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The Pennsylvania State University
University Park, PA 16802

(10) Principal Investigator: Francis H. Fenlon
Associate Professor of Engineering Science

Assistant Investigator: Joseph E. Fahy
Graduate Assistant, Department of Engineering Science and Mechanics

(12)

Parametric Acoustic Conversion-Efficiency Enhancement via Boundary Induced and Inherent Dispersivity

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Principal Investigator
Francis H. Fenlon

Francis H. Fenlon
Associate Professor of Engineering Science

John R. Mentzer

John R. Mentzer
Professor and Head, Department of Engineering Science and Mechanics

Assistant Investigator
Joseph E. Fahy

Graduate Assistant, Department of Engineering Science and Mechanics

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Abstract

A brief summary of work completed by the authors during the period 9/1/79-8/31/80 on the investigation of dispersive wave mechanisms for enhancing the conversion efficiency of parametric acoustic arrays is presented via (i) a detailed review of the physical concepts underlying the investigation, (ii) an in-depth review of the theoretical models upon which the investigation is based, and (iii) a summary of results obtained to date, followed by a statement of continuing objectives.

List of Symbols

x, y, z, t	Rectilinear Cartesian Coordinates and time
$\phi\ (x,y,z,t)$	Velocity Potential
$p'(x,y,z,t)$	Excess Pressure
$\omega_s = s\omega_o, s = 0 \pm 1,..$	Circular Frequency (ω_o being an arbitrary reference frequency)
$\omega_{\pm} = \omega_1 \pm \omega_2$	Sum and Difference Frequency
$\phi_\omega\ (x,y,z)$	Velocity Potential Spectrum
$p'_\omega\ (x,y,z)$	Excess Pressure Spectrum
ρ_o	Density
$c_o,\ c_\infty$	Low and High Frequency Speed of Sound in a Monorelaxing Medium, respectively
γ	Nonlinear Coefficient of the Equation-of-State
$\beta = (\gamma+1)/2$	Second-Order Nonlinear Coefficient of a Fluid ($\gamma=1+\frac{B}{A}$ in liquids)
ε^s_{mn}	Eigenmodes
$k_s = \omega_s/c_o$	Wavenumber at Frequency ω_s
$\kappa^{mn}_s = \kappa^{mn}_\omega$	Transverse Wavenumber
$\chi^{mn}_s = \chi^{mn}_{\omega_s}$	Axial Wavenumber
$A^s_{mn} = A^{\omega_s}_{mn}$	Velocity Potential Spectral Amplitudes
$B^{s,s''}_{mnm'n'm''n''}$	Velocity Potential Weighting Coefficients
$P^{\omega_s}_{11} = i\rho_o\omega_s A^{\omega_s}_{11}$	Fundamental Mode Primary Wave Pressure Amplitudes
$k_{T_{\pm}} = k_1 \pm k_2 - k_{\pm}$	Net Sum or Difference-Frequency Phase in a Dispersive Medium
α_s	Attenuation Coefficient at Frequency ω_s
$\alpha_{\pm}$	Sum or Difference-Frequency Attenuation Coefficient
$\alpha_{T_{\pm}} = \alpha_1 \pm \alpha_2 - \alpha_{\pm}$	Net Attenuation Coefficient
$\alpha_o = (\alpha_1+\alpha_2)/2$	Mean Primary-Wave Attenuation Coefficient

$z_{o_{\pm}}$, z_{os}	Sum, Difference, or Arbitrary Half Rayleigh Distances, respectively
$D_-(\theta)$	Difference-Frequency Directivity Function
θ	Beam Angle relative to the direction of Wave Propagation
A_o	Projector Area normal to the direction of Wave Propagation
$m = (c_\infty^2 - c_o^2)/c_\infty^2$	Dispersivity Parameter in a Monorelaxing Fluid
τ	Relaxation time in a Monorelaxing Fluid
η, η'	Shear and Dilatational Coefficients of Viscosity, respectively
K	Coefficient of Thermal Conductivity
C_p	Specific Heat at Constant Pressure
δ	Thermo-Viscous Absorption Coefficient
k_o, k_∞	Low and High-Frequency Wavenumbers respectively in a Monorelaxing Fluid.
$\chi_s = \chi_{\omega_s}$	Axial Wavenumber at frequency ω_s in a Monorelaxing Fluid
$P^s_{mn} = P^{\omega_s}_{mn}$	Modal Pressure Amplitudes at Frequency ω_s
$C^{s,s''}_{mnm'n'm''n''}$	Pressure Field Weighting Coefficients in a Monorelaxing Fluid
w_{os}	Gaussian Beam Spot Size at Frequency ω_s
$\delta_{mn} = 1, m = n$ $= 0, m \neq n$	Kronecker Delta
u	Particle Velocity
$\varepsilon = u/c_o$	Mach Number
$r_o = A_o/\lambda_o$	Rayleigh Distance at Mear Primary frequency
$\sigma_o = \beta\, \varepsilon_o k_o r_o$ $\equiv \beta\, p_o k_o r_o/\rho_o c_o^2$	Scaled Source Level Parameter
$r =$	Range
$\sigma' = \sigma_o(r/r_o)$	Plane wave Scaled Range Parameter

1. Introduction

Parametric Amplification, as originally envisaged by Cullen[1] and by Tien and Suhl[2], concerned the transfer of energy to a weak signal of frequency ω via sinusoidal perturbation of the parameters (e.g. inductances or capacitances) of an electrical transmission line at frequency 2ω. In effect, if a weak signal of frequency ω and a strong pump wave of frequency 2ω are simultaneously present in a nonlinear transmission line, interaction occurs between them giving rise to a difference-frequency component (i.e. $2\omega - \omega = \omega$) which augments and consequently amplifies the signal of frequency ω. As subsequently shown by Roe and Boyd[3] however, the presence of nonlinearly generated harmonics greater than 2ω significantly diminishes and ultimately undermines the amplification process. For example, nonlinear generation of the sum frequency (i.e. $2\omega + \omega = 3\omega$) component gives rise via degerative interaction with the second harmonic (i.e. $2\omega - 3\omega = -\omega$) to a depletion of the amplification gain at the signal frequency. In order to ensure the efficacy of Parametric Amplification therefore, filtering circuits must be applied to the transmission line to make the system appear as a low pass filter which will block frequencies greater than the second harmonic. In the case of nonlinear electromagnetic transmission lines such filtering is readily achieved. However, in the case of nonlinear optical and acoustical parametric wave interactions which occur in bulk media the problem of minimizing the influence of degenerative coupling becomes extremely difficult to realize. For this reason the development of low noise narrowband Parametric Acoustic receivers has been severely inhibited. Likewise, the conversion efficiency of Parametric Acoustic transmitters (which constitute a generalization of the basic form of Parametric Amplification described above) as envisaged by Westervelt,[4] where a difference-frequency signal ω_- is formed via nonlinear interaction of high intensity, high frequency pump waves of frequencies ω_1 and ω_2 in a bulk medium, is severely diminished by degenerative coupling effects. In this instance, the conversion efficiency is reduced both by pump depletion via energy transfer to nonlinearly generated higher harmonics (i.e. $2n\omega_1$, $2n\omega_2$)

and by degenerative coupling between these harmonics and the upper sideband intermodulation frequency components (i.e. $n\omega_1 \pm m\omega_2$). As shown by Tjotta[5], the most significant of the latter interactions is that which occurs between the comparatively strong sum-frequency component and the secondary pump wave harmonic $2\omega_2$.

One solution to the problem of blocking those parts of a nonlinearly generated spectrum that inhibit the resonant interaction of desired spectral components in bulk media is to inhibit the amplification of unwanted frequency components via dispersive processes. Since each spectral component travels at a different phase velocity in a dispersive medium, the amplitude variation of spectral components acquire the character of spatial beats due to the accumulation of relative phase shifts during the course of propagation. As the dispersivity increases the beat periods decrease, thus reducing the peak amplitudes of the spectral components. By ensuring that unwanted components of a nonlinearly generated spectrum occur in regions of strong dispersivity which at the same time appears virtually dispersionless to the frequencies of interest, only the latter spectral components are strongly coupled, thus ensuring the possibility of significant parametric amplification. On account of the strong 'inherent' dispersivity of most optical dielectrics this effect has been successfully exploited in nonlinear optical parametric amplifiers, as described by Bloembergen.[6] Since, on the contrary, most acoustical media are weakly dispersive, very few cases of 'dispersive wave filtering' have been realized with the exception of Shiren's[7,8] beautiful experimental induction of anomalous dispersion in MgO crystals containing N_i^{++} of f_e^{++} ions via applied magnetic fields. Another, exception is the Parametric Acoustic Amplifier realized by Ostrovskii and Papilova[9,10] via boundary-induced dispersion in fluid filled rectilinear waveguides. A related, but unrealized form of boundary-induced 'dispersive wave filtering' has been discussed by Zarembo, Serdobol'skaya, and Chernobai[11,12]. In this instance, plane pump waves propagating in a simple resonator of length L are reflected from frequency dependent termination impedances such that upon reflection from the impedance Z_L at L the resulting sum-frequency phase shift is equal to π radians. Since the amplitude of the sum. frequency component is therefore zero at points 2L, 4L, etc., its growth and resulting degenerative influence will be significantly reduced. Again, the use of inhomogeneties (e.g. bubbles) to effect 'dispersive wave filtering' as analysed by Zabolotskaya and Soluyan[13] appears to virtually exhaust the extent of investigations in fluids carried out to date. In solids, Parametric Amplification based on the interaction of pump waves propagating at oblique angles with respect to each other has been investigated by Zabolotskaya, Soluyan, and Khokhlov[14-16], Lord[17], and Ivanov and Pluzhnikov[18], based on earlier theoretical work[19-22]. As described by Rudenko and Soluyan[23], the required angle for 'resonant interaction' between interacting waves is determined by the synchronism conditions (i.e. $\omega_{nm} = n\omega_1 \pm m\omega_2$, $\underline{k}_{nm} = n\underline{k}_1 + m\underline{k}_2$). For waves propagating at an angle these conditions are not satisfied at all the intermodulation frequencies. Since longitudinal and transverse waves propagate in a solid with different velocities, it is possible therefore at certain intersection angles between the pump waves to satisfy the synchronism conditions at the difference frequency. In such instances, the synchronism conditions are violated at the sum frequency and hence the latter is effectively suppressed. This in turn ensures parametric efficiency enhancement via reduced degenerative coupling interactions.

Summarizing the limitations of the above papers, it should be noted that previous investigations of parametric gain enhancement via 'dispersive wave filtering' have been peacemeal and generally restricted to particular cases of lossless plane wave propagation. They have moreover, been based on the ubiquituous assumption that the effect of dispersivity is sufficient to reduce all nonlinear coupling to that of three frequency (pump, signal, and idler) interaction process.[6,23] No attempt has been made to address the problem of how much dispersivity is required in the presence of absorption and diffraction losses to effect varying degrees of parametric amplification. Of course, this is a much more comprehensive and difficult issue because it involves the analysis, not of three, but of all significant nonlinear wave interactions subject to absorption and diffraction losses, which must inevitably be carried out via numerical methods. It is our contention however, and the purpose of the present investigation to show that no meaningful progress can be made in utilizing 'dispersive wave filtering' to enhance the conversion efficiency of parametric acoustic arrays until this matter is resolved. This will be accomplished by computing solutions of dispersive non-linear acoustic wave equations in terms of dimensionless parameters. The secondary question of how much amplification can be realized in particular dispersive media, in particular frequence bands etc., can then be deduced as a consequence of these investigations.

2. Theory

In this section we will outline the methods and procedures that will form the basis of our investigation, including some new analytical results. Although related, we wish to make a distinction between 'boundary-induced' dispersion and 'inherent' dispersion, the former being self-evident, and the latter being due to relaxation mechanisms.

It should be noted that, with the exception of Ostrovskii and Papilova's[10] work, previous investigations[24-28] of finite-amplitude wave propagation in bounded media have not been concerned with the problem of exploiting dispersivity to enhance the process of parametric amplification. Likewise, in the case of inherently dispersive media none of the previous work[29-34] has been concerned with this question, but rather with the process of soliton formation[35], which although a topic of great interest does not concern us here. We will proceed therefore, to develop an analytical procedure that links the two dispersion mechanisms under consideration. This approach is similar to that adopted by Bloembergen[6], in his investigation of nonlinear optical wave interactions in dispersive media.

2.1 Boundary-Induced Dispersivity

In order to investigate the process of enhancing the conversion efficiency of parametric acoustic interactions via boundary-induced dispersivity we begin with the lossless form of the nonlinear wave equation and subsequently introduce losses in the frequency-domain by means of complex wavenumbers. As given by Blackstock[36] this equation, in terms of the velocity potential ϕ, assumes the form

$$(\nabla^2 - c_o^{-2}\,\partial_t^2)\phi = c_o^{-2}\left\{\partial_t\,(\nabla\phi.\nabla\phi) + (\gamma - 1)\,(\partial_t\phi)\,(\nabla^2\phi)\right\} + O\,(\phi^3) \tag{1}$$

$$= c_o^{-2}\,\partial_t\left\{\nabla\phi.\nabla\phi + \left(\frac{\gamma - 1}{2\,c_o^2}\right)(\partial_t\phi)^2\right\} + O\,(\phi^3) \tag{1a}$$

where the 'substitution corollary' has been invoked in deducing Eq. (1a) from Eq. (1) (i.e. $(\partial_t\phi)(\nabla^2\phi) \simeq c_o^{-2}(\partial_t\phi)(\partial_t^2\phi) = \frac{1}{2}\ c_o^{-2}\ (\partial_t\phi)^2$).

For the case of wave propagation in a rectilinear layer, such as that depicted in Fig. 1, if $\varepsilon_{mn}^{s}(x,y)$ represents the m,n linear eigenmode of the structure at an excitation frequence ω_s, and κ_s^{mn} is the corresponding transverse wavenumber for this mode, then by definition

$$(\nabla^2 + (\kappa_s^{mn})^2)\ \varepsilon_{mn}^{s} = 0\ ,\ \nabla^2 = \partial_x^2 + \partial_y^2 \tag{2}$$

where $\quad (\kappa_s^{mn})^2 = k_s^2 - (\chi_s^{mn})^2 \quad (3)$

In this notation $k_s = \omega_s/c_o$ and χ_s^{mn} is the axial wavenumber corresponding to κ_s^{mn}.

We now express ϕ in terms of a modal expansion of the form

$$\phi(x,y,z,t) = \frac{1}{2}\sum_{s=-\infty}^{\infty}\ \sum_{m,n=-\infty}^{\infty} A_{mn}^{s}(z)\ \varepsilon_{mn}^{s}(x,y)\ e^{i(\omega_s t - \chi_s^{mn} z)} \tag{4}$$

where the unknown coefficients $A_{mn}^{s}(z)$ are to be determined by substituting Eq. (4) into Eq. (1a). When this substitution is made in the left-hand-side of Eq. (1a) we obtain via Eqs.(2) and (3)

$$\begin{aligned}(\nabla^2 - c_o^{-2}\ \partial_t^2)\phi &= \frac{1}{2}\sum_s\sum_{m,n}\left[\left\{\partial_z^2\ A_{mn}^{s} - 2i\chi_s^{mn}\ \partial_z A_{mn}^{s} - (\chi_s^{mn})^2\ A_{mn}^{s}\right\}\varepsilon_{mn}^{s}\right.\\ &\quad \left. + \left\{\nabla^2\ \varepsilon_{mn}^{s} + k_s^2\ \varepsilon_{mn}^{s}\right\}\ A_{mn}^{s}\right] e^{i(\omega_s t - \chi_s^{mn} z)}\\ &= \frac{1}{2}\sum_s\sum_{m,n}\left\{\partial_z^2\ A_{mn}^{2} - 2i\chi_s^{mn}\ \partial_z\ A_{mn}^{s} + (k_s^2 - (\kappa_s^{mn})^2 - (\chi_s^{mn})^2)A_{mn}^{s}\right\} e^{i(\omega_s t - \chi_s\ z)}\\ &= \frac{1}{2}\sum_s\sum_{m,n}\left\{\partial_z^2\ A_{mn}^{s}\ - 2i\chi_s^{mn}\ \partial_z\ A_{mn}^{s}\right\}\ e^{i(\omega_s t - \chi_s^{mn} z)}\end{aligned}$$

Invoking the "slowly varying amplitude approximate"[6], this becomes

$$(\nabla^2 - c_o^{-2}\partial_t^2)\phi \simeq -i\sum_s\sum_{m,n}\chi_s^{mn}\ \partial_z\ A_{mn}^{s}\ e^{i(\omega_s t - \chi_s^{mn} z)} \tag{5}$$

where it has been assumed that $\left|\partial_z^2 A_{mn}^s\right| \ll \left|2\chi_s^{mn}\, \partial_z A_{mn}^s\right|$. Substituting Eq. (4) in the terms on the right-hand-side of Eq. (1a) gives

$$\partial_t\,(\nabla\phi.\nabla\phi) = \frac{1}{4}\,\partial_t \sum_{s'} \sum_{s''} \sum_{m',n'} \sum_{m'',n''} A_{m'n'}^{s'}\, A_{m''n''}^{s''}\, \nabla\varepsilon_{m'n'}^{s'} \cdot \nabla\varepsilon_{m''n''}^{s''}$$

$$e^{i\left\{(\omega_{s'}+\omega_{s''})t - (\chi_{s'}^{m'n'}+\chi_{s''}^{m''n''})z\right\}}$$

$$= \frac{i}{4} \sum_s \omega_s\, e^{i\omega_s t} \sum_{s''} \sum_{m',n'} \sum_{m'',n''} A_{m'n'}^{s-s''}\, A_{m''n''}^{s''}\, \varepsilon\nabla_{m'n'} \cdot \nabla\varepsilon_{m''n''}^{s''}\, e^{-i(\chi_{s-s''}^{m'n'} + \chi_{s''}^{m''n''})z} \tag{6}$$

where it has been assumed that $\omega_s = s\omega_o$, $s = 0, \pm 1, \pm 2, \ldots, \omega_o$ being an aribrary 'reference-frequency.'

In like manner

$$\partial_t\,(\partial_t\phi)^2 = -\frac{i}{4} \sum_s \omega_s\, e^{i\omega_s t} \sum_{s''} \sum_{m'n'} \sum_{m''n''} A_{m'n'}^{s-s''}\, A_{m''n''}^{s''}\, \omega_{s-s''}\, \omega_{s''}\, \varepsilon_{m'n'}^{s-s''}\, \varepsilon_{m''n''}^{s''}$$

$$e^{-i(\chi_{s-s''}^{m'n'} + \chi_{s''}^{m''n''})z} \tag{7}$$

Hence

$$c_o^{-2}\,\partial_t \left\{\nabla\phi.\nabla\phi + \frac{\gamma-1}{2c_o^2}\,(\partial_t\phi)^2\right\} = \frac{i}{4c_o^2} \sum_s \omega_s\, e^{i\omega_s t} \sum_{s''} \sum_{m'n'} \sum_{m''n''} A_{m'n'}^{s-s'}\, A_{m''n''}^{s''}$$

$$e^{-i(\chi_{s-s}^{m'n'} + \chi_{s''}^{m''n''})z} \times \left\{\nabla\varepsilon_{m'n'}^{s-s'}\ \nabla\varepsilon_{m''n''}^{s''} - \left(\frac{\gamma-1}{2c_o^2}\right) \omega_{s-s''}\ \omega_{s''}\ \varepsilon_{m'n'}^{s-s''}\ \varepsilon_{m''n''}^{s''}\right\} \tag{8}$$

Multiplying Eqs. (5) and (8) by $\varepsilon_{mn}^s\,(x,y)$, integrating over the transverse(x,y) plane and invoking the orthonormal relation

$$\iint_{-\infty}^{\infty} dxdy\ \varepsilon_{mn}^s\,(x,y)\ \varepsilon_{m'n'}^s\,(x,y) = \delta_{mm'}\,\delta_{nn'} \tag{9}$$

gives, upon equating the results

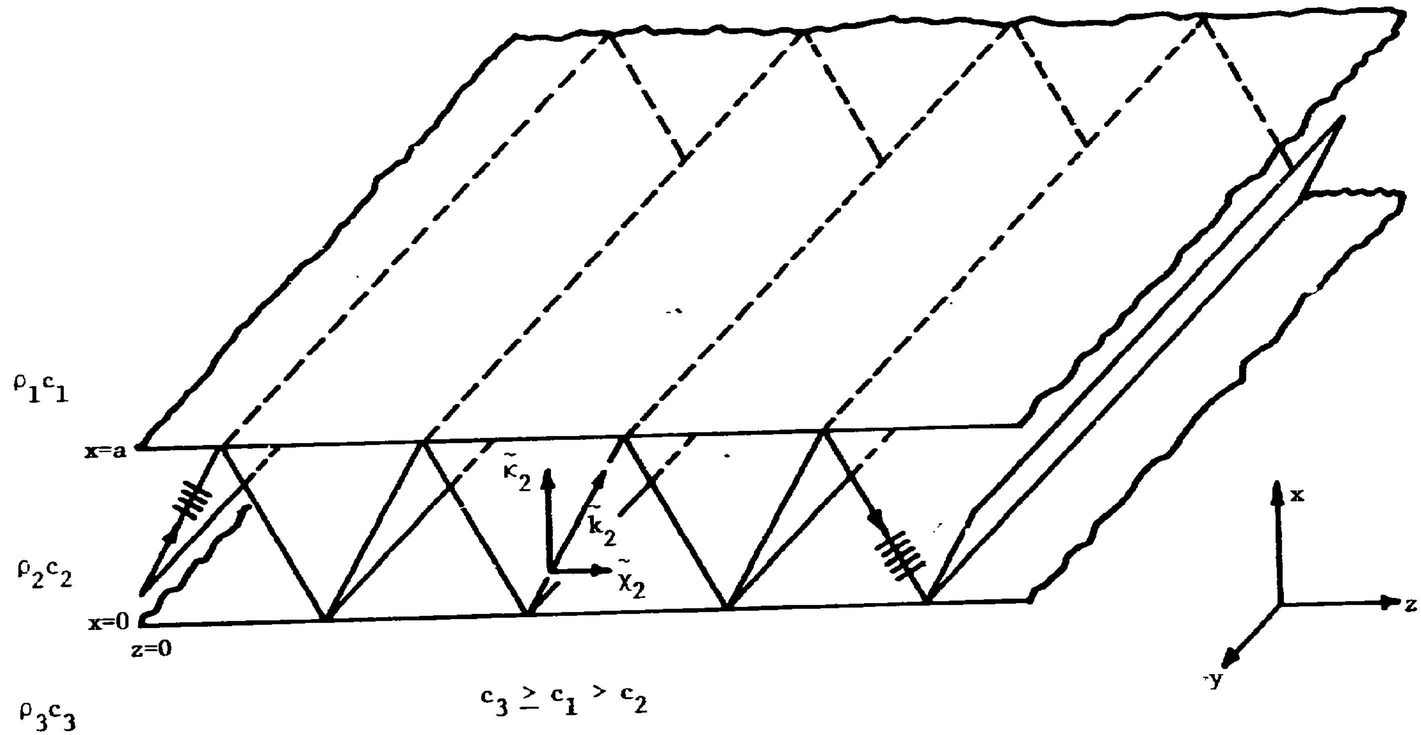


Figure 1. Propagation of plane wave trapped in medium II. Semi-infinite planes at x=0 and x=a represent fluid-fluid interface between media II and III and between II and I, respectively.

$$\frac{dA^s_{mn}}{dz} = \sum_{m',n'} \sum_{m'',n''} \left\{ \sum_{s''=1}^{s-1} B^{s,-s''}_{mnm'n'm''n''} A^{s-s''}_{m'n'} A^{s''}_{m''n''} + 2 \sum_{s=1}^{\infty} B^{s,s''}_{mnm'n'm''n''} A^{s+s''}_{m'n'} A^{s''*}_{m''n''} \right\} \quad (10)$$

$$\equiv B^{s,s''}_{mnm'n'm''n''} A^{s-s''}_{m'n'} A^{s''}_{m''n''} \text{ via the Einstein summation convention,} \quad (10a)$$

where $$B^{s,-s''}_{mnm'n'm''n''} = \frac{\omega_s}{4c_o^2 \chi^{mn}_s} e^{-i(\chi^{m'n'}_{s-s''} + \chi^{m''n''}_{s''} - \chi^{mn}_s)z}$$

$$x \iint_{-\infty}^{\infty} dxdy\, \varepsilon^s_{mn} \left\{ \nabla\varepsilon^{s-s''}_{m'n'} \cdot \nabla\varepsilon^{s''}_{m''n''} - \left(\frac{\gamma-1}{2c_o^2}\right) \omega_{s-s''}\omega_{s''}\varepsilon^{s-s''}_{m'n'}\, \varepsilon^{s''}_{m''n''} \right\} \quad (11a)$$

and $$B^{s,s''}_{mnm'n'm''n''} = \frac{\omega_s}{4c_o^2 \chi^{mn}_s} e^{-i(\chi^{m'n'}_{s+s''} - \chi^{m''n''*}_{s''} - \chi^{mn}_s)z}$$

$$x \iint_{-\infty}^{\infty} dxdy\, \varepsilon^s_{mn} \left\{ \nabla\varepsilon^{s+s''}_{m'n'} \cdot \nabla\varepsilon^{s''*}_{m''n''} + \frac{\gamma-1}{2c_o^2}\, \omega_{s+s''}\, \omega_{s''}\, \varepsilon^{s+s''}_{m'n'}\, \varepsilon^{s''*}_{m''n''} \right\} \quad (11b)$$

Once the eigenmodes ε^s_{mn} (x,y) have been specified for particular boundary conditions the $B^{s,\pm s''}_{mnm'n'm''n''}$ coefficients can readily be evaluated.

Since $p' = -\rho_o\, \partial_t \phi + O(\phi^3)$, the excess pressure spectrum p'_{ω_s} (x,y,z,) is given by Eq. (4) as $p'_{\omega_s}(x,y,z) = -i\omega_s \rho_o \phi_{\omega_s}(x,y,z)$

$$= -i\omega_s\, \rho_o \sum_{m,n} A^s_{mn}(z)\, \varepsilon^s_{mn}(x,y) e^{-i\chi_s z} \quad (12)$$

Assuming, in the case of parametric interaction between finite-amplitude primary waves of frequencies ω_1 and ω_2, that all nonlinearly generated waves other than the difference-frequency (i.e. $\omega_- = \omega_1 - \omega_2$) signal could be effectively suppressed via boundary-induced dispersion, Eq. (10) becomes

$$\frac{dA^{\omega_1}_{mn}}{dz} = 2B^{\omega_1,-\omega_-}_{mnm'n'm''n''} A^{\omega_2}_{m'n'} A^{\omega_-}_{m''n''} \quad (13a)$$

$$\frac{dA^{\omega_2}}{dz} = 2B^{\omega_2,\omega_-}_{mnm'n'} A^{\dot\omega_1}_{m'n'} A^{\omega_-*}_{m''n''} \quad (13b)$$

$$\frac{dA^{\omega_-}_{mn}}{dz} = 2B^{\omega_-,\omega_2}_{mnm'n'm''n''} A^{\omega_1}_{m'n'} A^{\omega_2 *}_{m''n''} \tag{13c}$$

where $A^{\omega_1}_{mn} \equiv A^{N_1}_{mn}$, $A^{\omega_2}_{mn} \equiv A^{N_2}_{mn}$, and $A^{\omega_-}_{mn} \equiv A^{N_-}_{mn}$.

If the lowest order primary wave modes alone are excited and $A^{\omega_1}_{11}$, $A^{\omega_2}_{11}$ are both constant, which in this instance assumes that nonlinear waveform distortion is small, or alternatively that the primary fields are only subject to absorption losses via the imaginary parts of their respective wavenumber, then Eq. (13c) becomes

$$A^{\omega_-}_{mn}(z) = 2\, A^{\omega_1}_{11} A^{\omega_2 *}_{11} \int_0^z dz'\, B^{\omega_-,\omega_2}_{mn1111}$$

$$= \frac{\omega_- A^{\omega_1}_{11} A^{\omega_2 *}_{11}}{2c_o^2 \chi^{mn}_{\omega_-}} \int_0^z dz' e^{-i(\chi^{11}_{\omega_1} - \chi^{11*}_{\omega_2} - \chi_{\omega_-})z'} \iint_{-\infty}^{\infty} dx'dy'\, \varepsilon^{\omega_-}_{mn}$$

$$\left(\nabla\varepsilon^{\omega_1}_{11} \cdot \nabla\varepsilon^{\omega_2 *}_{11} + \left(\frac{\gamma-1}{2c_o^2}\right) \omega_1\, \omega_2\, \varepsilon^{\omega_1}_{11} \varepsilon^{\omega_2 *}_{11} \right) \tag{14}$$

Substituting for $A^{\omega_-}_{mn}$ in Eq. (4) it follows that the difference-frequency pressure field is given by

$$p'_{\omega_-}(x,y,z) = \frac{-i\omega_-^2\, P^{\omega_1}_{11}\, P^{\omega_2^*}_{11}}{2\rho_o c_o^2} \sum_{m,n} \frac{e^{-i\chi^{mn}_{\omega_-} z}}{\chi^{mn}_{\omega_-}} \int_0^z dz' e^{-i(\chi^{11}_{\omega_1} - \chi^{11*}_{\omega_2} - \chi^{mn}_{\omega_-})z'}$$

$$\times \iint_{-\infty}^{\infty} dx'dy'\, \varepsilon^{\omega_-}_{mn}(x,y)\, \varepsilon^{\omega_-}_{mn}(x',y') \left\{ \nabla\varepsilon^{\omega_1}_{11}(x',y') \cdot \nabla\varepsilon^{\omega_2 *}(x',y') \right.$$

$$\left. + \left(\frac{\gamma-1}{2c_o^2}\right) \omega_1\, \omega_2\, \varepsilon^{\omega_1}_{11}(x',y')\, \varepsilon^{\omega_2 *}_{11}(x',y') \right\} \tag{15}$$

where $P_{11}^{\omega_1} = i\rho_o\omega_1 A_{11}^{\omega_1}$, and $P_{11}^{\omega_2^*} = i\rho_o\omega_2 A_{11}^{\omega_2^*}$

In order to test this weak finite-amplitude solution let us assume that the boundary surfaces of the layered medium are removed, but that the medium itself remains dispersive due to some inherent physical mechanism. The problem now corresponds to the case of plane wave parametric interaction in an inherently dispersive unbound medium. Thus, after some manipulation, Eq. (15) becomes

$$p'_{\omega_-}(x,y,z) \rightarrow \frac{-i\beta\omega_- P_{11}^{\omega_1} P_{11}^{\omega_2^*}}{2\rho_o c_o^3} e^{-i\chi_{\omega_-}^z z} \int_o^z dz' e^{-i(\chi_{\omega_1} - \chi_{\omega_2}^* - \chi_{\omega_-}^z)z'}$$

$$= -\frac{\beta\omega P_{11}^{\omega_1} P_{11}^{\omega_2^*}}{2\rho_o c_o^3} e^{-i\chi_{\omega_-}^z z} \left\{ \frac{1 - e^{-i(\chi_{\omega_1} \chi_{\omega_2}^* \chi_{\omega_-}^z)z}}{\chi_{\omega_1} - \chi_{\omega_2}^* - \chi_{\omega_-}^z} \right\} , \qquad (16)$$

where $\beta = \frac{\gamma+1}{2}$ is the nonlinear coefficient the fluid.

But
$$\chi_{\omega_1} - \chi_{\omega_2} - \chi_{\omega_-}^z = (k_1 - k_2 - k_{-z}) - i(\alpha_1 + \alpha_2 - \alpha_{-z})$$
$$= k_{T_-} - i\alpha_{T_-} \qquad (17)$$

Hence, Eq. (16) can be repressed as

$$p'_{\omega_-}(x,y,z) = \frac{i\beta\omega_- P_{11}^{\omega_1} P^{\omega_2^*}}{2\rho_o c_o^3} e^{-i\chi_{\omega}^z z} \left\{ \frac{1 - e^{-\alpha_{T-}z - ik_{T-}z}}{\alpha_{T-} + ik_{T_-}} \right\} \qquad (18a)$$

$$\rightarrow -\frac{i\beta\omega_- P_{11}^{\omega_1} P_{11}^{\omega_2^*}}{2\rho_o c_o^3} e^{-ik_{-z}z - ik_T z/2} z . \text{sinc}(k_{T_-}z/2) \text{ in a lossless dispersive medium} \qquad (18b)$$

$$\rightarrow -\frac{i\beta\omega_- P_{11}^{\omega_1} P_{11}^{\omega_2^*}}{4\alpha_o \rho_o c_o^3} e^{-i\chi_{\omega_-}^z z} \left\{ \frac{1 - e^{-\alpha_T z} \cdot e^{-i2k_- z \sin^2(\theta/2)}}{1 + i(k_-/\alpha_o)\sin^2(\theta/2)} \right\}$$

in a lossy dispersionless medium, (18c)

where $k_{T-} = k_1 - k_2 - k_- \cos\theta = 2k_- \sin^2(\theta/2)$, and $\alpha_{T_-} \simeq 2\alpha_o$ in the later case.

Eq. (18b) is the appropriate result for a weak parametric interaction in a lossless dispersive medium, being similar to that derived by Rudenko and Soluyan[23] for the second-harmonic field of an initially monotonic wave. Likewise, if the field described by Eq. (18c) is multiplied by $\frac{iz_{o-}}{z} D_{\omega_-}(\theta)$, corresponding to the case of a difference-frequency signal collimated over a Rayleigh distance $z_{o_-} = k_- A_o/2\pi$, where A_o is the area of the primary-wave projector and $D_{\omega_-}(\theta)$ is the diffraction pattern of the aperture at the difference-frequency, then in the far-field where $e^{-\alpha_T z} \cdot e^{-i2k_- z \sin^2(\theta/2)} = 0$, the resulting form of the difference-frequency signal reduces, as required, to Naze and Tjotta's[37] modification of Westervelts'[4] solution.

Although the solutions of Eqs. (13a) to (13c) considered in this section are only approximate, it should be noted that exact solutions expressed in terms of elliptic functions[38] exist for the case of constant coefficients as shown in Appendix A.

2.2 Inherent Dispersivity

In this section we deal with the case of parametric array formation via nonlinearly interacting colinear paraxial waves in inherently dispersive media. If such an interaction occurs, for example, in a monorelaxing, thermo-viscous fluid, the governing equation for the velocity potential ϕ, correct to second-order terms is given by the following modified form[39] of Eq. (1a):

$$(1 + \tau\partial_t) \left(\left([1 + 2\,\delta c_\infty \partial_t]\, \nabla^2 - c_\infty^{-2}\, \partial_t^2 \right) \phi - c_\infty^{-2}\, \partial_t \left[\nabla\phi\cdot\nabla\phi + \frac{\gamma-1}{2c_\infty^2} (\partial_t\phi)^2\right] \right)$$
$$= m c_o^{-2}\, \partial_t^2\, \phi + O(\phi^3) \qquad (19)$$

where $\delta = \frac{1}{2\rho_o c_\infty^3} \left\{ (2\eta + \eta') + (K/C_p)\,(\gamma - 1) \right\}$ is thermo-viscous coefficient, $m = (c_\infty^2 - c_o^2)/c_\infty^2$ is the dispersivity, and τ is the relaxation time. In this notation, c_o is the low frequency (i.e. $\omega\tau \ll 1$) speed-of-sound in the fluid, and c_∞ is the high-frequency (i.e. $\omega\tau \gg 1$) limit.

For the case of progressive finite-amplitude primary waves radiated in the z-direction by a single projector in an unbounded medium, previous work by Woodsum and Westervelt[40] justifies the approximation

$$\nabla\phi \cdot \nabla\phi \simeq (\partial_z\phi)^2 \approx c_\infty^{-2}(\partial_t\phi)^2$$

Substituting this approximation in Eq. (19) gives

$$(1 + \tau\partial_t)\left(\left([1 + 2\delta c_\infty\, \partial_t]\, \nabla^2 - c_\infty^{-2}\, \partial_t^2\right)\phi - \beta c_\infty^{-4}\, \partial_t(\partial_t\phi)^2\right) = mc_o^{-2}\, \partial_t^2\phi + 0(\phi^3) \tag{20}$$

where $\beta = (\gamma + 1)/2$.

Since the excess pressure $p' = -\rho_o\partial_t\phi + 0(\phi^3)$, it follows that

$$(1 + \tau\partial_t)\left(\left([1 + 2\delta c_\infty\partial_t]\nabla^2 - c_\infty^{-2}\, \partial_t^2\right) p' + \beta^{-1}\rho_o c_\infty^{-4}\, \partial_t^2\, p'^2\right) = mc_o^{-2}\partial_t^2\, p' + 0(p'^3) \tag{21}$$

$$\text{If } p'(x,y,z,t) = \frac{1}{2} \sum_{s=-\infty}^{\infty} \sum_{m,n=-\infty}^{\infty} P_{mn}^s(z)\, \varepsilon_{mn}^s\, (x,y,z)\, e^{i(\omega_s t - \chi_s z)} \tag{22}$$

$$\text{where } \left|\partial_z^2\, P_{mn}\right| << \left|2\chi_\omega\, \partial_z\, P_{mn}^s\right|$$

$$\text{and } \chi_s = k_s - i\alpha_s \tag{23}$$

$$k_s = k_\infty^2 + \frac{mk_o^2}{1+(\omega_s\tau)^2} \tag{24}$$

$$\alpha_s = \left(\delta + \frac{m\tau/2}{1+(\omega_s\tau)^2}\right)\omega_s^2 \tag{25}$$

$$\text{with } \partial_z\, \varepsilon_{mn}^s - \frac{i}{2\chi_s}\, \nabla_\perp^2\, \varepsilon_{mn}^s = 0 \tag{26}$$

then proceeding as before in section 2.1

$$\frac{dP_{mn}^s}{dz} = \sum_{m'n'} \sum_{m'',n''} \left\{ \sum_{s''=1}^{s-1} C_{mnm'n'm''n''}^{s,-s''}\, P_{m'n'}^{s-s''}\, P_{m''n''}^{s} + 2 \sum_{s''=1}^{\infty} C_{mnm'n'm''n''}^{s,s''}\, P_{m'n'}^{s+s''}\, P_{m''n''}^{s''*} \right\} \tag{27}$$

$$= C_{mnm'n'm''n''}^{s,s''}\, P_{m'n'}^{s-s''}\, P_{m''n''}^{s''} \text{ via the Einstein summation convention,} \tag{27a}$$

$$\text{where } C^{s,-s''}_{mnm'n'm''n''} = \frac{i\beta\,\omega_s^2}{4\rho_o c_\infty^4\,\chi_s}\, e^{-i(\chi_{s-s''}+\chi_{s''}-\chi_s)z} \iint_{-\infty}^{\infty} dxdy\; \varepsilon^s_{mn}\,\varepsilon^{s-s''}_{m'n'}\,\varepsilon^{s''}_{m''n''} \quad (28)$$

For the case of a square aperture radiator located in the plane $z = 0$ centered at $x = 0$, $y = 0$, the modes ε^s_{mn} are given by Kogelnik[41] and by Cook and Arnoult[42] as

$$\varepsilon^s_{mn}(x,y,z) = \frac{(1+iz/z_{os})^{(m+n)/2}}{(1-iz/z_{os})^{(m+n+2)/2}} \times \frac{e^{-\left[\frac{x^2+y^2}{w_{os}^2(1-iz/z_{os})}\right]}}{w_{os}(2^{m+n+1}\,m!n!\pi)^{1/2}} \times H_m\left(\frac{x}{w_s}\sqrt{2}\right)H_n\left(\frac{y}{w_s}\sqrt{2}\right) \quad (29)$$

where w_{os} is the 'spot size' at frequency ω_s, $w_s = w_{os}(1+z^2/z_{os}^2)^{1/2}$, and $z_{os} = k_s w_{os}^2/2$. The orthonormal relation satisfied by the modes is

$$\iint_{-\infty}^{\infty} dxdy\; \varepsilon^s_{mn}(x,y,z)\,\varepsilon^s_{m'n'}(x,y,z) = \delta_{mm'}\,\delta_{nn'}\,. \quad (30)$$

$$\text{and} \sum_{m,n=-\infty}^{\infty} \varepsilon^s_{mn}(x,y,z)\,\varepsilon^s_{mn}(x',y',z') = \frac{ik_s}{2\pi(z-z')}\, e^{-\frac{ik_s}{2}\left[\frac{(x-x')^2+(y-y')^2}{z-z'}\right]} \quad (31)$$

$$\rightarrow \delta(x-x')\,\delta(y-y'),\ \text{when } z=z' \quad (31a)$$

Assuming again, as in section 2.1, for the case of a weak parametric interaction between finite-amplitude primary waves of frequencies ω_1 and ω_2, that all non-linearly generated waves other than the difference-frequency (i.e. $\omega_- = \omega_1 - \omega_2$) signal could be effectively suppressed via inherent dispersion in the medium, Eq. (27) becomes

$$\frac{dP^{\omega_1}_{mn}}{dz} = 2\, C^{\omega_1,-\omega_-}_{mnm'n'm''n''}\, P^{\omega_2}_{m'n'}\, P^{\omega_-}_{m''n''} \quad (32a)$$

$$\frac{dP^{\omega_2}_{mn}}{dz} = 2\, C^{\omega_2,\omega_-}_{mnm'n'm''n''}\, P^{\omega_1}_{m'n'}\, P^{\omega_-*}_{m''n''} \quad (32b)$$

$$\frac{dP^{\omega_-}_{mn}}{dz} = 2\, C^{\omega_-,\omega_2}_{mnm'n'm''n''}\, P^{\omega_1}_{m'n'}\, P^{\omega_2 *}_{m''n''} \quad (32c)$$

where $P_{mn}^{\omega_1} \equiv P_{mn}^{N_1}$, $P_{mn}^{\omega_2} \equiv P_{mn}^{N_2}$, and $P_{mn}^{\omega_-} \equiv P_{mn}^{N_1-N_2}$ (33)

If, as in section 2.1, the fundamental primary wave modes alone are excited and both $P_{oo}^{\omega_1}$, $P_{oo}^{\omega_2}$ are constant, which implies that nonlinear waveform distortion is small, or alternatively, that the primary waves are only subject to viscous attenuation and spreading losses, then Eq. (32c) becomes

$$P_{mn}^{\omega_-}(z) = 2\, P_{oo}^{\omega_1} P_{oo}^{\omega_2 *} \int_0^z dz'\, C_{mnoooo}^{\omega_-,\omega_2}$$

$$\frac{i\beta\omega_-^2 P_{oo}^{\omega_1} P_{oo}^{\omega_2 *}}{2\rho_o c_\infty^4 \chi_{\omega_-}} \int_0^z dz' e^{-i(\chi_{\omega_1} - \chi_{\omega_2}^* - \chi_{\omega_-})z'} \iint_{-\infty}^{\infty} dx'dy'\, \varepsilon_{mn}^{\omega_-} \varepsilon_{oo}^{\omega_1} \varepsilon_{oo}^{\omega_2 *}$$

from Eq. (28) (34)

Taking the Fourier transform of Eq. (22) and substituting for $P_{mn}^{\omega_-}$ from Eq. (34), the difference-frequency field assumes the form

$$p_{\omega_-}(x,y,z) = e^{-i\chi_{\omega_-} z} \sum_{m,n=-\infty} P_{mn}^{\omega_-}(z)\, \varepsilon_{mn}^{\omega_-}(x,y,z)$$

$$= \frac{i\beta\omega_-^2 P_{oo}^{\omega_1} P_{oo}^{\omega_2 *}}{2\rho_o c_\infty^4 \chi_\omega} e^{-i\chi_{\omega_-} z} \sum_{m,n=-\infty}^{\infty} \varepsilon_{mn}^{\omega}(x,y,z) \int_0^z dz' e^{-i(\chi_{\omega_1} - \chi_{\omega_2}^* - \chi_{\omega_-})z}$$

$$\times \iint_{-\infty}^{\infty} dx'dy'\, \varepsilon_{mn}^{\omega_-}(x',y',z')\, \varepsilon_{oo}^{\omega_1}(x',y',z')\, \varepsilon_{oo}^{\omega_2 *}(x',y',z') \quad (35)$$

From Eq. (31) this becomes

$$p'_{\omega_-}(x,y,z) = \frac{i\beta\omega_- P_{oo}^{\omega_1} P_{oo}^{\omega_2 *}}{2\rho_o c_\infty^3} e^{-i\chi_{\omega_-} z} \int_0^z dz' \frac{ik_-}{2\pi(z-z')} e^{-i(\chi_{\omega_1} - \chi_{\omega_2}^* - \chi_{\omega_-})z'}$$

$$\times \iint_{-\infty}^{\infty} dx'dy'\, \varepsilon_{oo}^{\omega_1}(x',y',z')\, \varepsilon_{oo}^{\omega_2 *}(x',y',z')\, e^{-\frac{ik_-}{2}\left[\frac{(x-x')^2+(y-y')^2}{z-z'}\right]} \quad (36)$$

But Eq. (29) gives

$$\varepsilon_{oo}(x,y,z) = \frac{e^{-\left[\frac{x^2+y^2}{w_{os}^2(1-iz/z_{os})}\right]}}{(2\pi)^{1/2}\, w_{os}(1-iz/z_{os})} \tag{37}$$

Hence $$\varepsilon_{oo}^{\omega_1}(x',y',z')\, \varepsilon_{oo}^{\omega_2^*}(x',y',z') = \frac{e^{-(M_-/N_-)(x'^2+y'^2)}}{2\pi\, w_{01}w_{02}\, N_-} \tag{38}$$

where $$M_-(z') = (w_{01}^{-2} + w_{02}^{-2}) + i\, 2(w_{01}\, w_{02})^{-2}\, (k_2^{-1} - k_1^{-1})z' \tag{39}$$

and $$N_-(z') = 1 + i(z_{02}^{-1} - z_{01}^{-1})z' + (z_{01}\, z_{02})^{-1}\, z'^2 \tag{40}$$

Substituting Eq. (38) in Eq. (36) and making use of the identity[43]

$$\iint_{-\infty}^{\infty} dx'dy'\; e^{ia(x'^2+y'^2)-i(bx'+cy')} = \frac{i\pi}{a}\, e^{-\frac{i}{4a}(b^2+c^2)},$$

Eq. (36) becomes

$$p_{\omega_-}(x,y,z) = \frac{i\beta\omega_- P_{oo}^{\omega_1}\, P_{oo}^{\omega_2^*}}{4\pi\, w_{01}w_{02}\rho_o c_\infty^3} \times e^{-i\chi_{\omega_-} z} \;.$$

$$\int_0^z dz'\; \frac{e^{\frac{-M_-(z')(x^2+y^2)}{N_-(z')-i2M_-(z')(z-z')/k_-}}}{N_-(z')-i2M_-(z')(z-z')/k_-}\; e^{-i(\chi_{\omega_1}-\chi_{\omega_2}^*-\chi_{\omega_-})z'}$$

$$= \frac{i\beta k_- P_{oo}^{\omega_1}\, P_{oo}^{\omega_2^*}}{4\pi\, w_{01}w_{02}\rho_o c_\infty^2} \times e^{-i\chi_{\omega_-} z} \;.$$

$$\int_0^z dz'\; \frac{e^{\frac{-M_-(x^2+y^2)}{a_1^{(-)}+a_2^{(-)}z'+a_3^{(-)}z'^2}}}{a_1^{(-)}+a_2^{(-)}z'+a_3^{(-)}z'^2} \cdot e^{-i(\chi_{\omega_1}-\chi_{\omega_2}^*-\chi_{\omega_-})z'} \tag{41}$$

$$p'_+(x,y,z) = \frac{i\beta k_+ P_{oo}^{\omega_1} P_{oo}^{\omega_2}}{4\pi w_{01} w_{02} \rho_o c_\infty^2} e^{-i\chi_{\omega_+} z} \int_o^z dz \frac{e^{\frac{-M_+(x^2+y^2)}{a_1^{(+)}+a_2^{(+)}z'+a_3^{(+)}z'^2}}}{a_1^{(+)}+a_2^{(+)}z'+a_3^{(+)}z'^2}$$

$$\cdot\ e^{-i(\chi_{\omega_1}+\chi_{\omega_2}-\chi_{\omega+})z'} \tag{42}$$

where $M_+(z') = (w_{01}^{-2} + w_{02}^{-2}) - i2(w_{01}w_{02})^{-2}\ (k_2^{-1} + k_1^{-1})z'$ (43)

and $a_1^{(\pm)} = \left\{1 - \frac{i2}{k_\pm}\ (w_{01}^{-2} + w_{02}^{-2})z\right\}$ (44)

$$a_2^{(\pm)} = i\left\{\frac{2}{k_\pm}\ (w_{01}^{-2} + w_{02}^{-2}) \pm \left[\frac{(z_{01}\pm z_{02})-i\left(\frac{k_1\pm k_2}{k_\pm}\right)z}{z_{01}\ z_{02}}\right]\right.$$

$$a_3^{(\pm)} = \frac{4}{k_1 k_2 k_\pm w_{01}^2 w_{02}^2}\ \left\{k_\pm - (k_1 \pm k_2)\right\} \tag{46}$$

To the best of the authors knowledge Eqs. (41) and (42) is a new result for the propagation of a parametically generated wave in a dispersive medium. If this dispersivity is removed and $k_1 - k_2 = k_-$ the equation reverts to the form investigated by Fenlon[44-47] in his detailed analysis of near and far-field parametric acoustic array interactions. As it stands it can also be shown that in the far-field Eqs. (41) and (42) reveals a shift in the angular spectrum dependent on the amount of dispersivity which was first referred to by Novikov[30] and subsequently by Fenlon.[48] Without going into further details now however, it is our intention to fully evaluate Eqs. (41) and (42) numerically in order to determine the nature of the spreading losses so that the complete set of spectral equations may be solved along the beam axis. In this manner the effect of dispersivity in destroying unwanted spectral components will be investigated.

Finally, on other task that must be carried out is to elaborate on the formulation of Eqs.(41)and(42) by rederiving it for the case where the two primary waves are at an angle relative to each other. In this instance a resonant interaction should be possible at least in parts of the field at the frequencies of interest (note that the interaction is considerably more complicated than that due to simple plane wave fields) but less and less resonant at other frequencies in the nonlinearly generated spectrum.

3. Results and Statement of Continuing Research

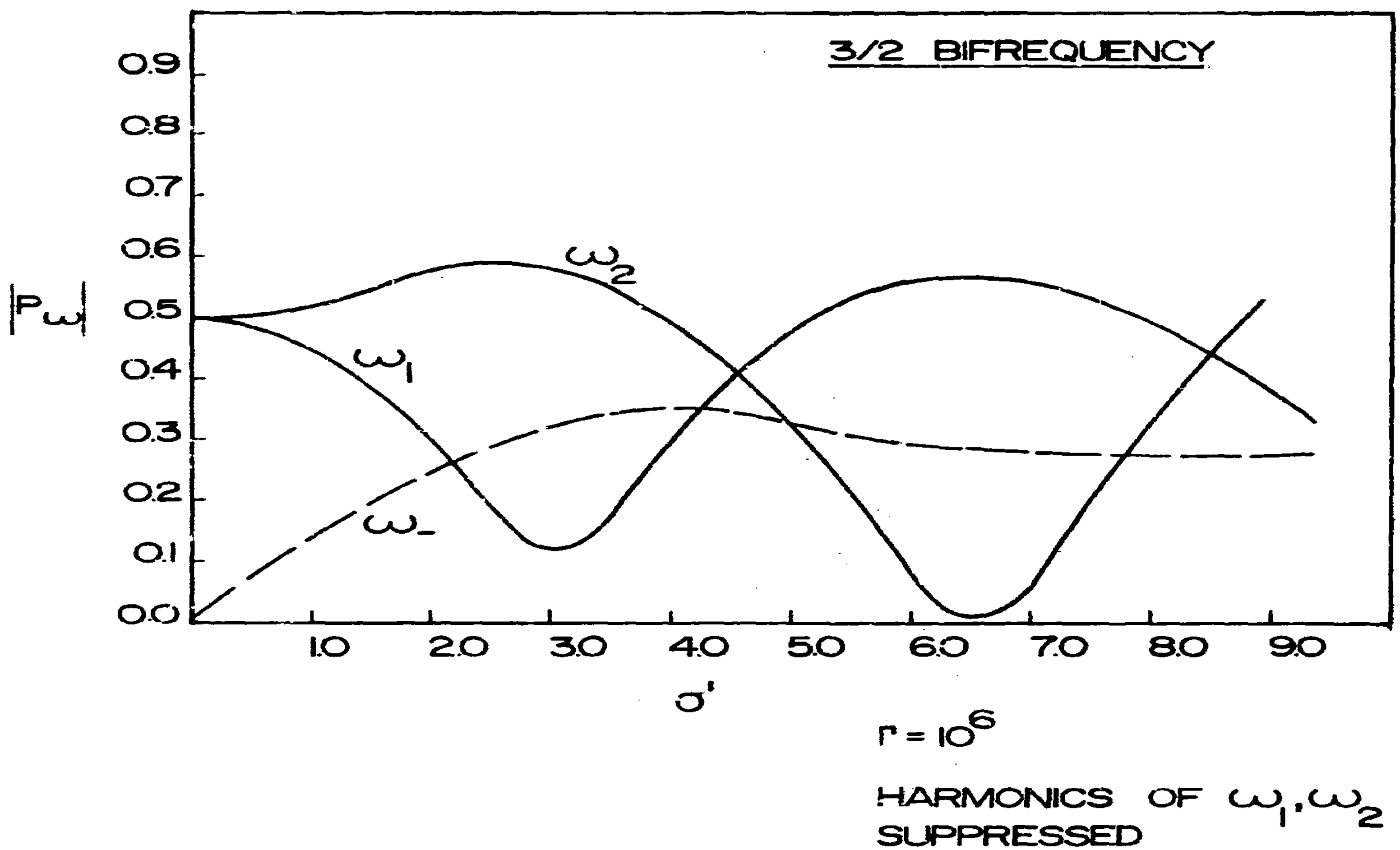
Before proceeding to a computer investigation of the mathematical models outlined in this report it was felt desirable to obtain upper bound estimates of the extent to which the conversion efficiency of a parametric array can be increased when all frequency components other than the primary waves and the difference-frequency are suppressed. Making use of a computer program developed by McKendree and Fenlon[49] which operates in the time domain but is subject to iterative spectrum analysis this comparison is shown in Figs. 2 and 3 for the case of a virtually lossless plane wave parametric interaction in which the frequency ratio of the primary waves is 3/2. In this instance, it can be seen from inspection that the three frequency case has a maximum difference-frequency amplitude nearly an order-of-magnitude than that of the all-frequency component spectrum. Moreover, the first peak of the difference-frequency in Fig. 2 occurs at $\sigma' = 3.5$ at $R = 3.5/\sigma_o$. Hence at a value of σ_o between 1 and 10 it can be seen from Fig. 4, which was first evaluated by Fenlon and McKendree,[47] that the beamwidth is almost fully established. This ensures that if the medium in which the three frequency interaction is terminated an ideal and much more efficient parametric array will be formed relative to that which would have existed under normal conditions. Similar but even more efficient comparisons are then shown in Figs. 5 and 6, for $\omega_1/\omega_2 = 6/5$ and in Figs. 7 and 8 $\omega_1/\omega_2 = 11/10$ respectively. These cases provide typical upper bounds to the maximum realizable gains in conversion efficiency achievable via absorption and dispersion mechanisms, both physical and artificial.

The the present time we are in the process of programming and testing frequency-domain computer methods to determine the effect of dispersive mechanisms in layered media, and using then to obtain the on-axis solutions of finite beams in relaxing fluids. The latter is particularly difficult to solve numerically because the diffraction losses are extremely difficult to include. These are however, defined paraxially by Eqs. (41) and (42) were derived within and therefore, further analysis and simplification of these results is expected to reveal the correct way to include them in a step by step wave propagation analysis.

Once the computer models are running, sets of nondimensional plots of the parametric fields in dispersive media can then be evaluated, plotted and studied in detail for subsequent relation to real and artificial acoustic media.

Before concluding this discussion it is interesting to check the accuracy of Figs. 2, 5, and 7 by comparison with the exact three frequency solutions given in Appendix A. As far as the first peaks of the latter are concerned they are in excellent agreement with the analogous at exact curves of Figs. (A1), (A2), and (A3). Subsequent deterioration of the computed curves is due both to the presence of small absorption losses (i.e. $\Gamma_o = 10^6$) and to "numerical error". However, in general only the first peak of the difference-frequency is of any interest because at this point the dispersive medium must be terminated and the amplified difference-frequency signal released into the unbounded surrounding medium. Hence the utility and validity of the numerical methods.

During the next phase of the contract as well as performing the numerical analysis referred to above we will carry out an analytical investigation of finite-amplitude waves of Gaussian cross-section intersecting at a fixed angle in dispersive fluids. Unlike the simple plane wave case investigated by Rudenko and Solugan[23] we do not know if this will result in a resonant interaction at all points of the field or only in the far-field of the interaction. This is one objective of the investigation. The other is to establish how rapidly resonant interaction falls off throughout the field at nonlinearly generated frequencies other than the difference-frequency, thus determining the extent to which the latter can grow due to dispersive coupling.



3/2 BIFREQUENCY
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$|P_\omega|$
ω_2
ω_1
ω_-
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
σ'
$r = 10^6$
HARMONICS OF ω_1, ω_2
SUPPRESSED

figure 2

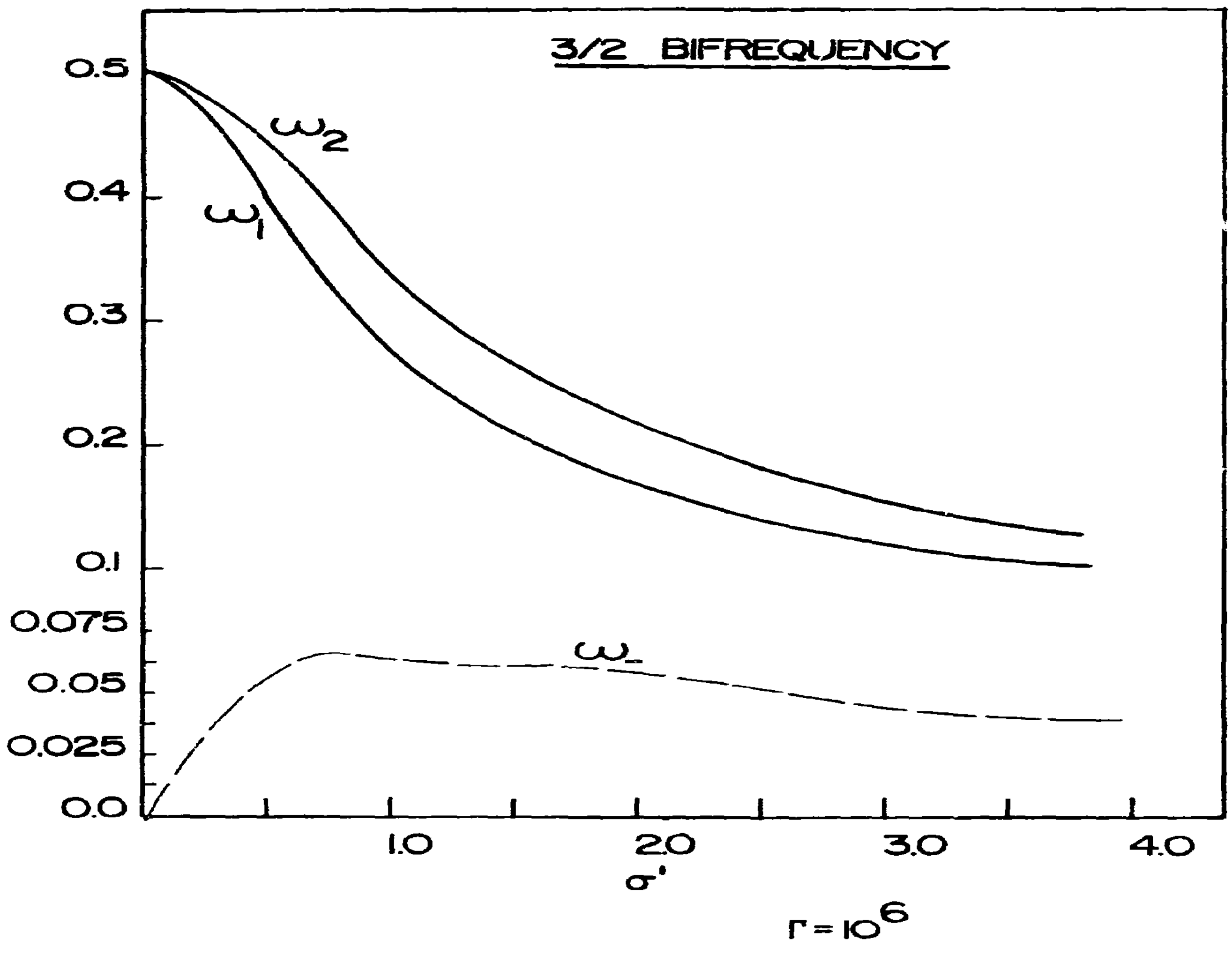
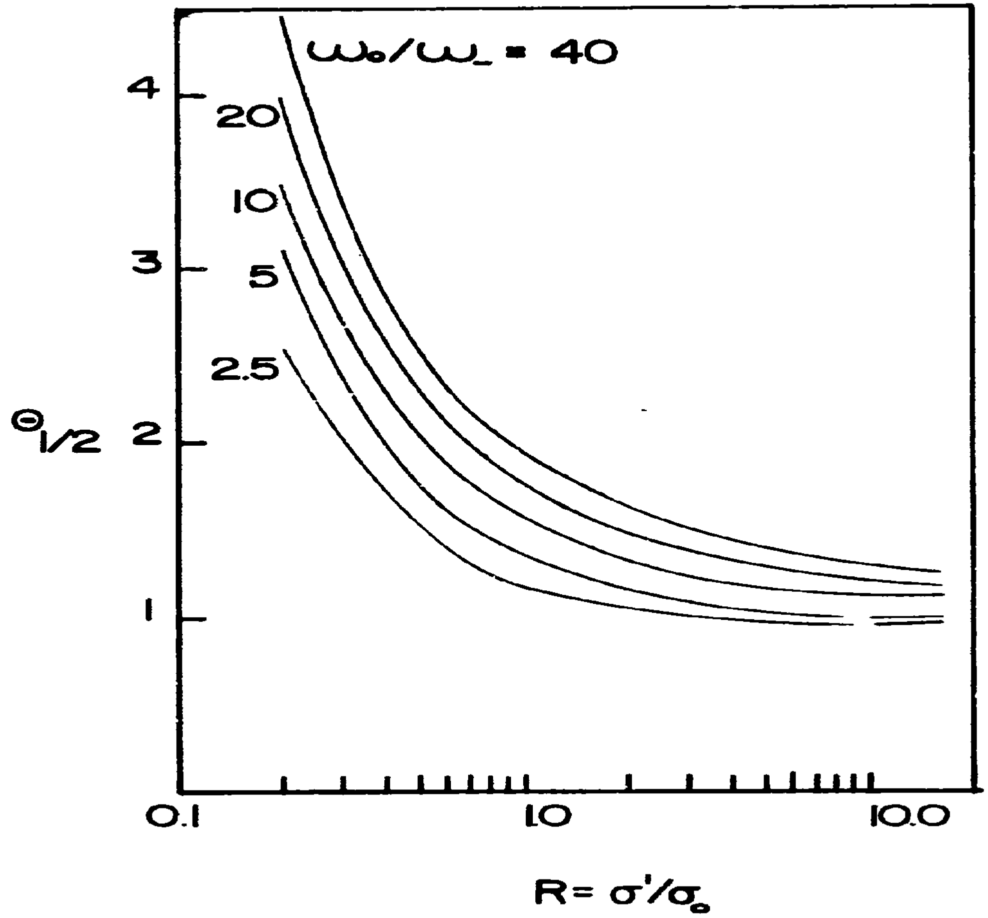

3/2 BIFREQUENCY
ω2
ω1
ω-
0.5
0.4
0.3
0.2
0.1
0.075
0.05
0.025
0.0
1.0
2.0
3.0
4.0
σ'
Γ = 10^6
NO HARMONIC
SUPRESSION

figure 3



$\omega_o/\omega_- = 40$
20
10
5
2.5
4
3
2
1
$\Theta_{1/2}$
0.1
1.0
10.0
$R = \sigma'/\sigma_o$

figure 4

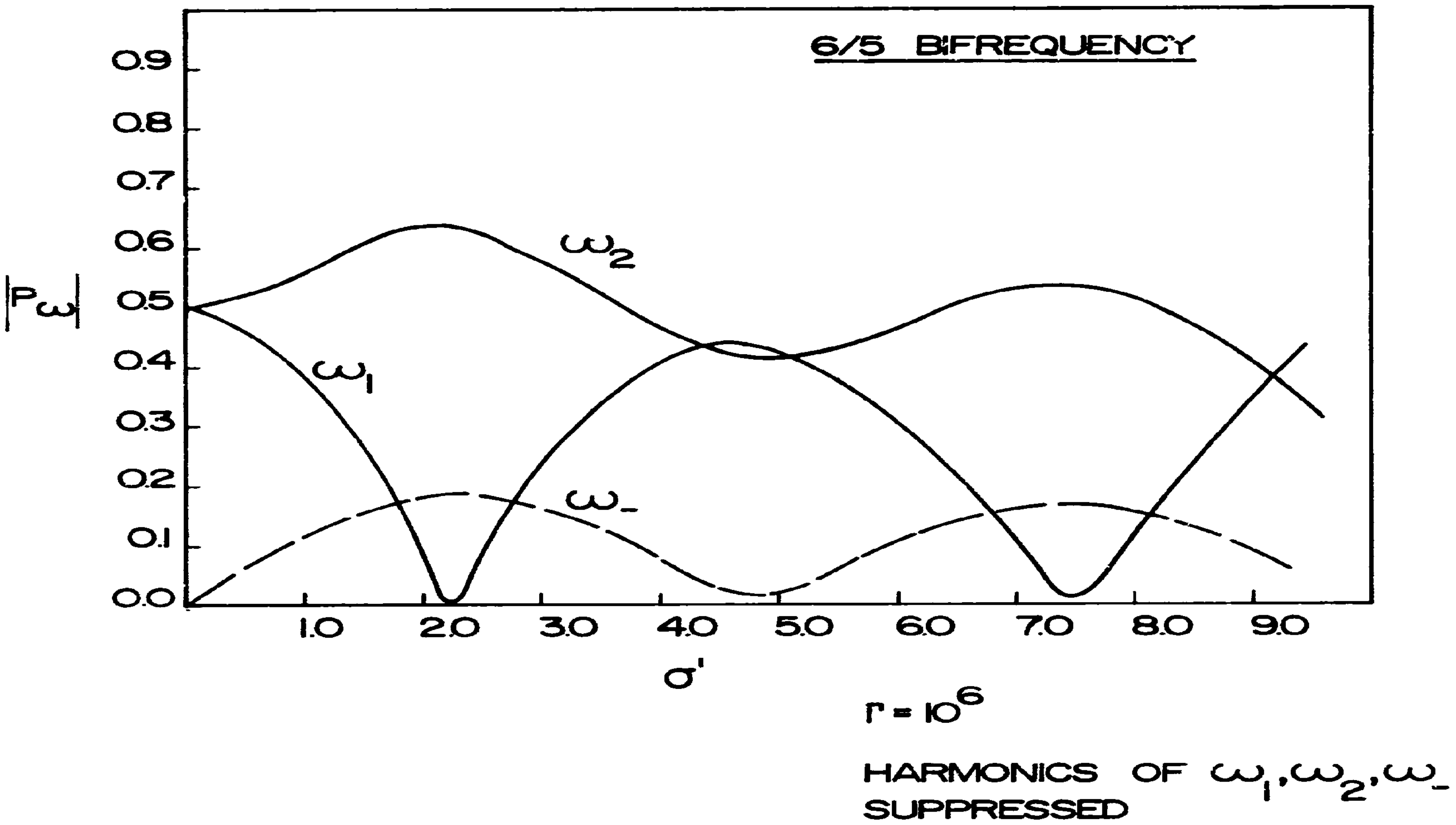

6/5 BIFREQUENCY
|Pω|
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
σ'
ω2
ω1
ω-
Γ = 10^6
HARMONICS OF ω1, ω2, ω- SUPPRESSED

figure 5

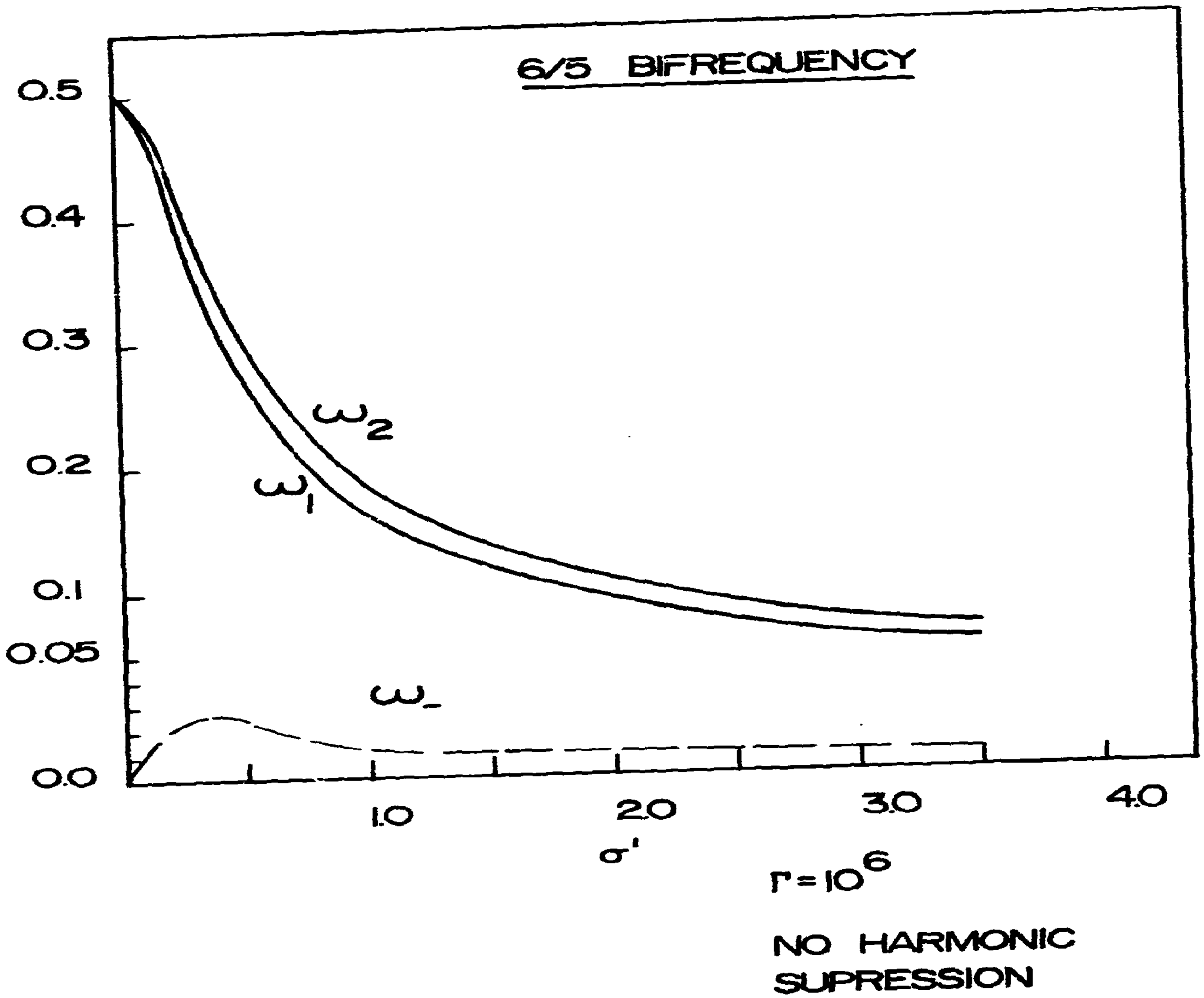
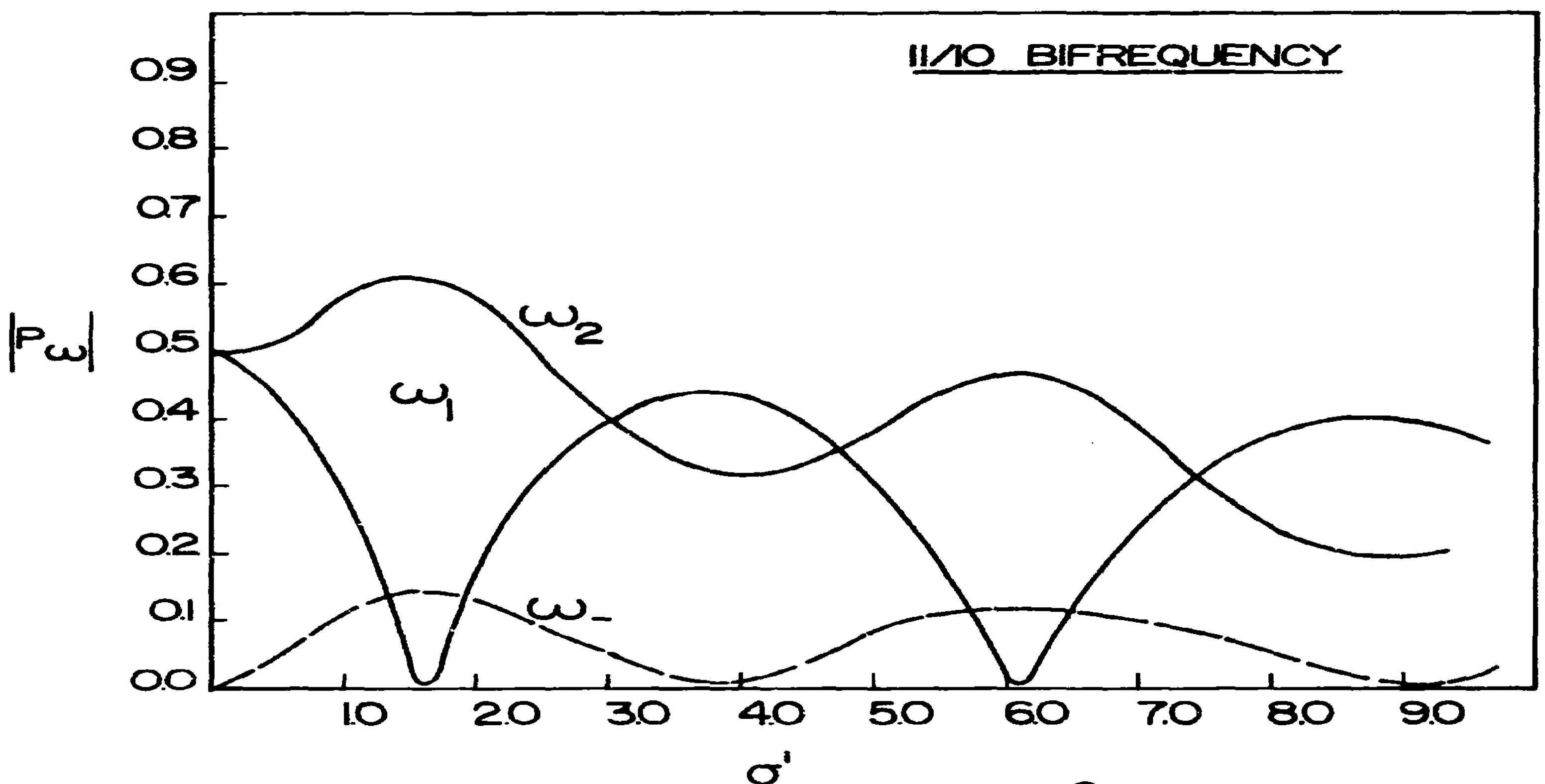

6/5 BIFREQUENCY
0.5
0.4
0.3
0.2
0.1
0.05
0.0
1.0
2.0
3.0
4.0
σ'
ω2
ω1
ω-
Γ = 10^6
NO HARMONIC
SUPRESSION

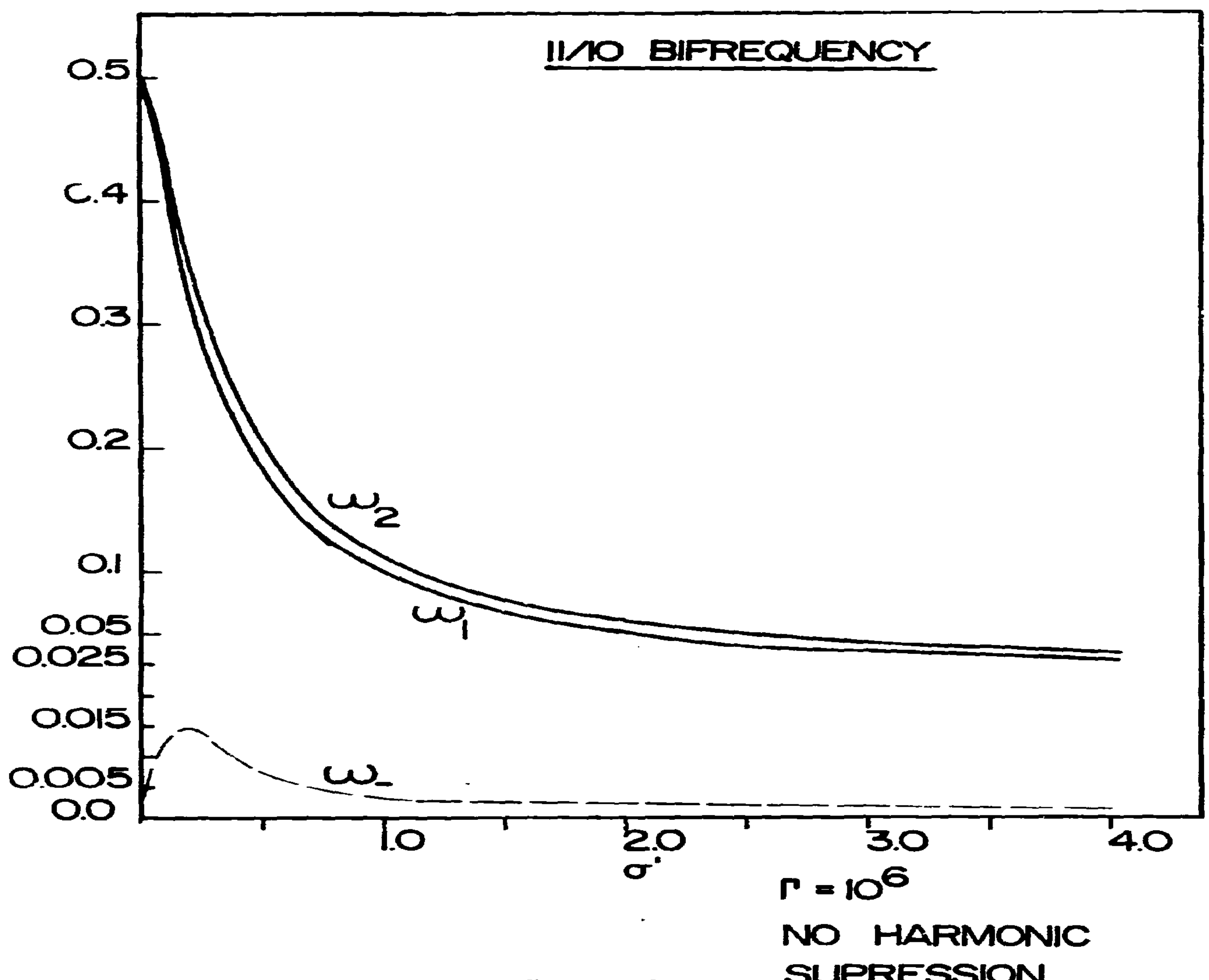
figure 6



$\Gamma = 10^6$

HARMONICS OF $\omega_1, \omega_2, \omega_-$ SUPPRESSED

figure 7



11/10 BIFREQUENCY
0.5
C.4
0.3
0.2
0.1
0.05
0.025
0.015
0.005
0.0
1.0
2.0
3.0
4.0
σ'
ω2
ω1
ω-
Γ = 10^6
NO HARMONIC
SUPRESSION

figure 8

Appendix A

If P_1 and P_2 are plane primary wave amplitudes of frequencies ω_1 and ω_2 respectively normalized with respect to P_{10} and P_- is the normalized difference-frequency (i.e. $\omega_- = \omega_1 - \omega_2$) field resulting from their nonlinear interaction then in the absence of losses and other frequency components if the three waves have a phase of $\pi/2$ then

$$\frac{dP_1}{d\sigma'} = \frac{iN_1}{2} P_2 P_- \tag{A1}$$

$$\frac{dP_2}{d\sigma'} = -\frac{iN_2}{2} P_1 P_- \tag{A2}$$

$$\frac{dP_-}{d\sigma'} = -\frac{iN_-}{2} P_1 P_2^* \tag{A3}$$

where
$$\sigma' = \left(\frac{2N_-}{N_1+N_2}\right) \sigma_o R, \tag{A4}$$

and
$$N_1 = \frac{\omega_1}{\omega_-}, \; N_2 = \frac{\omega_2}{\omega_-}, \; N_- = N_1 - N_2. \tag{A5}$$

The normalized range $R = \frac{z}{z_o}$, where z_o is the half-Rayleigh distance of the projector and $\sigma_o = \frac{z_o}{z_c}$, where $z_c = \frac{\rho_o \; c_o^2}{\beta P_{10} k_o}$ is the critical range for a mean wave-number $k_o = \frac{1}{2}(k_1+k_2)$. As shown by Bloembergen, Eqs. (A1) to (A3) have an exact analytical solution in terms of Jacobian Elliptic functions, summarized by Scott[50].

$$|P_1| = P_{10} \; cd(k_1\sigma', k_2) \tag{A6}$$

$$|P_2| = P_{20} \; nd(k_1\sigma', k_2) \tag{A7}$$

$$|P_-| = \left(\frac{N_-}{N_2}\right)^{1/2} k_2 \; P_{20} \; sd(k_1\sigma', k_2) \tag{A8}$$

where $cd = cn/dn,\ nd = 1/dn,\ sd = sn/dn$ (A9)

and $$k_1 = \frac{P_{10}}{2}\sqrt{N_2 N_-} \tag{A10}$$

with $$k_2 = \frac{1}{\sqrt{1 + \frac{N_1}{N_2}\left(\frac{P_{20}}{P_{10}}\right)^2}} \tag{A11}$$

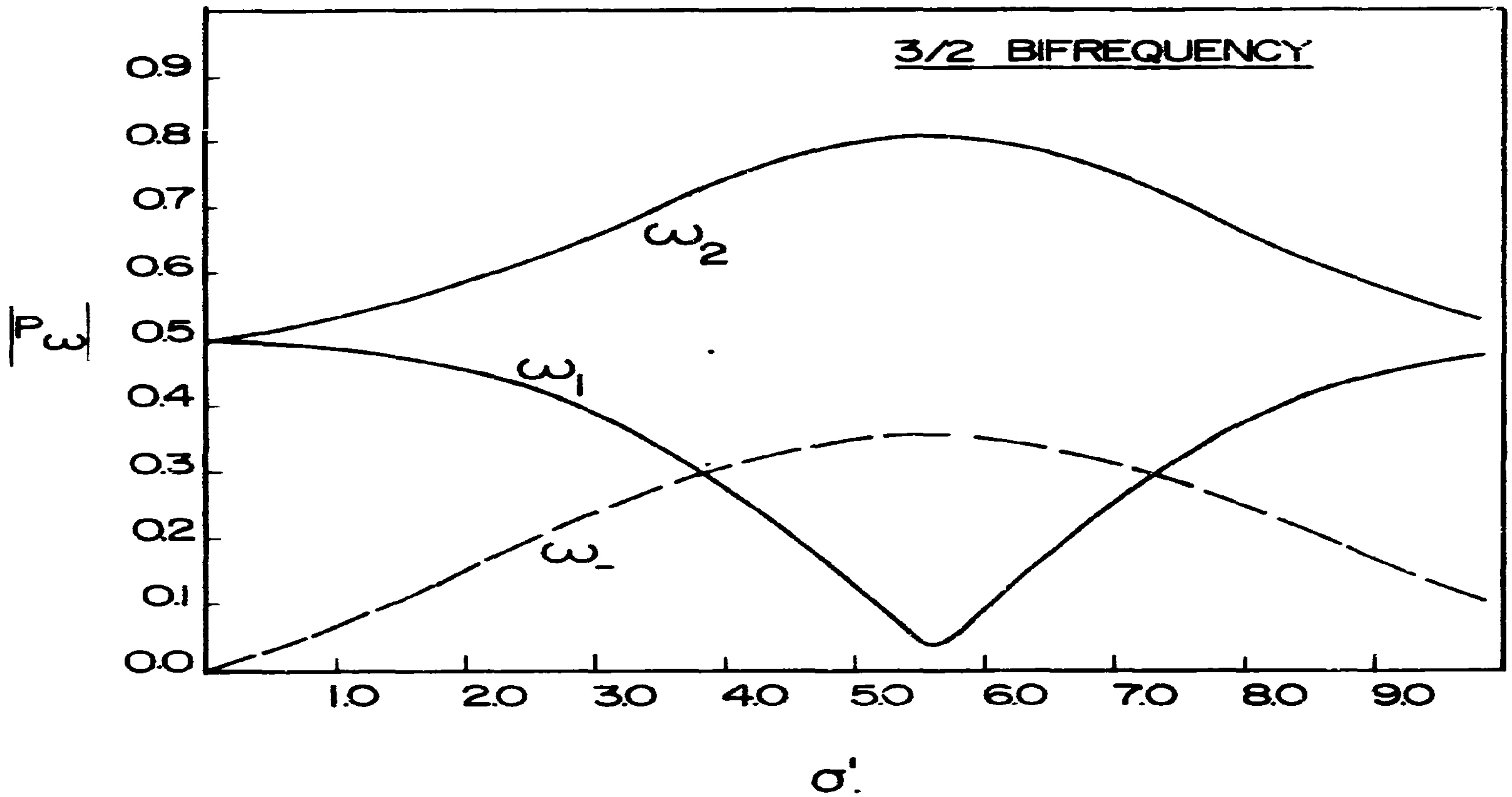

3/2 BIFREQUENCY
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
|Pω|
ω2
ω1
ω_
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
σ'

figure A1

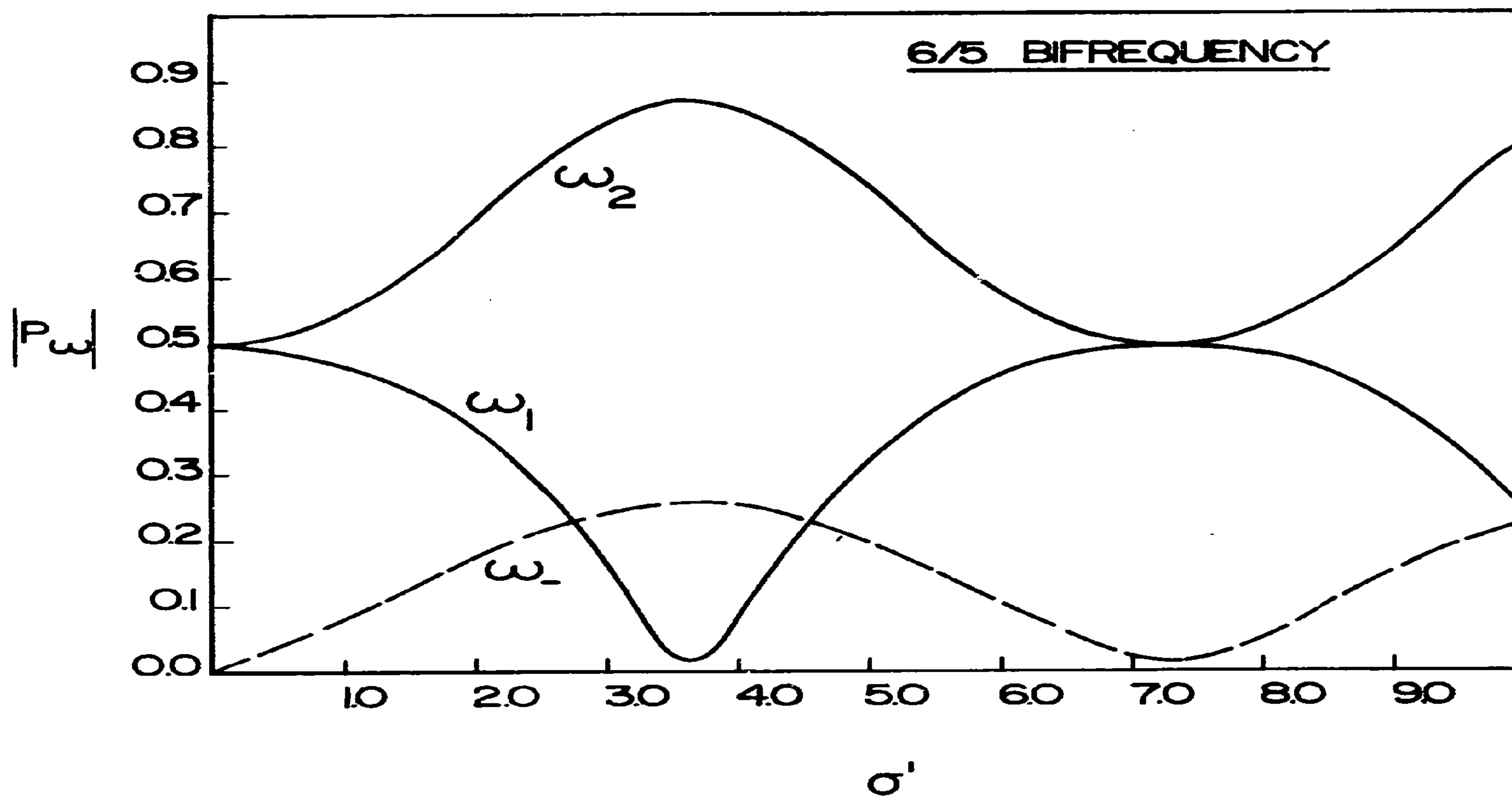

6/5 BIFREQUENCY
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
|Pω|
ω2
ω1
ω-
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
σ'

figure A2

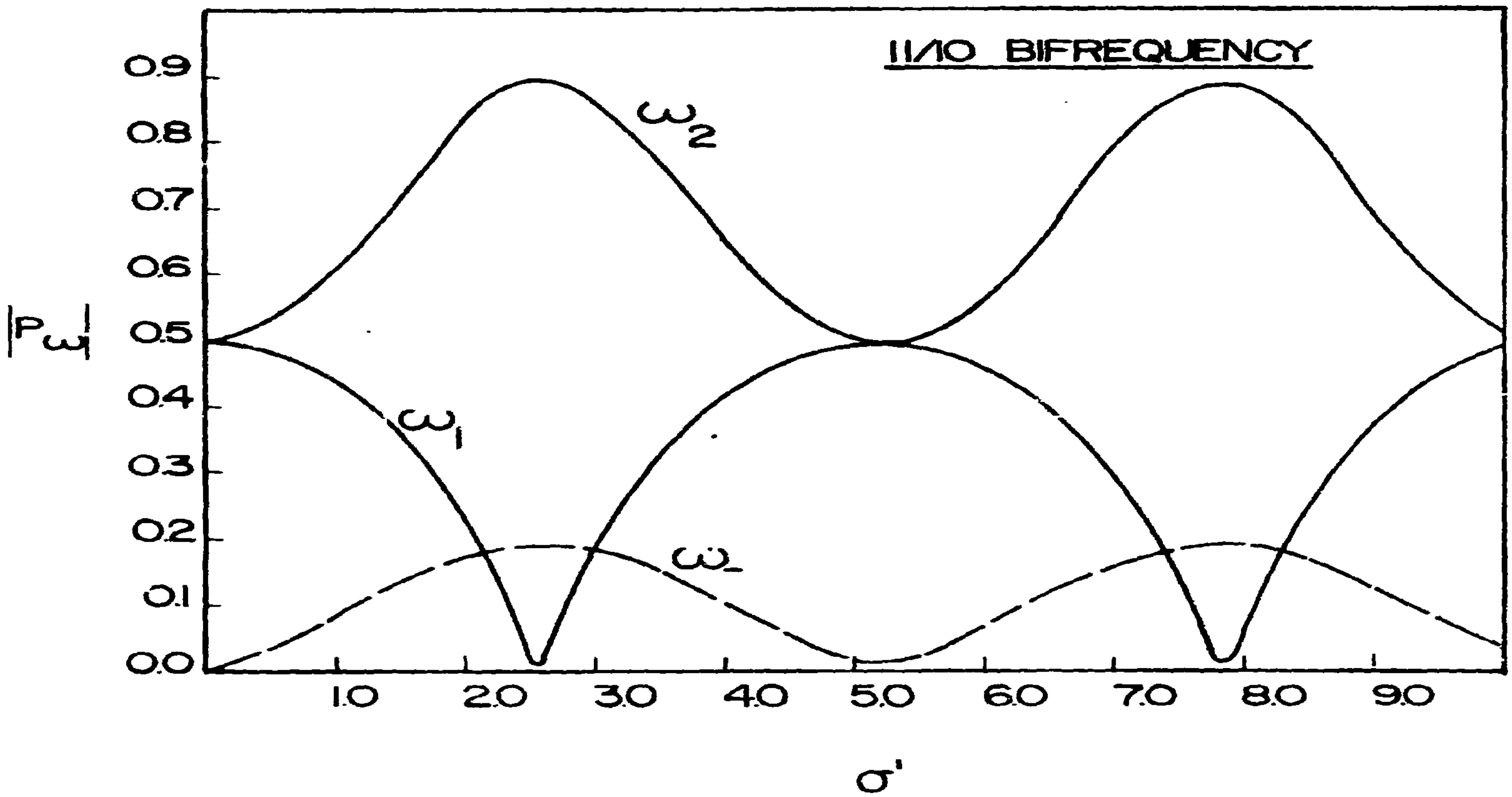

11/10 BIFREQUENCY
ω_2
ω_1
ω_-
$|P_\omega|$
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
σ'

figure A3

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