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Let $X_{N1},\ldots,X_{NN}$, $N \geq 1$ be independent random variables with continuous distribution functions $F_{N1},\ldots,F_{NN}$; and let R^+_{Ni} be the rank of $|X_{Ni}|$ among $|X_{N1}|,\ldots,|X_{NN}|$. Under different sets of conditions, we derive the asymptotic normality of simple linear signed rank statistic $S^+_N = \sum_{i=1}^{N} c_{Ni} a_N(R^+_{Ni}) \operatorname{sgn} X_{Ni}$ where (over)

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20. (cont.)

$c_{N1}, \ldots, c_{NN}$ are known regression constants, $\operatorname{sgn} x = 1$ if $x \geq 0$, $\operatorname{sgn} x = -1$ if $x < 0$, and $a_{N(1)}, \ldots, a_{N(N)}$ are scores generated by a function $\psi(t)$, $0 < t < 1$ which in contradistinction to the earlier literature is no longer assumed to be continuous. The results obtained are generalizations of the earlier results on limit theorems due to Hájek (1968, Ann. Math. Statist. 325-346) and Hušková (1970, Z. Wahrscheinlichkeitstheorie. Verw. Geb., 308-322), among others.

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Asymptotic normality of signed rank statistics with discontinuous score-generating function.

By Madan L. Puri

Indiana University, Bloomington

Dedicated to Professor C.R. Rao on the occassion of his 60th birthday.

The object of this paper is to derive the asymptotic distributions of simple linear signed rank statistic considered by Hájek (1968) and Hušková (1970) for the case when the score generating function is discontinuous.

1. Preliminaries

Let $X_{N1},\ldots,X_{NN}$, $N \geq 1$ be independent random variables with continuous distribution functions $F_{N1},\ldots,F_{NN}$; and let R_{Ni}^{+} be the rank of $|X_{Ni}|$ among $|X_{N1}|,\ldots,|X_{NN}|$. We shall be concerned with the asymptotic distribution of the statistic

$$S_N^+ = \sum_{i=1}^{N} c_{Ni}\, a_N(R_{Ni}^+)\ \mathrm{sgn}\ X_{Ni} \qquad (1.1)$$

where $c_{N1},\ldots,c_{NN}$ are known regression constants; $a_N(1),\ldots,a_N(N)$ are scores, and $\mathrm{sgn}\ x = 1$ if $x \geq 0$, $\mathrm{sgn}\ x = -1$ if $x < 0$. The results that we obtain are derived using the projection method together with a separate study of the case of the score generating function which has just one jump and is constant otherwise (see

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also Dupač and Hájek (1969).

We assume that the c_{Ni}'s satisfy the Noether condition

$$\lim_{N\to\infty} \max_{1\le i\le N} c_{Ni}^2 \Big/ \sum_{i=1}^{N} c_{Ni}^2 = 0 \qquad (1.2)$$

and the scores are generated by a function $\psi(t)$, $0 < t < 1$, either by interpolation

$$a_N(i) = \psi(i/_{(N+1)}) , \quad 1 \le i \le N \qquad (1.3)$$

or by a procedure satisfying

$$\sum_{i=1}^{N} |a_N(i) - \psi(i/_{(N+1)})| = O(1) \qquad (1.4)$$

Set

$$H_N^*(x) = N^{-1} \sum_{i=1}^{N} F_{Ni}^*(x) \qquad (1.5)$$

where F_{Ni}^* is the distribution function of $|X_i|$.

$$H_N^{*-1}(t) = \inf \{x : H_N^*(x) \ge t\} , \quad 0 < t < 1 \qquad (1.6)$$

$$L_{Ni}(t) = F_{Ni}(H_N^{*-1}(t)) , \quad 0 < t < 1 \qquad (1.7)$$

$$M_{Ni}(t) = -F_{Ni}(-H_N^{*-1}(t)) , \quad 0 < t < 1 \qquad (1.8)$$

$$G_{Ni}(t) = F_{Ni}^*(H_N^{*-1}(t)) = L_{Ni}(t) + M_{Ni}(t) , \quad 0 < t < 1 \qquad (1.9)$$

It is easy to check that

$$N^{-1} \sum_{i=1}^{N} G_{Ni}(t) = t \;,\; 0 \le t \le 1 \;,\; N > 1 \tag{1.10}$$

The price for allowing discontinuous scores is the following <u>differentiability conditions</u>:

Let v denote a jump point of the score generating function ψ ; $v \in (0,1)$. Then, for every $K > 0$, $K' > 0$, we impose the following conditions:

$$\max_{1 \le i \le N} \sup_{K N^{-\frac{1}{2}} \le |t-v| \le K' N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N} \left| \frac{L_{Ni}(t) - L_{Ni}(v)}{t - v} \right| = 0(1) \tag{1.11}$$

$$\max_{1 \le i \le N} \sup_{K N^{-\frac{1}{2}} \le |t-v| \le K' N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N} \left| \frac{M_{Ni}(t) - M_{Ni}(v)}{t - v} \right| = 0(1) \tag{1.12}$$

Furthermore, we assume that there exist real numbers $\ell_{Ni}(v)$, $m_{Ni}(v)$, $(1 \le i \le N)$ such that for every $K > 0$,

$$\max_{1 \le i \le N} \sup_{|t-v| \le K N^{-\frac{1}{2}}} |L_{Ni}(t) - L_{Ni}(v) - (t - v) \ell_{Ni}(v)| = o(N^{-\frac{1}{2}}) \tag{1.13}$$

$$\max_{1 \le i \le N} \sup_{|t-v| < K N^{-\frac{1}{2}}} |M_{Ni}(t) - M_{Ni}(v) - (t - v) m_{Ni}(v)| = o(N^{-\frac{1}{2}}) \tag{1.14}$$

From (1.11) - (1.14) it follows that $G_{Ni}(t)$ satisfies the same type of conditions with $g_{Ni}(v) = \ell_{Ni}(v) + m_{Ni}(v)$. Also it is easy to see that without loss of generality one can assume that $\ell_{Ni}(v) \ge 0$, $m_{Ni}(v) \ge 0$, and

$$N^{-1} \sum_{i=1}^{N} \ell_{Ni}(v) = 1 = N^{-1} \sum_{i=1}^{N} m_{Ni}(v)$$

Finally, note that the numbers $\ell_{Ni}(v)$, $m_{Ni}(v)$ considered as function of (i,N) , $1 \leq i \leq N$, are bounded. Another condition concerning the G_{Ni}'s that we use is

$$\liminf_{N\to\infty} N^{-1} \sum_{i=1}^{N} G_{Ni}(v)(1 - G_{Ni}(v)) > 0 \tag{1.15}$$

Some times, mainly for purposes of applicatons, we replace (1.11) - (1.15) by more feasible conditions easier to verify:

Suppose, for example, that each F_{Ni} has a density f_{Ni} such that for some α , $(0 < \alpha \leq \infty)$, we have

(a) $f_{Ni}(x) = 0$ for $x \notin (-\alpha,\alpha)$ if α is finite,

(b) $f_{Ni}(x)$ is continuous on every compact sub interval of $(-\alpha,\alpha)$ uniformly in (x, N, i) ,

(1.16)

(c) for every compact interval $C \subset (0,\alpha)$ there exists an $\varepsilon_C > 0$ such that for all $N \geq 1$,

$$N^{-1} \text{ Card } \{1 \leq i \leq N : \inf_{x\in C} f^*_{Ni}(x) > \varepsilon_C\} > \varepsilon_C ,$$

where f^*_{Ni} is the density of F^*_{Ni} .

(d) $0 < \liminf_{N\to\infty} H_N^{*-1}(t) \leq \limsup_{N\to\infty} H_N^{*-1}(t) < \alpha$ for all $t \in (0,1)$.

We will see that the condition (1.16) is satisfied in particular if

(a) $f_{Ni}(x) = e^{-d_{Ni}} f(x e^{-d_{Ni}})$

(b) f is uniformly continuous and positive on $(-\infty, \infty)$ (1.17)

(c) $\sup_N \max_{1 \leq i \leq N} |d_{Ni}| < \infty$

The last condition that we require concerns the nondegeneration of $\operatorname{Var} S_N^+$ in the form

$$\liminf_{N \to \infty} \operatorname{Var}(S_N^+) \Big/ \sum_{i=1}^{N} c_{Ni}^2 > 0 \tag{1.18}$$

2. Main Theorems.

Theorem 2.1. Consider the Statistic (1.1) with scores satisfying (1.4), where

$$\psi(t) = \begin{cases} 0 & \text{if} \quad 0 < t < v \\ 1 & \text{if} \quad v \leq t < 1 \end{cases}$$

Then S_N^+ is asymptotically normal with natural parameters $(E(S_N^+), \operatorname{Var}(S_N^+))$ if any of the following sets of conditions is satisfied:

(C_1^+) : (1.2), (1.11), (1.12), (1.13), (1.14), (1.15), (1.18)

(C_2^+) : (1.2), (1.16), (1.18)

(C_3^+) : (1.2), (1.17), (1.18)

Proof. We show that S_N^+ is asymptotically equivalent to its projection $\hat{S}_N^+$ onto the space of linear statistics and then that $\hat{S}_N^+$ is asymptotically equivalent to a sum of independent random variables to which the Lindeberg Central Limit theorem applies. (For the ease of convenience we shall from now suppress the first subscript N from X_{Ni}, R_{Ni}^+, etc).

First we would like to derive an upper bound for the residual variance $E(S_N^+ - \hat{S}_N^+)^2$, where

$$\hat{S}_N^+ = \sum_{i=1}^{N} E(S_N^+ | X_i) - (N-1)E(S_N^+) .$$

By the Residual Variance Lemma (see Hájek (1968)), we have

$$\begin{aligned} E(S_N^+ - \hat{S}_N^+)^2 \leq \sum_{i=1}^{N} c_i^2 \; & E(a(R_i^+) - E(a(R_i^+)|X_i))^2 \\ & + \sum_{i \neq j} c_i c_j \; \{E(\operatorname{sgn} X_i \operatorname{sgn} X_j \operatorname{Cov}(a(R_i^+), a(R_j^+)|X_i, X_j)) \\ & + E\{\operatorname{sgn} X_i \operatorname{sgn} X_j [E(a(R_i^+)|X_i, X_j) - E(a(R_i^+)|X_i)] \\ & \quad \times [E(a(R_j^+)|X_i, X_j) - E(a(R_j^+)|X_j)]\} \\ & - \sum_{k \neq i,j} \operatorname{Cov}\{E(\operatorname{sgn} X_i \, a(R_i^+)|X_k), E(\operatorname{sgn} X_j \, a(R_j^+)|X_k)\}\} \end{aligned}$$

We investigate each term in the above inequality. We begin by assuming that the scores are defined by (1.3) and that (C_1^+) holds. The proof is divided in several steps:

Lemma 2.1. *The functions* $L_{Ni}(t)$, $M_{Ni}(t)$, $G_{Ni}(t)$ *satisfy the following relations:*

$$|L_{Ni}(t) - L_{Ni}(s)| \leq N|t-s| ,$$

$$|M_{Ni}(t) - M_{Ni}(s)| \leq N|t-s| ;$$

and

$$|G_{Ni}(t) - G_{Ni}(s)| \leq N|t-s| ; \quad 0 < s, t < 1$$

Proof: It follows from the definitions and the fact that for $u > v > 0$ we have:

$$F_i(u) - F_i(v) \leq F_i^*(u) - F_i^*(v) , \quad F_i(-v) - F_i(-u) \leq F_i^*(u) - F_i^*(v)$$

where we set $u = H^{*-1}(t)$, $v = H^{*-1}(s)$.

Denote $V = [(N+1)v]$, $D^2 = N^{-1} \sum_{i=1}^{N} G_i(v)(1 - G_i(v))$

$([\cdot]$ = integer part$)$

Lemma 2.2. *Let* $x , y \in \mathbb{R}$. *Then to each* $k_1 > 2$ *there exist a* $k_2 > 1$ *such that for all* $N > N_0(k_1)$ *we have:*

(i) $v - H_N^*(|x|) > k_1 N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N \Rightarrow P(R_i^+ \geq V | X_i = x, X_j = y) < N^{-k_2}$

(ii) $v - H_N^*(|x|) < -k_1 N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N \Rightarrow P(R_i^+ \leq V | X_i = x, X_j = y) < N^{-k_2}$

Furthermore, (i) and (ii) *remain true even when the condition* $X_j = y$ *is ommited.*

Lemma 2.3. *Suppose that* $|v - H^*(|x|)| \leq k_3 N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N$. *Then, for sufficiently large* N, *we have*

$$\text{(i)} \quad \left| \sum_{i=1}^{N} F_i^*(|x|)(1 - F_i^*(|x|)) - ND^2 \right| \leq k_4 N^{\frac{1}{2}} Lg^{\frac{1}{2}} N$$

$$\text{(ii)} \quad \Big| \phi\Big(V ; \sum_{i=1}^{N} F_i^*(|x|) , \sum_{i=1}^{N} F_i^*(|x|)(1 - F_i^*(|x|))\Big)$$

$$- \phi\Big(Nv ; \sum_{i=1}^{N} F_i^*(|x|) , ND^2\Big) \Big| \leq k_5 N^{-1} Lg^{\frac{1}{2}} N$$

$$\text{(iii)} \quad \Big| \Phi\Big(V ; \sum_{i=1}^{N} F_i^*(|x|) , \sum_{i=1}^{N} F_i^*(|x|)(1 - F_i^*(|x|))\Big)$$

$$- \Phi\Big(Nv ; \sum_{i=1}^{N} F_i^*(|x|) , ND^2\Big) \Big| \leq k_6 N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N$$

where $\phi(x ; \mu, \sigma^2)$, $\Phi(x ; \mu, \sigma^2)$ denote the normal density, resp. the normal distribution function with parameters (μ, σ^2).

The proofs of lemmas 2.2 and 2.3 are analogous to those of Lemmas 5 and 6 of Dupač and Hájek (1969), and are therefore omitted.

Lemma 2.4. *For* $N \to \infty$, *we have*

$$E(a(R_i^+) - E(a(R_i^+) | X_i))^2 = o(1)$$

uniformly in $1 \leq i \leq N$.

Proof: Let $\Omega^+(X_i) = E(a(R_i^+)|X_i) - [(E(a(R_i^+)|X_i)]^2$

Then, by conditioning we obtain:

$$E[a(R_i^+) - E(a(R_i^+)|X_i)]^2 = E(\Omega^+(X_i))$$

Now, by definition:

$$E(a(R_i^+)|X_i = x) = P(R_i^+ > V|X_i = x) \ .$$

Thus

$$\Omega^+(X_i = x) = P(R_i^+ > V|X_i = x) \cdot P(R_i^+ \leq V|X_i = x)$$

Let $I = \{x \,\big|\, |H^*(|x|) - v| \leq k_1 \ N^{-\frac{1}{2}} \ Lg^{\frac{1}{2}} N\}$

By Lemma 2, if $x \notin I$ we have:

$$\Omega^+(X_i = x) < N^{-k_2} \ , \text{ for every } \ N \geq N_0(k_2) \ , \ k_2 > 1$$

On the other hand, if $x \in I$, then since $P(R_i^+ = k|X_i = x) = B^i(k, F_1^*(|x|), \ldots F_N^*(|x|))$ (in the notation used by Dupač and Hájek (1969)), we obtain using Lemmas 2.2 and 2.3, that

$$\Omega^+(X_i = x) = \{\sum_{k>V} B^i(k, F_1^*(|x|), \ldots F_N^*(|x|))\}$$

$$\times \{\sum_{\ell \leq V} B^i(\ell, F_1^*(|x|), \ldots, F_N^*(|x|))\}$$

$$= \ldots = \Phi\left(\frac{H^*(|x|) - v}{DN^{-\frac{1}{2}}}\right) \left\{1 - \Phi\left(\frac{H^*(|x|) - v}{DN^{-\frac{1}{2}}}\right)\right\} + \theta_1 N^{-\frac{1}{2}} \ Lg^{\frac{1}{2}} N$$

for sufficiently large N, $|\theta_1| \leq k_7$

We observe that the last equality remains true even if we enlarge I to

$$I' = \{x : |H^*(|x|) - v| \leq k_9 \, DN^{-\frac{1}{2}} Lg^{\frac{1}{2}} N\}$$

where k_9 is such that $k_9 = {}^{k}1/2k_8$ with $k_8 \leq D \leq \frac{1}{2}$ and $k_9 > 2$. (Here we use Condition (1.15)).

Now using (1.11) and (1.12) it is easy to show that

$$N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N \int_{I'} \theta_1 \, dF_i(x) = o(1)$$

and

$$\int_{I'} \Phi\left(\frac{H^*(|x|) - v}{DN^{-\frac{1}{2}}}\right) \left\{1 - \Phi\left(\frac{H^*(|x|) - v}{DN^{-\frac{1}{2}}}\right)\right\} dF_i(x) = o(1)$$

Hence $E(\Omega^+(X_i)) = o(1)$ uniformly in $1 \leq i \leq N$ and the proof follows.

Lemma 2.5. *For* $N \to \infty$ *we have:*

$$E\{\operatorname{sgn} X_i \operatorname{sgn} X_j [E(a(R_i^+)|X_i, X_j) - E(a(R_i^+)|X_i)] \cdot [E(a(R_j^+)|X_i, X_j)$$

$$- E(a(R_j^+)|X_j))\} = o(N^{-1})$$

uniformly in $1 \le i,\ j \le N$.

The proof of this Lemma is similar to that of Lemma 2.4 and is therefore omitted.

Lemma 2.6 For $N \to \infty$ we have:

$$E[\operatorname{sgn} X_i \operatorname{sgn} X_j \operatorname{Cov}(a(R_i^+), a(R_j^+)| X_i , X_j)]$$

$$= N^{-1} D^2 (\ell_i(v) - m_i(v))(\ell_j(v) - m_j(v)) + o(N^{-1})$$

uniformly in $1 \le i , j \le N$.

Proof: We have:

$$\Delta^+ = \operatorname{Cov}(a(R_i^+),\ a(R_j^+)|X_i = x , X_j = y)$$

$$= \begin{cases} P(R_i^+ > V|X_i = x,\ X_j = y) \cdot P(R_j^+ \le V|X_i = x,\ X_j = y) & \text{if } |x|<|y| \\ P(R_j^+ > V|X_i = x,\ X_j = y) \cdot P(R_i^+ \le V|X_i = x,\ X_j = y) & \text{if } |x| \ge |y| \end{cases}$$

Let $k_1 > 2$. Denote:

$$I = \{(x,y): |H^*(|x|) - v| \le k_1 N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N, |H^*(|y|) - v| \le k_1 N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N\}$$

By considerations as used in the derivation of (4.11) and (4.12) in Dupač and Hájek (1969), we obtain

$$\Delta^+(x,y)\begin{cases} < N^{-k_2} & \text{for } (x,y) \notin I,\ N > N_0(k_2) \\ = \phi\left(\dfrac{H^*(|x|)-v}{DN^{-\frac12}}\right)\left\{1-\phi\left(\dfrac{H^*(|y|)-v}{DN^{-\frac12}}\right)\right\} + \theta_2 N^{-\frac12} Lg^{\frac12} N & \\ \qquad \text{for } N \text{ sufficiently large, } (x,y)\in I,\ |x|<|y|, & \\ \qquad \text{and } |\theta_2| \le k_{10} & \\ = \phi\left(\dfrac{H^*(|y|)-v}{DN^{-\frac12}}\right)\left\{1-\phi\left(\dfrac{H^*(|x|)-v}{DN^{-\frac12}}\right)\right\} + \theta_3 N^{-\frac12} Lg^{\frac12} N & \\ \qquad \text{for } N \text{ sufficiently large, } (x,y)\in I,\ |x|\ge|y| & \\ \qquad \text{and } |\theta_3| \le k_{11} & \end{cases} \tag{2.1}$$

We note that the equality in (2.1) remains true even if we enlarge I to I'

$$I' = \{(x,y) : \max\{|H^*(|x|)-v|,\ |H^*(|y|)-v|\} \le k_1' DN^{-\frac12} Lg^{\frac12} N\}$$

where k_1' is such that $k_1' = k_1/2k_8$, $k_8 \le D \le \frac12$, $k_1' > 2$ (k_1' coincides, with k_9 in the notation of Lemma (2.4)).

We have, using (2.1) that

$$E(\operatorname{sgn} X_i \operatorname{sgn} X_j \operatorname{Cov}(a(R_i^+),\ a(R_j^+)|\ X_i,\ X_j))$$

$$= \iint\limits_{I'\cap\{|x|<|y|\}} \operatorname{sgn} x \operatorname{sgn} y\ \phi\left(\frac{H^*(|x|)-v}{DN^{-\frac12}}\right)\left\{1-\phi\left(\frac{H^*(|y|)-v}{DN^{-\frac12}}\right)\right\} dF_i(x)\, dF_j(y)$$

$$+ \iint\limits_{I'\cap\{|x|\ge|y|\}} \operatorname{sgn} x \operatorname{sgn} y\ \phi\left(\frac{H^*(|y|)-v}{DN^{-\frac12}}\right)\left\{1-\phi\left(\frac{H^*(|x|)-v}{DN^{-\frac12}}\right)\right\} dF_i(x)\, dF_j(y)$$

$$+ N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N \iint_{I'} \operatorname{sgn} x \operatorname{sgn} y \ \theta_4(x,y) \ dF_i(x) dF_j(y) + \theta_5 N^{-k_2} \tag{2.2}$$

with $|\theta_4| \leq k_{12}$, $|\theta_5| \leq 1$.

The last two terms are $o(N^{-1})$ uniformly in i,j as follows by using (1.11), (1.12) and $k_2 > 1$. It reamains to estimate the first two terms.

Denote the first term by T . Consider:

$$T_1 = \iint \Phi\left(\frac{H^*(x) - v}{DN^{-\frac{1}{2}}}\right)\left\{1 - \Phi\left(\frac{H^*(y) - v}{DN^{-\frac{1}{2}}}\right)\right\} dF_i(x) \ dF_j(y)$$

$$\left\{(x,y) : \begin{array}{c} x>0 \\ y>0 \\ x<y \end{array}, \ \max(|H^*(x) - v|, \ |H^*(y) - v|) \leq k_1' DN^{-\frac{1}{2}} Lg^{\frac{1}{2}} N\right\}$$

Set $p = \dfrac{H^*(x) - v}{DN^{-\frac{1}{2}}}$, $q = \dfrac{H^*(y) - v}{DN^{-\frac{1}{2}}}$ and

$I'' = \{(p,q) : \max(|p|, \ |q|) \leq k_1' \ Lg^{\frac{1}{2}} N\}$. Then

$$T_1 = \iint_{I'' \cap \{p<q\}} \Phi(p)(1 - \Phi(q)) \ dL_i(v + DN^{-\frac{1}{2}} p) d L_j(v + DN^{-\frac{1}{2}} q)$$

As in the proof of Lemma 7 of Dupač and Hájek (1969) one can easily show that

$$T_1 = \tfrac{1}{2} N^{-1} D^2 \ell_i(v) \ \ell_j(v) + o(N^{-1}) \tag{2.3}$$

uniformly in $1 \leq i, \ j \leq N$

Let

$$T_2 = \iint - \Phi\left(\frac{H^*(-x) - v}{DN^{-\frac{1}{2}}}\right)\left\{1 - \Phi\left(\frac{H^*(y) - v}{DN^{-\frac{1}{2}}}\right)\right\} dF_i(x) \ dF_j(y)$$

$$\left\{(x,y) : \begin{array}{c} x<0 \\ y>0 \\ -x<y \end{array}, \ \max \{|H^*(-x) - v|, |H^*(y) - v|\} \leq k_1' DN^{-\frac{1}{2}} Lg^{\frac{1}{2}} N\right\}$$

Then

$$-T_2 = \iint\limits_{I''\cap\{p<q\}} \phi(p)(1-\phi(q))dM_i(v+DN^{-\frac{1}{2}}p)dL_j(v+DN^{-\frac{1}{2}}q)$$

Divide I'' into J and $I''\backslash J$ where:

$$J = \{(p,q) : \max(|p|, |q|) \leq k_1^{*}\}$$

and in J make use of the expansions:

$$M_i(v+DN^{-\frac{1}{2}}p) = M_i(v) + m_i(v)DN^{-\frac{1}{2}}p + \Omega_i(p)$$

$$L_j(v+DN^{-\frac{1}{2}}q) = L_j(v) + \ell_j(v)DN^{-\frac{1}{2}}q + \Lambda_j(q)$$

where $\Omega_i(p)$ and $\Lambda_j(q)$ are absolutely continuous and are of order $o(N^{-\frac{1}{2}})$ in $[-k_1^{*}, k_1^{*}]$. This follows from (1.13) and (1.14).

Then considerations similar to the ones used in the derivation of (2.3) lead to:

$$T_2 = -\tfrac{1}{2}N^{-1}D^2 m_i(v)\,\ell_j(v) + o(N^{-1}) \quad \text{uniformly in} \quad 1 \leq i, j \leq N .$$

Consider now

$$T_3 = \iint \phi\left(\frac{H^{*}(-x)-v}{DN^{-\frac{1}{2}}}\right)\left\{1-\phi\left(\frac{H^{*}(-y)-v}{DN^{-\frac{1}{2}}}\right)\right\} dF_i(x)\, dF_j(y)$$

$$\left\{(x,y) : \begin{matrix} x<0 \\ y<0 \\ -x<-y \end{matrix}, \max(|H^{*}(-x)-v|, |H^{*}(-y)-v|) \leq k_1' DN^{-\frac{1}{2}} Lg^{\frac{1}{2}} N\right\}$$

$$T_4 = \iint -\phi\left(\frac{H^*(x)-v}{DN^{-\frac{1}{2}}}\right)\left\{1-\phi\left(\frac{H^*(-y)-v}{DN^{-\frac{1}{2}}}\right)\right\} dF_i(x)\, dF_j(y)$$

$$\left\{(x,y) : \begin{matrix} x>0 \\ y<0 \\ x<-y \end{matrix} ; \max(|H^*(x)-v|, |H^*(-y)-v|) < k_1' DN^{-\frac{1}{2}} Lg^{\frac{1}{2}} N\right\}$$

Proceding as before, (omitting the details of computations) it follows that

$$T_3 = \tfrac{1}{2}N^{-1} D^2 m_i(v) m_j(v) + o(N^{-1}) \quad \text{uniformly in } 1 \le i, j \le N$$

$$T_4 = -\tfrac{1}{2}N^{-1} D^2 \ell_i(v) m_j(v) + o(N^{-1}) \quad \text{uniformly in } 1 \le i, j \le N .$$

Thus,

$$T = T_1 + T_2 + T_3 + T_4 = \tfrac{1}{2} N^{-1} D^2 (\ell_i(v) - m_i(v))(\ell_j(v) - m_j(v))$$

$$+ o(N^{-1}) \quad \text{uniformly in } 1 \le i, j \le N . \qquad (2.4)$$

Proceding as above, it can be shown that the second term of (2.2) is the same as (2.4). The proof follows.

<u>Lemma</u> 2.7. <u>For</u> $N \to \infty$, <u>we have</u> (for $i \neq j$)

$$\sum_{k \neq i,j} \mathrm{Cov}\ \{E(\mathrm{sgn}\, X_i\ a(R_i^+) | X_k)\ ,\ E(\mathrm{sgn}\, X_j\ a(R_j^+) | X_k)\} =$$

$$= N^{-1} D^2 (\ell_i(v) - m_i(v))(\ell_j(v) - m_j(v)) + o(N^{-1})$$

<u>uniformly in</u> $1 \le i, j \le N$.

<u>Proof</u>: By Lemma 3.2 in Hájek (1968) we have:

$$E(a(R_i^+)\ \mathrm{sgn}\ X_i \mid X_i = x,\ X_k = z) - E(a(R_i^+)\ \mathrm{sgn}\ X_i \mid X_i = x)$$

$$= \mathrm{sgn}\ x[u(|x|-|z|) - F_k^*(|x|)]\ P(R_i^+ = V+1 \mid X_i = x,\ |X_k| = |x| - 1)$$

where $u(t) = 1$ for $t \geq 0$, and $u(t) = 0$ for $t < 0$.

From Lemma 2.2, we have

$$P(R_i^+ = V+1 \mid X_i = x,\ |X_k| = |x| - 1) < N^{-k_2} \tag{2.5}$$

for some $k_2 > 1$ and all $|H^*(|x|) - v| \geq k_1 N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N$

Furthermore Lemmas 2.2 and 2.3 imply

$$P(R_i^+ = V+1 \mid X_i = x\ ,\ |X_k| = |x| - 1) = \Phi(Nv;\ \sum_{j=1}^{N} F_j^*(|x|), ND^2)$$

$$+\ \theta_6 N^{-1} Lg^{\frac{1}{2}} N$$

for some $|\theta_6| \leq k_{13}$ and all $|H^*(|x|) - v| \leq k_1 N^{-\frac{1}{2}} Lg^{\frac{1}{2}} N$.

As before, the last equality remains true even if

$$|H^*(|x|) - v| \leq k_1' DN^{-\frac{1}{2}} Lg^{\frac{1}{2}} N$$

Let $I' = \{x : |H^*(|x|) - v| \leq k_1' DN^{-\frac{1}{2}} Lg^{\frac{1}{2}} N\}$.

Then

$$E(a(R_i^+)\ \mathrm{sgn}\, X_i | X_k = z) - E(a(R_i^+)\ \mathrm{sgn}\, X_i) =$$

$$= \int \mathrm{sgn}\, x[u(|x| - |z|) - F_k^*(|x|)]P(R_i^+ = V + 1 | X_i = x, |X_k| = |x| - 1)dF_i(x)$$

$$= \int_{I'} (\cdots)dF_i(x) + \int_{\mathbb{R}\setminus I'} (\cdots)dF_i(x)$$

The second integral is $o(N^{-1})$ by (2.5), while the first is equal to

$$\int_{I'} \mathrm{sgn}\ x[u(|x| - |z|) - F_k^*(|x|)]\phi(Nv, \sum_{j=1}^{N} F_j^*(|x|), ND^2)dF_i(x)$$

$$+ \int_{I'} \mathrm{sgn}\ x[u(|x| - |z|) - F_k^*(|x|)]\ \theta_6\ N^{-1} Lg^{\frac{1}{2}}N\ dF_i(x)$$

In the last expression, let us denote by T_5 the first term and T_6 the second.

Then, we have

$$T_5 = D^{-1} N^{-\frac{1}{2}} \int_{I'} \mathrm{sgn}\ x[u(|x| - |z|) - F_k^*(|x|)]\phi(\frac{H^*(|x|) - v}{DN^{-\frac{1}{2}}})\ dF_i(x)$$

Using the fact that

$$D^{-1} N^{-\frac{1}{2}} \int_{|p| \le k_1' Lg^{\frac{1}{2}}N} G_k(v + DN^{-\frac{1}{2}}p)\ \phi(p)dL_i(v + DN^{-\frac{1}{2}}\ p)$$

$$= D^{-1} N^{-\frac{1}{2}} \int_{|p| \le k_1' Lg^{\frac{1}{2}}N} G_k(v)\ \phi(p)\ dL_i(v + DN^{-\frac{1}{2}}p) + o(N^{-1})$$

uniformly in i and k (which follows from (1.11)), we can show that

$$D^{-1} N^{-\frac{1}{2}} \int\limits_{\{x>0\ :\ |H^*(x)-v| \le k_i' DN^{-\frac{1}{2}} Lg^{\frac{1}{2}} N\}} [u(x - |z|) - F_k^*(x)] \ \phi\left(\frac{H^*(x) - v}{DN^{-\frac{1}{2}}}\right) dF_i(x)$$

$$= N^{-1} \ell_i(v) [1 - \Phi(q) - G_k(v)] + o(N^{-1})$$

uniformly in z, $1 \le i \le N$, where $q = \dfrac{H^*(|z|) - v}{DN^{-\frac{1}{2}}}$

Similarly we estimate the integral over $\{x < 0 : |H^*(-x) - v| \le k_i' DN^{-\frac{1}{2}} Lg^{\frac{1}{2}} N\}$ and we obtain

$$E(a(R_i^+) \text{ sgn } X_i | X_k = z) - E(a(R_i^+) \text{ sgn } X_i)$$

$$= N^{-1} [1 - \Phi(q) - G_k(v)] (\ell_i(v) - m_i(v)) + o(N^{-1}) \quad (2.6)$$

uniformly in $-\infty < z < \infty$.

Thus, denoting

$$K_k = \text{Cov}\{E(a(R_i^+) \text{ sgn } X_i | X_k) , E(a(R_j^+) \text{ sgn } X_j | X_k)\} \quad (2.7)$$

we obtain

$$K_k = \int_{-\infty}^{\infty} [E(a(R_i^+) \text{ sgn } X_i | X_k = z) - E(a(R_i^+) \text{ sgn } X_i)]$$

$$\times [E(a(R_j^+) \text{ sgn } X_j | X_k = z) - E(a(R_j^+) \text{ sgn } X_j)] \, dF_k(z)$$

$$= N^{-2} (\ell_i(v) - m_i(v)) (\ell_j(v) - m_j(v))$$

$$\times \int_{-v/DN^{-\frac{1}{2}}}^{(1-v)/DN^{-\frac{1}{2}}} [1 - \Phi(q) - G_k(v)]^2 \, dG_k(v + DN^{-\frac{1}{2}} q) + o(N^{-2})$$

Note that

$$\int_{-v/DN^{-1/2}}^{(1-v)/DN^{-1/2}} (1 - \Phi(q))dG_k(v + DN^{-1/2}q) = G_k(v) + o(1)$$

and

$$\int_{-v/DN^{-1/2}}^{(1-v)/DN^{-1/2}} (1 - \Phi(q))^2 dG_k(v + DN^{-1/2}q) = G_k(v) + o(1)$$

Hence:

$$K_k = N^{-2}(\ell_i(v) - \ell_j(v))(m_i(v) - m_j(v))[G_k(v)(1 - G_k(v))]$$

$$+ N^{-2}(\ell_i(v) - \ell_j(v))(m_i(v) - m_j(v))o(1) + o(N^{-2})$$

Summing over $1 \le k \le N$, $k \ne i,j$ $(i \ne j)$, we finally obtain:

$$\sum_{k\ne i,j} K_k = N^{-1}(\ell_i(v) - \ell_j(v))(m_i(v) - m_j(v))D^2 + o(N^{-1})$$

uniformly in $1 \le i \ne j \le N$. The proof follows.

Using the Residual Variance Inequality and Lemmas 2.4 - 2.7, we obtain

Lemma 2.8. For $N \to \infty$, we have

$$E(S_N^+ - \hat{S}_N^+)^2 = o(\sum_{i=1}^{N} c_i^2) \quad . \tag{2.8}$$

Now from the definition of $\hat{S}_N^+$, we have

$$\hat{S}_N^+ - E(\hat{S}_N^+) = \sum_{i=1}^{N}\sum_{j=1}^{N} c_j\{E(a(R_j^+) \text{ sgn } X_j | X_i) - E(a(R_j^+) \text{ sgn } X_j)\}$$

Set

$$Y_i = \sum_{j=1}^{N} c_j\{E(a(R_j^+)\ \mathrm{sgn}\ X_j|X_i) - E(a(R_j^+)\ \mathrm{sgn}\ X_j)\}\ ,\ 1 \le i \le N$$

and note that the Y_i , $1 \le i \le N$ are independent random variables with $E(Y_i) = 0$ and $\mathrm{Var}(\hat{S}_N^+) = \sum_{i=1}^{N} \mathrm{Var}\ Y_i$.

Define

$$Z_i = N^{-1}\Big(\sum_{\substack{j=1\\ j\neq i}}^{N} c_j(\ell_j(v) - m_j(v))\Big)[u(v - H^*(|X_i|)) - G_i(v)]$$

$$+ c_i[E(\mathrm{sgn}\ X_i a(R_i^+)|X_i) - E(\mathrm{sgn}\ X_i a(R_i^+))]\ ,\ 1 \le i \le N$$

and note that the Z_i, $1 \le i \le N$ are independent random variables with $E(Z_i) = 0$, $1 \le i \le N$.

Now for $j \neq i$, $1 \le i,j \le N$ we have

$$E(\mathrm{sgn}\ X_j a(R_j^+)|X_i) - E(\mathrm{sgn}\ X_j\ a(R_j^+))$$

$$= \frac{1}{N}[\ell_j(v) - m_j(v)][\Phi\left(\frac{v - H^*(|X_i|)}{DN^{-\frac{1}{2}}}\right) - G_i(v)] + \eta_j'$$

where the η_j' are random variables such that $|\eta_j'| \le \varepsilon_N'$ for some sequence of constants ε_N' satisfying $N\varepsilon_N' \to 0$.

Now proceeding as in the derivation of (5.4) in Dupač and Hájek (1969), we obtain

$$E(Y_i - Z_i)^2 = o(N^{-1}\sum_{i=1}^{N} c_i^2)\ . \tag{2.9}$$

Using

$$\mathrm{Var}(S_N^+ - \sum_{i=1}^{N} Z_i) \le 2\sum_{i=1}^{N} E(Y_i - Z_i)^2 + 2E(S_N^+ - \hat{S}_N^+)^2\ ,$$

(2.8) and (2.9), we obtain

<u>Lemma 2.9</u>. $\mathrm{Var}(S_N^+ - \sum_{i=1}^{N} Z_i) = o(\sum_{j=1}^{N} c_j^2)$. (2.10)

Lemma 2.10. (1.18) *holds if and only if*

$$\liminf_{N\to\infty} \sigma_N^2 / \sum_{i=1}^{N} c_i^2 > 0 \tag{2.11}$$

where

$$\sigma_N^2 = \sum_{i=1}^{N} \mathrm{Var}(Z_i)$$

In this case, $\lim_{N\to\infty} \mathrm{Var}(S_N^+)/\sigma_N^2 = 1$.

Proof. It follows from the Minkowski inequality that

$$((\mathrm{Var}(U_1)/\mathrm{Var}(U_2))^{1/2} - 1)^2 \le \mathrm{Var}(U_1 - U_2)/\mathrm{Var}(U_2) \tag{2.12}$$

Let (1.18) be satisfied, then putting $U_1 = \sum_{i=1}^{N} Z_i$, $U_2 = S_N^+$ in (2.12) and using Lemma 2.9, we obtain (2.11). Let (2.11) be satisfied, then putting $U_1 = S_N^+$, $U_2 = \sum_{i=1}^{N} Z_i$, and using Lemma 2.9, we obtain (1.18).

Lemma 2.11. *The random variables* $\sum_{i=1}^{N} Z_i$ *are asymptotically normal with parameters* $(0,\sigma_N^2)$.

Proof. Since $\ell_i(v)$, $m_i(v)$ are bounded as functions of (i,N) , $1 \le i \le N$, it follows that

$$|Z_i| \le C \max_{1\le j\le N} |c_j| \quad \text{for some constant } C > 0 . \tag{2.12'}$$

Now (1.2) and (2.11) along with (2.12') imply

$$\max_{1\le i\le N} |Z_i|/\sigma_N = o(1) ,$$

which by the Markov inequality, implies the Lindeberg condition for asymptotic normality.

Finally, since we have proved that

$$\sum_{i=1}^{N} Z_i/\sigma_N \xrightarrow{D} N(0,1) \ , \ (S_N^+ - E(S_N^+) - \sum_{i=1}^{N} Z_i)/\sigma_N \xrightarrow{L^2} o \ , \tag{2.13}$$

and $\mathrm{Var}(S_N^+)/\sigma_N^2 \to 1$,

we obtain

$$(S_N^+ - E(S_N^+))/(\mathrm{Var}\ S_N^+)^{1/2} \xrightarrow{D} N(0,1) \tag{2.14}$$

Remark 1. Suppose we want to relax the condition (1.3) to (1.4). Let us denote the statistic corresponding to (1.3) by S_N^+ and the statistic corresponding to (1.4) by S_N^{+*} . Then using (1.2) and (1.4), it follows that $\mathrm{Var}(S_N^+ - S_N^{+*}) = o(\sum_{i=1}^{N} c_i^2)$. Consequently, the asymptotic normality of S_N^{+*} follows by using (2.11), (2.12), (2.13) and (2.14).

Remark 2. We have proved Theorem 2.1 under condition (C_1^+) . It remains to show that this set of conditions is implied by the conditions (C_2^+) and (C_3^+) . The proofs of these facts are similar to the implications $(C_3) \Rightarrow (C_1)$ and $(C_2) \Rightarrow (C_1)$ in Dupač and Hájek (1969, Section 5), and are therefore omitted.

The following theorem based on Theorem 2.1 and on Lemma 2 of Hušková (1970) combines unbounded c_{Ni} with a class of bounded score generating functions. The proof of this theorem is similar to that of Theorem 3 in Dupač and Hájek (1969) and is omitted.

Theorem 2.2. *Let* $S_N^+ = \sum_{i=1}^{N} c_{Ni} a_N(R_{Ni}^+)\ \mathrm{sgn}\ X_{Ni}$ *where*

$a_N(i) = \psi(i/(N+1))$. Assume that $\psi = \psi_1 + \psi_2$, where ψ_1 is constant but for a finite number of jumps, and ψ_2 has a bounded second derivative. Assume that any one set of the conditions (C_1^+) , (C_2^+) , (C_3^+) holds along with (1.2). Then S_N^+ is asymptotically normal with natural parameters $(ES_N^+, \operatorname{Var} S_N^+)$.

We now show that under slightly strengthened assumptions concerning the regression constants, S_N^+ is asymptotically normal with (simpler) parameters (μ_N^+, σ_N^2) where

$$\mu_N^+ = \sum_{i=1}^{N} c_i E[\operatorname{sgn} X_i \psi(H^*(|X_i|))] \qquad (2.15)$$

and

$$\sigma_N^2 = \sum_{i=1}^{N} \operatorname{Var} Z_i \qquad (2.16)$$

Theorem 2.3. Consider the statistic S_N^+ given by (1.1) with scores given by (1.4) where $\psi(t) = u(t-v)$. Assume that (C_1^+) or (C_2^+) or (C_3^+) holds. Then S_N^+ is asymptotically normal with parameters (μ_N^+, σ_N^2) defined in (2.15) and (2.16) if

$$\max_{1\le i\le N} c_i^2 \Big/ \sum_{i=1}^{N} c_i^2 = O(N^{-\delta-1/2}) \text{ for some } \delta > 0 . \qquad (2.17)$$

Proof. Define

$$\Delta_i(X_i) = \{E(\operatorname{sgn} X_i a(R_i^+) | X_i) - E(\operatorname{sgn} X_i a(R_i^+))\}$$
$$- \{\operatorname{sgn} X_i \psi(H^*(|X_i|)) - E(\operatorname{sgn} X_i \psi(H^*(|X_i|)))\}$$

Proceeding as in Dupač (1970), it can be shown (omitting the details of computation) that

$$E(\Delta_i^2) = O(N^{-1/2}) \text{ , where } \Delta_i = \Delta_i(X_i) .$$

This, together with (2.17) entails

$$c_i^2 \, E(\Delta_i^2) = o(N^{-1} \sum_{i=1}^{N} c_i^2)$$

Now, using the inequality,

$$(E(S_N^+) - \mu_N^+)^2 \le (\sum_{i=1}^{N} c_i^2)(\sum_{i=1}^{N} [E(\operatorname{sgn} X_i a(R_i^+)) - E(\operatorname{sgn} X_i \psi(H^*(|X_i|)))]^2) ,$$

we obtain

$$(E(S_N^+) - \mu_N^+)^2 = o(\sum_{i=1}^{N} c_i^2) .$$

Now writing

$$E((S_N^+ - \mu_N^+ - \sum_{i=1}^{N} Z_i)/\sigma_N)^2 \le 2E((S_N^+ - E(S_N^+) - \sum_{i=1}^{N} Z_i)/\sigma_N)^2 + 2(E(S_N^+) - \mu_N^+)^2/\sigma_N^2$$

and proceeding as in Theorem 2.1, the proof follows.

Theorem 2.4. *Consider the statistic* S_N^+ *given by* (1.1) *with the scores given by* (1.3). *Assume that* $\psi = \psi_1 + \psi_2$ *where* $\psi_1 = \sum_{j=1}^{k} \lambda_j \psi_{v_j}$ *where* $\psi_{v_j}(t) = u(t - v_j)$, $j = 1,\ldots,k,$ *and* ψ_2 *has a bounded second derivative*. *Let* $Z_i = Z_i^{\psi_2} + \sum_{\ell=1}^{k} \lambda_\ell Z_i^{\psi_{v_\ell}}$, *where*

$$Z_i^{\psi_{v_\ell}} = N^{-1} (\sum_{\substack{j=1 \\ j\ne i}}^{N} c_j(\ell_j(v_\ell)) - m_j(v_\ell)) \, (u(v_\ell - H^*(|X_i|)) - G_i(v_\ell)) + c_i[E(\operatorname{sgn} X_i a(R_i^+) | X_i) - E(\operatorname{sgn} X_i \, a(R_i^+))] ,$$

$$1 \le i \le N , \; \ell = 1,\ldots,k$$

and

$$Z_i^{\psi_2} = N^{-1} \sum_{j=1}^{N} c_j \int \operatorname{sgn} x[u(|x| - |X_i|) - F_i^*(|x|)]\psi_2'(H^*(|x|))dF_j(x)$$

$$+ c_i[\operatorname{sgn} X_i\psi_2(H^*(|X_i|)) - E(\operatorname{sgn} X_i\psi_2(H^*(|X_i|)))],$$

$$1 \le i \le N$$

(cf. Hušková (1970), p. 310)

Assume that (C_1^+) or (C_2^+) or (C_3^+) holds. Then the condition (2.17) implies the asymptotic normality of S_N^+ with parameters (μ_N^+, σ_N^2) where μ_N^+ and σ_N^2 are given by (2.15) and (2.16) respectively with Z_i given by (2.18).

The proof follows by combining Theorem 2.3, lemma 2 of Hušková (1970) and going through routine mathematical details.

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