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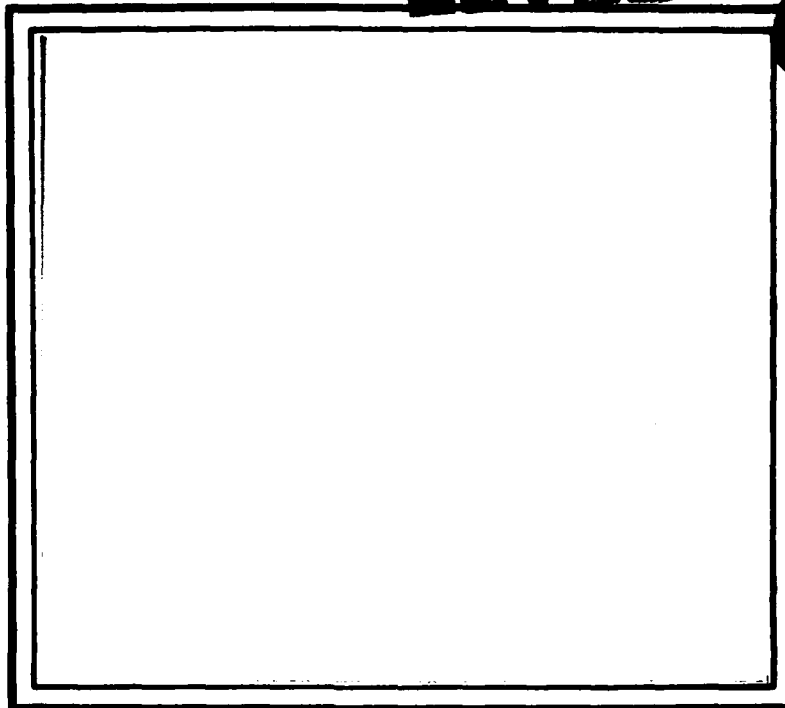
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STATIONARY WAITING TIME DISTRIBUTIONS
IN THE GI/PH/1 QUEUE

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ABSTRACT

It is known that the stable GI/PH/1 queue has an embedded Markov chain whose invariant probability vector is matrix-geometric with a rate matrix R. In terms of the matrix R, the stationary waiting time distributions at arrivals, at an arbitrary time point and until the customer's departure may be evaluated by solving finite, highly structured systems of linear differential equations with constant coefficients. Asymptotic results, useful in truncating the computations, are also obtained. The queue is assumed to follow the first-come, first-served discipline.

KEY WORDS

Matrix-geometric solutions, GI/PH/1 queue, waiting time distributions, Lindley's equation, computational probability.

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1. The Algorithmic Procedure

This paper is a sequel to [2]. All notations and definitions, which were introduced there, will also be used here and have the same meaning. The purpose of the present paper is to show that, for the GI/PH/1 queue, the stationary probability distributions of the waiting time of an arriving customer and of the virtual waiting time, as well as that of the time-in-system may be computed by the numerical integration of highly structured, finite systems of differential equations with constant coefficients. Only waiting times under the first-come, first-served discipline are considered.

These systems of differential equations require prior evaluation of the matrix R , which is shown in [2] to be the minimal, nonnegative solution to a nonlinear matrix equation of the form

$$(1) \quad R = \sum_{k=0}^{\infty} R^k A_k .$$

In the stable GI/PH/1 queue, the matrix R is positive and its Perron eigenvalue η satisfies $0 < \eta < 1$. The probabilistic significance of the matrix R is discussed in [3] and a detailed, general treatment of matrix-geometric solutions in stochastic models may be found in [5].

A. The Waiting Time at Arrivals

In Theorem 7 of [2], it is shown that the Laplace-Stieltjes transform $W^*(s)$ of the stationary waiting time distribution $W(\cdot)$ of an arriving customer

is given by

$$(2) \quad W^*(s) = C \underline{\alpha} \sum_{n=0}^{\infty} R^n [(sI-T)^{-1} T^0 A^0]^n e, \quad \text{for } \text{Re } s \geq 0 .$$

From Formula (2), the mean waiting time \bar{W} may be computed by routine differentiations. One obtains

$$(3) \quad \bar{W} = \lambda_1^{-1} L_2 - \mu_1^{-1} ,$$

where L_2 is the mean queue length at an arbitrary time. The quantity L_2 is explicitly given as a function of R and the parameters of the queue by Formula (92) of [2]. Upon rewriting (3) as $L_2 = \lambda_1^{-1} (\bar{W} + \mu_1^{-1})$, we recognize the classical relation, known as Little's Formula.

We now consider the matrix

$$(4) \quad \phi^*(s) = \sum_{n=0}^{\infty} R^n [(sI-T)^{-1} T^0 A^0]^n ,$$

whose entries are Laplace-Stieltjes transforms of mass-functions on $[0, \infty)$. The matrix $\phi^*(s)$ clearly satisfies

$$(5) \quad \phi^*(s) = I + R \phi^*(s) (sI-T)^{-1} T^0 A^0 .$$

In order to obtain the equation (5) in a more transparent form, we consider the vector $\underline{\phi}^*(s)$, obtained by forming the direct sum of the rows of $\phi^*(s)$. The vector \underline{v} is similarly obtained from the identity matrix I .

Equation (5) may then be equivalently written as

$$(6) \quad \underline{\phi}^*(s) = \underline{v} + \underline{\phi}^*(s) [R^T (sI-T)^{-1} T^0 A^0] ,$$

where \otimes denotes the Kronecker product of matrices and R^T is the transpose of R .

The latter equation may now be transformed, by using classical properties of the Kronecker product, into the following successive forms:

$$\underline{\phi}^*(s) = \underline{v} + \underline{\phi}^*(s) [I \otimes (sI - T)^{-1}] (R^T \otimes T^0 A^0),$$

$$\underline{\phi}^*(s) [I \otimes (sI - T)^{-1}] [sI \otimes I - I \otimes T - R^T \otimes T^0 A^0] = \underline{v},$$

and finally

$$(7) \quad \underline{\phi}^*(s) = \underline{v} + \underline{v} [sI \otimes I - I \otimes T - R^T \otimes T^0 A^0]^{-1} (R^T \otimes T^0 A^0)$$

The existence of the matrix inverse in (7) will be proved below.

Let $\underline{\phi}(\cdot)$ be the m^2 -vector of mass-functions, corresponding to the vector of Laplace-Stieltjes transforms $\underline{\phi}^*(s)$. It then readily follows from (7) that

$$(8) \quad \underline{\phi}(x) = \underline{v} + \underline{v} \int_0^x \exp [(I \otimes T + R^T \otimes T^0 A^0)u] du (R^T \otimes T^0 A^0)$$

$$= \underline{v} + \underline{v} (I \otimes T + R^T \otimes T^0 A^0)^{-1} \{ \exp [(I \otimes T + R^T \otimes T^0 A^0)x] - I \otimes I \} (R^T \otimes T^0 A^0),$$

for $x \geq 0$.

If we introduce the m^2 -vectors \underline{v}^0 and $\underline{\theta}(x)$ by setting

$$(9) \quad \underline{v}^0 = -\underline{v} (I \otimes T + R^T \otimes T^0 A^0)^{-1},$$

and

$$(10) \quad \underline{\theta}(x) = \underline{v}^0 \exp [(I \otimes T + R^T \otimes T^0 A^0)x], \quad \text{for } x \geq 0$$

then Formula (8) leads to

$$(11) \quad \underline{\phi}(x) = \underline{v} + \underline{v}^0 (R^T \otimes T^0 A^0) - \underline{\theta}(x) (R^T \otimes T^0 A^0), \quad \text{for } x \geq 0.$$

Let now $\underline{\phi}(\cdot)$, $\underline{\theta}(\cdot)$ and V^0 be the $m \times m$ matrices of which the direct sums of the rows are respectively the m^2 -vectors $\underline{\phi}(\cdot)$, $\underline{\theta}(\cdot)$ and \underline{v}^0 . Formulas

(11) and (9) may then be rewritten as

$$(12) \quad \underline{\phi}(x) = I + R V^0 T^0 A^0 - R \underline{\theta}(x) T^0 A^0, \quad \text{for } x \geq 0,$$

and

$$(13) \quad V^0 T + R V^0 T^0 A^0 = -I.$$

From Equation (10), it is clear that the vector $\underline{\theta}(\cdot)$ satisfies the matrix-differential equation

$$\underline{\theta}'(x) = \underline{\theta}(x) (I \otimes T + R^T \otimes T^0 A^0), \quad \underline{\theta}(0) = \underline{v}^0,$$

or equivalently

$$(14) \quad \underline{\theta}'(x) = \underline{\theta}(x) T + R \underline{\theta}(x) T^0 A^0, \quad \underline{\theta}(0) = \underline{v}^0.$$

The matrix V^0 may be explicitly expressed in terms of R . To see this, we postmultiply in (13) by the column vector \underline{e} and recall that $T \underline{e} = -T^0$. We then obtain

$$V^0 T^0 = (I - R)^{-1} \underline{e},$$

so that $V^0 T^0 A^0$ may be replaced by $(I - R)^{-1} \underline{e} \cdot \underline{a} = (I - R)^{-1} A^0 \underline{a}$. Doing so, readily yields

$$(15) \quad v^0 = (I-R)^{-1}(R-I-RA^{00})T^{-1} .$$

Equation (13) corresponds to an inhomogeneous system of m^2 linear equations in m^2 unknowns. The coefficient matrix $I \otimes T + R^T \otimes T^0 A^0$ of the latter is now clearly nonsingular. A more detailed result is proved in Lemma 2 below.

From Formulas (2) and (12), we now obtain that

$$W(x) = C + C \underline{\alpha} R v^0 \underline{T}^0 - C \underline{\alpha} R \theta(x) \underline{T}^0 , \quad \text{for } x \geq 0 .$$

Since however, $\underline{\alpha} R v^0 \underline{T}^0 = -\underline{\alpha} R (I-R)^{-1} \underline{e}$, and $C = [\underline{\alpha} (I-R)^{-1} \underline{e}]^{-1}$,

this expression may be further simplified to

$$(16) \quad W(x) = 1 - C \underline{\alpha} R \theta(x) \underline{T}^0 , \quad \text{for } x \geq 0 .$$

The results of this derivation may be summarized as follows.

Theorem 1

If for the stable GI/PH/1 queue, the matrix R is known, then the stationary waiting time distribution $W(\cdot)$ is given by Formula (16). The matrix $\theta(x)$ is obtained by numerical integration of the matrix-differential equation (14).

Remark

We note that Formula (16) provides a transparent solution to Lindley's equation for a wide subclass of GI/G/1 queues. For this subclass, the solution is more elementary and appropriate for numerical implementation than the classical approach, which is based on Wiener-Hopf techniques.

The stationary distribution $W_1(\cdot)$ of the time-in-system of an arriving customer may be easily computed along with the probability distribution $W(\cdot)$. One appropriate computational organization is discussed in Section 4.1 of [5]; we present a somewhat different one here.

The distribution $W_1(\cdot)$ is the convolution of $W(\cdot)$ and the service time distribution $H(\cdot)$. Since the latter is of phase type with irreducible representation $(\underline{\alpha}, T)$, we have

$$(17) \quad W_1(x) = \int_0^x W(u) \underline{\alpha} \exp[T(x-u)] T^0 du, \quad \text{for } x \geq 0.$$

If we set

$$(18) \quad \underline{p}(x) = \underline{\alpha} \int_0^x W(u) \exp[T(x-u)] du, \quad \text{for } x \geq 0,$$

then elementary calculations show that

$$(19) \quad \underline{p}'(x) = \underline{p}(x)T + W(x)\underline{\alpha}, \quad \text{for } x \geq 0,$$

with $\underline{p}(0) = \underline{0}$. In order to compute $W_1(\cdot)$, it therefore, suffices to integrate the system of differential equations (19) and to evaluate $W_1(x) = \underline{p}(x)T^0$, for $x \geq 0$. We note that this is a general procedure for the numerical convolution of two distributions on $[0, \infty)$, when one of them is of phase type.

B. The Virtual Waiting Time

In Theorem 8 of [2], the Laplace-Stieltjes transform $\hat{W}^*(s)$ of the virtual waiting time distribution $\hat{W}(\cdot)$ is given by

$$(20) \quad \hat{W}^*(s) = 1 - \rho + \sum_{i=1}^{\infty} C \underline{\alpha} R^{i-1} \psi[R] [(sI-T)^{-1} T^0 A^0]^i \underline{e}, \quad \text{for } \text{Re } s \geq 0.$$

The matrix $\Psi[R]$ is defined in Formula (74) of [2] and satisfies the equation

$$(21) \quad R = I + \lambda_1' \Psi[R] + \lambda_1' R \Psi[R] T^0 A^0 ,$$

proved in Lemma 9 of the same paper. The matrix $\Psi[R]$ may be explicitly obtained in terms of R . This was overlooked in [2] and the following lemma indeed yields the result of Theorem 6 more directly.

Lemma 1

The matrix $\Psi[R]$ is given by

$$(22) \quad \Psi[R] = \lambda_1'^{-1} (R - I - R A^{00}) T^{-1}.$$

Proof

Postmultiplication by \underline{e} in (21) yields

$$\underline{R} \underline{e} = \underline{e} - \lambda_1' \Psi[R] \underline{T}^0 + \lambda_1' R \Psi[R] \underline{T}^0.$$

Since $I - R$ is nonsingular, it follows that $\lambda_1' \Psi[R] \underline{T}^0 = \underline{e}$, and hence $\lambda_1' \Psi[R] T^0 A^0 = A^{00}$. Upon substitution into (21), we readily obtain Formula (22).

The transformation of Formula (20), required to obtain a convenient algorithm for the computation of the probability distribution $\hat{W}(\cdot)$, is entirely analogous to that given above. We shall only show the most important steps.

We may write (20) as

$$(23) \quad \hat{W}^*(s) = 1 - \rho + C \underline{\alpha} \hat{\phi}^*(s) (sI - T)^{-1} \underline{T}^0 ,$$

where $\hat{\phi}^*(s)$ is an $m \times m$ matrix of Laplace-Stieltjes transforms, satisfying

$$(24) \quad \hat{\phi}^*(s) = (R-I-RA^{00})T^{-1} + R \hat{\phi}^*(s)(sI-T)^{-1}T^0A^0 .$$

Performing the same manipulations as those following Equation (5), we obtain

$$(25) \quad \hat{\phi}(x) = (R-I-RA^{00})T^{-1} + R\hat{V}^0T^0A^0 - R\hat{\theta}(x)T^0A^0 ,$$

for $x \geq 0$. This equation corresponds to (12) above.

The matrix \hat{V}^0 , which now satisfies the equation

$$(26) \quad \hat{V}^0T + R\hat{V}^0T^0A^0 = -(R-I-RA^{00})T^{-1} ,$$

is explicitly given by

$$(27) \quad \hat{V}^0 = -(R-I-RA^{00})T^{-2} - R(I-R)^{-1}(R-I-RA^{00})T^{-1}A^{00}T^{-1} \\ = -V^0T^{-1} + RV^0(I-A^{00})T^{-1} ,$$

where V^0 is given in Formula (15).

The matrix $\hat{\theta}(\cdot)$ is obtained by solving the matrix-differential equation

$$(28) \quad \hat{\theta}'(x) = \hat{\theta}(x)T + R\hat{\theta}(x)T^0A^0 , \quad \hat{\theta}(0) = \hat{V}^0 .$$

As we compute the matrix $\hat{\phi}(\cdot)$, by use of (28) and (25), we simultaneously integrate the differential equation

$$(29) \quad \hat{\phi}_1'(x) = \hat{\phi}_1'(x)T + \hat{\phi}(x), \quad \hat{\phi}_1(0) = 0 .$$

This yields the inverse of the Laplace-Stieltjes transform $\hat{\phi}^*(s)(sI-T)^{-1}$.

The probability distribution $\hat{W}(\cdot)$ is then finally given by

$$(30) \quad \hat{W}(x) = 1 - \rho + \lambda_1^{-1} C \underline{a} \hat{\phi}_1(x) \underline{T}^0, \quad \text{for } x \geq 0.$$

The algorithm for $\hat{W}(\cdot)$ is now clear. We evaluate the matrix \hat{V}^0 and integrate the differential equations (28) and (29) by classical methods. These will usually give the computed value of $\hat{\phi}_1(x)$ at values of x , which are the successive multiples of an appropriately chosen step h . We note that it is not necessary to store all the preceding values of $\hat{\phi}(\cdot)$ and $\hat{\phi}_1(\cdot)$, but only those few that are needed by the integration procedure. For each computed value of $\hat{\phi}_1(\cdot)$, the corresponding value of $\hat{W}(\cdot)$ is evaluated and stored for printout.

The mean of $\hat{W}(\cdot)$ is obtained by a routine differentiation in Formula (20) and is given by

$$(31) \quad \hat{\bar{W}} = \rho \bar{W} + \frac{1}{2} \lambda_1^{-1} \mu_2'$$

where μ_2' is the second moment of the service time. The quantity \bar{W} is the stationary mean waiting time at arrivals.

2. Asymptotic Results for the Waiting Time Distributions

The preceding derivations also readily yield precise asymptotic results for the probability distributions $W(\cdot)$, $W_1(\cdot)$ and $\hat{W}(\cdot)$. We first discuss some preliminary matters.

The positive matrix R has the maximal eigenvalue η , satisfying $0 < \eta < 1$. Let the vectors \underline{u} and \underline{u}^0 be left and right eigenvectors, corresponding to η . Both vectors may be chosen to be positive and to satisfy $\underline{u} \underline{e} = \underline{u} \underline{u}^0 = 1$.

The matrix $T + \eta T^0 A^0$ is clearly an irreducible, stable matrix. It therefore has an eigenvalue $-\xi < 0$, which is simple and for which the

corresponding left and right eigenvectors \underline{z} and \underline{z}^0 may be chosen to be positive and to satisfy $\underline{z} \underline{e} = \underline{z}^0 \underline{e} = 1$. Any other eigenvalue ξ' satisfies $\text{Re}(\xi') < -\xi$. The eigenvalue $-\xi$ will be called the eigenvalue with maximum real part of $T + \eta T^0 A^0$.

Lemma 2

- a. The vectors \underline{u} and \underline{z} are identical.
- b. The matrix $I \otimes T + R^T \otimes T^0 A^0$ has nonnegative off-diagonal elements. Its eigenvalues ξ'' satisfy $\text{Re}(\xi'') \leq -\xi$, and the quantity $-\xi$ is its eigenvalue of maximum real part. The left and right eigenvectors corresponding to $-\xi$ are given by $\underline{u}^{0T} \otimes \underline{u}$ and $\underline{u}^T \otimes \underline{z}^0$, respectively. The inner product of these two positive vectors is equal to one.

Proof

We recall from [2] that, in the case of the GI/PH/1 queue, the matrix $A^*(z) = \sum_{v=0}^{\infty} A_v z^v$, $0 \leq z \leq 1$, is given by

$$(32) \quad A^*(z) = \int_0^{\infty} \exp[(T + zT^0 A^0)t] dF(t).$$

Moreover, η is also the maximal eigenvalue of the positive matrix $A^*(\eta)$ and the vector \underline{u} is the corresponding left eigenvector, whose components sum to one.

On the other hand, it readily follows from (32) that $\underline{z} A^*(\eta) = f^*(\xi) \underline{z}$, where $f^*(\cdot)$ denotes the Laplace-Stieltjes transform of $F(\cdot)$. The uniqueness of the Perron eigenvalue and of the (normalized) corresponding left eigenvectors now readily imply that $\underline{u} = \underline{z}$, and $\eta = f^*(\xi)$.

It is clear, from the definition of the matrix $I \otimes T + R^T \otimes T^0 A^0$, that its off-diagonal elements are nonnegative. Moreover, the positivity of R and the irreducibility of $T + T^0 A^0$ imply that this matrix is also irreducible.

We have

$$(\underline{u}^{0T} \otimes \underline{u})(I \otimes T + R^T \otimes T^0 A^0) = \underline{u}^{0T} \otimes \underline{u} T + \underline{u}^{0T} R^T \otimes \underline{u} T^0 A^0 =$$

$$\underline{u}^{0T} \otimes \underline{u} T + \eta \underline{u}^{0T} \otimes \underline{u} T^0 A^0 = \underline{u}^{0T} \otimes \underline{u} (T + \eta T^0 A^0) = -\xi (\underline{u}^{0T} \otimes \underline{u}) .$$

A similar calculation holds for the vector $\underline{u}^T \otimes \underline{z}^0$. Since the vectors $\underline{u}^{0T} \otimes \underline{u}$ and $\underline{u}^T \otimes \underline{z}^0$ are positive, they are respectively the left and right eigenvectors corresponding to the eigenvalue $-\xi$ of maximum real part of the matrix $I \otimes T + R^T \otimes T^0 A^0$. Finally $(\underline{u}^{0T} \otimes \underline{u})(\underline{u}^T \otimes \underline{z}^0) = (\underline{u} \underline{u}^0)^T \otimes (\underline{u} \underline{z}^0) = 1$.

It now follows from Formula (10) that

$$(33) \quad \underline{\theta}(x) = -\underline{v}(I \otimes T + R^T \otimes T^0 A^0)^{-1} [(\underline{u}^T \otimes \underline{z}^0) \cdot (\underline{u}^{0T} \otimes \underline{u})] e^{-\xi x} + o(e^{-\xi x}) ,$$

as $x \rightarrow \infty$. The coefficient of $e^{-\xi x}$ may be simplified as follows

$$\begin{aligned} & -\underline{v}(I \otimes T + R^T \otimes T^0 A^0)^{-1} [(\underline{u}^T \otimes \underline{z}^0) \cdot (\underline{u}^{0T} \otimes \underline{u})] \\ &= \frac{1}{\xi} \underline{v} [(\underline{u}^T \otimes \underline{z}^0) \cdot (\underline{u}^{0T} \otimes \underline{u})] \\ &= \frac{1}{\xi} (\underline{u} \underline{z}^0) (\underline{u}^{0T} \otimes \underline{u}) = \frac{1}{\xi} (\underline{u}^{0T} \otimes \underline{u}) . \end{aligned}$$

The m^2 -vector $\underline{u}^{0T} \otimes \underline{u}$ is the direct sum of the rows of the $m \times m$ matrix $\underline{u}^0 \underline{u}$, so that Formula (33) may be equivalently written as

$$(34) \quad \underline{\theta}(x) = \frac{1}{\xi} \underline{u}^0 \underline{u} e^{-\xi x} + o(e^{-\xi x}) , \quad \text{as } x \rightarrow \infty .$$

Theorem 2

The probability distribution $W(\cdot)$ satisfies

$$(35) \quad 1-W(x) = k e^{-\xi x} + o(e^{-\xi x}), \quad \text{as } x \rightarrow \infty.$$

The constant k is given by

$$(36) \quad k = C\eta(1-\eta)^{-1}(\underline{\alpha} \underline{u}^0).$$

Proof

Upon substitution of (34) into (16), we obtain

$$1-W(x) = \frac{C}{\xi} \underline{\alpha} R(\underline{u}^0 \underline{u}) \underline{T}^0 e^{-\xi x} + o(e^{-\xi x}), \quad \text{as } x \rightarrow \infty.$$

However, $R\underline{u}^0 = \eta\underline{u}^0$, and $\underline{u}\underline{T} + \eta(\underline{u}\underline{T}^0)\underline{\alpha} = -\xi\underline{u}$. Upon postmultiplication by \underline{e} in the latter equation, we obtain that $\underline{u} \underline{T}^0 = \xi(1-\eta)^{-1}$. The stated result is now immediate.

It is also clear from the last two equations in the preceding proof that $\underline{u} = -\xi\eta(1-\eta)\underline{\alpha}(\xi\underline{I} + \underline{T})^{-1}$. Upon postmultiplication by \underline{T}^0 , we obtain the useful formula

$$(37) \quad -\underline{\alpha}(\xi\underline{I} + \underline{T})^{-1} \underline{T}^0 = \frac{1}{\eta}.$$

Since $W_1(\cdot)$ satisfies the equation (17), one readily obtains that

$$(38) \quad \begin{aligned} 1-W_1(x) &= -\underline{\alpha}(\xi\underline{I} + \underline{T})\underline{T}^0 \cdot k e^{-\xi x} + o(e^{-\xi x}) \\ &= \eta^{-1}k e^{-\xi x} + o(e^{-\xi x}), \quad \text{as } x \rightarrow \infty. \end{aligned}$$

An entirely similar argument holds for the distribution $\hat{W}(\cdot)$. One obtains after some calculations that

$$(39) \quad 1 - \hat{W}(x) = \hat{k} e^{-\xi x} + o(e^{-\xi x}) ,$$

where \hat{k} is given by

$$(40) \quad \begin{aligned} \hat{k} &= -\lambda_1'^{-1} \xi^{-1} (1-\eta) \underline{\alpha} (\xi I + T)^{-1} \underline{T}^0 \cdot k \\ &= \lambda_1'^{-1} \xi^{-1} C(\underline{\alpha} \underline{u}^0) . \end{aligned}$$

Remarks

- a. The asymptotic results in Formulas (35), (38) and (39) correspond to the exact results for the GI/M/1 queue. For that simple case, the term $o(e^{-\xi x})$ may be omitted; the resulting formulas then hold for all $x \geq 0$.
- b. We shall show below how the quantities η and ξ may be computed by an elementary and efficient algorithm. We then see from the proof of Theorem 2 that

$$(41) \quad \underline{u} = -\xi \eta (1-\eta)^{-1} \underline{\alpha} (\xi I + T)^{-1} .$$

This formula provides us with a powerful accuracy check in the computation of the matrix R . The latter is obtained by iterative solution of Equation (1). The quality of the computed value for R may be assessed by seeing how closely the equation $\underline{u}R = \eta \underline{u}$, is satisfied. The amount of computation required to obtain ξ, η and \underline{u} is small, compared to that needed to get R .

c. It is impossible, in general, to compute the vector \underline{u}^0 without evaluating the matrix R . The coefficients k , k_1 and \hat{k} , therefore, require a fair amount of computation. Even without precise knowledge of these coefficients, the asymptotic results have a number of practical uses. We propose to discuss these in a forthcoming paper, which deals with similar asymptotic results of much greater generality.

The following result provides us with an elementary numerical procedure, which simultaneously yields the values of η and ξ . We note that the Laplace-Stieltjes transform of the service time distribution is given by $h(s) = \underline{a}(sI-T)^{-1}\underline{T}^0$. This transform is a rational function in s and has an abscissa of convergence $-\tau^* < 0$. The function $h(s)$ is strictly decreasing on the interval $(-\tau^*, \infty)$ and satisfies $h(0) = 1$.

We define $\psi(z)$ to be the unique real solution of the equation

$$(42) \quad \underline{a}[\psi(z)I-T]^{-1}\underline{T}^0 = \frac{1}{z},$$

where $0 < z \leq 1$. We see that $\psi(z)$ increases from $-\tau^*$ to zero, as z varies from 0 to 1.

Theorem 3

For the stable GI/PH/1 queue, the quantity η is the unique solution in $(0,1)$ to the equations

$$(43) \quad z = f^*[-\psi(z)], \quad \frac{1}{z} = h[\psi(z)],$$

and

$$(44) \quad \xi = -\psi(\eta).$$

Proof

The quantity $\psi(z)$ is the eigenvalue of maximum real part of the matrix $T+zT^0A^0$. To see this, we write

$$\underline{u}(z) [T+zT^0A^0] = \psi(z)\underline{u}(z) \quad , \quad 0 < z \leq 1 \quad ,$$

and normalize $\underline{u}(z)$ by $\underline{u}(z)\underline{e} = 1$.

We then readily obtain

$$\underline{u}(z)\underline{T}^0 = \frac{1}{z-1} \psi(z) \quad ,$$

and also

$$\underline{u}(z) = \frac{z}{z-1} \psi(z)\underline{u} [\psi(z)I-T]^{-1} \quad .$$

Postmultiplication by \underline{T}^0 yields (42). It is also elementary to verify that the vector $\underline{u}(z)$ is positive. The quantity $\psi(z)$ is therefore the eigenvalue of maximum real part of the stable matrix $T+zT^0A^0$, for $0 < z \leq 1$.

It is now immediate that

$$\underline{u}(z)A^*(z) = \underline{u}(z) \int_0^\infty \exp[(T+zT^0A^0)u] dF(u) = f^*[-\psi(z)] \underline{u}(z) \quad ,$$

so that $f^*[-\psi(z)]$ is the Perron eigenvalue of the positive matrix $A^*(z)$ for $0 < z \leq 1$.

We know however from [2] that, provided the queue is stable, η is the unique solution in $(0,1)$ of the equation $z = f^*[-\psi(z)]$. Finally, from the definition of $-\xi$, it is clear that $\xi = -\psi(\eta)$.

To compute η and ξ , we may either solve the equation $f^*(-s)h(s) = 1$, for its unique solution s^0 in $(-\tau^*, 0)$ and set $\xi = -s_0$, and $\eta = f^*(\xi)$ - or we may compute $\psi(z)$ for successive values of z by using (42) and select the latter to approximate the solution to $z = f^*[-\psi(z)]$ in $(0, 1)$. Any of the classical methods, such as the bisection, secant, or Newton's method may be implemented. Convergence is rapid and the required computation time is very small.

3. Asymptotic Behavior of the Quene Length Densities

The powers of the matrix R have the well-known asymptotic behavior

$$(45) \quad R^k = \eta^k \underline{u}^0 \cdot \underline{u} + o(\eta^k), \quad \text{as } k \rightarrow \infty.$$

Furthermore, it follows from (22) that

$$\underline{u} \psi[R] = \lambda_1^{-1} [(\eta-1)\underline{u} - \eta\underline{\alpha}] T^{-1}.$$

Using the expression for \underline{u} obtained in (41), we may routinely simplify the preceding formula to

$$(46) \quad \underline{u} \psi[R] = -\lambda_1^{-1} \eta\underline{\alpha}(\xi I + T)^{-1}.$$

The following results are now immediate.

Theorem 4

In the stable GI/PH/1 queue, the stationary queue length densities satisfy

$$(47) \quad \underline{x}_k = C \underline{a} R^k = C(\underline{a} \underline{u}^0) \eta^k \underline{u} + o(\eta^k)$$

$$\underline{x}_k \underline{e} = C(\underline{a} \underline{u}^0) \eta^k + o(\eta^k),$$

$$\sum_{v=k}^{\infty} \underline{x}_v = C(\underline{a} \underline{u}^0) (1-\eta)^{-1} \eta^k + o(\eta^k),$$

$$(48) \quad \underline{y}_k = C \underline{a} R^{k-1} \Psi[R] = -\lambda_1^{-1} C(\underline{a} \underline{u}^0) \eta^k \cdot \underline{a}(\xi I + T)^{-1} + o(\eta^k),$$

$$\underline{y}_k \underline{e} = -\lambda_1^{-1} C(\underline{a} \underline{u}^0) \eta^k \underline{a}(\xi I + T)^{-1} \underline{e} + o(\eta^k),$$

$$\sum_{v=k}^{\infty} \underline{y}_v = -\lambda_1^{-1} C(\underline{a} \underline{u}^0) (1-\eta)^{-1} \eta^k \cdot \underline{a}(\xi I + T)^{-1} + o(\eta^k)$$

$$(49) \quad \underline{z}_k = \left[\sum_{v=1}^{\infty} \underline{y}_v T^0 \right]^{-1} \underline{y}_{k+1} T^0 = C \underline{a} R^k \underline{e} = C(\underline{a} \underline{u}^0) \eta^k + o(\eta^k),$$

$$\sum_{v=k}^{\infty} \underline{z}_v = C(\underline{a} \underline{u}^0) (1-\eta)^{-1} \eta^k + o(\eta^k),$$

as $k \rightarrow \infty$.

Formulas 47-49 give asymptotic expressions for the stationary densities and distributions of the queue length, respectively at arrival epochs, at an arbitrary time and immediately after departure epochs.

Remarks and Acknowledgments

In Section 5 of [2], we described a recursive procedure to compute the matrices A_k , $k \geq 0$, for the PH/PH/1 queue. This procedure is particularly appealing as it completely avoids the use of numerical integration. We did

express concern, however, over the possible singularity of the matrix $I \otimes S + T \otimes I$, which appears in it. This concern is without foundation. As the Kronecker sum of the nonsingular matrices S and T , the matrix $I \otimes S + T \otimes I$ is nonsingular [1]. The recursive procedure for the computation of the matrices A_k has now been implemented and performs very well [4].

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