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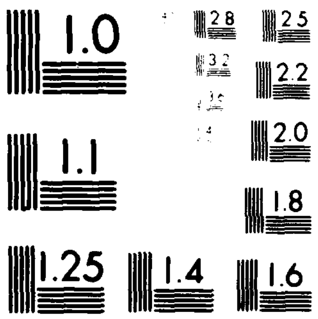
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A DECENTRALIZED TEAM DECISION PROBLEM WITH AN EXPONENTIAL COST CRITERION

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A static decentralized team is represented by the nodes of a network working together to optimize the expected value of an exponential of a quadratic function of the state and control variables. The information consists of known linear functions of the normally distributed state corrupted by additive Gaussian noise. For certain ranges of the system parameters, the stationary condition for optimality are satisfied by a linear decision rule operating on the available information. These

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stationary conditions reduce to a set of algebraic matrix equations and a matrix in equality condition from which the values of the decision gains are determined. Although the stationary conditions are necessary for the linear control law to be minimizing in the class of nonlinear control laws, sufficiency is obtained for our linear controller to be minimizing in the class of linear control laws. Since the quadratic performance criterion produces the only previously known closed form decentralized decision rule, the exponential criterion is an important generalization.

I. Introduction

A team can be visualized as the nodes of a network working together to optimize some common cost criterion. Controlling a system of nodes sometimes requires a decentralized decision-making function throughout the network. Each node is assumed to have assess to a limited amount of information and the control law for best processing this information must be determined.

In [1], Radner obtains conditions for a Bayes decision rule under some fairly general (but restrictive) assumptions. He shows explicitly that a quadratic cost criterion results in a Bayes decision rule for a decentralized information pattern when the a priori probability of the "state of the world" and measurement functions are jointly Gaussian. This result has formed the basis for the dynamic, nonclassical information controllers in [2,3]. Previously, only the quadratic cost criterion produced an implementable linear decision rule. It is shown here that the exponential of a quadratic function displays similar properties.

The quadratic cost criterion used as the basis of LQG control synthesis is an additive cost criterion. However, a simple representation of a multiplicative cost criterion can be formed by the exponential of a quadratic function, since the exponential has the property of being multiplicative when its argument is additive. The exponential form is quite flexiable. In many problems the cost function that is chosen to evaluate overall performance is of a probabilistic form where the conditional probabilities are determined from known constant probabilities whose exponents are functions of the state and control variables. This form can be easily converted to the exponential form.

A theory based upon the exponential cost criterion is given in [4] for the dynamic centralized control problem with linear dynamics and Gaussian noise. Our results for the static team problem with exponential cost show that under certain conditions the optimal decision rules are linear in the available observations. This decentralized decision process contains phenomena not present in the previous works. For example, there exists what has been called an "uncertainty threshold principle" [5]. This means that if the value of some of the system parameters are too large, then a solution may not exist.

We begin by formulating the team problem with exponential cost criterion. Next, a theorem of Radner which gives sufficent conditions for a decision rule to be Bayes (minimizing) is stated. The conditions of Radner's theorem are satisfied over the class of linear control laws for the positive exponential cost criterion, and these conditions are derived in detail. However, over the class of non-linear control laws one of Radner's conditions has not been verified and only stationarity for the linear control law is established. Finally, computation of the linear decision rule is illustrated for a two node network.

2. Problem Formulation

Consider the problem of finding the decision function u for a K node network which minimize the cost criterion:

$$J(u) = E\{\mu \exp \mu \psi/2\} \tag{1}$$

where

$$\psi = x^T Qx + u^T Nx + u^T Ru \tag{2}$$

where $x \varepsilon \mathbb{R}^n$ denotes the state space and $u \varepsilon \mathbb{R}^{\sum_{j=1}^{K} p^j}$ is the control vector over all nodes where j denotes the j^{th} node and p^j denotes the dimension of the control vector of the j^{th} node. The matrices R, N, and Q are given constant matrices, $N^T = [N_1^T, \ldots, N_K^T]$, and $R = [R_{ji}]$.

The state space is not observed directly at each node but through the noisy linear measurement

$$z^j = H_j x + v^j \quad ; \quad z^j \epsilon \mathbb{R}^{q^j} \quad , \quad j=1,\ldots,K \tag{3}$$

where q^j denotes the dimension of the measurement at the j^{th} node. It is assumed that at each node the control is based only on the information z^j and that the information at each node is <u>not</u> shared*. Therefore, the control at each node is confined to be of the form

$$u^j = \gamma^j(z^j) \tag{4}$$

It is to be shown that under certain conditions if the "state of the world" x and $\{v^j;j=1,\ldots,K\}$ are Gaussian, then the optimal decision rule is found and an algorithm for determining the gains of this decision rule is presented. We assume that the "state of the world" x and $\{v^j;j=1,\ldots K\}$ are normally distributed with zero mean and variances

$$E\{xx^T\} = P_o, \; E\{v^j(v^i)^T\} = v_j\delta_{ij}, \;\; E\{x(v^j)^T\} = 0 \tag{5}$$

The assumption of zero mean has a simplifying effect on the resulting algorithm for determining the decision gains but the inclusion of nonzero mean is straightforward.

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*This qualification is done for simplicity in presenting the theory but can easily be generalized to neighbors sharing data.

3. Conditions for Bayes Decision Rule

The decision functions $\hat{\gamma}^j(\cdot)$ defined in (4) are Bayes decision functions if they minimize the expected cost criterion $J(\gamma)$. We restate below a theorem due to Radner [1] which gives sufficient conditions for $\gamma = \hat{\gamma}$ to be optimal. First, following [1] we make a definition.

Definition: The cost functional $J(\gamma)$ is said to be locally finite at $\gamma = \hat{\gamma}$ if

1. $|J(\hat{\gamma})| < \infty$
2. For any admissible decision function δ such that $|J(\hat{\gamma} + \delta)| < \infty$, there exist a scalar $\beta > 0$ such that if $\sup_j |h_j| \leq \beta$ for $j=1,\ldots,K$ then

$$|J(\hat{\gamma}^1 + h_1\delta^1,\ldots,\hat{\gamma}^K + h_K\delta^K)| < \infty$$

We are now ready to state Radner's theorem specialized to our particular case:

Theorem (Radner): If D is the set of admissible controls $\hat{\gamma}\varepsilon D$, and

1. $\mu \exp [\mu\psi(x,u)/2]$ is convex and differentiable in u for almost every x,
2. $\inf_{\gamma\varepsilon D} J(\gamma) > -\infty$,
3. $\gamma = \hat{\gamma}$ is stationary,
4. $J(\gamma)$ is locally finite at $\hat{\gamma}$;

then $\hat{\gamma}$ is a Bayes decision rule.

Condition one is satisfied for $\mu > 0$ but if $\mu < 0$, then $\mu \exp [\mu\psi(x,u)/2]$ is not convex in u. Hence, the theorem does not apply for $\mu < 0$ and we leave open the question of sufficient conditions for optimality in this case. Condition two guarantees a finite minimum and conditions three and four together are sufficient to quarantee $J(\hat{\gamma})$ is optimal.

Unfortunately, condition (4) is difficult to verify for the general case. However, it is easy to verify that it is satisfied if we restrict the decision rules to be affine functions of the observations. Hence, in the remainder of this paper, it should be understood that we are referring only to control laws over the affine functions unless we specifically specify otherwise. Furthermore, this global sufficiency condition for this restricted class of controls produces the global optimal affine control law. We speculate that the optimal affine rule is, in fact, the optimal rule over all appropriately measureable functions but offer no proof. Nevertheless, the stationary condition, which will be seen to be satisfied by affine control laws, is an important necessary condition for optimality over the class of non-linear control laws and for the negative exponential cost criterion. Note that even in the class of affine control laws, a global minimum has not been established for the negative exponential cost criterion.

4. Stationary Conditions for Bayes Decision Rule

The stationary conditions for a minimum are now presented. Suppose that the decision function of all but one of the team members are fixed. Then, a one-person minimization is performed by assuming that the fixed decision functions of the other person are at their one-person minimum given by (4). The one-person cost criterion is the conditional expectation

$$E\{\mu \exp \mu/2\ \overline{\psi}_j(u^j)/z^j\} \triangleq E\{\mu \exp[\mu/2\ \psi(\hat{\gamma}^1(z^1),\ldots,\hat{\gamma}^{j-1}(z^{j-1}),\ u^j\ ,\ \hat{\gamma}^{j+1}(z^{j+1}),\ldots,\ \hat{\gamma}^K(z^K),x)]/z^j\} \tag{6}$$

where $E\{(\cdot)/z^j\}$ denotes conditional expection. Due to condition one and the monotone convergence theorem, the interchange of the operations of expectation and differentiation gives

$$\frac{\partial}{\partial u^j} E\{\mu\exp\mu/2\ \overline{\psi}_j(u^j)/z^j\} = E\{\mu\frac{\partial}{\partial u^j} \exp\mu/2\ \overline{\psi}(u^j)/z^j\} \qquad (17)$$

from which the following <u>set</u> of K <u>stationary conditions</u> arise as

$$E\{\mu \frac{\partial}{\partial u^j} \exp\mu/2\ \overline{\psi}_j(u^j)/z^j\} = 0 \qquad (8)$$

for $j=1,\ldots,K$.

By condition three of the theorem, a necessary condition that the decision function be Bayes is that u^j satisfies (8) and the cost criterion be finite. More explicitly, the stationary conditions (8) using (2) become the K set of equations

$$E\{[\frac{\mu}{2} N_j x + \mu \sum_{\substack{i=1\\ i\neq j}}^{K} R_{ij}\hat{\gamma}^i(z^i) + \mu R_{jj}u^j]e^{\mu\overline{\psi}_j/2}/z^j\} = 0 \qquad (9)$$

for $j=1,\ldots,K$ and the cost criterion (1) $J(\hat{\gamma}) < \infty$. This is the precise definition of stationarity required for the theorem.

5. <u>Linear Bayes Decision Rule for the Exponential Cost Criterion</u>

The results of this section for the exponential payoff parallel the results of Radner's theorem 5 [1] for the quadratic payoff. Observe that the minimum of the function $e^{\mu\psi/2}$ is

$$\gamma(x) = -R^{-1}Nx/2 \qquad (10)$$

It is shown here that if the a priori distributions induce a normal distribution on all the measurements z^j and the vector $\gamma(x)$, then the Bayes decision function can be linear in the measurements. This is done by explicitly assuming a linear decision rule and showing that the stationary conditions (9) are satisfied. If $J(\hat{\gamma}) < \infty$ is not satisfied by any linear decision rule satisfying (9), known results do not exclude the possibility of an optimal non-linear solution, although this seems unlikely.

5.1 Exponent of Exponential Function as an Explicity Function of the Linear Decision Rule

Suppose the linear decision rule is

$$\hat{\gamma}^j(z^j) = D_j z^j + C_j \tag{11}$$

where D_j and C_j are $p^j \times q^j$ and $p^j \times 1$ matrices, respectively, to be determined through the necessary conditions (9). Introducing (11) into (9) results in the K set of equations

$$E\{\frac{N_j}{2} x \, e^{\mu\Phi/2} + \sum_{i=1}^{K} R_{ij}(D_i z^i + C_i)e^{\mu\Phi/2}/z^j\} = 0; \; j=1,\ldots,K \tag{12}$$

where

$$\Phi = x^T Q x + \sum_{i=1}^{K} (D_i z^i + C_i)^T N_i x$$

$$+ \sum_{i=1}^{K} \sum_{k=1}^{K} (D_i z^i + C_i)^T R_{ik} (D_k z^k + C_k) \tag{13}$$

Since the a priori distributions induce normal densities conditioned on the measurements $\{z^j; j=1,\ldots,K\}$, the expectation in (12) can be determined in closed form. The stationary conditions then reduce to a coupled set of K

algebraic matrix equations. This is because the normal density is an exponential and, therefore, the integration implied by the expectation operation is performed by completing the square.

We now derive this set of coupled matrix algebraic equations. First, we note that since the expectation in (12) is conditioned on z^j, then the explicit form of $z^i; i \neq j$ (3) should be introduced. This defines

$$\Phi_j \triangleq x^T Q x + \sum_{i\neq j} (D_i H_i x + D_i v^i + C_i)^T N_i x + (D_j z^j + C_j)^T N_j x$$

$$+ \sum_{i\neq j} \sum_{k\neq j} (D_i H_i x + D_i v^i + C_i)^T R_{ik} (D_k H_k x + D_k v^k + C_k)$$

$$+ (D_j z^j + C_j)^T R_{jj} (D_j z^j + C_j) + \sum_{k\neq j} 2(D_j z^i + C_j)^T R_{jk} (D_k H_k x + D_k v^k + C_k) \qquad (14)$$

where $\sum_{k\neq j}$ denotes a sum from k=1 to K excluding k=j. Let

$$X_j^T \triangleq [x^T, (v^1)^T, \ldots, (v^{j-1})^T, (v^{j+1})^T, \ldots, (v^K)^T],\ C^T \triangleq [C_1^T, \ldots C_K^T] \qquad (15)$$

in which X_j^T represents the underlying random variables associated with the expectations of (12). Using (15), Φ_j can be rewritten in a more convenient form as

$$\Phi_j = X_j^T \underline{Q}_j X_j + C^T \underline{N}_j X_j + C^T R C + (z^j)^T [U_X^j X_j + U_C^j C + U_z^i z^i] \qquad (16)$$

Let $[W_{ik}]$ denote a matrix with matrix elements W_{ik} and i denoting block rows and k denoting columns. The notation $[W_{ik}]_{\substack{i\neq j\\ k\neq j}}$ denotes a matrix which excludes k=j and i=j elements. A matrix with a single subscript with or without fixed index j, i.e., $[W_{ji}]$, denotes a block row matrix. With this notation, the following matrixes are defined

$$\underline{Q}_j \triangleq \begin{bmatrix} Q + \sum_{i\neq j} (D_i H_i)^T N_i + \sum_{i\neq j} \sum_{k\neq j} (D_i H_i)^T R_{ik} D_k H_k , & \left[\frac{D_j^T N_i}{2} + \sum_{k\neq j} D_i^T R_{ik} D_k H_k \right]^T_{i\neq j} \\ \left[\frac{D_i^T N_i}{2} + \sum_{k\neq j} D_i^T R_{ik} D_k H_k \right]_{i\neq j} , & [D_i^T R_{ik} D_k]_{\substack{i\neq j\\ k\neq j}} \end{bmatrix} \tag{17}$$

$$\underline{N}_j \triangleq \begin{bmatrix} N + 2[\sum_{k\neq j} R_i D_k H_k] , & 2[R_{ik} D_k]_{k\neq j} \end{bmatrix} \tag{18}$$

$$U_X^j \triangleq \begin{bmatrix} D_j^T N_j + \sum_{k\neq j} 2D_j^T R_{jk} D_k H_k , & 2[D_j^T R_{jk} D_k]_{k\neq j} \end{bmatrix} \tag{19}$$

$$U_C^j \triangleq \begin{bmatrix} 2D_j^T R_{jk} \end{bmatrix} \tag{20}$$

$$U_z^j \triangleq \begin{bmatrix} D_j^T R_{jj} D_j \end{bmatrix} \tag{21}$$

5.2 Density Functions of the Random Variables

To perform the expectation in (12), the explicit forms of the probability densities are needed. Having assumed zero mean statistics for $\{v^j; j=1,\ldots,K\}$, the density for v^j using (5) is

$$p(v^1,\ldots,v^{j-1},v^{j+1},\ldots,v^K)$$

$$= \frac{1}{(2\pi)^{\sum_{i\neq j} q^i/2} \prod_{\substack{i=1\\ i\neq j}}^{K} |V_i|^{1/2}} \exp -1/2 \sum_{i\neq j} (v^i)^T V_i^{-1} v^i \tag{22}$$

where Π denotes the product operation and $|(\cdot)|$ denotes the determinant. Also, since the a priori mean of x is assumed zero, the posteriori conditional density is

$$p(x/z^j) = \frac{1}{(2\pi)^{n/2}|P_j|^{1/2}} \exp -1/2(x-K_jz_j)^T P_j^{-1}(x-K_jz_j) \tag{23}$$

where P_j is the posteriori error variance produced from

$$P_j^{-1} = P_o^{-1} + (H_j)^T V_j^{-1} H_j \tag{24}$$

and K_j is the Kalman filter gain producing the posteriori mean of the state as K_jz_j and is determined from

$$K_j = P_o(H_j)^T(H_jP_o(H_j)^T + V_j)^{-1} \tag{25}$$

5.3 Algebraic Equations for Determining the Decision Gains

By using (22) and (23), the expectation operation in (12) is explicitly written as (the integral sign denotes the implied multiple integration)

$$\int_{-\infty}^{\infty} \{\frac{N_j}{2} x + \sum_{i\neq j} R_{ij}(D_iH_ix + D_iv^i + C_i) + R_{jj}(D_jz^j + C_j)\}\times$$

$$\exp 1/2\ [\mu\Phi_j - \sum_{i\neq j} (v^i)^T V_i^{-1} v^i - (x-K_jz_j)^T P_j^{-1}(x-K_jz_i)]$$

$$\times\ dxdv^1,\ldots,dv^{j-1},dv^{j+1},\ldots dv^K = 0 \tag{26}$$

Again simplifying the notation by using (15), the integral is

$$\int_{-\infty}^{\infty} [L_j X_j + R_j\ C + R_{jj} D_j z^j)\ \exp\ 1/2\{X_j^T \tilde{Q}_j X_j + \mu C^T \underline{N}_j X_j$$

$$+ \mu C^T RC + (z^j)^T\ [\tilde{U}_X^j X_j + \mu U_C^j C + \tilde{U}_z^j z]\}\ dX_j = 0 \tag{27}$$

where $R_j \triangleq [R_{ji}]$ and

$$L_j = \left[\frac{N_j}{2} + \sum_{i \neq j} R_{ij} D_i H^i\ ,\ [R_{ij} D_i]_{i \neq j}\right] \tag{28}$$

$$Q_j \triangleq \underline{Q}_j - \left[\begin{array}{c|c} P_j^{-1} & 0 \\ \hline 0 & [V_i^{-1} \delta_{ik}]_{\substack{i \neq j \\ k \neq j}} \end{array}\right]$$

$$\tilde{U}_X^j \triangleq \mu U_X^j + [2K_j^T P_j^{-1}\ ,\ 0] \tag{30}$$

$$\tilde{U}_z^j = \mu U_z^j - K_j^T P_j^{-1} K_j \tag{31}$$

The procedure for the determination of the integral in (27) in closed form is to complete the square with respect to the vector X_j so that the argument of the exponent of the exponential is

$$1/2\ [X_j^T \tilde{Q}_j X_j + (\mu C^T \underline{N}_j + (z^j)^T \tilde{U}_X^j) X_j + (\mu CRC^T + (z^j)^T\ (\mu \tilde{U}_C^j C + U_z^j z^j))]$$

$$= 1/2\ [(X_j + G_j)^T \tilde{Q}_j (X_j + G_j) + (\mu CRC^T + (z^j)^T (\mu U_C^j C + \tilde{U}_z^j z^j)) - 1/4\ G_j^T \tilde{Q} G_j \tag{32}$$

where

$$G_j = \tilde{Q}^{-1} (\mu \underline{N}_j^T C + (\tilde{U}_X^j)^T z^j)/2 \tag{33}$$

In this new form a set of new variables of integration are suggested to put the integral in a well known form. Let

$$Y_j = X_j + G_j \tag{34}$$

and Y_i is the new variable of integration, where

$$dY_j = dX_j \tag{35}$$

Therefore, ignoring scalar coefficients which can be removed from under the integral sign, (27) is finally reduced to

$$\int_{-\infty}^{\infty} [L_j Y_j - L_j G_j + R_j C + R_{jj} D_j z^j] e^{1/2 \; Y_j^T \tilde{Q}_j Y_j} dY_j = 0 \tag{36}$$

For this integral to remain finite, the matrix $\tilde{Q}_j$ must be negative definite. Then

$$L_j \int_{-\infty}^{\infty} Y_j e^{1/2 \; Y_j^T \tilde{Q}_j Y_j} \, dY_j = 0 \tag{37}$$

The stationary condition reduces to the condition that

$$L_j G_j = R_j C + R_{jj} D_j z^j \quad ; \quad j=1,\ldots,K \tag{38}$$

This coupled set of algebraic equations can be decomposed to produce the gains C and $\{D_j ; j=1,\ldots,K\}$ which form the Bayes decision rule. Since z_j is arbitrary, then (37) becomes

$$1/2 \; L_j \tilde{Q}_j^{-1} [\tilde{U}_X^j]^T = R_{jj} D_j \quad ; \quad j=1,\ldots K \tag{39}$$

$$[1/2 \mu L_j \tilde{Q}_j^{-1} \underline{N}_j^T - R_j] \; C = 0 \quad ; \quad j=1,\ldots K \tag{40}$$

The stationary condition (40) is a homogeneous linear equation in C. If the zero mean assumption is relaxed, then a non-homogeneous linear equation for C results.

5.4 Conditions for Stationarity

For the stationary conditions (39) to yield a meaningful solution, the conditions $\tilde{Q}_j < 0$ for $j=1,\ldots,K$ must also be met. This restriction does not occur for the quadratic cost criterion. The concept here is that there are parameter values above which the cost criteria does not exist. This phenomena is referred to as the "uncertainty threshold principle." Furthermore, it should be pointed out that $\tilde{Q}_j < 0$; $j=1,\ldots K$ is only a necessary condition for the cost of (1) to be finite. Note that

$$J(\hat{\gamma}) = E\{\mu e^{\mu\psi(\hat{\gamma})/2}\} = E\{E\{\mu e^{\mu\psi(\hat{\gamma})/2}/z^j\}\} . \tag{45}$$

The existence of $E\{\mu e^{\mu\psi(\hat{\gamma})/2}/z^j\}$ is necessary for $J(\hat{\gamma})$ to be finite but not sufficient. This is satisfied by the condition $\tilde{Q}_j < 0$ for $j=1,\ldots K$. To quarantee that $J(\hat{\gamma})$ is finite, $\hat{\phi}$ of (13) is rewritten in terms of <u>all</u> the underlying random variables as

$$\hat{\phi} = X^T\underline{Q}X + C^T\underline{N}X + C^TRC \tag{42}$$

where

$$X^T \triangleq [x^T, v^{1^T}, \ldots, v^{N^T}] \tag{43}$$

and where $\underline{Q}$ and $\underline{N}$ are the same as $\underline{Q}_j$ and $\underline{N}_j$ in (17) and (18), respectively, <u>without</u> the exceptions on the sums and matrices for j. The requirement for stationarity $(J(\hat{\gamma})<\infty)$ is that

$$\tilde{Q} = \mu\underline{Q} - \left[\begin{array}{c:c} P_o^{-1} & 0 \\ \hdashline 0 & \left[V_i^{-1}\delta_{ik}\right] \end{array}\right] < 0 \tag{44}$$

Note that (44) implies $\tilde{Q}_j < 0$ for $j=1,\ldots,K$. Therefore, only (44) need be checked.

Although (39) is a complex function of the gains D_j for $j=1,\ldots K$, if any solution of these equations results which satisfies (44), this solution is minimizing by the theorem of Section 3 in the class of affine control laws. No other decision rule can reduce the cost criterion (1) any further. Note that for the zero mean assumption, $C=0$ is the unique solution. The zero mean assumption does not affect (39).

6. Two-Node Decentralized Decision Rule for Exponential Cost

The theory in Section 5 is illustrated for a two-node network with scalar decision or control variables and measurements located at each node. Referring back to the problem formulation of Section 2, the parameters have the dimensions; Q, P_o, V_1, V_2, H_1, H_2 are scalars, N is a 2-vector, and R is a 2x2 matrix. The posteriori error variance (24) and Kalman gain (25) are

$$P_j^{-1} = P_o^{-1} + H_j^2/V_j \quad , \quad K_j = P_oH_j/(H_j^2P_o + V_j) \quad ; \quad j=1,2 \qquad (45)$$

The linear decision rule, where $\mu=1$, is given by (11) where the scalars D_j for $j=1,2$ ($C=0$) are to be determined from the stationary conditions (39) where, from (29) and (17),

$$\tilde{Q}_1 = \begin{bmatrix} Q + D_2H_2N_2 + D_2^2R_{22}H_2^2 - P_1^{-1} \; , & \frac{N_2D_2}{2} + D_2^2R_{22}H_2 \\ \frac{N_2D_2}{2} + D_2^2R_{22}H_2 & , \; D_2^2R_{22} - V_2^{-1} \end{bmatrix} \qquad (46)$$

where from (30) and (19),

$$[\tilde{U}_X^1]^T = \begin{bmatrix} D_1N_1 + 2D_1R_{12}D_2H_2 + 2K_1P_1^{-1} \\ \\ 2D_1R_{12}D_2 \end{bmatrix} \qquad (47)$$

and where from (28),

$$L_1 = [\frac{N_1}{2} + R_{12}D_2H_2 \ , \ R_{12}D_2] \tag{48}$$

Similar expressions can be obtained for $\tilde{Q}_2$, U_X^2, and L_2.

In general, for this two-node problem, the set of two algebraic stationary equations reduce to a fifth order polynomial in either D_1 or D_2. At most, only one root will satisfy the negative definite requirements. If the parameter describing conditions at each node are the same, then the polynomial reduces to <u>third order</u>. In the following, it is this problem that we study.

6.1 Node Description Equal

If the parameters at each node are the same, then we write

$$\begin{gathered} H_1 = H_2 = H, \ V_1 = V_2 = V, \ N_1 = N_2 = N, \ R_{11} = R_{22} = R \\ R_{12} = R_{21} = \underline{R}, \ P_1 = P_2 = P, \ K_1 = K_2 = K \end{gathered} \tag{49}$$

Since there are no differences between the nodes, the resulting decision gains should be the same at each node, i.e.,

$$D_1 = D_2 = D \tag{50}$$

The stationary condition of (39) using (49) and (50) in (46), (47), and (48) becomes

$$\left[\frac{N}{2} + \underline{R}DH, \underline{R}D\right] \begin{bmatrix} Q + DHN + (DH)^2R - P^{-1} & , \frac{ND}{2} + D^2H\underline{R} \\ & \\ \frac{ND}{2} + D^2H\underline{R} & , D^2R - V^{-1} \end{bmatrix}^{-1} \begin{bmatrix} DN + 2D^2\underline{K}H^2 + 2KP^{-1} \\ \\ 2D^2\underline{R} \end{bmatrix} = QD \tag{51}$$

Performing the indicated operations, (51) reduces to a third-order polynomial from what had appeared to be a fifth-order polynomial (the coefficients of the fifth and fourth order terms cancel) as

$$
\begin{aligned}
&(R-\underline{R})[N^2 - 2(\underline{R}+R)(Q-P^{-1} - H^2V^{-1})]D^3 + 3NHV^{-1}(R-\underline{R})D^2 \\
&+ [2R(Q-P^{-1})V^{-1} - \frac{V^{-1}N^2}{2} - 2(V^{-1}H)^2\underline{R}]D - N(V^{-1})^2H = 0
\end{aligned}
\qquad (52)
$$

<u>Observation</u>: If $R=\underline{k}$, then (52) reduces to a linear equation in D. This decentralized result seems to be equivalent to a centralized control problem with a single scalar decision function of both measurements and using a control weighting R and N.

For simplicity let us now assume $\underline{R}=0$. This polynomial has been programmed. Three real roots have been found for certain choices of the parameters. Two roots are eliminated because $\tilde{Q}$ of (44), given here as

$$
\tilde{Q} = \begin{bmatrix}
(Q + 2DHN + 2(DH)^2 R - P_0^{-1}), & 1/2\ DN + D^2\ HR, & 1/2\ DN + D^2\ HR \\
1/2\ DN + D^2\ HR & , D^2R - V^{-1} & , 0 \\
1/2\ DN + D^2HR & , 0 & , D^2R - V^{-1}
\end{bmatrix}, \qquad (53)
$$

is not negative definite. Only one root satisfies this condition.

6.2 A Numerical Example

A simple example is presented where the polynomial of (52) reduces to a quadratic. This case occurs when

$$\pm (RV)^{-1/2} = - \frac{4NHV^{-1}}{N^2+4H^2RV^{-1}} \qquad (54)$$

This condition is satisfied by choosing $R = V = H = 1$, $N = 2$, where one root of (52) is always $D = -1$. For $P_0 = 1$, the resulting quadratic is

$$5D^2 - 2D - 1 = Q(D^2 - D) \qquad (55)$$

For various values of R, the following table gives the gain that satisfied all conditions except for $R=3$.

Q	0.	1.	2.	3.
D	-.29	-.39	-.58	-1.

When $Q \geq 3$, there is no solution. This is a simple illustration of the "uncertainty threshold principle".

Consider now a more complex example where $R = 1$, $Q = 2.6$, $P_o = 1.5$. In Figure 1 a log J versus $\underline{R}$ plot for $V = .75$ is presented * showing the relative performance of the control law with perfect information $(V=0)$,

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*The calculations for this problem were obtained by Fredrick Machell.

the control law with centralized information (z_1 and z_2 are available at each node), and the decentralized control law whose gains are determined from (52) and (53). In each case the curves rise as $\underline{R}$ increases and escape for positive values of $\underline{R}$. However, an interesting pathological development occurs when V is allowed to increase, say V = .95, as shown in Figure 2. For the decentralized control law the value of the cost criterion escapes for negatives values of $\underline{R}$ as well as positive values. Both the perfect information and centralized control laws still exhibit the same behavior as shown in Figure 1.

7. Discussion and Conclusion

Linear control laws for decentralized control has been presented which allows the cost criterion by being exponential to be in a multiplicative form. The sufficiency theorem of Radner is applicable to this cost criterion if the class of admissible control laws are restricted to be affine and to the positive exponential cost criterion. Otherwise, over the class of non-linear control laws or for the negative exponential cost criterion, the linear control law may only satisfy the necessary condition of stationarity. The exponential form can be motivated as a reasonable model for probabilistic costs. In practice many probability and density functions can be approximated by this functional form. An alternative viewpoint might be to consider the exponential as a type of membership function for which fuzzy set theory [6,7] could be applied.

The extension of our static results to dynamic decentralized problems is a logical step in the development of a complete control theory for the exponential cost criterion. The application of dynamic programming to dynamic decentralized problems is, in general, beyond the capability of current theory. However, the linear-exponential-Gaussian problem with one-step delayed information sharing pattern yields an implementable dynamic controller for the terminal cost problem (only the weighting on the terminal state is nonzero). This development parallels that for the LQG one-step delayed problem [3]. This extension depends heavily on the reproduction of the exponential cost functional form for the cost-to-go (the optimal return function) at each stage.

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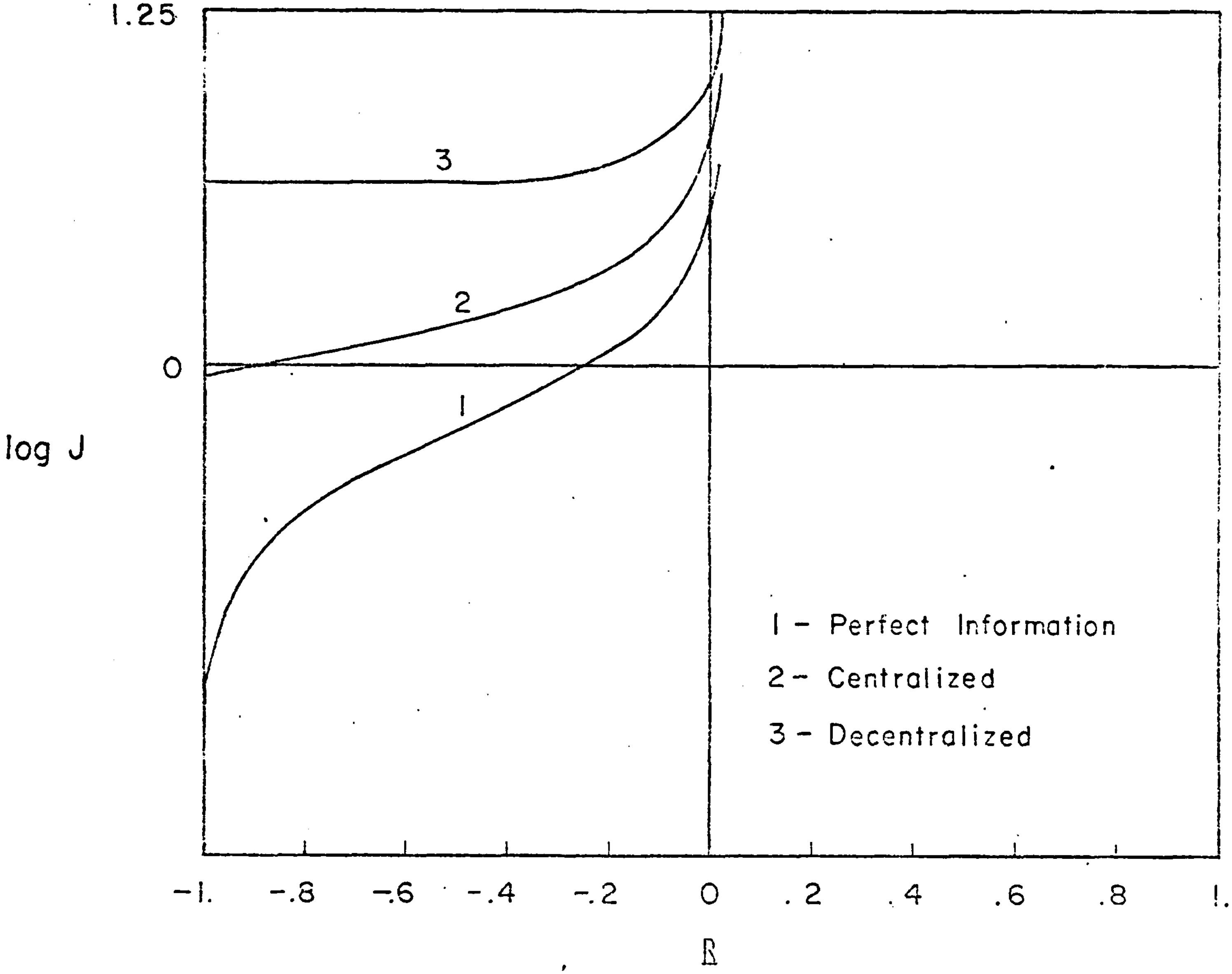

Figure 1: Log J vs. R for v = .75

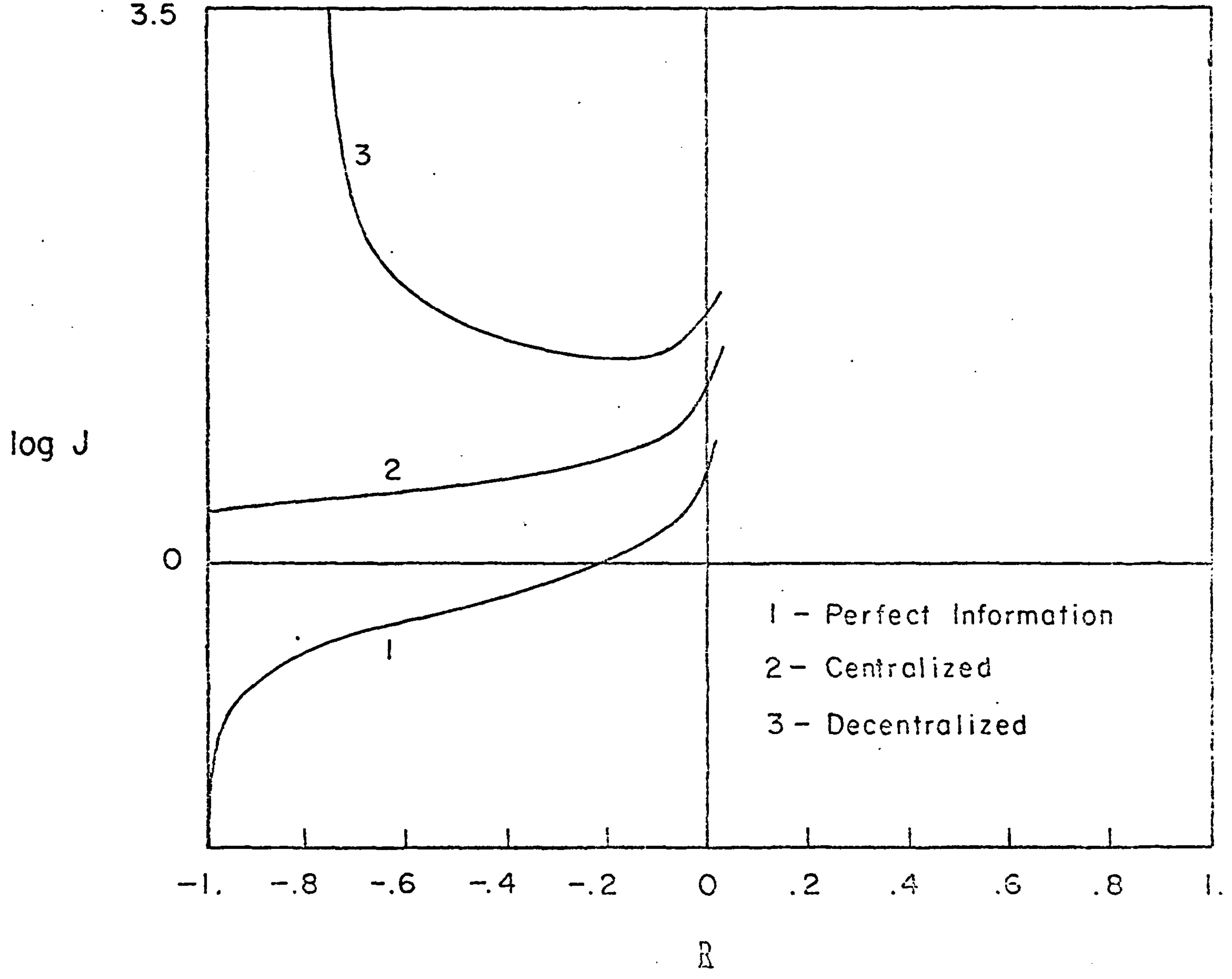

Figure 2: Log J vs R for v = .95