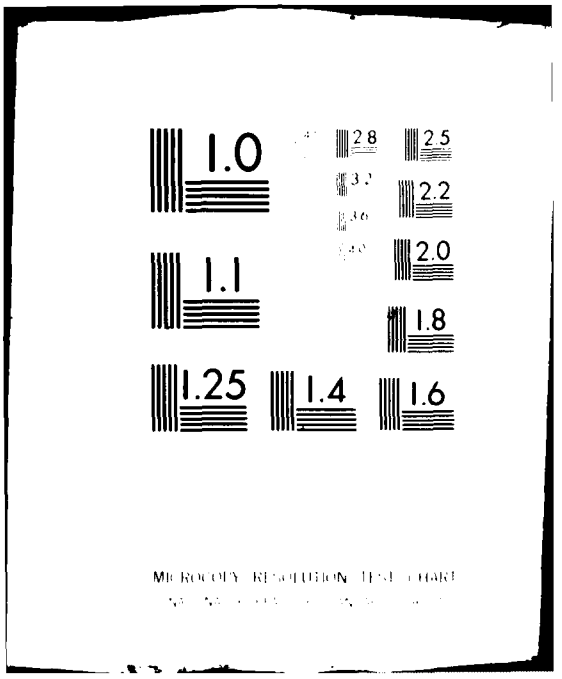
AD-A089 846 BROWN UNIV PROVIDENCE RI LEFSCHETZ CENTER FOR DYNAMI--ETC F/G 12/1
SPLINE APPROXIMATION FOR AUTONOMOUS NONLINEAR FUNCTIONAL DIFFER--ETC(U)
JUN 80 F KAPPEL DAAG29-79-C-0161

UNCLASSIFIED AFOSR-TR-80-0843 NL

1 OF 1
AD A089846

END
DATE FILMED
11-80
DTIC


1.0
2.8
2.5
3.2
2.2
3.6
2.0
1.1
1.8
1.25
1.4
1.6
MICROCOPY RESOLUTION TEST CHART

unclassified

REPORT DOCUMENTATION PAGE UNCLASSIFIED

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
AFOSR-TR- 80-0862		

4. TITLE (and Subtitle)

SPLINE APPROXIMATION FOR AUTONOMOUS NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

5. TYPE OF REPORT & PERIOD COVERED

Interim

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)

FRANZ KAPPEL

8. CONTRACT OR GRANT NUMBER(s)

AFOSR 76-3092

9. PERFORMING ORGANIZATION NAME AND ADDRESS

AIR FORCE OFFICE OF
DIVISION OF APPLIED MATHEMATICS
BROWN UNIVERSITY, PROVIDENCE, R.I. 02912

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

61102F 2304/A1

11. CONTROLLING OFFICE NAME AND ADDRESS

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
BOLLING AIR FORCE BASE
WASHINGTON, D.C.

12. REPORT DATE

June 18, 1980

13. NUMBER OF PAGES

22

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)

15. SECURITY CLASS. (of this report)

UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Based on abstract approximation results in semigroup theory ~~we develop~~ an approximation scheme for nonlinear autonomous functional-differential equations with globally Lipschitzean right-hand side (WAS DEVELOPED). The scheme can be realized by using spline approximation of the state.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE UNCLASSIFIED

LEVEL II

SPLINE APPROXIMATION FOR AUTONOMOUS NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS.

by

Franz Kappel*
Mathematisches Institut
Universität Graz
Elisabethstrasse 11
A 8010 Graz (Austria)

and

Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, R. I. 02912

* This research was supported in part by the Air Force Office of Scientific Research under contract AFOSR-76-3092, and in part by the United States Army Research Office under contract #ARO-DAAG29-79-C-0161.

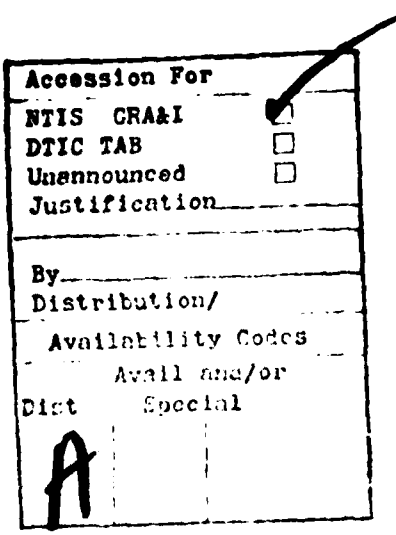
SPLINE APPROXIMATION FOR AUTONOMOUS NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

by

F. Kappel

ABSTRACT:

Based on abstract approximation results in semigroup theory we develop an approximation scheme for nonlinear autonomous functional-differential equations with globally Lipschitzean right-hand side. The scheme can be realized by using spline approximation of the state.



1. Introduction and Notation

In this paper we show that approximation techniques developed in [5] for linear autonomous functional-differential equations of retarded type are also applicable to a rather broad class of nonlinear equations. These techniques already have been successfully applied to linear autonomous equations of neutral type (see for instance [9], [10]). As in the linear case the approach here is based on abstract approximation results in semigroup theory and provides a sequence of ordinary differential systems of increasing dimension whose solutions approximate those of the original delay system. The same algorithm is obtained in [2] by a different method under the more restrictive condition of differentiability of the right-hand side. On the other hand the proof in [2] also works in the time-dependent case. The general idea of using abstract approximation results of semigroup theory in order to get algorithms for the numerical solution of optimal control problems involving delay systems goes back to [3], [4]. Here we restrict ourselves to the approximation of nonlinear autonomous delay equations. With respect to control problems for nonlinear delay systems see [1]. The assumptions of [1] in the autonomous case are stronger than those imposed here. In [11] the scheme of averaging projections developed in [4] was shown to be convergent also in the case of nonlinear autonomous equations with locally Lipschitzean right-hand side. For nonlinear time-dependent equations satisfying Caratheodory type conditions an approximation scheme based on interpolation by splines of first order was developed in [12].

Our state space will be $Z = \mathbb{R}^n \times L_2(-r,0;\mathbb{R}^n)$, $r > 0$, which is a Hilbert space with norm $|(\eta,\phi)|_Z^2 = |\eta|^2 + |\phi|_2^2$ and corresponding inner product.

$|\cdot|$ and $|\cdot|_2$ denote the Euclidean norm in $\mathbb{R}^n$ and the usual norm in $L_2(-r,0;\mathbb{R}^n)$, respectively. It will be necessary to endow Z with an equivalent norm $|(\eta,\phi)|_g^2 = |\eta|^2 + \int_{-r}^{0} |\phi(s)|^2 g(s)ds$, where g is a positive weighting function on $[-r,0]$. Z supplied with this equivalent norm and the corresponding inner product $\langle (\eta_1,\phi_1), (\eta_2,\phi_2) \rangle_g = \eta_1^T\eta_2 + \int_{-r}^{0} \phi_1(s)^T\phi_2(s)g(s)ds$ will be denoted by Z_g. $\mathscr{L}_2(-r,0;\mathbb{R}^n)$ is the linear space of square-integrable functions $[-r,0] \to \mathbb{R}^n$ (in contrast to $L_2(-r,0;\mathbb{R}^n)$ which is the Banach space of equivalence classes of such functions). $W^{1,2}$ denotes the space of absolutely continuous functions $\phi: [-r,0] \to \mathbb{R}^n$ such that $\dot{\phi}$ is square-integrable. $\mathscr{C}^k$, $k = 0,1,\ldots,$ is the subspace $\{(\phi(0),\phi) | \phi \in C^k\}$ of Z, where C^k is the space of k-times continuously differentiable functions $\phi: [-r,0] \to \mathbb{R}^n$. Finally, given a function $x: [-r,\alpha) \to \mathbb{R}^n$, $\alpha > 0$, the function x_t, $t \in [0,\alpha)$, is defined by $x_t(s) = x(t+s)$, $s \in [-r,0]$.

2. The Class of Autonomous Nonlinear Delay Equations

Let a map $f: \mathscr{L}_2(-r,0;\mathbb{R}^n) \to \mathbb{R}^n$ be given and assume that the following two conditions hold throughout the paper:

(H1) If, for some $\alpha > 0$, x is in $L_2(-r,\alpha;\mathbb{R}^n)$ and has a representation which is continuous on $[0,\alpha)$, then for any representation χ of x the map defined by $t \to f(\chi_t)$ on $[0,\alpha]$ is in the same equivalence class of $L_1(0,\alpha;\mathbb{R}^n)$.

(H2) There exist constants $L > 0$ and r_j, $j = 0,\ldots,m$, with $0 = r_0 < r_1 < \ldots < r_m = r$ such that for any ϕ,ψ in $\mathscr{L}_2(-r,0;\mathbb{R}^n)$
$|f(\phi) - f(\psi)| \leq L(\sum_{j=0}^{m} |\phi(-r_j) - \psi(-r_j)| + |\phi-\psi|_2)$.

The Cauchy problem we are concerned with in this section is

$$\dot{x}(t) = f(x_t), \quad t \geq 0, \tag{2.1}$$

$$x(0) = \eta \in \mathbb{R}^n, x(s) = \phi(s) \quad \text{a.e. on} \quad [-r,0], \tag{2.2}$$

where $\phi \in L_2(-r,0;\mathbb{R}^n)$. By a solution of (2.1), (2.2) we mean a function $x: [-r,\alpha) \to \mathbb{R}^n$, $\alpha > 0$, such that (2.2) holds and $x(t) = \eta + \int_0^t f(x_s)ds$ for $t \in [0,\alpha)$. Condition (H1) just assures integrability of $f(x_s)$ for all functions x which are candidates for a solution.

Lemma 2.1.

a) For any $(\eta,\phi) \in Z$ problem (2.1), (2.2) has a unique solution $x(t) = x(t;\eta,\phi)$ existing on $[-r,\infty)$. Moreover, this solution is continuously dependent on initial data.

b) The family $T(\cdot)$ of operators $Z \to Z$ defined by $T(t)(\eta,\phi) = (x(t;\eta,\phi), x_t(\eta,\phi))$, $t \geq 0$, $(\eta,\phi) \in Z$, has the following properties:

(i) $T(0) = I$.

(ii) $T(t+s) = T(t)T(s)$, $t,s \geq 0$.

(iii) For any $(\eta,\phi) \in Z$ the map defined by $t \to T(t)(\eta,\phi)$ is continuous on $[0,\infty)$.

(iv) There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $|T(t)z_1 - T(t)z_2|_Z \leq Me^{\omega t}|z_1-z_2|_Z$ for $t \geq 0$ and $z_i \in Z$, $i = 1,2$.

This lemma is a special case of results proved in [11] (Proposition 1.1, Remark 1.7, Proposition 1.2 and Proposition 2.1). Note, that (H1) and (H2) imply that the global Borisovič-Turbabin conditions as formulated in [11] hold.

In [11; Proposition 2.2] it was shown that the infinitesimal generator $\mathcal{A}$ of the semigroup $T(\cdot)$ defined in Lemma 2.1 is given by

$$\begin{aligned} \operatorname{dom}\mathcal{A} &= \{(\phi(0),\phi) \mid \phi \in W^{1,2}\}, \\ \mathcal{A}(\phi(0),\phi) &= (f(\phi),\dot{\phi}) \quad \text{for} \quad (\phi(0),\phi) \in \operatorname{dom}\mathcal{A}. \end{aligned} \tag{2.3}$$

We may decompose $\mathcal{A}$,

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1, \tag{2.4}$$

where $\mathcal{A}_0(\phi(0),\phi) = (0,\dot{\phi})$ and $\mathcal{A}_1(\phi(0),\phi) = (f(\phi),0)$. $\mathcal{A}_0$ is a closed linear operator with $\operatorname{dom}\mathcal{A}_0 = \operatorname{dom}\mathcal{A}$ dense in Z. $\mathcal{A}_0$ is the infinitesimal generator of the C_o-semigroup corresponding to the equation $\dot{x}(t) = 0$ (cf. [11]).

If $\mathcal{B}$ is a single valued nonlinear operator in a Hilbert space X, then $\mathcal{B} - \omega I$, $\omega \in \mathbb{R}$, is called dissipative if

$$\langle \mathcal{B}x_1 - \mathcal{B}x_2, x_1 - x_2 \rangle_X \leq \omega |x_1 - x_2|_X^2$$

for all $x_i \in \operatorname{dom}\mathcal{B}$, $i = 1,2$. In general, $\mathcal{A} - \omega I$ will not be dissipative for any $\omega \in \mathbb{R}$ ($\mathcal{A}$ the operator defined in (2.3)), because the constant M appearing in (iv) of Lemma 2.1, b) cannot be chosen as one. But following an idea given by Webb in [15] it is possible to renorm Z such that we get

dissipativeness. Corresponding to the numbers r_j in (H2) we choose a weighting function g satisfying

$$g(s) = m - j + 1 \quad \text{on} \quad (-r_j,-r_{j-1}), \quad j = 1,\dots,m.$$

Of course, $|\cdot|_g$ is a norm on Z equivalent to $|\cdot|_Z$.

Lemma 2.2.

a) There exists an $\omega \in \mathbb{R}$ such that $\mathscr{A} - \omega I$ is dissipative in Z_g.

b) There exists a $\lambda_0 > 0$ such that

$$\mathscr{R}(I - \lambda\mathscr{A}) = Z$$

for $\lambda \in (0,\lambda_0)$.

c) $T(t)z = \lim_{n\to\infty} (I - \frac{t}{n}\mathscr{A})^{-n}z$ for all $z \in Z$ uniformly with respect to t in bounded intervals.

Proof. For $(\phi(0),\phi)$ and $(\psi(0),\psi)$ in $\operatorname{dom}\mathscr{A}$ we obtain (using (2.3) and (H2))

$$\begin{aligned}
&\langle \mathscr{A}(\phi(0),\phi) - \mathscr{A}(\psi(0),\psi),\ (\phi(0)-\psi(0),\phi-\psi)\rangle_g \\
&\le L\Big(\sum_{j=0}^{m}\big|\phi(-r_j)-\psi(-r_j)\big|+\big|\phi-\psi\big|_2\Big)\big|\phi(0)-\psi(0)\big|+\sum_{j=1}^{m}(m-j+1)\int_{-r_j}^{-r_{j-1}}(\phi-\psi)^T(\phi-\psi)ds \\
&\le 2L\big|(\phi(0)-\psi(0),\phi-\psi)\big|_g^2 + \tfrac{m}{2}L^2\big|\phi(0)-\psi(0)\big|^2 + \tfrac{1}{2}\sum_{j=1}^{m}\big|\phi(-r_j)-\psi(-r_j)\big|^2 \\
&\quad + \tfrac{m}{2}\big|\phi(0)-\psi(0)\big|^2 - \tfrac{1}{2}\sum_{j=1}^{m}\big|\phi(-r_j)-\psi(-r_j)\big|^2 - \tfrac{1}{2}\big|\phi(-r_m)-\psi(-r_m)\big|^2 \\
&\le [2L + \tfrac{m}{2}(L^2+1)]\big|(\phi(0)-\psi(0),\phi-\psi)\big|_g^2 .
\end{aligned}$$

This proves a) with $\omega = 2L + \frac{m}{2}(L^2+1)$. In order to prove b) we show that

$$(I - \lambda\mathcal{A})(\phi(0),\phi) = (\eta,\psi) \tag{2.5}$$

has a solution $(\phi(0),\phi) \in \mathrm{dom}\,\mathcal{A}$ for any $(\eta,\psi) \in Z$ provided λ is sufficiently small. (2.5) is equivalent to

$$\phi(0) - \lambda f(\phi) = \eta \quad \text{and} \quad \phi - \lambda\dot{\phi} = \psi.$$

The second equation implies $\phi(s) = e^{s/\lambda}\phi(0) - \frac{1}{\lambda}\int_0^s e^{(s-\tau)/\lambda}\psi(\tau)d\tau$ which for any $\phi(0)$ is certainly in $W^{1,2}$. For $\phi(0)$ we get the fixed point equation $\phi(0) = h(\phi(0))$ in $\mathbb{R}^n$, where

$$h(a) = \lambda f(e^{\cdot/\lambda}a - \frac{1}{\lambda}\int_0^{\cdot} e^{(\cdot-\tau)/\lambda}\psi(\tau)d\tau) + \eta.$$

An easy calculation using (H2) shows that h has the Lipschitz constant $\lambda L(m+1+r^{1/2})$ on $\mathbb{R}^n$. This proves b) with $\lambda_0 = 1/L(m+1+r^{1/2})$.

In [11; Proposition 2.3] it is shown that $u(t) = T(t)z$ for $z \in \mathrm{dom}\,\mathcal{A}$ is a strong solution of the abstract Cauchy problem

$$\dot{u} = \mathcal{A}u, u(0) = z.$$

On the other hand any strong solution of this Cauchy problem is represented by $u(t) = \tilde{T}(t)z$, where $\tilde{T}(t)z = \lim_{n\to\infty}(I - \frac{t}{n}\mathcal{A})^{-n}z$ (see [8; Theorem II]). Therefore the semigroups $T(\cdot)$ and $\tilde{T}(\cdot)$ coincide on dom $\mathcal{A}$. Since dom $\mathcal{A}$ is dense in Z, we have $T(t) = \tilde{T}(t)$, $t \geq 0$.

We express the situation of Lemma 2.2, c) by saying that the semigroup $T(\cdot)$ is generated by $\mathcal{A}$.

For our approach we shall need

Lemma 2.3. The sets $\mathscr{C}^k$, $k = 0,1,\ldots$, and $(I - \lambda\mathscr{A})\mathscr{C}^k$, $k = 1,2,\ldots$, are dense in Z for λ sufficiently small. Moreover, $\mathscr{C}^k \subset \operatorname{dom}\mathscr{A}$ for $k = 1,2,\ldots$.

Proof. Define $\mathscr{D}^k = \{(\phi(0),\phi) \mid \phi \in C^k \text{ and } \dot{\phi}(0) = f(\phi)\}$, $k = 1,2,\ldots$. Then $(I - \lambda\mathscr{A})C^k \supset (I - \lambda\mathscr{A})\mathscr{D}^k = \mathscr{C}^{k-1} \supset \operatorname{dom}\mathscr{A}_0^k$ and everything follows from the fact that $\operatorname{dom}\mathscr{A}_0^k$ is dense in Z. Note, that $\mathscr{A}_0$ is the infinitesimal generator of a C_o-semigroup of bounded linear operators. It only remains to show $(I - \lambda\mathscr{A})\mathscr{D}^k = \mathscr{C}^{k-1}$. It is already established that equation (2.5) has a unique solution $(\phi(0),\phi)$ for $(\eta,\psi) = (\psi(0),\psi)$, $\psi \in C^{k-1}$ and $\lambda \in (0,\lambda_0)$. $\psi \in C^{k-1}$ immediately implies $\phi \in C^k$. From $\phi(0) - \lambda\dot{\phi}(0) = \psi(0)$ and $\phi(0) - \lambda f(\psi) = \psi(0)$ we get $\dot{\phi}(0) = f(\psi)$ which proves $(\phi(0),\phi) \in \mathscr{D}^k$. $\mathscr{C}^k \subset \operatorname{dom}\mathscr{A}$ is clear by (2.3).

Remark. From Lemma 2.2, a) and b) it follows that $(I - \lambda\mathscr{A})^{-1}$ exists for $\lambda \in (0,\lambda_0)$ and is a globally Lipschitzean operator $Z \to \operatorname{dom}\mathscr{A}$. Therefore $\mathscr{D}^k = (I - \lambda\mathscr{A})^{-1}\mathscr{C}^{k-1}$ for $\lambda \in (0,\lambda_0)$, which implies that $\mathscr{D}^k$ is also dense in Z, $k = 1,2,\ldots$.

3. The Approximation Scheme

Following the idea already used in [5] we choose a sequence Z_N of finite dimensional subspaces of Z such that

$$Z_N \subset \mathrm{dom}\,\mathscr{A}, \quad N = 1,2,\ldots .$$

Let $P_N: Z_g \to Z_N$ be the orthogonal projection onto Z_N and define $\mathscr{A}_N$ by

$$\mathscr{A}_N = P_N \mathscr{A} P_N, \quad N = 1,2,\ldots .$$

We call the sequence $\{Z_N, P_N, \mathscr{A}_N\}$ an approximation scheme for equation (2.1).

Lemma 3.1.

a) $\mathscr{A}_N - \omega I$ is dissipative in Z_g for all N, where ω is the constant of Lemma 2.2, a).

b) $\mathscr{A}_N$ is globally Lipschitzean on Z, $N = 1,2,\ldots$.

c) For all $\lambda > 0$ such that $\mathscr{R}(I - \lambda\mathscr{A}_N) = Z$

$$(I - \lambda\mathscr{A}_N)^{-1} Z_N \subset Z_N, \quad N = 1,2,\ldots .$$

Proof. Part a) is proved in the same way as in the linear case (see [5], Proof of Theorem 3.1). For the proof of part b) we recall that $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1$ (see (2.4)) and therefore $\mathscr{A}_N = \mathscr{A}_{N,0} + \mathscr{A}_{N,1}$, where $\mathscr{A}_{N,0} = P_N \mathscr{A}_0 P_N$ and $\mathscr{A}_{N,1} = P_N \mathscr{A}_1 P_N$. $\mathscr{A}_{N,0}$ is a bounded linear operator on Z (see [5], Proof of Theorem 3.1). We only need to estimate

$$\begin{aligned}
\left|\mathscr{A}_{N,1} z_1 - \mathscr{A}_{N,1} z_2\right|_Z &\le \left|\mathscr{A}_1 P_N z_1 - \mathscr{A}_1 P_N z_2\right|_Z \\
&= \left|f(\phi_N) - f(\psi_N)\right| \le L\Big(\sum_{j=0}^{m} \left|\phi_N(-r_j) - \psi_N(-r_j)\right| + \left|\phi_N - \psi_N\right|_2\Big) \\
&\le L\Big(m \sup_{-r \le s \le 0} \left|\phi_N(s) - \psi_N(s)\right| + \left|P_N z_1 - P_N z_2\right|_Z\Big),
\end{aligned}$$

where $z_1, z_2 \in Z$ and $P_N z_1 = (\phi_N(0), \phi_N)$, $P_N z_2 = (\psi_N(0), \psi_N)$. Since Z_N is finite dimensional all norms on Z_N are equivalent. Therefore there exists a constant $\sigma_N > 0$ such that

$$\left| \mathcal{A}_{N,1} z_1 - \mathcal{A}_{N,1} z_2 \right|_Z \leq \sigma_N \left| z_1 - z_2 \right|_Z$$

In order to prove part c) we recall that dissipativeness of $\mathcal{A}_N - \omega I$ implies the existence of $(I - \lambda \mathcal{A}_N)^{-1}$ on $\mathcal{R}(I - \lambda \mathcal{A}_N)$ for $\lambda \in (0, \frac{1}{\omega})$. Assume that $\mathcal{R}(I - \lambda \mathcal{A}_N) = Z$. We take $y \in Z_N$ and put $z = (I - \lambda \mathcal{A}_N)^{-1} y$. Then $z = z_1 + z_2$, where $z_1 \in Z_N$ and $z_2 \in Z_N^{\perp}$, and

$$z_1 + z_2 - \lambda \mathcal{A}_N (z_1 + z_2) = y \in Z_N$$

implies $z_2 = 0$, i.e., $z \in Z_N$. Note, that $\mathcal{A}_N Z \subset Z_N$.

Proposition 3.1.

a) $\mathcal{A}_N$ generates a semigroup $T_N(\cdot)$ of type ω on z (i.e., properties (i) - (iv) of Lemma 2.1, b) hold for $T_N(\cdot)$ with $M = 1$),

$$T_N(t) z = \lim_{n \to \infty} (I - \frac{t}{n} \mathcal{A}_N)^{-n} z, \quad z \in Z,$$

uniformly with respect to t in bounded intervals.

b) $T_N(t) Z_N \subset Z_N$ for $t \geq 0$, $N = 1, 2, \ldots$.

c) For any $z \in Z$ the function $u(t) = T_N(t) z$, $t \geq 0$ is continuously differentiable for $t \geq 0$ and

$$\dot{u}(t) = \mathcal{A}_N u(t), \quad t \geq 0,$$
$$u(0) = z. \tag{3.1}$$

Proof. Since by Lemma 3.1, b) $\mathcal{A}_N$ is globally Lipschitzean on Z, equation $(I - \lambda\mathcal{A}_N)\, z = y$ has for any $y \in Z$ a solution $z \in Z$ provided λ is sufficiently small, i.e., $\mathcal{R}(I - \lambda\mathcal{A}_N) = Z$ for those λ. This together with dissipativeness of $\mathcal{A}_N - \omega I$ (Lemma 3.1, a)) shows that we may use the Crandall-Liggett Theorem ([8; Theorem I]) in order to get a). Part b) then is an immediate consequence of Lemma 3.1, c). It is clear that equation (3.1) for any $z \in Z$ has a unique solution on $[0,\infty)$ which is continuously differentiable. By Theorem II of [8] we get $u(t) = T_N(t)z$.

In the proof of Proposition 3.1, a) we have seen that for each N $\mathcal{R}(I - \lambda\mathcal{A}_N) = Z$ for λ sufficiently small. For our convergence proof we shall need that this range condition holds uniformly with respect to N.

Proposition 3.2. Let ω be the constant of Lemma 3.1, a). Then

$$\mathcal{R}(I - \lambda\mathcal{A}_N) = Z$$

for all $\lambda \in (0,\frac{1}{\omega})$ and all N.

Proof. In the proof of Proposition 3.1, a) we have seen that for any N there exists a $\lambda_N > 0$ such that $\mathcal{R}(I - \lambda\mathcal{A}_N) = Z$ for all $\lambda \in (0,\lambda_N)$. Then for $\lambda \leq \min(\lambda_N,\frac{1}{\omega})$ we have

$$Z = \mathcal{R}(I - \lambda\mathcal{A}_N) = \mathcal{R}((1-\lambda\omega)(I - \frac{\lambda}{1-\lambda\omega}(\mathcal{A}_N - \omega I)))$$

$$= \mathcal{R}(I - \frac{\lambda}{1-\lambda\omega}(\mathcal{A}_N - \omega I)),$$

i.e., $\mathcal{R}(I - \mu(\mathcal{A}_N - \omega I)) = Z$ for μ sufficiently small. Since $\mathcal{A}_N - \omega I$

is dissipative, we immediately (see for instance [6; p.73]) get $\mathcal{R}(I - \mu(\mathcal{A}_N - \omega I)) = Z$ for all $\mu > 0$ or equivalently $\mathcal{R}(I - \lambda\mathcal{A}_N) = Z$ for all $\lambda \in (0,\frac{1}{\omega})$.

By Proposition 3.1 we obtain a sequence of ordinary differential equations on the finite dimensional spaces Z_N,

$$\dot{z}_N(t) = \tilde{\mathcal{A}}_N z_N(t), \quad t \geq 0, \tag{3.2}$$

where $\tilde{\mathcal{A}}_N$ denotes the restriction of $\mathcal{A}_N$ to Z_N. For any $z \in Z$ the solution of (3.2) with initial condition $z_N(0) = P_N z$ is given by $z_N(t) = T_N(t)P_N z$. In order to prove that $z_N(t) \to T(t)z$ as $N \to \infty$ we shall use the following nonlinear version of the Trotter-Kato Theorem:

Theorem 3.1. Let $\mathcal{B}_N$, $N = 1,2,\ldots$, and $\mathcal{B}$ be single-valued operators on a Banach space X such that $\operatorname{dom}\mathcal{B}_N \supset \operatorname{dom}\mathcal{B}$ for all N and $\overline{\operatorname{dom}\mathcal{B}} = X$. Moreover, assume that the following conditions are satisfied:

(i) There exists a $\lambda_0 > 0$ such that

$$\mathcal{R}(I - \lambda\mathcal{B}) = \mathcal{R}(I - \lambda\mathcal{B}_N) = X$$

for $N = 1,2,\ldots$ and all $\lambda \in (0,\lambda_0)$.

(ii) There exist real constants ω_N, $N = 1,2,\ldots$, and ω such that the sequence $\{\omega_N\}$ is bounded above and

$$\mathcal{B}_N - \omega_N I \quad \text{and} \quad \mathcal{B} - \omega I$$

are dissipative in X.

(iii) There exists a subset $\mathcal{D}$ of $\operatorname{dom}\mathcal{B}$ such that $\overline{(I - \lambda\mathcal{B})\mathcal{D}} = X$ for λ sufficiently small and

$$\mathcal{B}_N x \to \mathcal{B}x \quad \text{as} \quad N \to \infty$$

for all $x \in \mathcal{D}$.

Let $T_N(\cdot)$ and $T(\cdot)$ denote the semigroups generated by $\mathscr{B}_N$ and $\mathscr{B}$, respectively. Then for all $x \in X$

$$\lim_{N\to\infty} T_N(t)x = T(t)x$$

uniformly with respect to t in bounded intervals.

This theorem can quite easily be extracted from [7]. Using the same conclusions as in the proof of Theorem 4.1 in [7] one first shows that there exists a $\bar{\lambda} > 0$ such that $(I - \lambda\mathscr{B}_N)^{-1}x \to (I - \lambda\mathscr{B})^{-1}x$ as $N \to \infty$ for all $x \in X$ and all $\lambda \in (0,\bar{\lambda})$. The rest of the proof is the same as the proof of Theorem 3.1 in [7].

Theorem 3.2. Let $\{Z_N, P_N, \mathscr{A}_N\}$ be an approximation scheme for equation (2.1) and assume:

(i) $P_N z \to z$ for all $z \in Z$.

(ii) There exists a $k \geq 1$ such that $\phi \in C^k$ implies

$$\phi_N(0) \to \phi(0) \quad \text{and} \quad |\dot{\phi}_N - \dot{\phi}|_2 \to 0$$

as $N \to \infty$. $\phi_N \in W^{1,2}$ is defined by

$$P_N(\phi(0),\phi) = (\phi_N(0),\phi_N).$$

Then for any $z \in Z$

$$\lim_{N\to\infty} T_N(t)P_N z = T(t)z$$

uniformly on bounded t-intervals.

Proof. By assumption (i) it is sufficient to prove $\lim_{N\to\infty} T_N(t)z = T(t)z$. Conditions (i) and (ii) of Theorem 3.1 are satisfied because of Lemma 3.1, a) and Proposition 3.2. The verification of condition (iii) in Theorem 3.1 is done in the same way as in the linear case choosing $\mathcal{D} = \mathcal{C}^k$ (see [5], Proof of Theorem 3.1). Note, that under condition (ii) we have

$\sup_{-r\le s\le 0} |\phi_N(s)-\phi(s)| \to 0$ and therefore by (H2) $f(\phi_N) \to f(\phi)$ as $N \to \infty$.

Remark. The proof of Theorem 3.1 in [5] uses only $\mathcal{D} = \mathcal{C}^k$ and not $\mathcal{D} = \mathcal{D}^k$.

Let $k_N = \dim Z_N$. We choose a basis $\hat{\beta}_1^N,\dots,\hat{\beta}_{k_N}^N$ of Z_N, where $\hat{\beta}_j^N = (\beta_j^N(0),\beta_j^N),\beta_j^N \in W^{1,2}$, and define the $n \times k_N$ matrix

$$\beta^N = (\beta_1^N,\dots,\beta_{k_N}^N).$$

If we put $\hat{\beta}^N = (\beta^N(0),\beta^N)$, then for any $z_N \in Z_N$ we have

$$z_N = \beta^N\alpha_N = (\beta^N(0)\alpha_N,\beta^N\alpha_N),$$

where $\alpha_N \in \mathbb{R}^{k_N}$ is the coordinate vector of z_N. The same calculations as in [5] show that for $(\eta,\phi) \in Z$

$$P_N(\eta,\phi) = \hat{\beta}^N\alpha_N,$$

where α_N is the solution of

$$Q_N\alpha_N = k_N(\eta,\phi).$$

The $k_N \times k_N$-matrix Q_N and the k_N-vector $h^N(\eta,\phi)$ are given by

$$Q_N = \beta^N(0)^T\beta^N(0) + \int_{-r}^{0} \beta^N(s)^T\beta^N(s)g(s)ds,$$

$$h_N(\eta,\phi) = \beta^N(0)^T\eta + \int_{-r}^{0} \beta^N(s)^T\phi(s)g(s)ds.$$

The matrix representation $A_{N,0}$ of $\mathcal{A}_{N,0} = P_N\mathcal{A}_0P_N$ restricted to Z_N is given by

$$A_{N,0} = Q_N^{-1}H_N,$$

where $H_N = \int_{-r}^{0} \beta^N(s)^T\,\dot{\beta}^N(s)g(s)ds.$

For $z_N \in Z_N$ with coordinate vector α_N we denote the coordinate vector of $\mathcal{A}_{N,1}z_N = P_N\mathcal{A}_1P_Nz_N$ with $F_N(\alpha_N)$. From $\mathcal{A}_{N,1}z_N = P_N(f(\beta^N\alpha_N),0)$ we get that $F_N(\alpha_N)$ is the solution of $Q_NF_N(\alpha_N) = h^N((f(\beta^N\alpha_N),0)) = \beta^N(0)^Tf(\beta^N\alpha_N)$, i.e.,

$$F_N(\alpha_N) = Q_N^{-1}\,\beta^N(0)^Tf(\beta^N\alpha_N).$$

$z_N(t)$ is a solution of (3.2) with $z_N(0) = P_N(\eta,\phi)$ if and only if the coordinate vector $w_N(t)$ of $z_N(t)$, $z_N(t) = \hat{\beta}^Nw_N(t)$, is a solution of

$$\dot{w}_N(t) = A_{N,0}w_N(t) + F_N(w_N(t)),$$

$$w_N(0) = \alpha_N, \quad \text{where} \quad \hat{\beta}^N\alpha_N = P_N(\eta,\phi). \tag{3.3}$$

By Theorem 3.2 we have

$$\lim_{N\to\infty} \hat{\beta}^N w_N(t) = (x(t;\eta,\phi),\ x_t(\cdot,\eta,\phi)) \tag{3.4}$$

uniformly on bounded t-intervals.

4. Spline Approximation

A realization of the scheme presented in Section 3 can be obtained by using subspaces of spline functions. In order to do so take a sequence of partitions of $[-r,0]$ with meshpoints t_j^N, $N = 1,2,\ldots,$ such that $\max_j |t_j^N - t_{j-1}^N| \to 0$ as $N \to \infty$ and $\max_j |t_j^N - t_{j-1}^N| / \min_j |t_j^N - t_{j-1}^N| \leq \beta < \infty$, $N = 1,2,\ldots,$ for some constant $\beta > 0$. For instance we may take

$$t_j^N = -j\frac{r}{N}, \quad j = 0,\ldots,N,$$

or

$$t_{kN+j}^N = -r_k - j\,\frac{r_{k+1}-r_k}{N}, \quad k = 0,\ldots,m-1; \quad j = 1,\ldots,N.$$

In the second case $-r_k$, $k = 0,\ldots,m$, is always a meshpoint of the partition.

Theorem 4.1. Let Z_N be the subspace of all elements $(\phi(0),\phi)$ in Z such that ϕ is a first order, a cubic Hermite or a cubic spline with knots at t_j^N, respectively, and choose the corresponding approximation scheme $\{Z_N, P_N, \mathcal{A}_N\}$ for equation (2.1). Then for any $z \in Z$,

$$\lim_{N\to\infty} T_N(t)P_N z = T(t)z$$

uniformly on bounded t-intervals, where $z_N(t) = T_N(t)P_N z$ is the unique solution of (3.2) with $z_N(0) = P_N z$.

Proof. We have to verify conditions (i) and (ii) of Theorem 3.2. This is done in the same way as in the linear case using the nice convergence properties of spline functions (see [5], Proof of Theorem 4.1, or [9], Section 4).

For calculations one chooses B-splines as basis elements of Z_N (see for instance [14], Sections 2.1, 3.1 and 4.1). The matrices Q_N, H_N are calculated as indicated at the end of Section 3. Of course, system (3.3) is always solved in the form

$$Q_N \dot{w}_N(t) = H_N w_N(t) + \beta^N(0)^T f(\beta^N w_N(t)).$$

Example. The cubic spline scheme was used for the following initial value problem:

$$\dot{x}(t) = -x(t-1)(4-x^2(t)), \quad t \geq 0, \tag{4.1}$$

$$x(t) = \cos \tfrac{\pi}{2} t, \qquad -1 \leq t \leq 0. \tag{4.2}$$

Nussbaum proved [13; Corollary 2.1] that equation (4.1) has a unique slowly oscillating solution $x_0(t)$ such that $x_0(t) > 0$ on $[0,2]$, $x_0(-t) = -x_0(t)$ and $x_0(t+2) = -x_0(t)$ for all t. Moreover, $|x_0(t)| < 2$ for all t. The solution of (4.1), (4.2) tends to $x_0(t+c)$ for some $c \in \mathbb{R}$.

In Table 1 we give the values for $x(t)$ on the interval $[0,5]$,

which were obtained by stepwise integration of (4.1), (4.2) using a standard fourth order Runge-Kutta scheme, and the values of $\beta^N(0)w_N(t)$, $N = 16$, where $w_N(t)$ is the solution of (3.3) for $\eta = 1$, $\phi(t) = \cos \frac{\pi}{2} t$. Δ is the difference $x(t) - \beta^{16}(0)w_{16}(t)$.

TABLE I

t	x(t)	$\beta^{16}(0)w_{16}(t)$	$\Delta.10^{-5}$
0.0	1.	1.	0
0.2	0.90365	0.90365	0
0.4	0.59385	0.59384	1
0.6	0.04893	0.04890	3
0.8	-0.63787	-0.63791	4
1.0	-1.23865	-1.23869	4
1.2	-1.60894	-1.60897	3
1.4	-1.77853	-1.77855	2
1.6	-1.82899	-1.82901	2
1.8	-1.78650	-1.78653	3
2.0	-1.56826	-1.56829	3
2.2	-0.89005	-0.89006	1
2.4	0.40275	0.40279	-4
2.6	1.45999	1.46003	-4
2.8	1.85918	1.85920	-2
3.0	1.96289	1.96289	0
3.2	1.98664	1.98664	0
3.4	1.98925	1.98926	-1
3.6	1.97633	1.97634	-1
3.8	1.90934	1.90935	-1
4.0	1.61011	1.61013	-2
4.2	0.62214	0.62218	-4
4.4	-0.88263	-0.88258	-5
4.6	-1.70637	-1.70636	-1
4.8	-1.93456	-1.93456	0
5.0	-1.98423	-1.98422	-1

REFERENCES

[1] H.T. Banks, Approximation of nonlinear functional differential equation control systems, J. Optimization Theory Appl. 24(1979), 383-408.

[2] H.T. Banks, Identification of nonlinear delay systems using spline methods, to appear in Proc. Int. Conf. Nonl. Phenomena in Math. Sci., June 16-20, 1980, Arlington, Texas.

[3] H.T. Banks and J. A. Burns, An abstract framework for approximate solutions to optimal control problems governed by hereditary systems, in "International Conference on Differential Equations" (H. Antosiewicz, Ed.), Academic Press, New York 1975, pp.10-25.

[4] H.T. Banks and J.A. Burns, Hereditary control problems: Numerical methods based on averaging approximations, SIAM J. Control Optimization 16 (1978), 169-208.

[5] H.T. Banks and F. Kappel, Spline approximations for functional differential equations, J. Differential Eqs. 34(1979), 496-522.

[6] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff Int. Publ. 1976.

[7] H. Brezis and A. Pazy, Convergence and approximation of semigroups of nonlinear operators in Banach spaces, J. Funct. Analysis 9(1972), 63-74.

[8] M.G. Crandall and T.M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces, Am. J. Math. 93(1971), 265-298.

[9] F. Kappel, Approximation of neutral functional differential equations in the state space $R^n \times L^2$, to appear in "Colloquia Mathematica Societatis János Bolyai, Vol. 30: Qualitative Theory of Differential Equations", János Bolyai Math. Soc. and North Holland Publ. Comp.

[10] F. Kappel and K. Kunisch, Spline approximations for neutral functional differential equations, Inst. for Math., University of Graz, Preprint No. 5, July 1979.

[11] F. Kappel and W. Schappacher, Autonomous nonlinear functional differential equations and averaging approximations, Nonlinear Analysis: Theory, Methods and Appl. 2(1978), 391-422.

[12] F. Kappel and W. Schappacher, Non-linear functional differential equations and abstract integral equations, Proc. Royal Soc. Edinburgh 84A(1979), 71-91.

[13] R.D. Nussbaum, Uniqueness and nonuniqueness for periodic solutions of $x'(t) = -g(x(t-1))$, J. Differential Eqs. 34(1979), 25-54.

[14] M.H. Schultz, "Spline Analysis", Prentice-Hall, Englewood Cliffs, N.J., 1973.

[15] G.F. Webb, Functional differential equations and nonlinear semigroups in L^p-spaces, J. Differential Eqs. 20(1976), 71-89.