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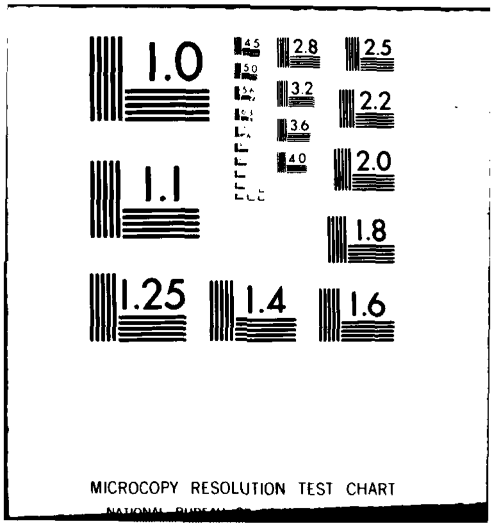
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Convergence in probability of $\sum_{k=1}^{n} a_{nk}X_k$ is obtained for random elements $\{X_k\}$ in Banach spaces satisfying various distributional conditions including independence, conditional independence, and unconditional semi-basic, and weights $\{a_{nk}\}$ such that $\sum_{k=1}^{n} |a_{nk}|^p \le 1$ for each n and $\max_{1\le k\le n} |a_{nk}| \to 0$ as $n\to\infty$. The constant

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p, ~~$1 \leq p \leq 2$~~, is related to a geometric property of the Banach space and to moment conditions. These results relax the usual hypothesis of identical distributions to tightness and are for conditionally independent and unconditionally semi-basic random elements which are more general than independent random elements with zero means.

MARCINKIEWICZ-ZYGMUND WEAK LAWS OF LARGE NUMBERS FOR UNCONDITIONAL RANDOM ELEMENTS IN BANACH SPACES

by

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1. Introduction. The stochastic convergence of $\frac{1}{n}\sum_{k=1}^{n} X_k$ or $\sum_{k=1}^{n} a_{nk}X_k$ has been extensively studied where $\{X_k\}$ are random elements, usually independent, in a Banach space and $\{a_{nk}\}$ are Toeplitz weights satisfying $\lim_{n\to\infty} a_{nk} = 0$ for each k and $\sum_{k=1}^{n} |a_{nk}| \le 1$ for each n. In this paper convergence in probability of $\sum_{k=1}^{n} a_{nk}X_k$ is obtained for random elements satisfying various distributional conditions, including independence, conditional independence, and unconditional semi-basic, and weights $\{a_{nk}\}$ such that $\sum_{k=1}^{n} |a_{nk}|^p \le 1$ for each n and $\max_{1\le k\le n} |a_{nk}| \to 0$ as $n \to \infty$. As in the results of Wei and Taylor (1978), Woyczynski (1978), Marcus and Woyczynski (1979), and Howell, Taylor, and Woyczynski (preprint) the constant p, $1 \le p < 2$, is related to a geometric property of the Banach space and to moment conditions. The results in Section 3 relax the usual hypothesis of identical distributions to tightness. In addition, results are exhibited for conditionally independent and unconditionally semi-basic random elements which are more general than independent mean zero random elements (as is shown in Section 2). Finally, where previous results require that the Banach space be of type $p+\delta$ it is shown, in Theorem 3.3, that type p is sufficient. Section 2 provides a brief but detailed development of the concept of unconditional semi-basic sequences of random elements. In particular, the critical p^{th} - moment inequality for type p Banach spaces is obtained.

In this paper E will denote a real separable Banach space and E* will denote the dual space of E. The usual Banach space definitions for random elements, moments and distributional conditions are assumed. Finally, a family of random elements $\{X_\alpha : \alpha \epsilon A\}$ is said to be <u>tight</u> if for each $\varepsilon > 0$ there exists a compact K_ε such that $P[X_\alpha \epsilon K_\varepsilon] > 1-\varepsilon$ for all $\alpha \epsilon A$.

2. <u>Dependent Random Elements</u>. In this section we examine several classes of dependent random elements. Recall that a sequence $\{X_n\}$ of E-valued random elements is <u>conditionally independent</u> if for every permutation of the X_n's the resulting partial sums form a martingale. Furthermore, $\{X_n\}$ is <u>orthogonal</u> in <u>$L^p(E)$</u>, $1 \le p < \infty$, if $E\|\sum_{i=1}^{n} a_{\sigma(i)} X_{\sigma(i)}\|^p \le E\|\sum_{i=1}^{n+m} a_{\sigma(i)} X_{\sigma(i)}\|^p$ for all n and m, all numbers $a_{\sigma(1)}, \ldots, a_{\sigma(n+m)}$, and all permutations σ of $\mathbb{N}$. Next, we introduce the following definition.

<u>Definition 2.1</u> A sequence of random elements $\{X_n\}$, $X_n \epsilon L^p(E)$, is <u>unconditional semi-basic in $L^p(E)$</u> (ucsb in $L^p(E)$), $1 \le p < \infty$, if there exists a constant M, $1 \le M < \infty$ such that

$$E\|\sum_{i=1}^{n} a_{\sigma(i)} X_{\sigma(i)}\|^p \le M\|\sum_{i=1}^{n+m} a_{\sigma(i)} X_{\sigma(i)}\|^p \quad \text{for all n and m,}$$

all numbers $a_{\sigma(1)}, \ldots, a_{\sigma(n+m)}$, and all permutation σ of $\mathbb{N}$. Also, $\{X_n\}$ is <u>weakly unconditional semi-basic in $L^p(E)$</u> (wuscb) if $X_n \epsilon L^p(E)$ and for each $f \epsilon E^*$ $\{f(X_n)\}$ is ucsb in L^p.

If in Definition 2.1 we stipulated that $X_n \neq 0 \in L^p(E)$, then the sequence would be unconditional basic. We use the phrase 'semi-basic' to indicate that while the non-zero elements of the sequence are basic, not all of the elements need be non-zero.

It is easily seen that if a sequence is independent mean zero then it is conditionally independent. Since martingale differences form a monotone basis in $L^p(E)$ [assuming $X_n \in L^p(E)$] conditional independence implies orthogonality. Finally a comparison of the definitions shows that orthogonality implies unconditional semi-basic. Easy examples show that none of the converse implications hold. Also, ucsb neither implies nor is implied by wucsb. Example 2.1 will illustrate some of these concepts.

<u>Example 2.1</u> Let $E=\ell^p$, $1<p<2$, and let $e_k = \{\delta_{k\ell}\}_{\ell=1}^{\infty}$ be the standard vector basis of ℓ^p. Let Y be any non-trivial random variable with $E|Y|^p < \infty$ and define $X_k(\omega)=Y(\omega)e_k$. First, $\{X_k\}$ is ucsb in $L^p(E)$ since

$$E\|\textstyle\sum_{i=1}^n a_{p_i} X_{p_i}\|^p = E\|\sum_{i=1}^n a_{p_i} Ye_{p_i}\|^p$$

$$= E(\textstyle\sum_{i=1}^n |a_{p_i} Y|^p)$$

$$\leq E(\textstyle\sum_{i=1}^n |a_{p_i} Y|^p + \sum_{i=1}^m |a_{q_i} Y|^p)$$

$$= E\|\textstyle\sum_{i=1}^n a_{p_i} X_{p_i} + \sum_{i=1}^m a_{q_i} X_{q_i}\|^p .$$

The X_k's are neither independent, weakly orthogonal, or wucsb (although they are orthogonal) and need not have zero means.

The following fundamental inequality which is needed to obtain laws of large numbers in Banach spaces for ucsb sequences is strictly tied to the geometry of the Banach space.

<u>Proposition 2.1</u> The following conditions are equivalent

(i) E is of R-type p,

(ii) for each $\{X_n\}$ ucsb in $L^p(E)$ there exists a constant C such that, for all n,

$$E\|\sum_{i=1}^n X_i\|^p \le C\sum_{i=1}^n E\|X_i\|^p.$$

<u>Proof</u>: It is well known that if $E \in$ R-type p then $L^p(E) \in$ R-type p. So, assuming (i) holds and denoting the $L^p(E)$ norm by $\|\cdot\|_p$, we have

$$E\|\sum_{i=1}^n r_i X_i\|_p^p \le A\sum_{i=1}^n \|X_i\|_p^p$$

for all $X_1,\ldots,X_n \in L^p(E)$ where $0<A<\infty$ is the R-type constant of $L^p(E)$ and $\{r_i\}$ denote the Rademacher random variables. If $\{X_n\}$ is ucsb in $L^{p'}(E)$, then there exists a constant $1 \le M < \infty$ such that

$$\|\sum_{i=1}^n |a_i| X_i\|_p^p \le M\|\sum_{i=1}^n a_i X_i\|_p^p$$

for all n and all numbers $a_1,\ldots,a_n$. Since $|r_i| = 1$, we have

$$\|\sum_{i=1}^n X_i\|_p^p \le MA\sum_{i=1}^n \|X_i\|_p^p$$

which is

$$E\|\sum_{i=1}^n X_i\|^p \le C\sum_{i=1}^n E\|X_i\|^p$$

where $0 < C = MA < \infty$. Thus (i) implies (ii).

Conversely, suppose (ii) holds. Then (ii) must hold for each sequence of independent mean zero random elements in $L^p(E)$. Thus, by a theorem of Hoffmann - Jørgensen and Pisier (1976), E is of R-type p. ///

The following lemmas will be used in Section 3. In these lemmas we assume that E has a Schauder basis $\{b_i\}$ with coefficient functionals $\{f_i\}$. U_t and Q_t are the continuous linear operators on E defined by $U_t(x) = \sum_{i=1}^{t} f_i(x) b_i$ and $Q_t(x) = \sum_{i=t+1}^{\infty} f_i(x) b_i$.

<u>Lemma 2.2</u> If $\{X_n\}$ is wucsb in $L^p(E)$, then $\{U_t(X_n)\}$ is wucsb in $L^p(U_t(E))$ and $\{Q_t(X_n)\}$ is wucsb in $L^p(Q_t(E))$ for each t.

<u>Proof</u>: For each $f \in (U_t(E))^*$, $(f \circ U_t) \in E^*$. Thus the result follows for $\{U_t(X_n)\}$. Similarly, $\{Q_t(X_n)\}$ is wucsb. ///

<u>Lemma 2.3</u> If $\{X_n\}$ is ucsb and wucsb in $L^p(E)$, then $\{Q_t(X_n)\}$ is ucsb in $L^p(Q_t(E))$.

<u>Proof</u>: Without loss of generality assume, for all n, $X_n \neq 0 \in L^p(E)$. Then $\{X_n\}$ is a basis for $[X_n]$ and it follows that $\{Q_t(X_n)\}$ is a semi-basis for $[Q_t(X_n)]$. Since $\{X_n\}$ is wucsb, by Lemma 2.2 $\{Q_t(X_n)\}$ is wucsb. Since each $f \in E^*$ may be regarded as a continuous linear mapping of $L^p(E)$ into L^p, and since unconditionality is preserved under isomorphism, it suffices to show that there exists an $f \in E^*$ which is 1-1 on $[Q_t(X_n)]$. Assuming, without loss of generality, that, for all n,

$Q_t(X_n) \neq 0 \in L^p(Q_t(E))$, it suffices to show that there is an f such that, for all n, $f(Q_t(X_n)) \neq 0 \in L^p$. Suppose that this is not the case and let $S_n = \{f \in (Q_t(E))^*: f(Q_t(X_n)) = 0 \text{ a.s}\}$. Each S_n is a linear subspace of $(Q_t(E))^*$. If $S_n = (Q_t(E))^*$, then it follows that $Q_t(X_n) = 0$ a.s., a contradiction. Thus, S_n is a proper linear subspace of $(Q_t(E))^*$. It follows that for each n there exists a hyperplane through the origin, H_n, such that $S_n \subseteq H_n$. Since $(Q_t(E))^*$ is not the union of a countable number of hyper-planes, we have that $(Q_t(E))^* \neq \bigcup_{n=1}^{\infty} S_n$, i.e., there exists an $f \in (Q_t(E))^*$ such that $f \notin \bigcup_{n=1}^{\infty} S_n$, which completes the proof. ///

Remark: Since conditional independence implies ucsb and wuscb, conditional independence is sufficient for $\{Q_t(X_n)\}$ to be uscb when the appropiate moments hold.

3. Marcinkiewicz - Zygmund's Type Weak Laws of Large Numbers. Weak laws of large numbers for Banach space-valued random variables are obtained in this section using the various concepts of independence, conditional independence, unconditional semi-basic (ucsb), and weak unconditional semi-basic (wucsb). First, a family $\{X_\alpha: \alpha \in A\}$ of random elements in a Banach space E is said to have uniformly bounded tail probabilities by tail probabilities of a real-valued random variable X, denoted by $\{X_\alpha\} \prec X$, if

$$P[\|X_\alpha\| > t] \le P[X > t]$$

for each $\alpha \in A$ and for each $t \ge 0$. Next, recall that $\{a_{nk}\}$ denotes an array of real numbers such that

$$\sum_{k=1}^{n} |a_{nk}|^p \le 1 \quad \text{for each } n \tag{3.1}$$

where $1 \le p < 2$ (and p will relate to a geometric condition on the space) and

$$\max_{1 \le k \le n} |a_{nk}| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.2}$$

Theorem 3.1 If $\{X_n\}$ are random elements in a separable Banach space of R-type q, $1 \le q \le 2$, which are ucsb in $L^q(E)$ and if

$$\sum_{k=1}^{n} |a_{nk}|^q \, E\| X_k \|^q \to 0 \quad \text{as} \quad n \to \infty, \tag{3.3}$$

then

$$\| \sum_{k=1}^{n} a_{nk} X_k \| \to 0 \text{ in probability.}$$

Proof: Let $\varepsilon > 0$ be given. For each n

$$P[\| \sum_{k=1}^{n} a_{nk} X_k \| > \varepsilon] \le \frac{1}{\varepsilon^q} E[\| \sum_{k=1}^{n} a_{nk} X_k \|^q]$$

$$\le \frac{C}{\varepsilon^q} \sum_{k=1}^{n} |a_{nk}|^q \, E\| X_k \|^q$$

from Proposition 2.1. Thus, the result follows from (ii). ///

The major use of Theorem 3.1 will be for the real line which is of R-type 2. In particular, in relating the geometry condition to the constant p in condition (3.1) (see Theorems 3.2 and 3.3), Theorem 3.1 can be used to show that the weak law of large numbers holds for each $f \in E^*$.

Condition (3.3) is easily satisfied when $\sup_k E\| X_k \|^q = \Gamma < \infty$ where $q > p$ since

$$\sum_{k=1}^{n} |a_{nk}|^q E\| X_k \|^q \le (\max_{1 \le k \le n} |a_{nk}|)^{q-p} \, \Gamma \sum_{k=1}^{n} |a_{nk}|^p$$

goes to 0 by (3.1) and (3.2). In general, results using the concept of ucsb or wucsb will require a slightly higher moment condition than results using independence. This fact is illustrated in the following theorem which is patterned after Theorem 2 of Wei and Taylor (1978) and Theorem 3.1 of Marcus and Woyczynski (1979).

Theorem 3.2 Let $\{X_{nk}\}$ be random elements in a separable Banach space of type $p+\delta$, $1<p<2$, for some $\delta>0$. If

(i) $\{X_{nk}: k=1,2,\dots\}$ are independent for each n,

(ii) $EX_{nk}=0$ for each n and k, and

(iii) $\{X_{nk}\} \prec X$ with $n^p\, P[X>n] \to 0$ as $n\to\infty$,

then

$$\|\textstyle\sum_{k=1}^{n} a_{nk}\, X_{nk}\| \to 0 \text{ in probability.}$$

Proof: Let

$$S_n = \textstyle\sum_{k=1}^{n} a_{nk}\, X_{nk} I_{[\|X_{nk}\| \le |a_{nk}|^{-1}]}$$

(let $|a_{nk}|^{-1}=\infty$ when $a_{nk}=0$). For each n

$$P[\textstyle\sum_{k=1}^{n} a_{nk}X_{nk} \neq S_n] \le \sum_{k=1}^{n} P[\|X_{nk}\| > |a_{nk}|^{-1}]$$

$$\le \textstyle\sum_{k=1}^{n} |a_{nk}|^p (|a_{nk}|^{-p} P[X>|a_{nk}|^{-1}]) \qquad (3.4)$$

which goes to 0 by (iii) since $|a_{nk}|^{-1} \le (\max_{1\le k\le n} |a_{nk}|)^{-1} \to \infty$. Hence, for $\varepsilon>0$ it suffices to show that $\|S_n\| \xrightarrow{p} 0$. For each n and k and for each $T>0$,

$$\int_0^T x^{p+\delta} dP[\|X_{nk}\| \le x]$$

$$= T^{p+\delta} P[\| X_{nk} \| \le T] - (p+\delta) \int_0^T x^{p+\delta-1} P[\| X_{nk} \| \le x] dx$$

$$\le (p+\delta) \int_0^T x^{p+\delta-1} P[\| X_{nk} \| > x] dx. \tag{3.5}$$

Given $\eta > 0$, by (iii) choose $B > 0$ such that

$$P[X > x] \le \eta x^{-p} \tag{3.6}$$

for all $x \ge B$. Then,

$$T^{-\delta} \int_0^T x^{p+\delta-1} P[X > x] dx$$

$$\le T^{-\delta} \left(\int_0^B x^{p-1} P[X > x] dx + \eta \int_B^T x^{\delta-1} dx \right)$$

$$\le T^{-\delta} (B^p + \eta T^{\delta}) < 2\eta \tag{3.7}$$

for T sufficiently large. If $\| E S_n \| \nrightarrow 0$, then from (ii)

$$\| \sum_{k=1}^n a_{nk} E(X_{nk} I_{[\| X_{nk} \| \le |a_{nk}|^{-1}]}) \|$$

$$\le \sum_{k=1}^n |a_{nk}| E \| X_{nk} I_{[\| X_{nk} \| > |a_{nk}|^{-1}]} \|$$

$$= \sum_{k=1}^n |a_{nk}| \left(\int_0^{|a_{nk}|^{-1}} P[\| X_{nk} \| > |a_{nk}|^{-1}] dt \right.$$

$$\left. + \int_{|a_{nk}|^{-1}}^{\infty} P[\| X_{nk} \| > t] dt \right)$$

$$\le \sum_{k=1}^n |a_{nk}| \left(|a_{nk}|^{-1} \eta (|a_{nk}|^{-1})^{-p} + \int_{|a_{nk}|^{-1}}^{\infty} \eta t^{-p} dt \right)$$

$$= \eta \left(1 + \frac{1}{p-1}\right) \tag{3.8}$$

for sufficiently large n from (3.6) and (3.3). Thus, for sufficiently large n

$$P[\| S_n \| > 2\varepsilon] \le P[\| S_n - ES_n \| > \varepsilon]$$

$$\le \varepsilon^{-(p+\delta)} C \sum_{k=1}^{n} E\| a_{nk} X_{nk} I_{[\| X_{nk} \| \le |a_{nk}|^{-1}]}$$

$$-E(a_{nk} X_{nk} I_{[\| X_{nk} \| \le |a_{nk}|^{-1}]}) \|^{p+\delta}$$

$$\le \varepsilon^{-(p+\delta)} C 2^{(p+\delta)} \sum_{k=1}^{n} |a_{nk}|^{p+\delta} \int_0^{|a_{nk}|^{-1}} x^{p+\delta} dP[\| X_{nk} \| \le x]$$

$$\le (\varepsilon/2)^{-(p+\delta)} C \sum_{k=1}^{n} |a_{nk}|^{p+\delta} (p+\delta) \int_0^{|a_{nk}|^{-1}} x^{p+\delta-1} P[\| X_{nk} \| > x] dx$$

$$\le (\varepsilon/2)^{-(p+\delta)} C (p+\delta) \sum_{k=1}^{n} |a_{nk}|^{p} (|a_{nk}|^{\delta} \int_0^{|a_{nk}|^{-1}} x^{p+\delta-1} P[X > x] dx)$$

$$< (\varepsilon/2)^{-(p+\delta)} C(2\eta) \tag{3.9}$$

from (3.5) and (3.7). ///

If $EX^p < \infty$ or if the truncated means are zero, (for example, if the random elements are symmetric), then it easy to see that $p=1$ holds in Theorem 3.2. For $p<1$, existing real-valued results suffice since

$$P[\| \sum_{k=1}^{n} a_{nk} X_k \| > \varepsilon] \le P[\sum_{k=1}^{n} |a_{nk}| \| X_k \| > \varepsilon]$$

and zero means for $\{\| X_k \|\}$ are not required since the cancellation is acheived by the strong restriction of $\sup_n \sum_{k=1}^{\infty} |a_{nk}|^p \le 1$. The condition $\{X_{nk}\} \prec X$ is easily satisfied by moment conditions, and in particular, $\{X_n\} \prec \| X_1 \|$ for identically distributed random elements. An important special set of weights satisfying (3.2) and (3.3) are the uniform weights defined by

$$a_{nk} = \begin{cases} n^{-\frac{1}{p}} & \text{if } 1 \le k \le n \\ 0 & \text{if } k > n. \end{cases}$$

Thus, the condition of symmetry in Theorem 3.1 of Marcus and Woyczynski (1979) and identical distributions has been replaced by the uniform boundedness of the tail probabilities. Also, since the conditions of Section V.9.1 of Woyczynski (1978) are satisfied, the geometric condition of type $p+\delta$ is also necessary in Theorem 3.2. The geometric condition $p+\delta$ is relaxed to p in Theorem 3.3 by assuming tightness and $EX^p<\infty$.

Basis techniques will be used in the next two theorems and their proof. In particular, for Theorem 3.3 the Banach space E will be assumed to have a Schauder basis $\{b_i, f_i\}$, and U_t and Q_t will denote the linear operators defined on E by

$$U_t(x) = \sum_{i=1}^t f_i(x) b_i \quad \text{and} \quad Q_t(x) = x - U_t(x).$$

Finally, let m denote the basic constant such that

$$\| U_t \| \le m \quad \text{and} \quad \| Q_t \| \le m \quad \text{for all } t.$$

<u>Theorem 3.3</u> Let $\{X_{nk}\}$ be tight random elements in a separable Banach space which is of R-type p, $1 \le p < 2$ and which has a Schauder basis. If

(i) $\{Q_t(X_{nk}): k=1,2,\ldots\}$ are ucsb in $L^p(E)$ (with the same constant M) for all n and t, and

(ii) $\{X_{nk}\} \prec X$ with $EX^p < \infty$, then

$$\| \sum_{k=1}^n a_{nk} X_{nk} \| \to 0 \quad \text{in probability}$$

if and only if for each $f \epsilon E^*$

$$\left| \sum_{k=1}^{n} a_{nk} f(X_{nk}) \right| \to 0 \quad \text{in probability.}$$

Proof: For each T and each compact set K,

$$\begin{aligned} E(\| X_{nk}\|^p I_{[X_{nk} \notin K]}) &\le \int_0^\infty P[\|X_{nk}\|^p I_{[\| X_{nk}\|^p \ge T]} \ge t]dt + TP[X_{nk} \notin K] \\ &\le TP[X^p \ge T] + \int_T^\infty P[X^p \ge t]dt + TP[X_{nk} \notin K] \end{aligned}$$

which can be made arbitrarily small by first choosing T from $EX^p = \int_0^\infty P[X^p \ge t]dt < \infty$ and then choosing the compact set K by tightness. Fix $\varepsilon > 0$ and $\delta > 0$. Choose a compact set K so that

$$E\| X_{nk}\|^p I_{[X_{nk} \notin K]} < \frac{\delta \varepsilon^p}{mC2^{p+2}} \tag{3.10}$$

for all n and k. Since $\sup_{x \epsilon K} \| Q_t(x)\|^p \to 0$ as $t \to \infty$, t_o can be chosen so that

$$E\| Q_{t_o}(X_{nk})\|^p < \frac{\delta \varepsilon^p}{C2^{p+1}} \tag{3.11}$$

for all n and k. For each n

$$\begin{aligned} P[\| \sum_{k=1}^{n} a_{nk} Q_{t_o}(X_{nk})\| > \tfrac{\varepsilon}{2}] \\ \le \frac{2^p C}{\varepsilon^p} \sum_{k=1}^{n} |a_{nk}|^p E\|Q_{t_o}(X_{nk})\|^p < \frac{\delta}{2} \end{aligned} \tag{3.12}$$

from Proposition 2.4 and (3.11). Next,

$$\begin{aligned} P[\| \sum_{k=1}^{n} a_{nk} U_{t_o}(X_{nk})\| > \tfrac{\varepsilon}{2}] \\ \le \sum_{i=1}^{t_o} P[| \sum_{k=1}^{n} a_{nk} f_i(X_{nk})| > \frac{\varepsilon}{2t_o\| b_i\|}] \\ \to 0 \quad \text{as} \quad n \to \infty \end{aligned} \tag{3.13}$$

since for each $i=1,\ldots,t_o$

$$|\sum_{k=1}^{n} a_{nk} f_i(X_{nk})| \to 0$$

in probability. From (3.12) and (3.13)

$$P[\|\sum_{k=1}^{n} a_{nk} X_{nk}\| > \varepsilon] < \delta$$

for all $n \le N(\varepsilon,\delta)$. ///

In Theorem 3.3 the space is of R-type p instead of $p+\delta$ as in Theorem 3.2. If the random elements are assumed to be row-wise independent with zero means, then $\{Q_t(X_{nk})\}$ are row-wise independent and hence row-wise ucsb in $L^p(E)$ with the common constant being the R-type p constant. Since an isomorphic embedding preserves R-type p and independence, the assumption that the space has a Schauder basis is not needed. Moreover, $\{f(X_{nk})\}$ are row-wise independent, zero means, random variables for each $f \in E^*$, and there exists $\delta>0$ such that $p+\delta \le 2$ since $p<2$. Hence, Theorem 3.2 provides that

$$|\sum_{k=1}^{n} a_{nk} f(X_{nk})| \to 0 \quad \text{in probability}$$

and hence

$$\|\sum_{k=1}^{n} a_{nk} X_{nk}\| \to 0 \qquad \text{in probability.}$$

Thus, Theorem 3.3 achieved a reduction of $p+\delta$ to p in the geometric condition at the expense of the moment condition $nP[X^p > n] \to 0$ to $EX^p < \infty$.

Theorem 3.3 has wider application than row-wise independence since in Section 2 it was shown that either condition independence or wucsb & ucsb for $\{X_n\} \in L^p(E)$ would imply $\{Q_t(X_n)\}$ is ucsb. It is not clear at this time what is the best possible formulation of Condition (i) in Theorem 3.3. However, Theorem 3.3 has the appealing feature of relating the weak laws of large numbers in R-type p spaces to orthogonality (similar to the real-valued case) instead of independence.

To illustrate the necessity of tightness in Theorem 3.3 and R-type $p+\delta$ in Theorem 3.2, let $e_1=(1,0,0,\ldots)$, $e_2=(0,1,0,\ldots)$,...denote the standard basis in ℓ^p, $1 \le p < 2$, and let $\{X_n\}$ be independent random elements in ℓ^p defined by $X_n = \pm e_n$ with probability $\frac{1}{2}$ each. Then taking the special case of Theorems 3.2 and 3.3 where each row of the array is the same sequence, $\{X_n\} \prec X \equiv 1$ and $EX_n = 0$ for all n. However, $\| n^{-\frac{1}{p}}(X_1+\ldots+X_n) \| \equiv 1$ for all n, and the weak law of large numbers fails.

The case $p=1$ is interesting since it includes all separable Banach spaces. In this case Condition (i) of Theorem 3.3 is not needed.

<u>Theorem 3.4</u> Let $\{X_{nk}\}$ be tight random elements in a separable Banach space E such that $\{X_{nk}\} \prec X$ and $EX < \infty$. For each $f \in E^*$

$$| \textstyle\sum_{k=1}^n a_{nk} f(X_{nk}) | \to 0 \qquad \text{in probability}$$

if and only if

$$\| \textstyle\sum_{k=1}^n a_{nk} X_{nk} \| \to 0 \qquad \text{in probability.}$$

Proof: First, wlog E can be assumed to have a Schauder basis. For $C=1$ and $p=1$, Inequalities (3.10) and (3.11) of the proof of Theorem 3.3 follow in a similar manner. Inequality (3.12) can be accomplished by the Markov inequality and the triangle inequality of the norm, and the remainder of the proof is identical to the proof of Theorem 3.3. ///

Theorem 3.4 contains the previous results of Wei and Taylor (1978b) for Toeplitz weights. In addition to being applicable for arrays, Theorem 3.4 has a weaker moment condition. Previous weak laws of large numbers for weakly uncorrelated random elements are contained in these results since weakly uncorrelated random elements are wucsb and for $f \epsilon E^*$

$$\frac{1}{n^q} \sum_{k=1}^{n} E|f(X_k)|^q \to 0$$

for some q, $1 \le q \le 2$, is sufficient for the weak law of large numbers by Theorem 3.1.

For independent, identically distributed random elements $\{X_n\}$, R-type $p+\delta$ was reduced to R-type p by assuming $E\|X_1\|^p < \infty$ instead of $n^p P[\|X_1\| > n] \to 0$. The question of the necessity of type p in Theorem 3.3 is interesting since if $n^p P[\|X_1\| > n] \to 0$ (or $n^p P[X>n] \to 0$ in the general case) sufficed then E must be of R-type $p+\delta$ for some $\delta>0$.

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