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The third boundary value problem, or Robin problem, for the Helmholtz equation in an exterior domain in R_u^3 may be cast with Green's theorem, as a boundary integral equation. This equation is not equivalent to the original problem because it has eigenvalues at wave numbers for which the exterior problem is soluble. However, a second boundary equation is derived and it is shown that the pair is equivalent to the original boundary value problem. The boundary condition is of the form $\frac{u}{n} + \sigma u = g$ where both σ and g are in L^∞ on the boundary and

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20. Abstract cont.

it is shown that the coupled boundary integral equations as well as the original boundary value problem have a unique solution for all real values of wave number. The second boundary integral equation involves the normal derivative of the double layer and in order to show that this operator is always well defined on the class of functions of interest, a trace theorem is proved for single and double layer potentials with L_2 densities.

Boundary Integral Equations for the
Helmholtz Equation;
The Third Boundary Value Problem*

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The present work extends boundary integral equation techniques, developed in an earlier paper for the Dirichlet and Neumann problems [6], to the boundary value problem of the third kind (also referred to as the Robin problem) associated with the Helmholtz equation in exterior domains. While the existence and uniqueness of solutions of the exterior Robin problem have been established previously (see e.g. Leis [9]; Danilova [2]), the application of the Helmholtz representation and the subsequent derivation of boundary integral equations does not appear to have been studied until now.

The considerations below will show that the familiar difficulties at interior eigenvalues arise in the case of the Robin boundary conditions. The non-uniqueness of solutions of the boundary integral equation at interior eigenvalues is, of course, an artifact of the method chosen to solve the original boundary value problem. As explained in [6], supplementary conditions of some form are needed. Either the fundamental solution may be modified, as was done by Danilova [2] in her layer theoretic treatment of the third boundary value problem with an oblique derivative, or an additional equation is needed if the fundamental solution is unaltered. In the present work we employ the Green's theorem approach to boundary integral equations with an unmodified fundamental solution and show that, unlike the situation for the exterior Dirichlet and Neumann problems, the supplementary condition appears as an operator equation of the second kind rather than one of the first kind.

Rather than repeat the bibliographic details reviewed by
Kleinman and Roach [6], we refer the reader to the introduction
of that paper. Moreover, we will adopt the notational conventions
of that paper. Section 2, below, briefly reviews those conventions
as they are appropriate to the present considerations. However,
the boundary value problem discussed here, in contrast to those
in [6] and, in the papers of Leis [9] and Danilova [2], is
generalized so that the data and the multiplier which appears in
the boundary condition may be specified in $L_{\infty}(\partial D)$ rather than
$C_0(\partial D)$ where ∂D is the boundary of the exterior domain.

This more general setting requires some extensions of
theorems which appear in the literature, in particular extensions
of a regularity theorem appearing in Mikhlin [10] and of a
uniqueness theorem due to Leis [9]. More significantly, it
requires a careful formulation of a trace theorem suitable for
functions described by single and double layer potentials. Standard
trace theorems, e.g. Miranda [11] were not sufficient to ensure
existence of the normal derivatives of double layer potentials
required for application in the present setting. These matters are
discussed carefully in section 3 and in the appendix which contains
some computational details collected there so as not to interrupt
the flow of the main argument.

Having discussed these background matters, the main results
concerning the boundary integral equation formulation of the Robin
problem are presented in section 4 and the equivalence of this
formulation with the original boundary value problem is proven.
Section 4 concludes with a result pertaining to the scattering
problem for imperfectly reflecting surfaces.

2. Formulation of boundary integral equations.

Let D_- denote a bounded domain in $\mathbb{R}^3$ with boundary ∂D which is a closed Lyapunov surface with index 1. We denote the exterior of ∂D (i.e. the complement of $D_- \cup \partial D$ in $\mathbb{R}^3$) by D_+.

Let $R \equiv R(P,Q)$ denote the distance between two typical points P and Q in R^3. A fundamental solution of the Helmholtz equation $(\Delta + k^2)u = 0$, $\text{Im}\,k \geq 0$, is a two-point function of position, $\gamma(P,Q)$, which for convenience we write in the form

$$(2.1) \qquad \gamma(P,Q) := -e^{ikR}/2\pi R.$$

Complex conjugates will be denoted by a bar, thus

$$(2.2) \qquad \bar{\gamma}(P,Q) := -e^{-ikR}/2\pi R.$$

We denote by $\partial/\partial n_p$ differentiation in the direction of the unit vector $\hat{n}_p$ normal to ∂D at the point $p \in \partial D$. A point $P \in D_+$ is assumed to have spherical polar coordinates (r_p, θ_p, ϕ_p) relative to a Cartesian coordinate system erected with origin in D_-. We emphasize that throughout we shall assume that $\hat{n}_p$ is the outward drawn normal with respect to D_-, that is $\hat{n}_p$ is directed from ∂D into D_+. Further, we shall write $\partial/\partial n_p^-$ and $\partial/\partial n_p^+$ to denote the normal derivative when $P \to p \in \partial D$ from D_- and D_+ respectively.

Let $(S\mu)(P)$ and $(D\mu)(P)$ denote the effect at the point $P \in \mathbb{R}^3$ of single layer and double layer distributions respectively on ∂D of density μ as follows:

$$(2.3) \qquad (S\mu)(P) := \int_{\partial D} \mu(q)\gamma(P,q)\,dS_q, \quad P \epsilon R^3$$

and

$$(2.4) \qquad (D\mu)(P) := \int_{\partial D} \mu(q)\frac{\partial\gamma}{\partial n_q}(P,q)\,dS_q, \quad P \epsilon R^3.$$

Further, for $p \epsilon \partial D$ we define

$$(2.5) \qquad (K\mu)(p) := \int_{\partial D} \mu(q)\frac{\partial\gamma}{\partial n_p}(p,q)\,dS_q.$$

Notice that K is a completely continous operator on $L_2(\partial D)$ (Mikhlin [10]). Jump conditions for the single and double layer potentials on ∂D can be expressed in terms of K and its $L_2(\partial D)$ adjoint K* which is given by

$$(2.6) \qquad (K^*\mu)(p) = \int_{\partial D} \mu(q)\frac{\partial\overline{\gamma}}{\partial n_q}(p,q)\,dS_q.$$

These jump conditions as well as continuity properties of the single and double layers depend on the smoothness of the density μ and will be discussed in Section 3.

Finally, we remark that the single and double layer operators with $P \epsilon D_-$, can be considered as operators with range in $L_2(D_-)$ and, as such, are continuous operators from $L_2(\partial D)$ to $L_2(D_-)$ (see e.g. Miranda [11]).

With this notation, representations of solutions of the Helmholtz equation obtained by applying Green's Theorem or the Helmholtz representation have the following forms. For radiating wave functions ϕ_+ satisfying

$$(\Delta+k^2)\phi_+(P) = 0, \qquad P\epsilon D_+,$$

$$(2.7)$$

$$\lim_{r_p\to\infty} \left\{ r_p \left[\frac{\partial\phi_+}{\partial r_p}(P) - ik\phi_+(P) \right] \right\} = 0,$$

we obtain

$$(2.8) \qquad \int_{\partial D} \left\{ \frac{\partial\phi_+}{\partial n_q}(q)\gamma(P,q) - \phi_+(q)\frac{\partial\gamma}{\partial n_q}(P,q) \right\} dS_q$$

$$= \left(S\frac{\partial\phi_+}{\partial n} \right)(P) - (D\phi_+)(P) = \begin{cases} 2\phi_+(P), & P\epsilon D_+, \\ \phi_+(p), & p\epsilon\partial D, \\ 0, & P\epsilon D_-. \end{cases}$$

whilst for any solution ϕ_- of the Helmholtz equation in D_-, we have

$$\int_{\partial D} \left\{ \phi_-(q)\frac{\partial\gamma}{\partial n_q}(P,q) - \frac{\partial\phi_-}{\partial n_q}(q)\gamma(P,q) \right\} dS_q$$

$$(2.9)$$

$$= (D\phi_-)(P) - \left(S\frac{\partial\phi_-}{\partial n} \right)(P) = \begin{cases} 0, & P\epsilon D_+, \\ \phi_-(p), & p\epsilon\partial D, \\ 2\phi_-(P), & P\epsilon D_-. \end{cases}$$

We will be concerned with the following boundary value problem:

Exterior Robin Problem:

$$(2.10) \qquad (\nabla^2 + k^2)\, u_+(P) = 0, \quad P \varepsilon D_+$$

$$(2.11) \qquad \frac{\partial u_+}{\partial n_p} + \sigma(P)\, u_+(P) = g_+(P), \quad p \varepsilon \partial D$$

$$(2.12) \qquad \lim_{r_p \to \infty} r_p \left(\frac{\partial u_+}{\partial r_p}(P) - iku_+(P) \right) = 0$$

where $g_+ \varepsilon L_\infty(\partial D)$ and $\sigma \varepsilon L_\infty(\partial D)$. In the exterior scattering problem, q_+ is replaced by $-\frac{\partial u^i}{\partial n} - \sigma u^i$ where u^i is the known incident field.

Proceeding formally to obtain boundary integral equations whose solutions will subsequently be used to construct solutions to the boundary value problem we apply the Robin condition to the representation formula (2.8) which we write as

$$(2.13) \qquad \left(S \frac{\partial u_+}{\partial n} \right)(p) - (Du_+)P = \begin{cases} 2u_+(P), & P \varepsilon D_+, \\ u_+(p), & P \varepsilon \partial D, \\ 0, & P \varepsilon D_-. \end{cases}$$

to obtain

$$(2.14) \qquad (I + \bar{K}^* + S\sigma)\, u_+(P) = (Sg_+)(P), \quad p \varepsilon \partial D$$

where we have also used the fact that on ∂D

$$Du_+ = \bar{K}^* u_+.$$

If this boundary integral equation has a solution $u_+ \epsilon L_2(\partial D)$, then equation (2.13) may be employed to define u_+ in the exterior domain D_+ as

$$(2.15) \qquad u_+(P) = \frac{1}{2} S(g_+ - \sigma u_+)(P) - \frac{1}{2} (Du_+)(P), \quad P\epsilon D_+.$$

This function is clearly a solution of the Helmholtz equation in D_+. To insure that this solution in fact satisfies the boundary conditions, we require that the function u_+ satisfy an additional relation which we derive from the Helmholtz representation by computing normal derivatives as follows:

$$(2.16) \qquad \lim_{P\to p^-} [\frac{\partial}{\partial n_p} S(\frac{\partial u_+}{\partial n_q}) - \frac{\partial}{\partial n_p} D(u_+)](P) = 0$$

or, using the relation (3.16) below,

$$(2.17) \qquad -\frac{\partial u_+}{\partial n_p}(p) + K(\frac{\partial u_+}{\partial n_q})(p) - \lim_{P\to p^-} \frac{\partial}{\partial n_p} D(u_+)(P) = 0.$$

By utilizing the boundary condition we have

$$(2.18) \qquad (-g_+ + \sigma u_+ + K(g_+ - \sigma u_+) - D_n u_+)(p) = 0, \quad p\epsilon \partial D,$$

or, rearranging terms,

$$(2.19) \qquad (-\sigma + K\sigma + D_n) u_+ = (K-I)g_+,$$

where

$$(2.20) \quad (D_n u_+)(p) := \lim_{P \to p^-} \frac{\partial}{\partial n_p} (D u_+)(P).$$

Care must be exercised to ensure existence of this limit. We deal with this question (which depends on both the smoothness of the density and the degree of regularity of ∂D) in Lemma 4.1.

As with the integral equations derived in [6], we will see that the representation (2.15) affords a solution of the exterior boundary value problem (the Robin problem in the present case) provided $u_+ \epsilon L_2(\partial D)$ satisfies the pair of boundary integral equations. In fact, as we shall see below, solutions of the system (2.14) and (2.19) will necessarily be more regular than merely square-integrable.

Moreover we will see that, as in [6], this pair of boundary integral equations is redundant unless the homogeneous adjoint of equation (2.14) has non-trivial solutions. In such a case, however, the pair of integral equations will have a unique solution; it is this result we wish to establish in the following pages. Note, however, that unlike the cases discussed in [6], the second of these equations is an operator equation of the second kind.

The spectrum of K is clearly of vital importance in discussing the solutions of these equations and, since K may be considered as an operator-valued function of the parameter k, it is convenient to treat separately the operators $(-K)$ and $(+K)$ rather than introduce another parameter, λ, in order to obtain the standard form $(I - \lambda K)$. To this end, we adopt the following definition. Those

real values of k for which the equation $\overset{o}{w}-K\overset{o}{w} = 0$ has nontrivial

solutions, $\overset{o}{w}$, will be called characterisitc values of K.

Similarly, those real values of k for which $\overset{o}{w} + K\overset{o}{w} = 0$ has non-

trivial solutions, $\overset{o}{w}$, will be called characteristic values of

(-K). We note that if k is a characteristic value of K, then it

is also a characteristic value of K* [[5] Chap. XIII, §1.3], and

trivially it is also a characteristic value of $\bar{K}$ and $\bar{K}^*$. Similarly,

(-K), (-K*), (-$\bar{K}$),(-$\bar{K}^*$) have the same characteristic values, which

may differ from the preceding. Further, if $\overset{o}{w}$ is a nontrivial

solution of $\overset{o}{w} \pm K\overset{o}{w} = 0$, then $\overset{o}{\bar{w}}$ is a nontrivial solution of $\overset{o}{\bar{w}} \pm \bar{K}\overset{o}{\bar{w}}=0$,

and $\overset{o}{w}^*+K^*\overset{o}{w}^* = 0$ implies $\bar{w}^*+\bar{K}^*\bar{w}^* = 0$. Hereafter, an

elevated index zero, for example $\overset{o}{w}$, will indicate that the function

concerned is a solution of a homogeneous equation.

The values of k for which there exist nontrivial solutions

of the homogeneous Helmholtz equation in D_ with vanishing normal

derivative on ∂D will be called eigenvalues of the interior

Neumann problem. Similarly, eigenvalues of the exterior Neumann

and exterior and interior Dirichlet problems are those values of

k for which there exist nontrivial solutions of the corresponding

boundary value problems.

It is well known [[14] vol. IV, §229] that there are no

eigenvalues of the exterior problems, but there are eigenvalues

of the interior problems, and furthermore, these eigenvalues are

real (Stakgold [[15] vol. II, p. 137]). The relationship between

characteristic values of K and eigenvalues of the interior problems

is clarified in the following theorem, which appears in this form

in [5, p. 222].

Theorem 2.1:

(A) k is a characteristic value of K if and only if k is an eigenvalue of the interior Neumann problem.

(B) k is a characteristic value of (-K) if and only if k is an eigenvalue of the interior Dirichlet problem.

The interested reader is referred to [6] for the proof of this result. We remark only that an immediate consequence is that if k is an eigenvalue of the interior Dirichlet problem, then k is a characteristic value of (-K) and hence (-K*).

To conclude this section, we wish to point out that the analysis presented here is applicable to the scattering problem for imperfectly reflecting surfaces. Under appropriate hypotheses, among which is one requiring the radius of curvature to be large relative to the penetration depth, the exact transmission conditions may be approximated by a boundary condition of the third kind with purely imaginary σ which may interpreted as an acoustic impedance [13]. The analogous problem in electromagnetics is discussed by Senior [12]. It is possible, then, to reduce the scattering problem to an equivalent pair of boundary integral equations. These equations differ from equations (2.14) and (2.19) only in their right hand members and, as we shall see, for sufficiently smooth incident fields the analysis of Section 4 will be valid for the scattering problem.

Specifically, we may proceed as follows. Denoting the incident and scattered fields, respectively by u^i and u^s and recognizing that the incident field must be a solution of the Helmholtz equation on the interior domain, we may use the Helmholtz representations (2.8) and (2.9) to show that, on the boundary ∂D

$$(2.21) \quad u^S = S(\partial u^S/\partial n) - \bar{K}*u^S$$

$$(2.22) \quad u^i = \bar{K}*u^i - S(\partial u^i/\partial n).$$

Consequently,

$$u = u^S + u^i = 2u^i + S(\partial u/\partial n) - \bar{K}*u.$$

Invoking the boundary condition $\partial u/\partial n + \sigma u = 0$, this last relation may be rewritten as

$$(2.23) \quad (I + S\sigma + \bar{K}*)u = 2u^i$$

where, we emphasize, u^i is the known incident field.

Likewise, since

$$(2.24) \quad u^S(P) = (1/2)[S(\partial u^S/\partial n) - Du^S](P), \quad P\varepsilon D_+,$$

and

$$(2.25) \quad u^i(P) = (1/2)[Du^i - S(\partial u^i/\partial n)](P), \quad P\varepsilon D_-,$$

we have, taking normal derivatives and taking account of the jump conditions, that for $p\varepsilon\partial D$,

$$\partial u^S/\partial n_p = (1/2)[\partial u^S/\partial n_p + K \partial u^S/\partial n_p] - (1/2) D_n u^S,$$

and

$$\partial u^i/\partial n_p = (1/2)[\partial u^i/\partial n_p - K \partial u^i/\partial n_p] + (1/2)D_n u^i.$$

It is easy to check, from these last two relations, that since $\partial u/\partial n_p = -\sigma u$ on ∂D, the total field u must satisfy the equation

$$(2.26) \quad (-\sigma + K\sigma + D_n)u = 2\ (\partial u^i/\partial n_p).$$

Equations (2.23) and (2.26) are thus the particular forms of (2.14) and (2.19) applicable to the scattering problem.

3. *Auxillary Results*

In this section we collect some known results and their extensions as well as some new results which will be used in the subsequent discussion. For more details, the interested reader should consult the cited references; we have attempted to select not original references to these particular theorems but rather those which are readily accessible.

We begin with the result alluded to above which we will refer to, in the sequel, as the regularity theorem. The particular version of the results presented here is, on the one hand, a specialization to weakly singular surface integrals of a theorem of Mikhlin [10] for closed, bounded sets in $\mathbb{R}^n$ which, on the other hand, is a minor generalization of Mikhlin's theorem because of the occurrence of the factor μ which lies in $L_\infty(\ D)$.

Theorem 3.1:

Let ∂D be a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces). If A: $\partial D \times \partial D \rightarrow \mathbb{C}$ and f: $\partial D \rightarrow \mathbb{C}$ are continuous, $\mu \epsilon L_\infty(\partial D)$ and $\alpha < 2$, then any solution of

$$(3.1) \quad u(p_0) - \int_{\partial D} \frac{A(p,p_0)}{R^\alpha(p,p_0)} \, \mu(p) \, u(p) \, dS_p = f(p_0)$$

which belongs to $L_2(\partial D)$ is also continuous on ∂D.

Proof: For any $\epsilon > 0$, choose $\eta(t)$ to be a continuous, real-valued, non-increasing function of $t \geq 0$, with the property that $\eta(t) = 1$, $0 \leq t \leq \epsilon/2$, and $\eta(t) = 0$, $t \geq \epsilon$. Then, setting

$$(3.2) \quad K_1(p,p_0) := \frac{A(p,p_0) \, \eta(R(p,p_0))}{R^\alpha(p,p_0)}$$

and

$$(3.2) \quad K_2(p,p_0) := \frac{A(p,p_0) \, [1-\eta(R(p,p_0))]}{R^\alpha(p,p_0)}$$

where $K_2(p,p_0)$ is continuous on $\partial D \times \partial D$, the integral equation (3.1) may be written as

$$(3.4) \quad u(p_0) - (K_1 \mu u)(p_0) = F(p_0)$$

where

$$(3.5) \quad (K_1 \mu u)(p_0) := \int_{\partial D} K_1(p,p_0) \mu(p) \, u(p) \, dS_p.$$

and

$$(3.6) \quad F(p_0) = f(p_0) + \int_{\partial D} K_2(p,p_0) \mu(p) u(p) \, dS_p.$$

The function F is continous since f is continuous and

$$(3.7) \quad \left| \int_{\partial D} [K_2(p,p_0) - K_2(p,p_1)] \, \mu(p) u(p) \, dS_p \right| \leq$$

$$\left\{ \int_{\partial D} |K_2(p,p_0) - K_2(p,p_1)|^2 \, dS_p \right\}^{1/2} ||\mu u||_{L_2(\partial D)}$$

where K_2 is continuous in both variables. Note that $\mu u \varepsilon L_2(\partial D)$ since $u \varepsilon L_2(\partial D)$ and $\mu \varepsilon L_\infty(\partial D)$.

Now observe that, with the Schwartz inequality and the fact that $\eta \leq 1$, and writing R for $R(p,p_0)$,

$$(3.8) \quad ||K_1 \mu u||^2_{L_2(\partial D)} = \int_{\partial D} \left| \int_{\partial D} \frac{A(p,p_0) \eta(R) \mu(p)}{R^{\alpha/2}} \cdot \frac{u(p)}{R^{\alpha/2}} dS_p \right|^2 dS_{p_0}$$

$$\leq \int_{\partial D} \left\{ \int_{\partial D} \frac{|A(p,p_0)|^2 \eta^2(R) |\mu(p)|^2}{R^\alpha} dS_p \cdot \int_{\partial D} \frac{|u(p)|^2}{R^\alpha} dS_p \right\} dS_{p_0}$$

$$\leq c^2 \int_{\partial D} \left\{ \int_{\partial D \cap B_\varepsilon(p_0)} \frac{dS_p}{R^\alpha} \int_{\partial D} \frac{|u(p)|^2}{R^\alpha} dS_p \right\} dS_{p_0}$$

where $c^2 = ||A||^2_{C_0(\partial D \times \partial D)} \cdot ||\mu||^2_{L_\infty(\partial D)}$ and $B_\varepsilon(p_0)$ is a ball of radius ε centered at p_0, outside of which $\eta(R(p,p_0)) = 0$.

15

If d is the radius of the Lyapunov sphere associated with ∂D and $\varepsilon < d$, we have the following estimates:

$$(3.9) \quad \int_{\partial D \cap B_\varepsilon(p_0)} \frac{dS_p}{R^\alpha} \leq c' \int_0^\varepsilon \rho^{1-\alpha} d\rho = \frac{c'\varepsilon^{2-\alpha}}{2-\alpha}$$

for some c' independent of p_0 and ε and

$$(3.10) \quad \int_{\partial D} \frac{dS_{p_0}}{R^\alpha} = \int_{\partial D \cap B_d(p_0)} \frac{dS_{p_0}}{R^\alpha} + \int_{\partial D \setminus \partial D \cap B_d(p_0)} \frac{dS_{p_0}}{R^\alpha} \leq \frac{c'd^{2-\alpha}}{2-\alpha} + \frac{S}{d^\alpha}$$

where S is the surface area of ∂D.

Employing these estimates in (3.8) we find that

$$(3.11) \quad ||K_1\mu u||^2_{L_2(\partial D)} \leq c^2 c' \frac{\varepsilon^{2-\alpha}}{2-\alpha} \left(\frac{c'd^{2-\alpha}}{2-\alpha} + \frac{S}{d^\alpha}\right) ||u||_{L_2(\partial D)}.$$

Thus for ε sufficiently small

$$(3.12) \quad ||K_1\mu||_{L_2(\partial D)} < 1$$

and the solution of (3.4) in $L_2(\partial D)$ is given by

$$(3.13) \quad u(p_0) = \sum_{n=0}^\infty ([K_1\mu]^n F)(p_0).$$

In fact, this series is uniformly convergent hence $u(p_0)$ is continuous. This follows because F is continuous, hence bounded on ∂D, and, using arguments similar to the preceeding (in particular using (3.9)) we may establish

$$(3.14) \quad |(K_1 \mu F)(p_0)| = \left| \int_{\partial D} \frac{A(p,p_0)}{R^\alpha} \, \eta(R) \, \mu(p) \, F(p) \, dS_p \right|$$

$$\leq c||F||_{C_0(\partial D)} \int_{\partial D \cap B_\varepsilon(p_0)} \frac{dS_p}{R^\alpha} \leq \frac{cc'}{2-\alpha} \, \varepsilon^{2-\alpha} ||F||_{C(\partial D)}.$$

Thus we have the estimate

$$(3.15) \quad ||(K_1 \mu)^n F||_{C_0(\partial D)} \leq \frac{cc'}{2-\alpha} \varepsilon^{2-\alpha} ||(K_1 \mu)^{n-1} F||_{C(\partial D)}$$

$$\leq \left(\frac{cc'}{2-\alpha} \varepsilon^{2-\alpha}\right)^n ||F||_{C_0(\partial D)}$$

which, for ε sufficiently small, establishes the uniform convergence of (3.13) and hence the continuity of u.

The next three results may be found in the book of Günter [4] and relate to the behavior of single and double layer potentials. The results on potentials with integrable densities, due originally to Fichera [3], are proven in [4] for the case $k=0$ but remain valid for $k \neq 0$ since the singular part of the kernels concerned remains unaltered. The first of these results establishes continuity properties of single and double layer potentials.

Theorem 3.2:

If ∂D is a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces) and if $\mu\epsilon L_\infty(\partial D)$ then the single layer potential with density μ is Hölder continuous in the entire space, while the double layer potential with the same density is Hölder continuous on ∂D. Moreover, both are infinitely differentiable off the boundary.

Remarks: The statement of this theorem may be summarized by the assertion: $\mu\epsilon L_\infty(\partial D)$ implies (a) $S\mu\epsilon \ C_{0,\alpha}(\mathbf{R}^3) \ \cap C_\infty(\mathbf{R}^3\setminus\partial D)$ and (b) $\bar{K}^*\mu \ \epsilon \ C_{0,\alpha}(\partial D)$, $D\mu\epsilon \ C_\infty(\mathbf{R}^3\setminus\partial D)$. The statement here differs from that appearing in Günter in that the assumption that $\mu\epsilon L_\infty(\partial D)$ replaces the assumption that μ is bounded on ∂D. The estimates necessary for the proof are identical in the present case to those appearing in Günter.

Turning our attention to the jump relation, we may state the the following result. Recall that if μ is integrable, then almost every point of its domain is a Lebesgue point.

Theorem 3.3:

If ∂D is a surface as described above, $\mu\epsilon L_1(\partial D)$ and if $p\epsilon\partial D^-$ is a Lebesgue point of μ, then

$$(3.16) \quad \lim_{P\to p^-} \frac{\partial}{\partial n_p} \ (S\mu)(P) = (-\mu+K\mu)(p),$$

$$(3.17) \quad \lim_{P\to p^+} \frac{\partial}{\partial n_p} \ (S\mu)(P) = (\mu+K\mu)(p),$$

and, for the double layer,

$$(3.18) \quad \lim_{P \to p^-} (D\mu)(P) = (\mu + \bar{K}*\mu)(p),$$

and

$$(3.19) \quad \lim_{P \to p^+} (D\mu)(P) = (-\mu + \bar{K}*\mu)(p).$$

Moreover, the limits in each of the four cases is integrable.

We complete our discussion of the properties of single and double layer potnetials with the following:

Theorem 3.4:

Let ∂D be a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces) with index 1 (i.e. $\hat{n}_{p_1} \cdot \hat{n}_{p_2} < ER(p, p_2)$ where $\hat{n}_{p_i}$, $i = 1, 2$, is the exterior unit normal to ∂D at the point p_i and E is a constant independent of P_1, P_2 and let μ be continuous on ∂D. If the double layer potential (2.4) possesses one of the derivatives

$$(3.20) \quad \frac{\partial}{\partial n_p^-} D\mu(p) := D_u^- \mu(p) \quad \text{or} \quad \frac{\partial}{\partial n_p^+} D\mu(p) := D_n^+ \mu(p)$$

then both derivatives exist and are equal at the point $p \in \partial D$. Their common value will be denoted by $D_n \mu(p)$.

Next, we present a version of the divergence theorem which is essential to what follows. This treatment allows the development of an L_2 theory of surface potentials which is of interest for its own sake as well as the present application. It is closely allied to the treatment of the volume integral as an improper integral presented in Smirnov [14]. We remark that an alternative generalized divergence theorem involving potentials with continuous rather than square integrable densities and based on the interpretation of the normal derivative as a distribution or "boundary flow" is available [1], [8] and has been used in an integral equation approach to the Neumann problem for the Helmholtz equation [7]. That treatment does not address the non-uniqueness problem at characteristic values of the integral operator in contrast to the present approach.

Theorem 3.5:

Let ∂D be a Lyapunov surface of index 1 and let $\{S_\delta^-\}$ be a family of "parallel" surfaces in D_- such that (i) for fixed δ, S_δ^- is the image of ∂D under a continuously differentiable transformation $F_\delta^-:\ \partial D \to S_\delta^-$; (ii) $F_\delta^-(p) := P_\delta = p + 0(\delta)$ and if $p(u,v)$ is a parameterization of ∂D, $\dfrac{\partial P_\delta}{\partial u} = \dfrac{\partial p}{\partial u} + 0(\delta)$ and $\dfrac{\partial P}{\partial v} =$

$\dfrac{\partial p}{\partial v} + 0(\delta)$; (iii) $\displaystyle\int_{\Sigma_\delta} dS_{P_\delta} = \int_\Sigma dS_p + 0(\delta)$ for every $\Sigma \subset \partial D$ where $\Sigma_\delta = F_\delta^-(\Sigma)$.

Now, if $\Phi \in C_1(D_-)$ is a vector-valued function with limiting values in $L_2(\partial D)$ by which is meant

$$(3.21) \qquad \int_{\partial D} |\Phi(F_\delta^-(p)) - \Phi(p)|^2 dS_p = o(\delta^0)$$

then

$$(3.22) \qquad \int_{\partial D} \hat{n}_p \cdot \Phi(P) dS_p = \int_{D_-} \nabla \cdot \Phi(P) d\tau_p.$$

Remark: The order symbol $O(\delta)$ denotes a vector whose magnitude decreases uniformly with respect to δ. While it is not necessary to require that $|P_\delta - p|$ be independent of p for fixed δ, we do require that the surface S_δ not coincide with ∂D at any point, i.e.,

$$(3.23) \qquad \inf_{p, q \in \partial D} |F_\delta(p) - q| \geq \theta \, \delta$$

for some $\theta > 0$, independent of δ.

Proof: Since $\Phi \varepsilon C_1(D_-)$ and $S_\delta \subset D_-$, the divergence theorem in its classical form yields

$$(3.24) \qquad \int_{S_\delta^-} \hat{n}_{P_\delta} \cdot \Phi(P_\delta) \, dS_{P_\delta} = \int_{D_{\delta_-}} \nabla \cdot \Phi(P) \, d\tau.$$

where D_{δ_-} denotes the interior of S_δ^-. Consider, first, the integral on the left hand side. The surface ∂D may be decomposed into a finite number of patches, $\partial D = \bigcup_{i=1}^{N} \Sigma_i$ where each Σ_i, $i = 1, \ldots, n$, lies in a Lyapunov sphere. Let $\Sigma_{i\delta} = F_\delta^-(\Sigma_i)$. Then

$$(3.25) \qquad \int_{S_\delta^-} \hat{n}_{P_\delta} \cdot \Phi(P_\delta) \, dS_{P_\delta} = \sum_{i=1}^{N} \int_{\Sigma_{i\delta}} \hat{n}_{P_\delta} \cdot \Phi(P_\delta) \, dS_{P_\delta}.$$

Points on Σ_i may be represented in local coordinates $(\xi, \eta, \zeta(\xi, \eta))$ where Σ_i is the image of some domain $D_{i\xi, \eta}$ in the (ξ, η)-plane. Similarly, (ξ, η) may be chosen to parameterize $\Sigma_{i\delta}$. An element of area on Σ_i is $dS_p = \sqrt{1 + \zeta_\xi^2 + \zeta_\eta^2} \; d\xi d\eta$ whereas an element of area on $\Sigma_{i\delta}$ is, using condition (ii), $dS_{P_\delta} = [\sqrt{1 + \zeta_\xi^2 + \zeta_\eta^2} + O(\delta)] d\xi d\eta$ or $dS_p = dS_{P_\delta} + O(\delta)$.

Similarly, on Σ_i, $\hat{n}_p = \dfrac{(-\zeta_\xi, -\zeta_\eta, 1)}{\sqrt{1+\zeta_\xi^2+\zeta_\eta^2}}$, while on $\Sigma_{i\delta}$, $\hat{n}_{P_\delta} = \hat{n}_p + 0(\delta)$.

Hence, with $\Phi(P)$ and $\Phi(F_\delta^-(p)) \varepsilon L_2(\partial D)$, we have

$$(3.26) \quad \int_{\Sigma_{i\delta}} \hat{n}_{P_\delta} \cdot \Phi(P_\delta) \, dS_{P_\delta} = \int_{\Sigma_i} \hat{n}_p \cdot \Phi(F_\delta^-(p)) \, dS_p + 0(\delta) \ .$$

Also, in light of formula (3.21), we have the estimate

$$(3.27) \quad \left| \int_{\Sigma_i} \hat{n}_p \cdot \Phi(F_\delta^-(p)) \, dS_p - \int_{\Sigma_i} \hat{n}_p \cdot \Phi(P) \, dS_p \right| \leq$$

$$\left\{ \int_{\Sigma_i} |\Phi(F_\delta^-(p)) - \Phi(p)|^2 \, dS_p \right\}^{1/2} + 0(\delta) \ \leq 0(\delta) \ .$$

Therefore

$$(3.28) \quad \lim_{\delta \to 0} \int_{\Sigma_{i\delta}} \hat{n}_{P_\delta} \cdot \Phi(P_\delta) \, dS_{P_\delta} = \int_{\Sigma_i} \hat{n}_p \cdot \Phi(p) \, dS_p.$$

Thus the limit on the left hand side of (3.24) exists as $\delta \to 0$, is equal to the left hand member of the relation (3.22) and hence the limit of the right hand side in relation (3.24) exists and equals $\int_{\partial D} \hat{n}_p \cdot \Phi(P) \, dS_p$. Thus the integral $\int_{D_-} \nabla \cdot \Phi(P) \, dS_p$ has a meaning as an improper integral and, in this sense, we have established the equality (3.22).

Remark: The parallel surfaces used here are similar to,
but more restrictive than those discussed by Burago, Maz'ja and
Sapozhnikova in [1].   The differentiability of the transforma-
tion ensures the convergence of normals and surface elements
and, while the assumption of such convergence may, in fact,
insure differentiability of the transformation, we have not
established this assertion.   We also wish to point out that the
construction of parallel surfaces given by Smirnov in [14] does
not guarantee that the normals converge unless the surface ∂D is
in fact C_2, i.e. the normal is differentiable rather than merely
Lipschitz continuous.

Theorem 3.5 has two immediate consequences which we state
here.

Corollary 1:   If $w \in C_2(D_-)$ and if w and $\frac{\partial w}{\partial n}$ have limits in
$L_2(\partial D)$ from D_- in the sense of Theorem 3.5, then

$$(3.29) \quad \int_{\partial D} w(p) \, \frac{\partial \bar{w}}{\partial n}(p) \, dS_p = \int_{D_-} (\,|\nabla w(P)|^2 + w(P) \, \Delta \, \bar{w}(P)\,) \, d\tau_p.$$

Corollary 2:   If $w \in C_2(D_+)$, if w and $\frac{\partial w}{\partial n}$ have limits in $L_2(\partial D)$
from D_+ in the sense of Theorem 3.5, and if B_R is a ball of
radius R containing D_- in its interior, then

$$(3.30) \quad \int_{\partial D} w(p) \frac{\partial \bar{w}}{\partial n}(p) \, dS_p = - \int_{D_+ \cap B_R} (\,|\nabla w(P)|^2 + w(P) \, \Delta \, \bar{w}(P)\,) \, d\tau_p + \int_{\partial B_R} w(p) \frac{\partial \bar{w}}{\partial n}(p) \, dS_p$$

The first of these corollaries follows immediately from
Theorem 3.5 by identifying Φ in that theorem with the function $w\nabla\bar{w}$.
The second follows from the first by applying the statement to the

interior domain bounded by ∂D and ∂B_R.

We conclude this section with a uniqueness theorem for the exterior Robin problem for the Helmholtz equation. The proof is based on Rellich's lemma and was given by Leis [9] in the classical case for real, continuous σ. The extension presented here takes into account complex-valued σ in $L_\infty(\partial D)$ and considers a more general class of solutions. Specifically, we consider solutions of the Helmholtz equation in D_+ which, together with their normal derivatives take on boundary values, square-integrable on ∂D, in the sense of Theorem 3.5. This notion is precisely the notion of generalized boundary conditions described by Miranda [8; §29]. We refer the reader to that work for a discussion of the history of this interpretation of boundary conditions for elliptic equations. It is however vital to the results of section IV to know that single and double layer distributions assume their boundary values in this generalized sense which we state in the following:

Theorem 3.6: If ∂D is a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces) and $\mu\epsilon L_2(\partial D)$ then $S\mu$, $D\mu$, and $\dfrac{\partial S\mu}{\partial n^{\pm}}$ assume boundary values in $L_2(\partial D)$ in the sense of Theorem 3.5.

The proof of this theorem is deferred to the appendix. Throughout the remainder of this paper, the boundary value problem will be understood in this generalized sense and the statement of the boundary value problem (2.10)-(2.12) should be so interpreted. With this understanding we state

Theorem 3.7:

Let ∂D be a Lyapunov surface of index 1, let $k\epsilon\mathbb{C}$ with $0\le\arg k\le\pi$ (Im$k\ge 0$) and let $\sigma\epsilon L_{\infty}(\partial D)$ with $\arg k\le \arg \sigma\le\arg k+\pi$. If $w\epsilon C_2(D_+)$ is such that

$$(3.31) \quad (\Delta +k^2)w(P)=0 \qquad , \quad P\epsilon D_+ , \quad \cdot$$

$$(3.32) \quad \frac{\partial w}{\partial r} - ikw = o(1/r), \quad r=|P|,$$

and

$$(3.33) \quad \frac{\partial w}{\partial n_p^+}(p) + \sigma w(p)= 0, \quad p\epsilon\partial D ,$$

then $w=0$ almost everywhere in $D_+\cup\partial D$.

Remark: We repeat for emphasis that the boundary condition (3.33) is to be interpreted in the generalized sense and the equality, consequently, holds only almost everywhere on ∂D.

Proof: Let B_R be a ball with boundary ∂B_R, center in D_- and radius R sufficiently large so that $D_-\subset B_R$. Assuming that w satisfies the hypothesis of the theorem, we may apply Green's theorem to the function w and its complex conjugate $\bar{w}$ in the region $D_+\cap B_R$ obtaining, using Theorem 3.5 and the relation (3.30) the integral identity

$$(3.34) \quad \int_{D_+\cap B_R} (w\Delta \bar{w}-\bar{w}\Delta w)\,d\tau=\int_{\partial B_R} (w\frac{\partial\bar{w}}{\partial n} - \bar{w} \frac{\partial w}{\partial n})\,dS$$

$$-\int_{\partial D} (w\frac{\partial\bar{w}}{\partial n} - \bar{w} \frac{\partial w}{\partial n})\,dS.$$

The radiation condition (3.32) shows that

$$(3.35) \quad (\frac{\partial w}{\partial n} - ikw)(\frac{\partial \bar{w}}{\partial n} + i\bar{k}\bar{w}) = o(\frac{1}{r^2})$$

from which we may deduce that, on ∂B_R,

$$(3.36) \quad w\frac{\partial \bar{w}}{\partial n} - \bar{w}\frac{\partial w}{\partial r} = \frac{1}{i\mathrm{Re}k}\{|\frac{\partial w}{\partial n}|^2 + |kw|^2 + (\mathrm{Im}k)(\bar{w}\frac{\partial w}{\partial n} + w\frac{\partial \bar{w}}{\partial n})\} + o(\frac{1}{R^2}).$$

Substituting this result in the identity (3.34) and making use of the fact that both w and $\bar{w}$ are solutions of the Helmholtz equation and satisfy homogeneous boundary conditions (equations (3.31 and (3.33) and their conjugates) we obtain

$$(3.37) \quad (k^2 - \bar{k}^2)\int_{D_+ \cap B_R}|w|^2 d\tau = \frac{1}{i\mathrm{Re}k}\int_{\partial B_R}\{|\frac{\partial w}{\partial n}|^2 + |kw|^2 + (\mathrm{Im}k)(\bar{w}\frac{\partial w}{\partial n} + w\frac{\partial \bar{w}}{\partial n})\}dS$$

$$+ \int_{\partial D}(\bar{\sigma} - \sigma)|w|^2 dS + o(1).$$

However, Theorem 3.5 may be applied to $\bar{w}\Delta w + w\Delta \bar{w}$ and, again using the boundary conditions, we obtain

$$(3.38) \quad \int_{\partial B_R}(\bar{w}\frac{\partial w}{\partial n} + w\frac{\partial \bar{w}}{\partial n})dS = \int_{D_+ \cap B_R}(2|\nabla w|^2 - (k^2 + \bar{k}^2)|w|^2)d\tau - \int_{\partial D}(\sigma + \bar{\sigma})|w|^2 dS.$$

Substituting this result in (3.37) and making use of the facts

$$(3.39) \quad i(\mathrm{Re}k)(\bar{k}^2 - k^2) - (\mathrm{Im}k)(k^2 + \bar{k}^2) = 2|k|^2(\mathrm{Im}k)$$

and

(3.40) $(\text{Re}k)(\text{Im}\sigma) - (\text{Im}k)(\text{Re}\sigma) = \text{Im}(\overline{k}\sigma)$

we obtain

(3.41) $\int_{\partial B_R} (|\frac{\partial w}{\partial n}|^2 + |kw|^2)\,dS + 2(\text{Im}k) \int_{D_+ \cap B_R} (|kw|^2 + |\nabla w|^2)\,d\tau$

$+ 2 \int_{\partial D} (\text{Im}\overline{k}\sigma)|w|^2\,dS = o(1).$

All the terms on the left are easily seen to be non-negative if $(\text{Im}k) \geq 0$ and $(\text{Im}\overline{k}\sigma) \geq 0$, this last condition being equivalent to the condition arg $k \leq$ arg $\sigma <$ arg $k+\pi$. It follows, then, from Rellich's Lemma that $w=0$.

Note that the space in which w lies allows us to conclude that $w=0$ in D_+ whereas we can assert only that $w=0$ almost everywhere on ∂D. It should also be noted that the classical solution space, $C_2(D_+) \cap C_1(D_+ \cup \partial D)$ is a subspace of the space of functions specified in the statement of the theorem hence in the classical case $w=0$ everywhere in $D_+ \cup \partial D$.

4. *Existence and Uniqueness of Solutions*

The formal calculations of section 2 have given us a pair of boundary integral equations for a function $u \varepsilon L_2(\partial D)$, namely the equations

(4.1) $(I + \overline{K}^* + S\sigma)u = Sg_+$

(4.2) $(-\sigma + K\sigma + D_n)\,u = Kg_+ - g_+.$

Our object in this section is to demonstrate the existence of a
unique solution of this pair of equations and to establish the
equivalence of this problem to the original boundary value problem.
The proof will proceed by means of a series of lemmas. Through-
out this section, unless explicitly mentioned to the contrary,
(e.g. Lemma 4.3) the functions g_+ and σ will be assumed to lie
in the space $L_\infty(\partial D)$. Furthermore, the boundary ∂D will be assumed
to satisfy the hypothesis of theorem 3.5.

We remark, first, that by applying the regularity theorem
(Theorem 3.1) to any solution of equation (4.1), we may conclude
that any L_2-solution of equation (4.1) must, in fact, be continuous.
Note that the regularity theorem is applicable here since, under
the hypotheses on g_+, the function Sg_+ is continuous everywhere.
(Theorem 3.2), and the kernels of the integral operators $\bar{R}^*$ and $S\dot{\sigma}$
are products of weakly singular, essentially bounded and continuous
functions. In fact it is to ensure the continuity of Sg_+ and the
subsequent applicability of the regularity theorem that we restrict
g_+ to lie in L_∞ rather than L_2. As we shall see below the fact
that L_2 solutions of (4.1) must also be continuous will be of
crucial importance in ensuring that certain functions used in the
construction of the required solution of the system (4.1)-(4.2)
will lie in the domain of the operator D_n. Indeed, we will begin
with the following basic result relating the set of solutions of
equation (4.1) and the domain of this operator.

Lemma 4.1: Suppose that the wave number k is not an eigen-
value of the interior Dirichlet problem for the Helmholtz
equation. Then, for any solution u of equation (4.1) in $L_2(\partial D)$
$D_n^- u$, and hence $D_n^+ u$, exists and, moreover, $D_n^- u = D_n^+ u := D_n u$.

Proof: Let u be any solution of the equation (4.1). In light of the theorem of Günter (Theorem 3.4 above) it will be sufficient to show that $D_n^- u$ exists, at least almost everywhere on ∂D.

To this end, define the function v on D_- by the equation

(4.3) $v(P) := (Du + S\sigma u)(P) - Sg_+(P), \quad P \epsilon D_-$.

Then v takes on boundary values in the sense of Theorem 3.5. Since the function u is a solution of the integral equation (4.1), the regularity theorem, 3.1, assures that u is continuous on ∂D. Therefore u is bounded on ∂D and hence the function σu is essentially bounded, measurable, and so integrable on ∂D. Likewise, the function $\sigma u - g_+$ is essentially bounded and integrable on ∂D, and so the single layer with density $\sigma u - g_+$ is continuous everywhere by Theorem 3.2. Recalling the form of the jump conditions for double layer with continuous density (theorem 3.5), we have for all $p \epsilon \partial D$,

(4.4) $\lim_{P \to p^-} v(P) = (u + \bar{K}^* u + S\sigma u)(p) - (Sg_+)(p)$.

Since, by assumption, the function u is a solution of the equation (4.1) we may conclude that $\lim_{P \to p^-} v(P) = 0$ for all points $p \epsilon \partial D$. The function v, therefore, is a solution of the interior homogeneous Dirichlet problem with zero boundary data. Since k is <u>not</u> an eigenvalue of this problem, the function v must vanish identically

in $\bar{D}_-$ and so $\partial v / \partial n_p^- = 0$ on ∂D.

We now rewrite the relation defining v in D_- as follows:

(4.5) $(Du)(P) = v(P) + S[g_+ - \sigma u](P), \quad P \epsilon D_-.$

Again, since density of the single layer operator in this last relation is integrable, we may use Theorem 3.3 to conclude that

(4.6) $\lim\limits_{P \to p^-} \dfrac{\partial}{\partial n_p^-} S[g_+ - \sigma u] (P) = [-g_+ + Kg_+ + \sigma u - K\sigma u] (p)$

almost everywhere on ∂D. Since we have just shown that $\partial v / \partial n_p^- = 0$ for all $p \epsilon \partial D$, we may conclude that $D_n^- u$ exists for almost all $p \epsilon \partial D$ which is what we wished to demonstrate. Moreover at such points p,

(4.7) $D_n^- u(p) = (-g_+ + Kg_+ + \sigma u - K\sigma u) (p),$

which relates the solutions of equation (4.1) to those of equation (4.2). More precisely, we have established the following result.

Lemma 4.2: If k is not an eigenvalue of the interior homogeneous Dirichlet problem and if the function u is a solution of equation (4.1) in $L_2(\partial D)$, then the function u is likewise a solution of equation (4.2).

We may now turn to a discussion of the existence and uniqueness of solutions.

Lemma 4.3: For all choices of $g_+ \epsilon L_2(\partial D)$, the equation (4.1) has a solution in $L_2(\partial D)$.

Proof: The integral operators in (4.1), $\bar{K}^*$ and $S\sigma$, are completely continuous on $L_2(\partial D)$ (Mikhlin [10]) hence Fredholm's Alternative applies to (4.1). Observe that the operator $S\sigma$ should be viewed as a composition of multiplication by the essentially bounded function σ which maps $L_2(\partial D)$ into itself and the single layer which is weakly singular. Fredholm's Alternative implies that for any g_+ either equation (4.1) has a solution or the homogeneous problem has non-trivial solutions, in which case, equation (4.1) has a solution if and only if the function Sg_+ is orthogonal to all solutions of the homogeneous adjoint equation which may be written $(I+\bar{K}+\bar{\sigma}S^*)\overset{0}{w} = 0$ or, equivalantly, $(I+\bar{K}+\bar{\sigma}\bar{S})\overset{0}{w} = 0$. In fact, we will show that, for all such $\overset{0}{w}$, $(Sg_+, \overset{0}{w}) = 0$ for any choice of $g_+ \epsilon L_2(\partial D)$.

To this end note that $(Sg_+, \overset{0}{w}) = (g_+, \bar{S}\overset{0}{w})$ and define a function v in D_+ by

$$(4.8) \quad v(P) := (\bar{S}\overset{0}{w})(P), \quad P \epsilon D_+.$$

Then, since $\overset{0}{w} \epsilon C_0(\partial D)$, the jump relations yield, for almost all $p \epsilon \partial D$,

$$(4.9) \quad \partial v/\partial n_p^+ (p) = (\overset{0}{w} + \bar{K}\overset{0}{w})(p).$$

But the function $\overset{0}{w}$ satisfies the equation $(I+\bar{K}+\bar{\sigma}\bar{S})\overset{0}{w} = 0$ and so we may write

(4.10) $\partial v / \partial n_p^+ (p) = -\bar{\sigma}\bar{S}\overset{0}{\hat{w}} (p) = -\bar{\sigma}v(p)$, for almost all $p\epsilon\partial D$.

Hence, the function v is a solution of the exterior homogeneous Robin problem which assumes boundary values in the sense of Theorem 3.5 and so vanishes identically according to the uniqueness theorem discussed above. We may conclude $\bar{S}\overset{0}{\hat{w}} = 0$ and this establishes the lemma.

This last result, together with the preceeding lemma, shows that if k is not an eigenvalue of the interior homogeneous Dirichlet problem, then the pair of integral equations (4.1) - (4.2) has a simultaneous solution. Moreover, there can be at most one such solution as we now show.

Lemma 4.4: **If** $u\epsilon L_2$ (∂D) is a solution of the homogeneous system of boundary integral equations

(4.11) $(I+\bar{K}^*+S\sigma)u = 0$

(4.12) $(-\sigma+K\sigma+D_n)u = 0$.

then u $=$ 0, $p\epsilon\partial D$

Note that since the function u is assumed to satisfy the equation (4.11), we may again invoke the regularity theorem to establish the continuity of u. Indeed, it may well be smoother since we assume as an hypothesis that it lies in the domain of the operator D_n.

Proof: Again, define the function v on D_- by the relation

$v(P) := (D + S\sigma)u(P)$, $P\epsilon D_-$

where u is a solution of the homogeneous system (4.11)-(4.12).
Then, as before, v takes on boundary values in the sense of
Theorem 3.5 and, since u is an L_2-solution of equation (4.11)
which has a continuous right hand member, it is also continuous.
Taking limits as the point $P \epsilon D_-$ approaches the boundary of D,
and using the jump conditions for the double layer operator with
continuous density, we have

$$(4.13) \quad v(p) = u(p) + \bar{K}*u(p) + (S\sigma)u(p)$$

and so v(p) = 0 for all $p \epsilon \partial D$ since u is a solution of equation
(4.11). Hence the function v is a solution of the homogeneous
Dirichlet problem in D_-. There are, then, two possibilities:
either (i) v vanishes identically in D_-, or (ii) v is a non-
trivial eigenfunction of the interior homogeneous Dirichlet
porblem for the Helmholtz equation. In case (ii) the function
v has regular normal derivative in the sense of Lyapunov (see
Smirnov [14] p. 586 and p. 675).

In the first case, we must have that (Du + Sσu)(P) = 0
for all $P \epsilon D_-$. Now define the function v_+ in D_+ by

$$(4.14) \quad v_+(P) := (Du + S\sigma u)(P), \quad P \epsilon D_+,$$

which, like v_-, takes boundary values in the sense of Theorem 3.5.
Using the jump relations for the double layer operator as the
point P approaches ∂D from D_+, we may write

$$(4.15) \quad v_+ (p) = -u(p) + \bar{K}*u(p) + S(\sigma u)(p), \quad p \varepsilon \partial D.$$

Since, for merely bounded and measurable densities, the jump relations for the normal derivative of the single layer operator hold only almost everywhere on ∂D, we may argue in a manner similar to the proof of Lemma 4.1 and conclude that, for almost all $p \varepsilon \partial D$,

$$(4.16) \quad \partial v_+/\partial n_p^+ = (\sigma u)(p) + (K\sigma u)(p) + D_n u(p).$$

Note that since k is not an eigenvalue of the interior Dirichlet problem, Lemma 4.1 assures the existence of $D_n u(p)$. But, from equation (4.11), we have $\bar{K}*u + S\sigma u = -u$ and so, with (4.15), $v_+(p) = -2u(p)$, $p \varepsilon \partial D$, while from equation (4.12) we have that, with (4.16), $\partial v_+/\partial n_p^+ (p) = 2(\sigma u)(p)$ almost everywhere on ∂D. Comparing these last two results, we see that the relation

$$(4.17) \quad \partial v_+/\partial n_p^+(p) + (\sigma v_+)(p) = 0$$

holds almost everywhere on ∂D and we may invoke the uniqueness theorem to conclude that v_+ vanishes identically in D_+ and that $v_+(p) = 0$ almost everywhere on ∂D. It follows since $v_+ = -2u$ that $u(p) = 0$ almost everywhere on ∂D. But the function u is continuous on ∂D and consequently vanishes on ∂D, which establishes the result in case (i).

If the second possibility obtains, then the function v is not identically zero. But, as before,

$$(4.18) \quad \partial v / \partial n_p^- \ (p) = (D_n u - \sigma u + K \sigma u) \ (p)$$

for almost all $p \epsilon \partial D$, and so, since the function u satisfies equation (4.12), $\partial v / \partial n_p^- \ (p) = 0$ almost everywhere on ∂D. But then the function v is a solution to the interior Helmholtz problem which vanishes on the boundary and whose normal derivative vanishes almost everywhere on ∂D. This leads to a contradiction and completes the proof.

It remains for us to establish that, if k is an eigenvalue of the interior homogeneous Dirichlet problem, then the system (4.1)-(4.2) of boundary integral equations admits a simultaneous solution.

We recall the following result from [6, p. 224 and p. 277].

Lemma 4.5: There exists a non-trivial function $\overset{0}{w}$ such that

$$(4.19) \quad \overset{0}{w} + K \overset{0}{w} = 0$$

if and only if $(S \overset{0}{w}) (p) = 0$, $p \epsilon \partial D$. Moreover such a function exists if and only if k is an eigenvalue of the interior Dirichlet problem. Finally, $\overset{0}{v}$ is an eigenfunction if and only if it may be represented in the form $\overset{0}{v}(P) = -1/2 (S \overset{0}{w}) (P)$, $P \epsilon D_-$ where $\overset{0}{w} + K \overset{0}{w} = 0$.

Clearly such a function $\overset{0}{w}$ satisfies the boundary integral equation

(4.20) $\overset{o}{w} + K\overset{o}{w} + \sigma S\overset{o}{w} = 0,$

and having made this observation, we may prove the following result which is analogous to Theorem 4.3 of [6].

Lemma 4.6: The function $\overset{o}{v}$ is an eigenfunction of the interior Dirichlet problem if and only if $\overset{o}{v}$ can be represented in the form

(4.21) $\overset{o}{v} = D\overset{o}{\bar{w}}{}^* + S\sigma\overset{o}{\bar{w}}{}^*, \quad P\epsilon D_-$

where $\overset{o}{\bar{w}}{}^*$ satisfies the homogeneous boundary integral equation

(4.22) $(I + \bar{K}^* + S\sigma) \overset{o}{w}{}^* = 0.$

Proof: Assume, first, that $\overset{o}{\bar{w}}{}^*$ is a non-trivial solution of the given boundary integral equation and hence with the regularity Theorem 3.1, is continuous. Define a function $\overset{o}{v}$ in D_- by

(4.23) $\overset{o}{v}(P) := (D\overset{o}{\bar{w}}{}^* + S\sigma\overset{o}{\bar{w}}{}^*)(P), \quad P\epsilon D_-,$

which takes boundary values in the sense of Theorem 3.5. Then, as in the proof of Lemma 4.1, the function $\overset{o}{v}$ vanishes on ∂D and hence either vanishes identically in D_- or is a non-trivial eigenfunction of the interior Dirichlet problem. To see that $\overset{o}{v}$ is not identically zero, notice that if this were the case then its normal derivative would vanish on ∂D. But, then, for almost all $p\epsilon\partial D$, we would have, using the jump conditions as in previous proofs,

$$(4.24) \quad \partial \overset{o}{v}/\partial n_p^- (p) = (D_n^- \overset{o}{\overline{w}}* + K\sigma \overset{o}{\overline{w}}* - \sigma \overset{o}{\overline{w}}*)(p) = 0$$

where $D_n^- \overset{o}{\overline{w}}*$ exists since $\dfrac{\partial \overset{o}{v}}{\partial n_p^-}$ and $\dfrac{\partial}{dn^-} S\sigma \overset{o}{\overline{w}}*$ exist and with Theorem 3.4, which applies because $\overset{o}{\overline{w}}*$ is continuous, $D_n \overset{o}{\overline{w}}*$ exists. Thus the function $\overset{o}{\overline{w}}$ is a solution of the homogeneous boundary intergral equation

$$(4.25) \quad (D_n + K\sigma - \sigma) \overset{o}{\overline{w}}* = 0$$

Now define a function $\overset{o}{u}$ on D_+ by

$$(4.26) \quad u(P) := (D\overset{o}{\overline{w}}* + S\sigma \overset{o}{\overline{w}}*), \quad P \varepsilon D_+,$$

which, again, takes on boundary values in the sense of Theorem 3.5. Again, using the jump conditions, we have for $P \rightarrow p^+$,

$$(4.27) \quad \overset{o}{u}(p) := (-\overset{o}{\overline{w}}* + \overline{K}*\overset{o}{\overline{w}}* + S\sigma \overset{o}{\overline{w}}*)(p), \quad p \varepsilon \partial D,$$

and

$$(4.28) \quad \partial \overset{o}{u}/\partial n_p^+ (p) = (\sigma \overset{o}{\overline{w}}* + K\sigma \overset{o}{\overline{w}}* + D_n \overset{o}{\overline{w}}*)(p), \quad \text{a.e. on } \partial D.$$

But since $(I + \overline{K}*S\sigma)\overset{o}{\overline{w}}* = 0$ we have with (4.27) that $\overset{o}{u}(p) = -2\overset{o}{\overline{w}}*(p)$ for all $p \varepsilon \partial D$, while from (4.25) and (4.28) we conclude that $\partial \overset{o}{u}/\partial n_p^+(p) = 2(\sigma \overset{o}{\overline{w}}*)(p)$ almost everywhere on ∂D. The function $\overset{o}{u}$ is thus a solution to the exterior homogeneous Robin problem and, by the uniqueness theorem, must vanish everywhere. Thus $\overset{o}{\overline{w}}*(p) = 0$ for all $p \varepsilon \partial D$ which contradicts the choice of $\overset{o}{\overline{w}}*$. Hence $\overset{o}{v}$ must be a non-trivial eigenfunction.

Conversely, assume that $\overset{o}{v}$ is an eigenfunction of the interior
Dirichlet problem and let $\overset{o}{v}_i$, i=1,2,...,n, be a basis of eigen-
functions. Thus there exists scalars, $\alpha_1,...,\alpha_n$, such that
$\overset{o}{v} = \overset{n}{\underset{i=1}{\Sigma}} \alpha_i v_i$. By lemma 4.5, to each $\overset{o}{v}_i$ there corresponds a function
$\overset{o}{w}_i$ such that $\overset{o}{w}_i$ satisfies

(4.29) $(I+K+\sigma S)\overset{o}{w}_i(p) = 0$ $p\epsilon\partial D$

(4.30) $\overset{o}{v}_i(P) = \frac{1}{2}(S\overset{o}{w}_i)(P)$, $P\epsilon D_-$.

Let $W=\text{span}\{\overset{o}{w}_i\}$. Note that the $\overset{o}{w}_i$ i=1,2,...,n, are linearly independent
since the $\overset{o}{v}_i(p)$, i=1,2,...,n, are chosen as a basis for the
eigenspace. Hence $\dim<\overset{o}{v}_n>=\dim W=n$ and $W\subset\ker(I+K+\sigma S)$. But,

(4.31) $\dim \ker (I+K+\sigma S) = \dim \ker (I+K+\sigma S)*$
                           $= \dim \ker (I+K*+\overline{S\sigma})$
                           $= \dim \ker (I+\overline{K}*+S\sigma)$.

Let $\{\overset{o}{w}_i^*\}$, i=1,2,...,n, be a basis for $\ker(I+\overline{K}*+S\sigma)$ and define for
each i=1,...,n

(4.32) $\overset{o}{u}_i(P) = (D+S\sigma)\overset{o}{w}*_i(P)$, $P\epsilon D_-$.

Since $\overset{o}{w}_i^*$ are non-trivial solutions of $(I+\overline{K}*+S\sigma)\overset{o}{w}*=0$, the first
part of the proof shows that the functions $\overset{o}{u}_i$, i=1...,n are
eigenfunctions of the interior Dirichlet problem. Moreover,
they are linearly independent for if β_i, i=1,...,n, are scalars
such that $\Sigma\beta_i u_i=0$, then

$$(4.33) \quad (D+S\sigma)(\Sigma \beta_i \overset{\text{o}}{\overline{w}}{}^*_i) = 0$$

which implies, as in the first part of the proof, that

$$(4.34) \quad \Sigma \beta_i \overset{\text{o}}{\overline{w}}{}^*_i = 0$$

and hence $\beta_i = 0$ for all $i=1,\ldots,n$.

Thus every eigenfunction $\overset{\text{o}}{v}$ of the interior Dirichlet problem can be written in the form

$$(4.35) \quad \overset{\text{o}}{v} = \sum_{i=1}^{n} \alpha_i (D+S\sigma) \overset{\text{o}}{\overline{w}}{}^*_i = (D+S\sigma) \sum_{i=1}^{n} \alpha_i \overset{\text{o}}{\overline{w}}{}^*_i = (D+S\sigma) \overset{\text{o}}{\overline{w}}{}^*$$

for α_i, $i=1,\ldots,n$, appropriately chosen scalars, which completes the proof.

Finally, we may establish the following existence theorem for solutions of the system of boundary integral equations (4.1)-(4.2).

Theorem 4.1

Let $\sigma \varepsilon L_\infty(\partial D)$. Then there exists a unique solution of the system of boundary integral equations

$$(4.1) \quad (I + \overline{K}^* + S\sigma)u = Sg_+$$

$$(4.2) \quad (-\sigma + K\sigma + D_n)u = Kg_+ - g_+$$

for each choice of $g_+ \varepsilon L_\infty(\partial D)$.

Proof: If k is not an eigenvalue of the interior Dirichlet problem for the Helmholtz equation, then Lemmas 4.3, 4.2, and 4.4 ensure that a unique solution of the system exists since $g_+ \epsilon L_\infty(\partial D)$ implies that $g_+ \epsilon L_2(\partial D)$.

If k is an eigenvalue, then while Lemma 4.3 still guarantees the existence of at least one solution of equation (4.1), we cannot show immediately that such a solution will satisfy the second equation. Nor can we make any statement concerning uniqueness since solutions to equation (4.1) in this case are not unique.

Suppose, then, that $\tilde{w}_1$ is a solution of equation (4.1) and define a function v on D_- by

$$(4.36) \quad v(P) := (D\tilde{w}_1 + S\sigma\tilde{w}_1 - Sg_+)(P), \quad P\epsilon D_-.$$

Then, as in the proof of Lemma 4.1, v is either identically zero or is a non-trivial solution of the interior homogeneous Dirichlet problem. In either case, according to Lemma 4.6, we may represent the function v in the form $v = (D+S\sigma)\overset{o}{\overset{=}{w}}*$ for some solution $\overset{o}{\overset{=}{w}}*$ of the homogeneous boundary integral equation $(I+\overline{K}*+S\sigma)\overset{o}{\overset{=}{w}}*=0$. Hence, for $P\epsilon D_-$,

$$(4.37) \quad (D+S\sigma)\overset{o}{\overset{=}{w}}*(P) = (D\tilde{w}_1)(P) + (S\sigma)\tilde{w}_1(P) - (Sg_+)(P).$$

Now, define a function w on ∂D by

$$(4.38) \quad w := \tilde{w}_1 - \overset{o}{\overset{=}{w}}*.$$

Then

$$(4.39) \quad w + \bar{K} * w + S\sigma w = \tilde{w}_1 - \overset{o}{\overline{w}} * + \bar{K} * \tilde{w}_1 - \bar{K} * \overset{o}{\overline{w}} * + S\sigma \tilde{w}_1 - S\sigma \overset{o}{\overline{w}} *$$

$$= \tilde{w}_1 + \bar{K} * \tilde{w}_1 + S\sigma \tilde{w}_1$$

$$= Sg_+$$

and so the function w satisfies the equation (4.1). Moreover, for all $P \epsilon D_-$, with (4.37),

$$(4.40) \quad (Dw + S\sigma w)(P) = (D\tilde{w}_1 - D\overset{o}{\overline{w}} * + S\sigma \tilde{w}_1 - S\sigma \overset{o}{\overline{w}} *)(P)$$

$$= Sg_+(P)$$

or equivalantly,

$$(4.41) \quad (Dw)(P) = S(g_+ - \sigma w)(P), \text{ for all } P \epsilon D_-.$$

The function $\tilde{w}_1$ and $\overset{o}{\overline{w}} *$ are not only L_2-solutions of the non-homogeneous and homogeneous integral equations, respectively, but are also, according to the regularity theorem, continuous functions on ∂D. Hence they are both bounded there and, consequently, the functions w and $g_+ - \sigma w$ are essentially bounded and integrable on ∂D. So again, we may use Theorem 3.4 to conclude that

(4.42) $\lim_{P \to p} \partial/\partial n_p \, S(g_+ - \sigma w)(P) = (-g_+ + Kg_+ + \sigma w - K\sigma w)(p)$

almost everywhere on ∂D. And so we have that $D_n^- w(p)$ exists almost everywhere on ∂D and $(D_n w - \sigma w + K\sigma w)(p) = (-g_+ + Kg_+)(p)$, almost everywhere on ∂D. Consequently, the function $w \epsilon L_2(\partial D)$ satisfies the system of boundary integral equations. To finish the proof, we appeal to Lemma 4.4 for the uniqueness statement.

Next we establish the equivalence of the Robin Problem and its boundary integral equation formulation.

Theorem 4.2:

Let g_+ and σ be in $L_\infty(\partial D)$, Im $k \geq 0$, and Im $\bar{k}\sigma \geq 0$. Then u_+ is a solution of the Exterior Robin Problem,

1. $u_+ \, \epsilon C_2(D_+)$; u_+, $\frac{\partial u_+}{\partial n} + \epsilon L_2(\partial D)$ in the sense of theorem 3.5

2. $(\Delta + k^2) \, u_+ = 0$, $P \epsilon D_+$

3. $\frac{\partial u_+}{\partial r} - iku_+ = o(1/r)$

4. $\frac{\partial u_+}{\partial n_p} + \sigma u_+ = g_+$ almost everywhere on ∂D,

if and only if

(4.43) $u_+ = \frac{1}{2} S(g_+ - \sigma u) - \frac{1}{2} Du$, $P \epsilon D_+$

where

(4.1) $(I + \bar{K}^* + S\sigma) \, u = Sg_+$

(4.2) $(-\sigma + K\sigma + D_n) \, u = Kg_+ - g_+.$

Proof: First assume u_+ is a solution of the Robin problem 1-4. Then Theorem 3.5 may be employed to obtain the Representation Theorem (2.13) which is the same as (4.43) above with u_+ replacing u. Then the formal calculations by which (2.14) and (2.19) which are identical with (4.1) and (4.2) were obtained may be repeated. Note that (4.1) follows from the continuity of Sg_+ and $S\sigma u_+$ (Theorem 3.2) and the jump relation for Du_+ which holds for $u_+ \epsilon L_2(\partial D)$ (Theorem 3.3). Then the regularity Theorem 3.1 ensures that $u_+ \epsilon C_0(\partial D)$. Hence with Theorem 3.4 and the fact that u_+ and $S\frac{\partial u_+}{\partial n}$ have normal derivatives in $L_2(\partial D)$ the representation (2.13) may be differentiated and $D_n u_+$ must exist. Thus u_+ is a simultaneous solution of (4.1) and (4.2) hence by Theorem 4.1 is the unique solution.

Conversely assume u is the unique solution of (4.1) and (4.2) and u_+ is defined by (4.43). Clearly u_+ satifies the Helmholtz equation and the radiation condition in D_+.

The regularity Theorem 3.1 ensures that u is continuous on ∂D, hence $S\sigma u$ is continuous and has a normal derivative in $L_2(\partial D)$ (Theorems 3.2, 3.3, and 3.6). Moreover u must be in the domain of D_n. Hence u_+ and $\frac{\partial u_+}{\partial n}$ are in $L_2(\partial D)$. Then taking limits as $P \to p^+$ we find that on ∂D

$$(4.44) \quad u_+ = \frac{1}{2} S(g_+ - \sigma u) + \frac{u}{2} - \frac{1}{2} \bar{K}*u$$

and with (4.1)

$$(4.45) \quad u_+ = u \quad \text{on } \partial D.$$

Also

$$(4.46) \quad \frac{\partial u_+}{\partial n} = \frac{1}{2} g_+ + \frac{1}{2} K g_+ - \frac{\sigma u}{2} - \frac{1}{2} K \sigma u - \frac{1}{2} D_n u$$

and with (4.2)

$$(4.47) \quad \frac{\partial u_+}{\partial n} = g_+ - \sigma u \quad \text{on } \partial D$$

hence

$$(4.48) \quad \frac{\partial u_+}{\partial n} + \sigma u_+ = g_+ \quad \text{on } \partial D.$$

and u_+ satisfies the Robin problem.

In conclusion, we remind the reader of the closing remarks of Section 2. There we derived, formally, boundary integral equations, of the same form as equations (4.1) and (4.2), but with different right hand members which were suitable for describing the total field for the scattering problem. In that derivation, the right hand sides represent, up to a constant factor, the incident field and its normal derivative respectively. As the results of this section do not depend on the particular form of the function appearing on the right hand side, we see that for sufficiently smooth incident fields the analysis of this section is valid and we may state the following theorem.

Theorem 4.3:

Let u^i denote an incident field and $u=u^i+u^s$ the total field when u^s denoted the field scattered by the obstacle D_-. Then, for any $\sigma \epsilon L_\infty(\partial D)$, there exists a unique solution of the system of boundary integral equations

(4.49) $(I + S\sigma + \bar{K}*)u = 2u^i$

(4.50) $(-\sigma + K\sigma + D_n)u = 2(\partial u^i / \partial n_p)$.

The equivalence of solutions of this pair of boundary integral equations and the scattering problem follows, mutatis mutandis, from Theorem 4.2.

APPENDIX - *On the generalized boundary values*
of single and double layers.

In Theorem 3.5 the notion of a family of parallel surfaces $\{S_\delta^-\}$ contained in D_- was introduced. We may define another family $\{S_\delta^+\}$ contained in D_+ which converges to ∂D in the same sense as $\{S_\delta^-\}$. With the understanding that the layers may approach their boundary values from D^+ or D^- we state and prove Theorem 3.6.

Theorem:

If ∂D is a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces) and $\mu \epsilon L_2(\partial D)$ then $S\mu$, $D\mu$ and $\frac{\partial}{\partial n} S\mu$ assume boundary values in the sense of Theorem 3.5, namely;

$A-1$
$$\int_{\partial D} |(S\mu)(F_\delta^\pm(p)) - (S\mu)(p)|^2 dS_p = o(\delta^0),$$

$A-2$
$$\int_{\partial D} |(D\mu)(F_\delta^\pm(p)) - (D\mu)(p)|^2 dS_p = o(\delta^0),$$

and

$A-3$
$$\int_\partial |\frac{\partial}{\partial n_p^\pm}(S\mu)(F_\delta^\pm(p)) - \frac{\partial}{\partial n_p}(S\mu)(p)|^2 dS_p = o(\delta^0)$$

Proof:

Considering equation A-1 first, we denote by P_δ the image on $S_\delta^\pm$ of a point p on ∂D, i.e., $P_\delta = F_\delta^\pm(p)$. Recall that

$A-4$ $|P_\delta - p| := R(P_\delta, p) = 0(\delta)$ uniformly in p.

With (2.3) the integrand in A-1 becomes

A-5 $|(S\mu)(P_\delta) - (S\mu)(p)|^2 = |\int_{\partial D} \mu(q)[\gamma(P_\delta,q) - \gamma(p,q)]dS_q|^2$

$$= \frac{1}{4\pi^2} |\int_{\partial D} \mu(q) \left[\frac{e^{ikR(P_\delta,q)}}{R(P_\delta,q)} - \frac{e^{ikR(p,q)}}{R(p,q)}\right] dS_q|^2.$$

It may be shown that for $Imk \geq 0$,

A-6 $|\frac{e^{ikR(P_\delta,q)}}{R(P_\delta,q)} - \frac{e^{ikR(p,q)}}{R(p,q)}| \leq \frac{(1+|k|D)R(P_\delta,p)}{R(P_\delta,q)R(p,q)}$

where D is the diameter of ∂D. From the definition of S_δ it follows that there is a constant c such that,

A-7 $R(P_\delta,p) \leq c\delta$.

Thus there is a constant c, independent of δ, such that A-5 gives rise to

A-8 $|(S\mu)(P_\delta) - (S\mu)(p)|^2 \leq c\delta^2 \left(\int_{\partial D} \frac{|\mu(q)|dS_q}{R(P_\delta,q)R(p,q)}\right)^2$

$$\leq c\delta^2 \int_{\partial D} \frac{dS_q}{R(P_\delta,q)R(p,q)} \int_{\partial D} \frac{|\mu(q)|^2}{R(P_\delta,q)R(p,q)}dS_q$$

where Schwartz' inequality has been used. It also may be established that for $R(P_\delta, p) < d/2$ where d is sufficiently small (depending on the Lyapunov constants of the surface)

$$A\text{-}9 \quad \frac{1}{R(P_\delta, q)} \leq \frac{2}{R(p,q)} \quad , \quad \forall p, q \in \partial D$$

and

$$A\text{-}10 \quad \frac{1}{R(P_\delta, q)} \leq \frac{2}{R(P_\delta, p)} \quad , \quad \forall p, q \in \partial D .$$

Employing these relations selectively we find, for any $\alpha(0,1)$,

$$A\text{-}11 \quad |(S\mu)(P_\delta) - (S\mu)(p)|^2 \leq c\delta^\alpha \left(\int_{\partial D} \frac{dS_q}{R^{1+\alpha}(p,q)} \right) \int_{\partial D} \frac{|\mu(q)|^2}{R(p,q)} \, dS_q .$$

The term $\int_{\partial D} \frac{S_q}{R^{1+\alpha}(p,q)}$ is an integral with a weak singularity

and differentiable density (=1) hence is a continuous function of p on ∂D while the term $\int_{\partial D} \frac{|\mu(q)|^2}{R(p,q)} \, dS_q$ is a single layer with

integrable density hence is integrable ([4], p. 110) and the validity of A-1 follows.

Next consider A-2 which involves somewhat different estimates. With (2.4), (2.5), (3.18) and (3.19) we have

$A-12 \quad |(D\mu)(P_\delta)-(D^\pm\mu)(p)|^2 =$

$$| \int_{\partial D} \mu(q)\frac{\partial}{\partial n_q}[\gamma(P_\delta,q)-\gamma(p,q)]dS_q \pm\mu(p)|^2, \quad P_\delta\epsilon D^\pm .$$

Introducing Gauss' integral which has the form

$$A-13 \quad \int_{\partial D} \frac{\partial}{\partial n_q} \gamma_0(P_\delta,q)dS_q = \begin{cases} 0, & P_\delta\epsilon D_+ \\ 1, & P_\delta\epsilon\partial D \\ 2, & P_\delta\epsilon D_- \end{cases}$$

where

$$A-14 \quad \gamma_0(P_\delta,q) = -\frac{1}{2\pi R(P_\delta,q)} ,$$

equation A-12 may be written

$$A-15 \quad |(D\mu)(P_\delta)-(D^\pm\mu)(p)|^2 = | \int_{\partial D}\{\mu(q)\frac{\partial}{\partial n_q}[\gamma(p_\delta,q)-\gamma(p,q)]$$

$$-\mu(p)\frac{\partial}{\partial n_q}[\gamma_0(P_\delta,q)-\gamma_0(p,q)]\}dS_q|^2$$

$$\leq 2|\int_{\partial D}\mu(q)\frac{\partial}{\partial n_q}[\gamma(P_\delta,q)-\gamma(p,q)-\gamma_0(P_\delta,q)+\gamma_0(p,q)]dS_q|^2$$

$$+2|\int_{\partial D}[\mu(q)-\mu(p)]\frac{\partial}{\partial n_q}[\gamma_0(P_\delta,q)-\gamma_0(p,q)]dS_q|^2.$$

The triangle inequality was employed in obtaining the last inequality.

It may be shown that for $\mathrm{Im} k \geq 0$

$A-16$ $\left| \frac{\partial}{\partial n_q} [\gamma(P_\delta, q) - \gamma_o(P_\delta, q) + \gamma_o(p, q) - \gamma(p, q)] \right| =$

$$= \frac{1}{2\pi} \left| \frac{\partial}{\partial n_q} \left\{ \frac{e^{ikR(P_\delta, q)} - 1}{R(P_\delta, q)} - \frac{e^{ikR(p, q)} - 1}{R(p, q)} \right\} \right| \leq \frac{c\,\delta}{R(P_\delta, q)\,R(p, q)}$$

where c is a constant which depends on k and the diameter of ∂D but is independent of p and δ. Hence the first term on the right hand side of A-15 may be estimated as

$$A-17 \quad 2 \left| \int_{\partial D} \mu(q) \frac{\partial}{\partial n_q} [\gamma(P_\delta, q) - \gamma_o(P_\delta, q) + \gamma_o(p, q) - \gamma(p, q)] dS_q \right|^2$$

$$\leq c_1 \delta^2 \left(\int_{\partial D} \frac{|\mu(q)|}{R(P_\delta, q)\,R(p, q)} dS_q \right)^2$$

which is exactly the same as the estimate A-7 hence the subsequent analysis applies here as well. The second term on the right of A-15 is analysed with the help of the following estimate which may be shown to hold on a Lyapunov of order α:

$$A-18 \quad \left| \frac{\partial \gamma_o(P_\delta, q)}{\partial n_q} - \frac{\partial \gamma_o(p, q)}{\partial n_q} \right| \leq \frac{c_1 R(P_\delta, p)}{R^3(P_\delta, q)} + \frac{c_2 R(P_\delta, p)}{R(P_\delta, q)\,R^2(p, q)}$$

where c_1 and c_2 are constants. Then

A-19 $\left| \int_{\partial D} [\mu(q) - \mu(p)] \frac{\partial}{\partial n_q} [\gamma_o(P_\delta, q) - \gamma_o(p, q)] \, dS_q \right|^2$

$$\leq \delta^2 \left(\int_{\partial D} |\mu(q) - \mu(p)| \left(\frac{c_1}{R^3(P_\delta, q)} + \frac{c_2}{R(P_\delta, q) R^{2-\alpha}(p, q)} \right) dS_q \right)^2$$

$$\leq a_1 \delta^2 \left(\int_{\partial D} \frac{|\mu(q) - \mu(p)|}{R^3(P_\delta, q)} \, dS_q \right)^2 +$$

$$a_2 \delta^2 \left(\int_{\partial D} \frac{|\mu(q) - \mu(p)|}{R(P_\delta, q) R^{2-\alpha}(p, q)} \, dS_q \right)^2$$

for modified constants a_1 and a_2. The first integral on the right of A-19, with Schwartz' inequality, becomes

A-20 $\left(\int_{\partial D} \frac{|\mu(q) - \mu(p)|}{R^3(p_\delta, q)} dS_q \right)^2 \leq \int_{\partial D} \frac{dS_q}{R^3(P_\delta, q)} \int_{\partial D} \frac{|\mu(q) - \mu(p)|^2}{R^3(P_\delta, q)} \, dS_q .$

Defining $\Sigma_d(p)$, a patch on ∂D as,

A-21 $\Sigma_d(p) = \{ q | \begin{array}{c} q \epsilon \partial D \\ R(p,q) < d \end{array} | \}$

where d is sufficiently smaller than the Lyapunov radius associated with ∂D so that A-9 and A-10 may be used, it follows that

A-22 $\int_{\partial D} \frac{dS_q}{R^3(P_\delta, q)} = \int_{\partial D \setminus \Sigma_d(p)} \frac{dS_q}{R^3(P_\delta, q)} + \int_{\Sigma_d(p)} \frac{dS_q}{R^3(P_\delta, q)}$

$$\leq \frac{8A}{d^3} + c \int_0^d d\rho \int_0^{2\pi} d\phi \frac{\rho}{(R^2(P_\delta, p) + \rho^2)^{3/2}} .$$

The constants A and c are of no particular importance for our present purpose but we note that A is the surface area of ∂D. In the second integral in A-22 we have introduced polar coordinates (ρ, ϕ) on the disk tangent to ∂D at p and used the inequalities (e.q. Günter[4])

$$A\text{-}23 \quad dS_q \leq 2\rho d\rho d\phi \;, q\varepsilon\Sigma_d(p)$$

and

$$A\text{-}24 \quad R(P_\delta, q) \geq \frac{1}{2}\sqrt{R^2(P_\delta, p) + R^2(p,q)} \geq \frac{1}{2}\sqrt{R^2(P_\delta, p) + \rho^2} \quad .$$

The integral in A-22 is easily evaluated and we have

$$A\text{-}25 \quad \int_{\partial D} \frac{dS_q}{R^3(P_\delta, q)} \leq \frac{8A}{d^3} + 2\pi c \left[\frac{1}{R(P_\delta, p)} - \frac{1}{(R^2(P_\delta, p) + d^2)^{1/2}} \right] < \frac{c_1}{\delta}$$

for some constant c_1. Similarly the second integral in (A-20) may be written

$$A\text{-}26 \quad \int_{\partial D} \frac{|u(q) - u(p)|^2}{R^3(P_\delta, q)} dS_q \leq 16 \int_{\partial D \backslash \Sigma_d(p)} \frac{|u(q)|^2 + |u(p)|^2}{d^3} dS_q$$

$$+ 8\int_{\Sigma_d(p)} \frac{|u(q) - u(p)|^2}{(\theta^2\delta^2 + R^2(p,q))^{3/2}} dS_q$$

$$\leq c_1 ||u||^2_{L_2(\partial D)} + c_2 |u(p)|^2 + c_3 \int_0^d d\rho \int_0^{2\pi} d\phi \frac{\rho|u(p+q_1) - u(p)|^2}{(\theta^2\delta^2 + \rho^2)^{3/2}}$$

where again we use polar coordinates on the plane targent to p to parameterize $\Sigma_d(p)$ and employ (A-23), (A-24) and (3.21). The vector q_1 in local rectangular coordinates centered at p is

A-27 $q_1 = (\xi, \eta, \zeta(\xi, \eta))$, $\xi = \rho\cos\phi$, $\eta = \rho\sin\phi$

where ζ is a local representation of the surface, hence depends on p, but also satisfies, since d is smaller than the Lyapunov radius

A-28 $|\zeta| < a\rho^{1+\alpha}$, α = Lyapunov index.

Hence for d sufficiently small

A-29 $|q_1| < 2\rho$.

Substituting these results in A-20 we have

A-30 $\left(\int_{\partial D} \frac{|\mu(q) - \mu(p)|}{R^3(P_\delta, q)} \, dS_q \right)^2 \leq \frac{1}{\delta} \left\{ c_1 ||\mu||^2_{L_2(\partial D)} + c_2 |\mu(p)|^2 \right.$

$\left. + c_3 \int_0^d d\rho \int_0^{2\pi} d\phi \quad \frac{\rho |\mu(p+q_1) - \mu(p)|^2}{(\theta^2 \delta^2 + \rho^2)^{3/2}} \right\}$.

The second integral on the right in A-19 is treated similarly yielding

$A\text{-}31$
$$\left(\int_{\partial D} \frac{|\mu(q)-\mu(p)|}{R(P_\delta,q)\,R^{2-\alpha}(p,q)}\,dS_q \right)^2 \le \frac{1}{\delta^{1-\alpha}}\left(b_1\|\mu\|^2_{L_2(\partial D)} + b_2|\mu(p)|^2 \right.$$

$$\left. + b_3 \int_0^d \!d\rho \int_0^{2\pi}\!d\phi\; \frac{|\mu(p+q_1)-\mu(p)|^2}{\rho^{1-\alpha}\sqrt{\theta^2\delta^2+\rho^2}} \right)$$

Incorporating A-30 and A-31 into A-19 we have, for appropriately redefined constants,

$A\text{-}32$
$$\left| \int_{\partial D}[\mu(q)-\mu(p)]\,\frac{\partial}{\partial n_q}[\gamma_o(P_\delta,q)-\gamma_o(p,q)]\,dS_q \right|^2 \le c_1\delta\|\mu\|^2_{L_2(\partial D)}$$

$$+ c_2\,\delta|\mu(p)|^2$$

$$+ c_3\,\delta \int_0^d \!d\rho \int_0^{2\pi}\!d\phi\; \frac{\rho|\mu(p+q_1)-\mu(p)|^2}{(\theta^2\delta^2+\rho^2)^{3/2}} + c_4\delta^{1+\alpha}\int_0^d \!d\rho \int_0^{2\pi}\!d\phi\; \frac{|\mu(p+q_1)-\mu(p)|^2}{\rho^{1-\alpha}\sqrt{\theta^2\delta^2+\rho^2}}$$

Integrating over ∂D and employing Fubini's theorem, which applies since $\mu\epsilon L_2(\partial D)$ and both $\dfrac{\rho}{(\theta^2\delta^2+\rho^2)^{3/2}}$ and $\dfrac{1}{\rho^{1-\alpha}\sqrt{\theta^2\delta^2+\rho^2}}$ are

integrable in the ρ,ϕ variables for $\delta>0$, we obtain

$A\text{-}33$
$$\int_{\partial D}\left| \int_{\partial D}[\mu(q)-\mu(p)]\frac{\partial}{\partial n_q}[\gamma_o(P_\delta,q)-\gamma_o(p,q)]\,dS_q \right|^2 dS_p$$

$$\le (c_1+c_2)\delta\|\mu\|^2_{L_2(\partial D)}$$

$$+ c_3\delta \int_0^d \rho \int_0^{2\pi}\!d\phi\; \frac{\rho}{(\theta^2\delta^2+\rho^2)^{3/2}} \left(\int_{\partial D}|\mu(p+q_1)-\mu(p)|^2 dS_p \right)$$

$$+ c_4 \, \delta^{1+\alpha} \int_0^d d\rho \int_0^{2\pi} d\phi \, \frac{1}{\rho^{1-\alpha} \sqrt{\theta^2 \delta^2 + \rho^2}} \left(\int_{\partial D} |\mu(p+q_1) - \mu(p)|^2 dS_p \right)$$

The integral $\int_{\partial D} |\mu(p+q_1) - \mu(p)|^2 dS_p$ is the L_2 modulus of continuity of μ which is $0(|q_1|^2)$ (Stein [16]) although for present purposes it is sufficient to have $0(|q_1|^\beta)$, $\beta > 0$. Then with (A-29)

$$A\text{-}34 \quad \int_{\partial D} |\mu(p+q_1) - \mu(p)|^2 dS_p \; < \; c \, \rho^\beta$$

and (A-33) becomes

$$A\text{-}35 \quad \int_{\partial D} \left| \int_{\partial D} [\mu(q) - \mu(p)] \frac{\partial}{\partial n_q} [\gamma_0(p_\delta, q) - \gamma_0(p, q)] dS_q \right|^2 dS_p$$

$$\leq \; \tilde{c}_1 \, \delta \, ||\mu||^2_{L_2(\partial D)}$$

$$+ \tilde{c}_2 \, \delta \int_0^d \frac{\rho^{1+\beta}}{(\theta^2 \delta^2 + \rho^2)^{3/2}} d\rho \; + \tilde{c}_3 \delta^{1+\alpha} \int_0^d \frac{d\rho}{\rho^{1-\alpha-\beta} \sqrt{\theta^2 \delta^2 + \rho^2}}$$

$$\leq \; \tilde{c} \, \delta \, ||\mu||^2_{L_2(\partial D)} + \tilde{c}_2 \delta^\beta \int_0^\infty \frac{s^{1+\beta}}{(\theta^2 + s^2)^{3/2}} ds + \tilde{c}_3 \delta^{2\alpha+\beta} \int_0^\infty \frac{ds}{s^{1-\alpha-\beta} \sqrt{\theta^2 + s^2}}, \beta < 1$$

When $\beta < 1$, both integrals are bounded and the right hand side is $0(\delta^\beta)$. If $\beta \geq 1$ even more rapid decay of (A-34) with δ is easily shown. In all cases the validity of (A-2) is thus ensured.

The proof of A-3 involves an argument similar to that used in establishing A-2 which we merely suggest. With (2.3) (3.16), (3.17) and (A-13) we obtain

$$A\text{-}36 \quad \left|\frac{\partial}{\partial n_p^{\pm}}(S\mu)(P_\delta)-\frac{\partial}{\partial n_p^{\pm}}(S\mu)(p)\right|^2 \leq 4\left(\int_{\partial D}|\mu(q)|\left|\frac{\partial}{\partial n_p}[\gamma(P_\delta,q)-\gamma_0(P_\delta,q)\right.\right.$$

$$\left.+\gamma_0(p,q)-\gamma(p,q)]\right|dS_q\Bigg)^2$$

$$+4\left(\int_{\partial D}|\mu(q)|\ \left|\frac{\partial}{\partial n_p}[\gamma_0(P_\delta,q)-\gamma_0(p,q)]+\frac{\partial}{\partial n_q}[\gamma_0(P_\delta,q)\right.\right.$$

$$\left.-\gamma_0(p,q)]\right|dS_q\Bigg)^2$$

$$+4\left(\int_{\partial D}|\mu(p)-\mu(q)|\ \left|\frac{\partial}{\partial n_q}[\gamma_0(P_\delta,q)-\gamma_0(p,q)]\right|dS_q\right)^2.$$

The first integral on the right may be estimated in the same way as (A-17) since (A-16) remains valid when $\frac{\partial}{\partial n_q}$ is replaced by $\frac{\partial}{\partial n_p}$. The third integral is precisely the same as (A-19) and is treated identically. The second integral is estimated using the following inequality

$$A\text{-}37 \quad \left|\frac{\partial}{\partial n_p}[\gamma_0(P_\delta,q)-\gamma_0(p,q)]+\frac{\partial}{\partial n_q}[\gamma_0(P_\delta,q)-\gamma_0(p,q)]\right|$$

$$\leq \frac{c\ R(P_\delta,p)}{R(P_\delta,q)\ R^{2-\alpha}(p,q)}$$

which enables us to write

$$A\text{-}38 \quad \left(\int_{\partial D} |\mu(q)| \ \left| \frac{\partial}{\partial n_p} [\gamma_0(P_\delta,q) - \gamma_0(p,q)] + \frac{\partial}{\partial n_q} [\gamma_0(P_\delta,q) - \gamma_0(p,q)] \right| dS_q \right)$$

$$\leq c \ \delta^2 \left(\int_\partial \frac{|\mu(q)|}{R(P_\delta,q) R^{2-\alpha}(p,q)} \, dS_q \right)^2$$

$$\leq 2c \ \delta^2 \left(\int_{\partial D} \frac{|\mu(q) - \mu(p)|}{R(P_\delta,q) R^{2-\alpha}(p,q)} dS_q \right)^2 +$$

$$2 \ c\delta^2 |\mu(p)|^2 \left(\int_{\partial D} \frac{dS_q}{R(P_\delta,q) R^{2-\alpha}(p,q)} \right)^2 .$$

The first term in the right also appears in A-19 and is treated as it was there. The second term is $o(\delta^0)$ since the integral may be shown to be $0(\frac{1}{\delta^{1-\alpha}})$. This completes the proof of Theorem 3.6.

We remark that while we have proven the theorem for Lyapunov surfaces of index α, the proof is contingent upon the existence of families of parallel surfaces, which we have only asserted for index 1. This is sufficient for our use of the theorem in Section IV, however should the families exist under the weaker hypothesis (see Burago, Maz'ja and Sapozhnikova [1]) then Theorem 3.6 will hold in that case as well.

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