

AD-AU90 176

DELAWARE UNIV NEWARK APPLIED MATHEMATICS INST F/G 20/3
SOME OPTIMAL CONTROL PROBLEMS FOR THE HELMHOLTZ EQUATION. (U)
JUL 80 T S ANGELL AFOSR-79-0085

UNCLASSIFIED

AFOSR-TR-80-0962

NL

1 of 1
AD-AU90 176

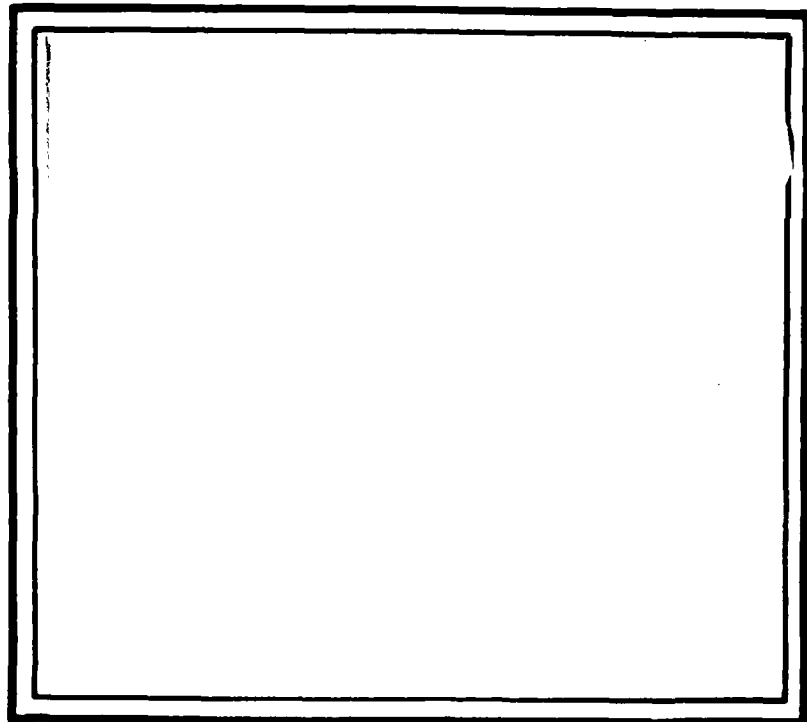


END
DATE
FILMED
11-80
DTIC

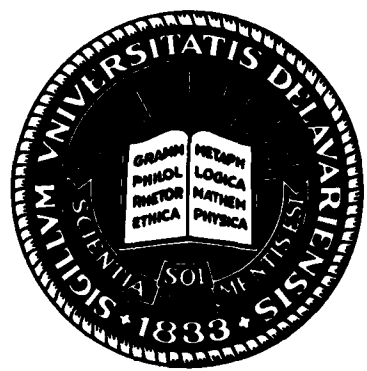
F

(11)

AD A090176



LEVEL II



DTIC
ELECTE
OCT 10 1980
D
C

**APPLIED
MATHEMATICS INSTITUTE**

**University of Delaware ✓
Newark, Delaware**

DDC FILE COPY

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

14 TR-87A

1. REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
2. REPORT NUMBER	7. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER	
18 AFOSR TR-80-0962	AD-A090176		
4. TITLE (and Subtitle)		5. TYPE OF REPORT & PERIOD COVERED	
6 SOME OPTIMAL CONTROL PROBLEMS FOR THE HELMHOLTZ EQUATION.		9 Interim rept.	
7. AUTHOR(s)		8. CONTRACT OR GRANT NUMBER(s)	
10 Thomas S. Angell		15 AFOSR-79-0085 NEW	
9. PERFORMING ORGANIZATION NAME AND ADDRESS		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	
University of Delaware Dept of Mathematical Sciences Newark, DE 19711		16 61102F 2304/A4	
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE	
Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332		11 Jul 80	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES	
12 17		16	
		15. SECURITY CLASS. (of this report)	
		UNCLASSIFIED	
16. DISTRIBUTION STATEMENT (of this Report)		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
Approved for public release; distribution unlimited.			
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)			
18. SUPPLEMENTARY NOTES			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)			
Radiation and scattering problems are formulated as optimal control problems in which either a current or surface impedance is sought from a class of admissible functions which optimizes a functional of the scattered far field. In both cases the existence of an optimal solution is proven. In the linear (radiation) case constructive algorithms for finding the optimal solution are presented.			

411278 PM UNCLASSIFIED

11

SOME OPTIMAL CONTROL PROBLEMS FOR
THE HELMHOLTZ EQUATION

Thomas S. Angell

Department of Mathematical Sciences
University of Delaware
Newark, Delaware

July, 1980

Applied Mathematics Institute
Technical Report No. 87A ✓

DUPLICATE
SELLING
OCT 10 1980

*Dedicated to Professor Lamberto Cesari on the occasion
of his seventieth birthday.*

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF TRANSMITTAL TO DDC
This technical report has been reviewed and is
approved for public release IAW AFR 190-12 (7b).
Distribution is unlimited.
A. D. BLOSE
Technical Information Officer

*International Conference on Nonlinear Phenomena in Mathematical
Sciences, June 16-20, 1980, University of Texas at Arlington.*

411278 DM

Abstract

Radiation and scattering problems are formulated as optimal control problems in which either a current or surface impedance is sought from a class of admissible functions which optimizes a functional of the scattered far field. In both cases the existence of an optimal solution is proven. In the linear (radiation) case constructive algorithms for finding the optimal solution are presented.

Accession For	
NTIS STAB	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/ _____	
Availability Codes	
Dist	Avail and/or Special
A	

SOME OPTIMAL CONTROL PROBLEMS FOR
THE HELMHOLTZ EQUATION

Thomas S. Angell

Department of Mathematical Sciences
University of Delaware
Newark, Delaware

Dedicated to Professor Lamberto Cesari on the occasion
of his seventieth birthday.

1. INTRODUCTION

In this paper, we review results obtained by the author in close collaboration with R.E. Kleinman [1], [2], [3], [4]. The class of problems referred to in the title arise in the study of time harmonic electromagnetic and acoustic radiation and scattering. As such, these problems are problems in exterior domains in contrast to the more common problems with bounded domains usually encountered in the literature of distributed parameter control systems. Such problems require, in addition to the usual boundary condition, a so-called "radiation condition" in order to guarantee that the problem has a unique solution. This condition at infinity can then be used, as in classical potential theory, to recast the problem which is originally stated in differential form, into a problem involving integral operators acting on appropriate spaces of functions defined on the boundary of the scattering obstacle or radiating structure. It is this formulation of the control problems in terms of boundary integral equations that we will use in our discussion of the scattering problem.

First, however, we will consider a simpler linear radiation problem with Dirichlet conditions under the assumption that we have available a Green's function appropriate to the exterior

domain. This will help to focus the nature of the boundary control problems we consider and will allow us to mention some numerical results for the control of radiation patterns. In the last section, we will turn to the more complicated non linear optimization problem with a Robin boundary condition and without the assumption that the appropriate Green's function is available. In our treatment of this non linear problem of optimal control, we will use a pair of boundary integral equations derived in [3], the unique solution to which affords a solution of the exterior boundary value problem. That we deal with a pair of boundary integral equations rather than a single equation is a result of the well-known non-uniqueness of solutions of the first of these equations at interior eigenvalues of the homogeneous Dirichlet problem. It must be emphasized that these exceptional values of the wave number do not correspond to exceptional values for the exterior scattering problem but are the result of the particular reformulation of the exterior boundary value problem in terms of integral equations. Various means have been proposed to circumvent this difficulty (see e.g. [6] and [7]). Here, we follow [8] by introducing a second equation involving the normal derivative of the double layer potential which restores uniqueness for real wave numbers.

2. FORMULATION OF THE LINEAR PROBLEM

Let Γ be a simple closed curve, sufficiently smooth (C^2 is more than sufficient for our purposes) in E^2 , let Ω be the unbounded region determined by Γ and let Ω^c denote the complementary region. In particular, the curve Γ will have a

a unit normal $\hat{n}(q)$ which varies continuously on Γ . We emphasize that we shall take $\hat{n}(q)$ to be the outward drawn normal with respect to Ω^c . Further, we shall write $\partial/\partial n_p^-$ and $\partial/\partial n_p^+$ to denote the normal derivative when $P \rightarrow p \in \Gamma$ from Ω^c and Ω respectively.

We choose the origin of a rectangular coordinate system in the interior of Ω^c and let (r, θ) be polar coordinates of an arbitrary point $P \in \Omega$. The three classical boundary value problems, Dirichlet (D), Neumann (N), and Robin (R) problems for the Helmholtz equation in the exterior domain Ω consist of finding a function u , defined in Ω , satisfying (1,i) and (1,iii) below as well as the appropriate boundary condition in (1,ii):

$$(1) \quad \begin{cases} (i) & (\nabla^2 + k^2) u = 0 \quad \text{in } \Omega; \\ (ii) & \begin{cases} (D) & u = h \\ (N) & \partial u / \partial n = h \\ (R) & \partial u / \partial n + \sigma u = h \end{cases} \quad \text{on } \Gamma; \\ (iii) & \lim_{r \rightarrow \infty} \int_{S_r} |\partial u / \partial r - iku|^2 dS = 0 \end{cases}$$

where S_r is a circle of radius r lying entirely in Ω . This last condition is the radiation condition referred to above.

For the problem of Dirichlet (D) the Green's function has the form

$$(2) \quad G(P, Q) = (1/4) H_0^{(1)}(k|P-Q|) + g(P, Q)$$

where $g(P, Q) \in C^2(\Omega)$ and $H_0^{(1)}$ is the Hankel function of the first kind, the fundamental solution of the Helmholtz equation consistent with the radiation condition. Since Γ is bounded one may employ the asymptotic properties of G for large r viz. $G(P, Q) = (e^{ikr}/r^2) \hat{G}(\theta, \varphi) + o(1/r)$ to represent the solution u in the far field as

$$(3) \quad u(P;h) = (e^{ikr}/r^{\frac{1}{2}}) \int_{\Gamma} h(q) \partial/\partial n_q \hat{G}(\theta, q) d\Gamma_q + O(1/r).$$

The regularity of \hat{G} allows us to define the far field pattern $f \in L^2(0, 2\pi)$ in terms of the Hilbert-Schmidt integral operator $K : L^2(\Gamma) \rightarrow L^2(0, 2\pi)$ generated by the kernel $\partial/\partial n_q \hat{G}(\theta, q)$

$$(4) \quad f(\theta) := \int_{\Gamma} h(q) \partial/\partial n_q \hat{G}(\theta, q) d\Gamma_q.$$

Two of the problems commonly discussed in the engineering literature of antenna theory fit naturally into the present context. We refer to them as the identification problem and the synthesis problem. In the former, we are given an actual far field pattern and are asked to find the boundary data h whose associated solution has f as its far field. In the latter, we are given a desired far field pattern g and are asked to find the boundary data h producing a far field closest to the desired pattern in the L^2 sense. In terms of the integral operator K introduced above, the first of these is the problem of solving the integral equation of the first kind $Kh = f$ with f given in the range of K , while the second is that of finding an absolute minimum for the form $\|Kh - g\|$. Both of these problems fall into the class of problems called ill-posed in the sense of Hadamard. The interested reader may refer to the paper of Angell and Washed [5] for a complete discussion of these problems as they arise in antenna theory.

Here, it is our purpose to present a different optimization problem in which the quantity to be optimized is the power radiated into a specified portion of the far field. Specifically,

Specifically, we define the far field power in a measurable set $\alpha \subset [0, 2\pi]$ by

$$(5) \quad Q_{\alpha}(h) := \int_0^{2\pi} \alpha(\theta) |r(\theta)|^2 d\theta = \int_0^{2\pi} \alpha(\theta) |Kh|^2(\theta) d\theta$$

where $\alpha(\theta)$ is the characteristic function of the set α .

Having introduced this quadratic functional, we may pose a meaningful optimization problem.

Let $U \subset L^2(\Gamma)$ be a closed, bounded, convex set.

We will refer to the elements of U as admissible controls.

The optimal control problem is now to find an $h_0 \in U$ giving the functional $Q_{\alpha}(h)$ its maximum value i.e. find $h_0 \in U$ such that $Q_{\alpha}(h_0) \geq Q_{\alpha}(h)$, for all $h \in U$.

Since the form $Q_{\alpha}(h) = (\alpha Kh, Kh)_{L^2(0, 2\pi)}$ is both convex and weakly continuous, it is easy to prove the following.

Theorem 1: If $U \subset L^2(0, 2\pi)$ is closed, bounded and convex then there exists an element $h_0 \in \text{bdy } U$ which is optimal with respect to the performance index Q_{α} defined by (5).

We remark that this statement not only asserts the existence of an optimal solution, but also gives an additional condition which leads to a "bang-bang" result for certain special choices of the class U of admissible controls (see the appendix in [1] for details). The requirements that the set U be a closed, convex and bounded set in $L^2(\Gamma)$ still allows enough flexibility to allow us to model a number of constraints suggested by physical or design considerations. In the engineering literature, the former are usually referred

to as realizability conditions, and include constraint sets of the following forms:

$$(a) \quad U = \{h \in L^2(\Gamma) \mid \|h\| \leq 1\} \quad (\text{inputs with bounded energy});$$

$$(b) \quad U = \{h \in H^1(\Gamma) \mid \|h\|_1 \leq 1\} \quad (\text{inputs with bounded energy and oscillation});$$

$$(c) \quad U = \{h \in L^2(\Gamma) \mid \psi_0(q) \leq h(q) \leq \psi_1(q), \text{ a.e., } \psi_0, \psi_1 \in C(\Gamma)\};$$

while design considerations may lead to control domains defined by state constraints, e.g.

$$(d) \quad U = \{h \in H^1(\Gamma) \mid \|(1 - \alpha(\theta)) f(\theta)\| \leq 1\} \quad (\text{bounded side lobes}).$$

We remark that it is case (c) for which we can show that there exists a maximizing sequence of "bang-bang" controls since the extreme points of the set U in (c) are functions of the form

$$\psi_0 \chi_{E_0} + \psi_1 \chi_{E_1} \quad \text{where } \{E_0, E_1\} \text{ is a measurable partition}$$

of Γ . For full proofs the reader is referred to [1].

In the case that U is the unit sphere in $L^2(\Gamma)$, the observation that the functional \mathcal{J}_α can be expressed in terms of the symmetric operator $R := K^* \alpha K$, where K^* is the adjoint of K , leads to the following result.

Theorem 2: If λ_0 is the largest eigenvalue of the operator R and h_0 is a corresponding normalized eigenvector, then

$$\sup_{\|h\|=1} \mathcal{J}_\alpha(h) = \sup_{\|h\|=1} (Rh, h)_{L^2(\Gamma)} = \lambda_0 = \mathcal{J}_\alpha(h_0).$$

One may now implement constructive methods to compute the optimal control by considering a family of subspaces E_n of $L^2(\Gamma)$, ultimately dense in this function space, and considering the finite dimensional problems $P_n(\lambda u - Ru) = 0$, $n = 1, 2, \dots$, where P_n is the projection associated with the

subspace \mathcal{H}_n . Standard estimates for the rate of convergence are available e.g. one has the estimate

$$(6) \quad |\lambda_0 - \lambda_{0,n}| \leq \sqrt{2} \|(I - P_n)R\|$$

where $\lambda_{0,n}$ is the eigenvalue for the finite dimensional problem and λ_0 is the true eigenvalue (see Mikhlin [9]). This method has been implemented on the Burrough's 6800 at the University of Delaware using standard ISM eigenvalue routines to produce highly directed single and double beam patterns for a circular antenna of radius a for various values of ka from 1 to 20.

3. THE NONLINEAR PROBLEM

In this section, we will consider, not a radiation problem, but a problem in scattering wherein we assume that an object, again with a sufficiently regular boundary, in \mathbb{R}^2 , interacts with a known incident field, u^i , to produce a scattered field u^s . In this case, u^s is a solution of the Helmholtz equation in the unbounded region Ω which satisfies the radiation condition, while u^i is assumed to be a solution of the Helmholtz equation in Ω^c .

The nonlinear character of the problem arises from the form of the boundary condition. Under suitable restrictions on the geometry of the scatterer and constitutive parameters, among which is the requirement that the radius of curvature be large relative to the skin depth (see e.g. [10]) the transition conditions at the surface of an imperfectly conducting scatterer may be replaced by what is usually called an impedance boundary condition. Here, such a condition takes the form of the

Robin boundary condition $\partial u / \partial n + \sigma u = 0$, where $\sigma \in L^\infty(\Gamma)$ represents the surface impedance. It is this surface impedance which we take as the control and, by requesting that σ lie in a preassigned closed, bounded and convex (and hence weak*-sequentially compact) subset of $L^\infty(\Gamma)$, we may again consider the constrained optimization problem of finding that admissible control which affords an absolute maximum to the functional J , defined by (5).

In this problem, the expression for the far field depends explicitly on the product of the control and the scattered field and it is here that the nonlinearity arises. Our treatment is more complicated than that followed in the previous case not only due to this nonlinearity but also in that we do not assume that we have explicit knowledge of the Green's function for the region. Instead, we rely on the Helmholtz representation, knowing the fundamental solution $\gamma(P, Q) = (-i/2) H_0^{(1)}(k|P-Q|)$, to derive boundary integral equations whose unique solution may then be used to construct the scattered field in the exterior domain Ω .

The scattered and incident fields being regular solutions of the Helmholtz equation in Ω and Ω^c respectively, we may write

$$(7) \quad u^S = S(\partial u^S / \partial n) - D(u^S) \quad \text{and} \quad u^I = D(u^I) - S(\partial u^I / \partial n)$$

where S and D are the usual single and double layer operators with kernels $\gamma(P, q)$ and $\partial \gamma / \partial n_q(P, q)$ respectively. If K denotes the integral operator with kernel $\partial \gamma / \partial n_p(p, q)$, then K^* is the integral operator for D and the usual jump relations

for the normal derivative of the single layer and for the double layer with density μ , namely

$$(8) \quad \begin{aligned} \partial/\partial n_p^+(S\mu)(p) &= (I + K)\mu(p) \\ \partial/\partial n_p^-(S\mu)(p) &= (-I + K)\mu(p) \\ \lim_{P \rightarrow p^+}(D\mu)(P) &= (-I + \bar{K}^*)\mu(p) \\ \lim_{P \rightarrow p^-}(D\mu)(P) &= (I + \bar{K}^*)\mu(p) \end{aligned}$$

together with the relation (7) and the boundary condition lead easily to a pair of integral equations which may be written as

$$(9) \quad \begin{aligned} (I + S\sigma + \bar{K}^*)u &= 2u^1 \\ (-\sigma I + \bar{K}^*\sigma + D_n)u &= 2\partial u^1/\partial n \end{aligned}$$

where the operator D_n is defined by

$$(10) \quad (D_n\mu)(p) := \partial/\partial n_p \int_r \mu(q) \partial\gamma/\partial n_q(p,q) d\Gamma_q.$$

Note that the integral operators all depend on the wave number k although we have not explicitly indicated such dependence. The values of the wave number for which there exist non-trivial solutions of $w + kw = 0$ will be called characteristic values of $(-K)$ and a classical result is that k is such a characteristic value if and only if it is an eigenvalue of the interior homogeneous Dirichlet problem. As described above, while the exterior Robin problem has a unique solution, at interior eigenvalues of the Dirichlet problem the first of these boundary integral equations has a multiplicity of solutions and it is for this reason that we consider the pair of equations (9).

The analysis in [3] shows that, for appropriate choices of k , and in particular for k real, the system (9) has a

unique solution which can then be used to construct a solution of the original scattering problem. Specifically, we have the following result [3]:

Theorem 3: Let $\sigma \in L^\infty(\Gamma)$ and suppose that k satisfies the conditions $\text{Im } k \geq 0$, $\text{Im}(\bar{k}\sigma) \geq 0$. Then there exists a unique solution $\hat{u} \in L^2(\Gamma)$ of the pair of boundary integral equations (9). Moreover, the function $u = u^i + u^s$ is a solution of the exterior Robin problem

- (i) $u \in C^2(\Omega)$, $u, \partial u / \partial n \in L^2(\Gamma)$;
- (ii) $(\nabla^2 + k^2) u^s = 0$ in Ω ,
 $(\nabla^2 + k^2) u^i = 0$ in Ω^c ;
- (iii) $\lim_{r \rightarrow \infty} \int_{S_r} |\partial u^s / \partial r - iku^s|^2 dS_r = 0$;
- (iv) $\partial u / \partial n + \sigma u = 0$ a.e. on Γ ;

if and only if

$$u^s = \frac{1}{2}S(-\partial u^i / \partial n - \sigma \hat{u}) - \frac{1}{2}D(\hat{u} - u^i)$$

where \hat{u} is the unique solution of the system of boundary integral equations (9).

Returning to the optimization problem, it is possible to represent the far scattered field in the form

$$(11) \quad f(\theta) = K_1(\partial u^i / \partial n + \sigma u) - K_2 u^s$$

where K_1 and K_2 are compact operators from $L^2(\Gamma)$ to $L^2(0, 2\pi)$.

Using the cost functional given in (5), we can pose the following optimization problem:

Given $U \subset L^\infty(\Gamma)$ a closed, bounded, convex set, find $\sigma_0 \in U$ for which $J_\alpha(\sigma_0) \leq J_\alpha(\sigma)$ for all $\sigma \in U$.

Note that the results described above guarantee that, given $\sigma \in U$ there exists a unique solution $u(\sigma)$ of the system (9).

If we let A be the set of all pairs $\{\sigma, u(\sigma)\}$, $\sigma \in U$, we may prove the following theorems (see [4] for complete details).

Theorem 4: The set $A \subset L^\infty(\Gamma) \times L^2(\Gamma)$ is bounded in the product topology.

Theorem 5: The set A is closed relative to weak*-convergence of the σ and strong convergence of the u .

Theorem 6: The map $\sigma \mapsto f$ is continuous from the weak*-topology of $L^\infty(\Gamma)$ to the strong topology of $L^2(0, 2\pi)$.

Theorem 7: There exists an element $\sigma_0 \in U$ (and consequently a pair $\{\sigma_0, u(\sigma_0)\} \in A$) such that $Q_\alpha(\sigma_0) \geq Q_\alpha(\sigma)$ for all $\sigma \in U$.

Clearly, this last theorem follows immediately from Theorem 6 if one takes into account the form of the cost functional. The techniques used to establish Theorems 4, 5, and 6 are similar and we conclude our discussion with a sketch of the proof of Theorem 4 in the hope that it will give the reader sufficient insight.

Proof of Theorem 4: Suppose that the set A were not bounded. Then, since U is bounded, we may choose a sequence $\{(\sigma_m, u_m)\} \subset A$ for which the set $\{\sigma_m\}$ is bounded but $\|u_m\| \rightarrow \infty$. Let $\psi_m := u_m / \|u_m\|$, and choose a subsequence for which

(relabeling) $\sigma_m \xrightarrow{w^*} \sigma$, $\psi_m \xrightarrow{w} \psi$, and $\sigma_m \psi_m \xrightarrow{w} \rho \in L^2(\Gamma)$.

Then the functions ψ_m satisfy the integral equation

$$(12) \quad (I + S\sigma_m + D)\psi_m = 2u^1 / \|u_m\|.$$

Now the right-hand member converges strongly to 0 in $L^2(\Gamma)$

since the incident field u^1 is fixed, while the compactness of the operators D and S insure that $D\psi_m \xrightarrow{S} D\psi$ and $S\sigma_m\psi_m \xrightarrow{S} S\rho$. But then, the relation (12) can be rewritten as

$$(13) \quad \psi_m = 2u^1/\|u_m\| - S\sigma_m\psi_m - D\psi_m$$

which shows that $\psi_m \xrightarrow{S} \psi$. Moreover, this strong convergence of the ψ_m is enough to insure that the products $\sigma_m\psi_m \xrightarrow{W} \sigma\psi$ in $L^2(\Gamma)$ and, consequently, that the function ψ satisfies

$$(14) \quad (I + S\sigma + D)\psi = 0.$$

Now, consider the second boundary integral equation

$$(15) \quad (-\sigma I + \bar{K}^*\sigma + D_n)u = 2\partial u^1/\partial n$$

and look at the sequence

$$(16) \quad D_n\psi_m = 2(\partial u^1/\partial n)\|u_m\|^{-1} - (-\sigma_m + \bar{K}^*\sigma_m)\psi_m.$$

Since the ψ_m converge to ψ strongly and the products $\sigma_m\psi_m$ converge to $\sigma\psi$ weakly in $L^2(\Gamma)$, the functions $D_n\psi_m$ defined by (16) must converge weakly in this space to $\sigma\psi - \bar{K}^*\sigma\psi$.

This fact, together with the estimates which appear in [3], shows that the function ψ , as a solution of the first equation (12), must lie in the domain of the operator D_n and must satisfy the homogeneous equation

$$(17) \quad (-\sigma I + \bar{K}^*\sigma + D_n)\psi = 0.$$

Hence the function ψ , as the strong limit of elements of norm 1, must be a non-trivial solution to the homogeneous system of boundary integral equations, which contradicts the unique solvability of the system (9).

ACKNOWLEDGMENT: This work was supported by the U.S. Air Force under grant AFOSR 79-0085.

References

1. T. S. Angell and R. E. Kleinman, Optimal Control Problems in Radiation and Scattering, Proceedings of Symposium on Applied Mathematics dedicated to the late Professor R. Timman, A. J. Hermans and M. W. C. Oosterveld, ed., Delft University Press, Sijthoff and Noordhoff, (1978) 78-90.
2. T. S. Angell and R. E. Kleinman, Optimally Directed Antennas, in Information Linkage Between Applied Mathematics and Industry, P. C. C. Wang, Ed., Academic Press, (1979) 503-512.
3. T. S. Angell and R. E. Kleinman, Boundary Integral Equations for the Helmholtz Equation; the Third Boundary Value Problem, University of Delaware Applied Mathematics Institute Technical Report 73A, (1980).
4. T. S. Angell and R. E. Kleinman, Scattering Control by Impedance Loading, University of Delaware Applied Mathematics Technical Report 81A, (1980) (to be included in Proceedings of International URSI Symposium 1980 on Electromagnetic Waves, Munich).
5. T. S. Angell and M. Z. Nashed, Operator-theoretic and computational aspects of ill-posed problems in antenna theory, International Symposium on the Mathematical Theory of Networks and Systems, Vol. 3, P. Dewilde, ed., Delft University Press, Western Periodicals, (1979).
6. H. Brakhage and P. Werner, Über das Dirichetsche Aussenraum problem für die Helmholtzsche Schwingungsgleichung, Arch. Math. (Basel) 16, (1965) 325-329.
7. D. S. Jones, Integral Equations for the Exterior Acoustic Problem, Q. Journ. Appl. Math. XXVII, pt. I (1974), 129-142.
8. R. E. Kleinman and G. F. Roach, Boundary Integral Equations for the Three Dimensional Helmholtz Equation, SIAM Review, 16, (1974) 214-236.
9. S. G. Mikhailin, The Numerical Performance of Variational Methods, R. S. Anderssen tr., Wolters-Noordhoff, Groningen (1971).
10. T. B. A. Senior, Impedance Boundary Conditions for Imperfectly Conducting Surfaces, Appl. Sci., Res. B, (1960) 418-436.