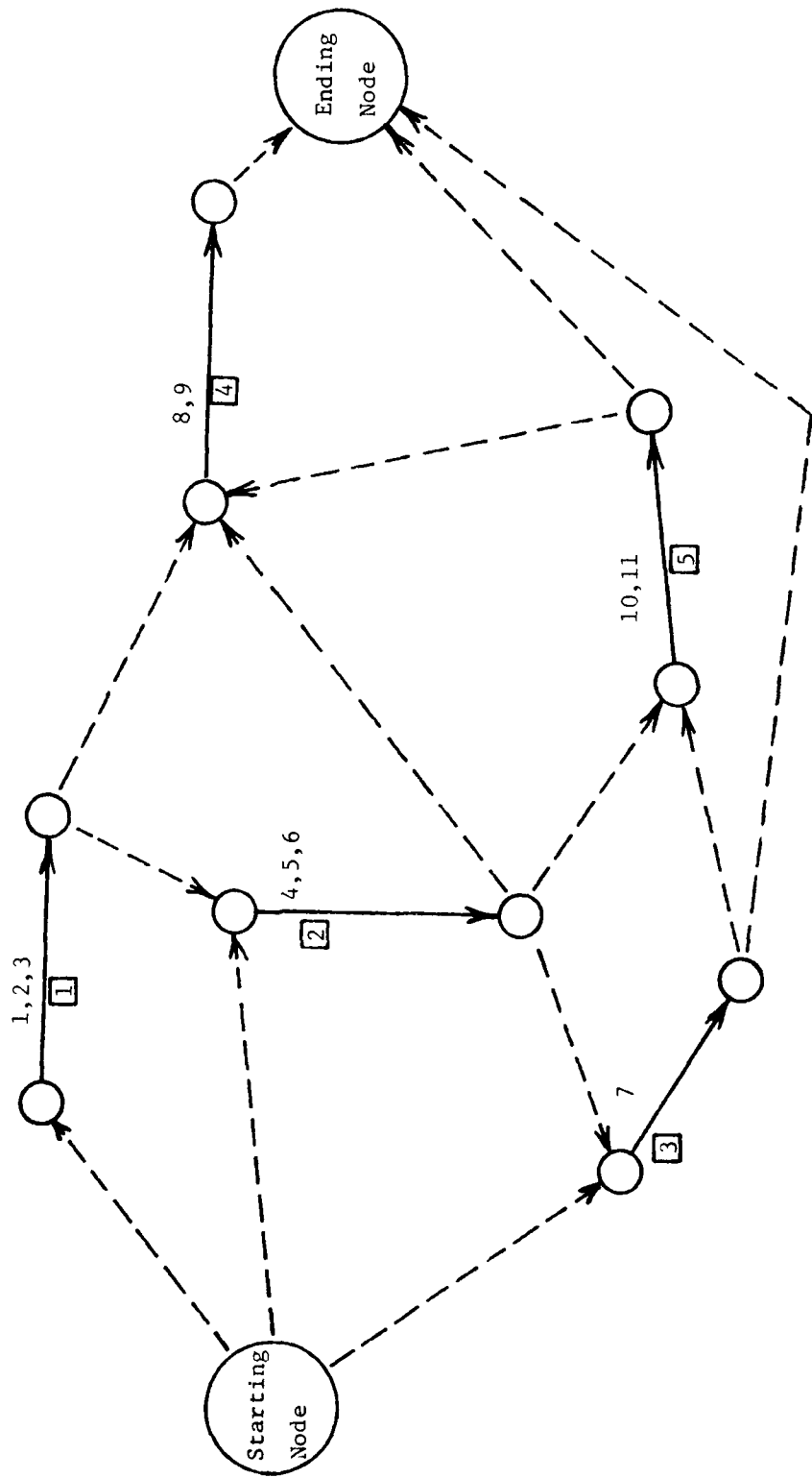
Figure 10. The Network Flow Problem for Chaining of the Clusters in Figure 7

we associate a column of the partitioning matrix for each feasible chain
(vehicle route) and a row for each trip. Column j contains 1 in row
i if trip i is serviced by chain (vehicle route) j. Otherwise, it
contains a 0 (zero). The zero-one variable associated with the
columns provide indications of which chains were selected in the optimal
partitioning solution. Table 3 illustrates a partitioning matrix
for several chains in Figure 10.

Besides solving the partitioning model optimally (which we would not
likely do), the obvious difficulty with such a model is in generating
the candidate chains (columns). To aide in this column generation
process, we could employ (1) the human (which would certainly be part
of any interactive process), (2) a savings approach to combine clusters
into chains, (3) a network flow model or (4) some other method. The
first two of these column generation approaches should be reasonably
intuitive to develop. We shall briefly describe the third one.
Suppose we have identified some candidate columns (chains) for the
partitioning problem (e.g. the individual clusters from the clustering
process). Then, the partitioning model will yield (1) an optimal
solution and (2) a set of row prices. As before, a column j not in the
solution appears favorable if

$$\sum_i p_i a_{ij} - c_j > 0.$$

| | clusters 1 & 2 | clusters 1 & 4 | clusters 2 & 5 | clusters 1, 2 & 5 | clusters 3, 4 & 5 | clusters 1, 2 & 4 | clusters 1, 2, 3 & 5 |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 1 | | 1 | | 1 | 1 |
| 2 | 1 | 1 | | 1 | | 1 | 1 |
| 3 | 1 | 1 | | 1 | | 1 | 1 |
| 4 | 1 | | 1 | 1 | | 1 | 1 |
| 5 | 1 | | 1 | 1 | | 1 | 1 |
| 6 | 1 | | 1 | 1 | | 1 | 1 |
| 7 | | | | | 1 | | 1 |
| 8 | | 1 | | | 1 | 1 | |
| 9 | | 1 | | | 1 | 1 | |
| 10 | | | 1 | 1 | 1 | | 1 |
| 11 | | | 1 | 1 | 1 | | 1 |

Table 3   An Example Covering Model
for Certain Chains in Figure 10

Now, $c_i$ is the cost (distance) of servicing the chain. The $\sum_i p_i a_{ij}$ is the cost of servicing the given trips in the current partitioning solution. Consider how these two terms might be represented in the network of Figure 10.

Suppose we associate with each dashed arc the negative of whatever costs that are incurred in traversing the arc (this is the same as before with a sign change). Associated with each clustering are the quantity $\sum p - c$, where the term $\sum p$ represents the sum of row prices for trips in the cluster and $c$ represents the cost of the cluster. With these costs defined on the network, we seek a path (or paths) in the network, from the starting to the ending node, which has positive total cost. It should be clear that such a path satisfies the condition for a potentially improving chain, and can be added to the partitioning problem.

Again, with this network flow model (a shortest path model for a single chain) we have the difficulties associated with circuits and with the same trip serviced several times by a single chain. The phenomenon of a trip being serviced several times seems to occur infrequently since in seeking a least cost solution, the model tends to avoid such an instance. When it does occur, an attractive heuristic is to simply delete one of the conflicting clusters from the column being generated. We are currently utilizing the human interactor to handle the case where the flow problem contains circuits. The human breaks the circuits and patches the paths back together. If we re-

strict the procedure to locating a single improving path each time,
the resulting shortest path model lends itself more intuitively to
the development of good heuristic procedures for solving both of the
major difficulties associated with the flow mode. In particular it
is much easier for the human to break the circuits in a single path
than in multipaths.

12.   CONCLUSIONS

An interactive dial-a-ride system based on the concepts presented
here has been implemented on a Chromatics color graphics terminal inter-
faced with a CYBER 74 mainframe computer.   Although the system is in
many ways very rudimentary, it dramatically indicates the potential
for this kind of interactive optimization.   We are in the process of
making extensive modifications in the software and are testing a
variety of new models and heuristics to aid in route generation.

## ACKNOWLEDGMENT

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