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MULTISTATE RELIABILITY MODELS: A SURVEY.(U)

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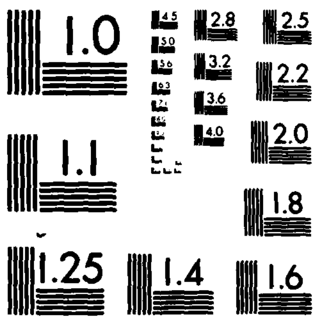

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MULTISTATE RELIABILITY MODELS: A SURVEY

by

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* Research sponsored by the Air Force Office of Scientific Research under Gra[redacted] AFOSR-76-305[redacted].

Abstract

The vast majority of reliability analyses assume that components and system are in either of two states: functioning or failed. In many real life situations one is capable of distinguishing between various "levels of performance" for both the system and its components. For such cases, the existing dichotomous model is a gross oversimplification of the real situation and models representing multistate systems and components are more adequate.

In the present paper, a survey is made of the recent papers which treat the more sophisticated and more realistic situations in which components and systems may assume many states ranging from perfect functioning to complete failure. The present survey updates and complements a previous survey given by El-Neweihi and Proschan.

1. Introduction

The theory of binary coherent systems has served as a unifying foundation for a mathematical and statistical theory of reliability. In such models systems and components are assumed to be in one of two states: functioning or failed. In many real-life situations, however, the systems and their components are capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. In order to describe more adequately the performance of such "degradable" systems and components we need to develop the theory of multistate coherent systems.

Until recently, little work has been done on this more general problem of multistate systems. However, a growing interest in this area is indicated by the increasing number of research papers written on this subject. In this paper a survey is made of the recent treatments of multistate models included in the work that has been performed by Barlow and Wu [2], Block and Savits [3], El-Neweihi, Proschan and Sethuraman [4], El-Neweihi [6], Griffith [7] and Ross [10]. This survey updates and complements a previous one given by El-Neweihi and Proschan [5].

We now summarize the contents of this paper. Our formulation and treatment are similar to that of Barlow and Proschan [1] for the binary case. In section 2 we present the notation and terminology used throughout the paper. In section 3 deterministic models of multistate systems are presented. For the system and for each of its components we can distinguish among different "levels of performance" represented by a totally ordered set S called the state space. The vector $\underline{x} = (x_1,\dots,x_n)$ representing the states of the n components takes its values in S^n, where S^n is the n^{th} cartesian power of S. The state of the system is determined by a function $\phi: S^n \longrightarrow S$. Various choices of state space S, and various definitions of the structure function ϕ presented in the different treatments of multistate models are then discussed and compared.

In section 4, the states of the n components are assumed to be random and are consequently represented by the random vector $\underline{X} = (X_1,\dots,X_n)$. The random variable $\phi(\underline{X})$ represents the state of the system itself. Stochastic relationship between the performance of the system and the performances of its components are studied. For instance, system performance is, as expected, a monotone increasing function of component performances. Bounds on system performance are provided when the exact value of system performance is difficult to determine.

Finally in section 5, we survey dynamic aspects of multistate system. The stochastic processes $\{X_i(t),\ t \geq 0\}$, $i = 1,\dots,n$ $\{\phi(\underline{X})),\ t \geq 0\}$ describe the states of the components and system at different points in time. Classes of decreasing stochastic processes generalizing know classes of life distributions are introduced. Closure of such classes under formation of multistate coherent systems are introduced.

2. Notation, Definitions and Terminology.

The vector $\underline{x} = (x_1,\ldots,x_n)$ denotes the vector of states of components $1,\ldots,n$.

$C = \{1,\ldots,n\}$ denotes the set of component indices.

$(j_i,\underline{x}) = (x_1,\ldots,x_{i-1},j,x_{i+1},\ldots,x_n)$, where $j = 0,1,\ldots,M$.

$(\cdot_i,\underline{x}) = (x_1,\ldots,x_{i-1},\cdot,x_{i+1},\ldots,x_n)$.

$\underline{j} = (j,\ldots,j)$.

$\underline{y} \le \underline{x}$ means that $y_i \le x_i$, $i = 1,\ldots,n$.

$\underline{y} < \underline{x}$ means that $y_i \le x_i$, $i = 1,\ldots,n$ and $y_i < x_i$ for some i.

$\underline{\alpha} = (\alpha_0,\alpha_1,\ldots,\alpha_M)$ is a <u>probability vector</u> means that $\alpha_j \ge 0$, $j = 0,\ldots,M$ and $\sum_{j=0}^{M} \alpha_j = 1$.

$\underline{\alpha} \overset{st}{\le} \underline{\alpha}'$ where both $\underline{\alpha}$, $\underline{\alpha}'$ are probability vectors means that $\sum_{j=\ell}^{M} \alpha_j \le \sum_{j=\ell}^{M} \alpha'_j$, $\ell = 0,1,\ldots,M$.

A subset $U \subset R^n$ is said to be an <u>upper set</u> if $\underline{x} \in A$ and $\underline{x} \le \underline{y}$ implies that $\underline{y} \in A$.

A subset $L \subset R^n$ is said to be a <u>lower set</u> if $\underline{x} \in L$ and $\underline{y} \le \underline{x}$ implies $\underline{y} \in L$.

$x \vee y$ denotes $\max(x,y)$.

$\underline{x} \vee \underline{y} = (x_1 \vee y_1,\ldots,x_n \vee y_n)$.

$x \wedge y$ denotes $\min(x,y)$.

$\underline{x} \wedge \underline{y} = (x_1 \wedge y_1,\ldots,x_n \wedge y_n)$.

"Increasing " is used in place of "nondecreasing" and "decreasing" is used in place of "nonincreasing". When we say $f(x_1,\ldots,x_n)$ is increasing we mean f is increasing in each argument.

Given a univariate distribution F, its complement $1-F$ is denoted by $\bar{F}$.

Given a set S, S^n denotes its $n^{\underline{th}}$ Cartesian power. As usual R denotes the set of real numbers.

3. Deterministic Models for Multistate Coherent Systems.

First let us recall the definition of a binary coherent system of n components. The vector $\underline{x} = (x_1,\ldots,x_n)$ represent the states of the n components where each x_i is either o or 1, $i = 1,\ldots,n$. The state of the system is determined by a structure function $\phi:\{o,1\}^n \longrightarrow \{o,1\}$. The structure function ϕ satisfies certain conditions that represent intuitively reasonable properties of systems encountered in practice. The following two conditions are required for a binary system to be a coherent structure [1,Def.2.1,p.6]:

(i) The function $\phi(\underline{x})$ is increasing.

(ii) For each i there exists a vector $(\cdot_i,\underline{x})$ such that $\phi(1_i,\underline{x}) \geq \phi(o_i,\underline{x})$. This means that the function ϕ is not constant in any of its arguments i, $i = 1,\ldots,n$.

Condition (i) expresses the reasonable assumption that improving component performances should not harm the system performance. Condition (ii) asserts that each component is relevant to the system performance, thus eliminating from consideration components which have no effect on system performance. It follows from (i) and (ii) that

(iii) $\phi(\underline{1}) = 1$ and $\phi(\underline{o}) = o$.

The binary model however, is an oversimplification in describing a situation in which both the system and its components are capable of a whole range of levels of performance varying from perfect functioning to total failure. For such case a larger state space S is needed to describe the situation more adequately. Also axioms defining <u>multistate</u> <u>structure</u> <u>functions</u> should also be presented to serve as a unifying foundation for a mathematical and statistical theory of reliability in the multistate case.

Most of the earlier treatments that dealt with multistate situations investigated only very special applications. (See for example Hirsch et al [8] and Postelnicu [9]).

More recent and more comprehensive research in multistate systems has been performed by Barlow and Wu [2], Block and Savits [3], El-Neweihi, Proschan and Sethuraman [4] (hereafter referred to as EPS [4]), Griffith [7] and Ross [10]. The definition given by Barlow and Wu [2] for the multistate structure is set-theoretical based on the concept of min path sets and min cut sets of binary coherent structures. Consider a system of n components. Assume that the state space for each of the components as well as for the system is the set $S = \{o,1,\ldots,M\}$, where o denotes the failed state and M denotes the perfect state. Let $P_1,\ldots,P_r$ be non-empty subsets of C such that $\bigcup_{j=1}^{r} P_j = C$ and $P_i \not\subset P_j$, $i \neq j$. The structure function $\phi: S^n \longrightarrow S$ is defined by

$$\phi(\underline{x}) = \max_{1\leq j\leq r} \min_{i\in P_j} x_i \tag{3.1}$$

where $\underline{x} \in S^n$ is the vector representing the states of components $1,2,\ldots,n$. In the binary case the structure function given in (3.1) is the most general coherent structure [1,Ch.1], and the sets $P_1,\ldots,P_r$ are called the min path sets of the system. Let ϕ' be the binary coherent structure associated with $P_1,\ldots,P_r$. The multistate coherent structure ϕ specified in (3.1) can then be expressed in terms of ϕ' as follows: For each $i = 1,\ldots,n$, let

$$y_{ij} = \begin{cases} 1 & \text{if } x_i \geq j \\ 0 & \text{o.w.} \end{cases}$$

and let $\underline{y}_j = (y_{1j},\ldots,y_{nj})$, $j = 0,1,\ldots,M$. It is fairly easy to see that $\phi(\underline{x}) \geq j$ iff $\phi'(\underline{y}_j) = 1$, and

$$\phi(\underline{x}) = \sum_{j=1}^{M} \phi'(\underline{y}_j). \tag{3.2}$$

Thus the multistate coherent structure given by Barlow and Wu [2], is very closely related to a corresponding binary coherent structure. Exploiting this relationship makes it easy to extend results from the binary case to the multistate case.

A more general approach has been taken by EPS [4] to define multistate coherent structures. The common state space for each of the components and the system is taken to be the set $S = \{0,1,\ldots,M\}$. The structure function $\phi : S^n \longrightarrow S$ is assumed to satisfy three conditions.

<u>Definition 3.1.</u> A system of n components is said to be a <u>multistate coherent system</u> (MCS) if its structure function ϕ satisfies:

(i)′ ϕ is increasing.

(ii)′ for level j and component i, there exists a vector $(\cdot_i, \underline{x})$ such that $\phi(j_i, \underline{x}) = j$ while $\phi(\ell_i, \underline{x}) \neq j$ for $\ell \neq j$, $i = 1,\ldots,n$ and $j = 0,\ldots,M$.

(iii)′ $\phi(\underline{j}) = j$ for $j = 0,1,\ldots,M$.

Note that conditions (i)′ and (ii)′ generalized conditions (i) and (ii) in the binary case. Condition (iii)′ is automatically satisfied in the binary case, but is not implied by (i)′ and (ii)′ in the present multistate case. The structure function specified in (3.1) is an MCS. However the class of MCS's is much larger than the one specified by (3.1).

In definition 3.1, condition (ii)′ is referred to as the relevance condition for the components of the system. This lends to a type of coherence which is called by Griffith [7], <u>strong coherence</u>. The following two successively weaker types of coherence have been introduced by Griffith [7]:

(ii)″ for any component i and state $j \geq 1$, there exists $\underline{x}$ such that $\phi((j-1)_i,\underline{x}) < \phi(j_i,\underline{x})$.

(ii)‴ for any component i, there exists $\underline{x}$ such that $\phi(o_i,\underline{x}) < \phi(M_i,\underline{x})$.

A structure function ϕ that satisfies (ii)″ is called <u>coherent</u>, and that which satisfies (ii)‴ is called <u>weakly coherent</u>. Let $\phi(\underline{x}) = \lfloor \frac{1}{n} \sum_{i=1}^{n} x_i \rfloor$, where $\lfloor o \rfloor$ is the greatest integer function. It is easy to verify that $\phi(\underline{x})$ is coherent but not strongly coherent. Also consider $\phi(\underline{x})$ defined by : $\phi(o,o) = o$, $\phi(1,o) = o$, $\phi(2,o) = 2$, $\phi(2,1) = 2$, $\phi(2,2) = 2$, $\phi(1,2) = 1$, $\phi(o,2) = 1$, $\phi(o,1) = 1$, $\phi(o,o) = o$, $\phi(1,1) = 1$. Then it easy to verify that ϕ is weakly coherent but not coherent. Thus the classes specified by (ii)′, (ii)″, and (ii)‴ are successively larger.

The definition given by Ross [10] for a multistate system is less structured than the ones given by Barlow and Wu [2], EPS [4] and Griffith [7]. The state space S is taken to be $[o,\infty)$ and the structure function ϕ is any increasing function from $[o,\infty)^n$ into $[o,\infty)$. Ross [10] concentrates on the stochastic properties of his model when observed either at a fixed point in time, or when observed at different points in time (dynamic models). Results of this type will be surveyed in the next two sections.

In the remainder of this section we present various structural properties of the multistate structures given by Barlow and Wu [2], EPS [4] and Griffith [7]. These properties extend well-known results in the binary case [1,Chapter 1] to the more general multistate case.

Theorem 3.1. Let ϕ be the structure function of a weakly coherent system of n components. Then

$$\min_{1\le i\le n} x_i \le \phi(\underline{x}) \le \max_{1\le i\le n} x_i \tag{3.3}$$

Theorem 3.1 states that a parallel system yields the best performance of a weakly coherent system, and a series system yields the worst performance.

The following lemma in EPS [4], gives a decomposition identity useful in carrying out inductive proofs. It holds for any multistate structure.

Lemma 3.1. The following identity holds for any n-component structure function ϕ:

$$\phi(\underline{x}) = \sum_{j=0}^{M} \phi(j_i,\underline{x}) I_{[x_i=j]}, \quad \text{for } i = 1,\dots,n \tag{3.4}$$

where

$$I_{[x_i=j]} = \begin{cases} 1 & \text{if } x_i = j \\ 0 & \text{o.w.,} \end{cases}$$

As in the binary case, one can define a dual structure for each multistate structure.

<u>Definition 3.2</u>. Let ϕ be the structure function of a multistate system. The dual structure function ϕ^D is given by:

$$\phi^D(\underline{x}) = M - \phi(M - x_1,\ldots,M - x_n). \qquad (3.5)$$

It is easy to verify that the dual inherits the same type of coherence possessed by the original structure.

Design engineers have used the well-known principle that redundancy at the component level is preferable to redundancy at the system level. This principle is presented by EPS [4] in mathematical form in (i) of the following theorem; (ii) is a dual result. Extension of this result to the class of coherent structures is given by Griffith [7].

<u>Theorem 3.2</u>. Let ϕ be the structure function of a coherent system. Then

(i) $\phi(\underline{x}\vee\underline{y}) \geq \phi(\underline{x}) \vee \phi(\underline{y})$.

(ii) $\phi(\underline{x}\wedge\underline{y}) \leq \phi(\underline{x}) \wedge \phi(\underline{y})$.

Equality holds in (i) ((ii)) for all $\underline{x}$ and all $\underline{y}$ iff the system is parallel (series).

Parts (i) and (ii) of theorem 3.2 are also proved by Barlow and Wu [2].

In binary coherent structures the concepts of minimal path vectors and minimal cut vectors play a crucial role. The analogue in MCS theory is the concept of critical connection vectors. This concept is defined by EPS [4] in the following:

<u>Definition 3.3</u>. A vector $\underline{x}$ is said to be a <u>connection vector to level j</u> if $\phi(\underline{x}) = j$, $j = 0,1,\ldots,M$.

<u>Definition 3.4.</u> A vector $\underline{x}$ is said to be an <u>upper critical conncection vector to level j</u> if $\phi(\underline{x}) = j$ and $\underline{y} < \underline{x}$ implies $\phi(\underline{y}) < j$, $j = 1,\ldots,M$.

A lower critical connection vector to level j can be defined in a dual manner, $j = 0,\ldots,M-1$.

The existence of such critical connection vectors is guaranteed by the conditions of definition 3.1.

For $j = 1,\ldots,M$, Let $\underline{y}_1^j,\ldots,\underline{y}_{n_j}^j$ be the upper critical connection vectors to level j, where $\underline{y}_r^j = (y_{1r}^j,\ldots,y_{nr}^j)$, $1 \leq r \leq n_j$. The following theorem by EPS [4], expresses the state of an MCS using its upper critical connection vectors.

<u>Theorem 3.3</u>. Let ϕ be the structure function of an MCS. Let $\underline{y}_1^j,\ldots,\underline{y}_{n_j}^j$ be its upper critical connection vectors to level j, $j = 1,\ldots,M$. Then $\phi(\underline{x}) \geq j$ iff $\underline{x} \geq y_\ell^t$ for some $j \leq t \leq M$ and some $1 \leq \ell \leq n_t$.

4. Stochastic Properties of Multistate Coherent Systems.

Having discussed some deterministic aspects of multistate systems, we now turn to the probabilistic aspects. In this section we survey important relationships between the stochastic performance of the system and the stochastic performances of its components which are assumed to be statistically independent.

Let X_i denotes the random state of component i, $i = 1,\ldots,n$. Let $\underline{X} = (X_1,\ldots,X_n)$ be the random vector representing the states of components $1,\ldots,n$. Then $\phi(\underline{X})$ is the random variable representing the state of the system. In the models described by Barlow and Wu [2], EPS [4] and Griffith [7], the random variables $X_1,\ldots,X_n$ and $\phi(\underline{X})$ assume their values in $S = \{o,1,\ldots,M\}$, with

$$\begin{aligned} P[X_i = j] &= p_{ij} , \qquad & P[\phi(\underline{X}) = j] &= p_j \\ P[X_i \leq j] &= P_i(j) , \qquad & P[\phi(\underline{X}) \leq j] &= P(j), \end{aligned} \tag{4.1}$$

for $j = o,1,\ldots,M$ and $i = 1,\ldots,n$. P_i (P) represents the performance distribution of component i (system). Clearly,

$$P_i(j) = \sum_{k=o}^{j} p_{ik} , \quad P_i(M) = 1,$$

for $i = 1,\ldots,n$. Similar relationships hold for P. Let $h = E(\phi(\underline{X}))$; we may express h as follows: $h = h(P_1,\ldots,P_n)$, since h is a function of the $P_1,\ldots,P_n$. Alternatively, we may express h as follow $h = h(\underline{p}_1,\ldots,\underline{p}_n)$, where $\underline{p}_i = (p_{io},\ldots,p_{iM})$ for $i = 1,\ldots,n$.

Using lemma 3.1, EPS [4], expresses the performance function h of a system of n components in terms of performance functions of system of $n-1$ components.

<u>Lemma 4.1.</u> The following identity holds for h:

$$h(\underline{p}_1,\ldots,\underline{p}_n) = \sum_{j=0}^{M} p_{ij} h(j_i,\underline{p}_1,\ldots,\underline{p}_n), \quad i = 1,\ldots,n, \quad (4.2)$$

where $h(j_i,\underline{p}_1,\ldots,\underline{p}_n) = E\,\phi(j_i,\underline{X})$.

The following theorem due to EPS [4] show that h is strictly increasing in each p_{ij}, $j > o$. This property generalizes the well known property of h in the binary case.

Theorem 4.1. Let $h(\underline{p}_1,\ldots,\underline{p}_n)$ be the performance function of an MCS. Let $o < p_{ij} < 1$ for $i = 1,\ldots,n$, $j = o,1,\ldots,M$. Then $h(\underline{p}_1,\ldots,\underline{p}_n)$ is strictly increasing in p_{ij}, $i = 1,\ldots,n$, $j = 1,\ldots,M$.

Properties of h as a function of $P_1,\ldots,P_n$ are also investigated by Barlow and Wu [2], EPS [4] and Griffith [7]. The following theorem due to EPS [4], shows that $h(P_1,\ldots,P_n)$ is increasing with respect to stochastic ordering. A similar result is proved by Barlow and Wu [2] for their subclass using a different proof. The same property is also proved by Ross [10] for his multistate model.

Theorem 4.2. Let P_i, P'_i be two performance distribution for component i, $i = 1,\ldots,n$. Assume $P_i(j) \geq P'_i(j)$ for $j = 0,\ldots,M$, $i = 1,\ldots,n$. Let P (P') be the corresponding system performance distribution. Then

$$\text{(i)} \quad P(j) \geq P'(j) \quad \text{for} \quad j = o,1,\ldots,M, \qquad (4.3)$$
$$\text{(ii)} \quad h(P_1,\ldots,P_n) \leq h(P'_1,\ldots,P'_n).$$

Griffith [7] shows that the above results hold for the classes of coherent and weakly coherent systems. He also introduces the concept of a utility function which is simply expressed as $Ef(\phi(\underline{X}))$ where f is a nonnegative increasing function representing a "reward" associated with various levels of performances.

The concept of upper connection critical vectors introduced by EPS [4] is exploited to obtain further bounds on P and h. Let $\underline{y}_1^j,\ldots,\underline{y}_{n_j}^j$ be the upper critical connection vectors to level j, $j = 1,\ldots,M$. Let A_r^j denote the event $[\underline{X} \geq \underline{y}_r^j]$, $r = 1,\ldots,n_j$. By Theorem 3.3

$$P[\phi(\underline{X}) \geq j] = P\left(\bigcup_{t=j}^{M} \bigcup_{r=1}^{n_t} A_r^t\right).$$

Now using the well known inclusion-exclusion principle the authors establish upper and lower bounds on $P[\phi(\underline{X}) \geq j]$. Note that $P(A_r^j) = P[\underline{X} \geq y_r^j] = \prod_{i=1}^{n} P[X_i \geq y_{ir}^j]$ for $1 \leq r \leq n_j$ and $j = 1,\ldots,M$.

An interesting generalization of the Moor-Shannon Theorem [1,Theorem 5.4] is obtained by Barlow and Wu [2]. In view of (3.2), it is easily verified that

$$P[\phi(\underline{X}) \geq j] = E\,\phi'(\underline{Y}_j) = h'(\underline{q}_j), \qquad (4.5)$$

where $\underline{q}_j = (q_{1j},\ldots,q_{nj})$ and $q_{ij} = \sum_{k=j}^{M} p_{ik}$, $i = 1,\ldots,n$.

Recall that Moore and Shannon show that if all components have the same reliability p, then either $h(\underline{p}) \geq p$ or $h(\underline{p}) \leq p$ for all $0 \leq p \leq 1$, or there exists $0 \leq p_0 < 1$ such that $h(\underline{p}) \leq p$ for $0 \leq p \leq p_0$, while $h(\underline{p}) \geq p$ for $1 \geq p \geq p_0$. Barlow and

Wu [2] give a natural generalization of this result to the multistate case with respect to stochastic ordering.

<u>Theorem 4.3.</u> Let $\underline{p}_i = \underline{\alpha} = (\alpha_o,\ldots,\alpha_M)$ for $i = 1,\ldots,n$. Assume $h'(p_o) = p_o$ $(0 < p_o < 1)$ Let $\underline{\alpha}^* = (1-p_o,o,\ldots,o,p_o)$. Then

$$\underline{\alpha} \overset{st}{\leq} \underline{\alpha}^* \quad \text{implies that} \quad \underline{p} \overset{st}{\leq} \underline{\alpha}\,,$$
$$\underline{\alpha} \overset{st}{\leq} \underline{\alpha}^* \quad \text{implies that} \quad \underline{p} \overset{st}{\geq} \underline{\alpha},$$

where $\underline{p} = (p_o,\ldots,p_M)$, $p_i = P[\phi(\underline{X}) = i]$, $i = o,\ldots,M$.
Note that (4.5) is central to the proof of the above theorem.

Finally, in the model proposed by Ross [10], X_i, $i = 1,\ldots,n$ and $\phi(\underline{X})$ are nonnegative random variables with distribution functions F_i, $i = 1,\ldots,n$, and F respectively. The function $r(\bar{F}_1,\ldots,\bar{F}_n)$ is defined by $r(F_1,\ldots,F_n) = E\,\phi(\underline{X})$.

Using an extension of Lemma 2.3, of Barlow and Proschan [1], Ross [10] proves the following:

<u>Theorem 4.4.</u> If ϕ is a binary increasing function then

$$r(F_1^{\alpha},\ldots,F^{\alpha}) \geq [r(F_1,\ldots,F_n)]^{\alpha} \tag{4.6}$$

for all $o \leq \alpha \leq 1$.

As a consequence of the above theorem, Ross [10] proves:

<u>Corollary 4.1</u>. Let $X_1,\ldots,X_n$ be independent IFRA random variables. Then

$$\sum_{i=1}^{n} X_i \text{ is IFRA} \qquad \text{(a)}$$

$$P\left\{\prod_{i=1}^{n} X_i > a\,\alpha^n\right\} \geq \left(P\left\{\prod_{i=1}^{n} X_i > a\right\}\right)^{\alpha}, \quad o \leq \alpha \leq 1 \qquad \text{(b)}$$

Observe that part (a) of Corollary 4.1 represents the well known property of the closure of the IFRA distribution under the convolution operation.

5. Dynamic Models for Multistate Coherent Systems.

In the binary reliability models, the length of time during which a component or system functions is called the lifelength of the component or system; these lifelengths are nonnegative random variables. Classes of lifelength distributions based on various notions of aging have been introduced and studied. See, e.g., [1]. Two of the important classes of life distributions are the increasing failure rate average (IFRA) class and the new better than used (NBU) class. Closure of these classes under basic reliability operations, such as convolution of distributions and formation of coherent systems, have been established. The counterparts of these concepts in the multistate case have been first investigated by Barlow and Wu [2], EPS [4], and Ross [10]. More recently, Block and Savits [3] and El-Neweihi [6] introduced general multivariate versions of these concepts.

Let $\{X_i(t), t \geq o\}$ denote the decreasing and right continuous stochastic process representing the state of component i at time

t, where t ranges over the nonnegative real numbers for $i = 1,\ldots,n$. The processes $\{X_i(t),\ t \geq o\}$, $i = 1,\ldots,n$, are assumed to be mutually independent. The stochastic process $\{\phi(\underline{X}(t)),\ t \geq o\}$ is also decreasing and right continuous and represents the corresponding system state as time varies, where $\underline{X}(t) = (X_1(t),\ldots,X_n(t))$, $t \geq o$.

In the model of Barlow and Wu [2] the state space is $\{o,1,\ldots,M\}$. Let us call $\{j,j+1,\ldots,M\}$ the "good" states. Assume that $[P(X_i(t) \geq j)]^{1/t}$ is decreasing in $t \geq o$ for fixed j, $i = 1,\ldots,n$. It is easily verified that $[P(\phi(\underline{X}(t)) \geq j)]^{1/t}$ is decreasing in $t \geq o$ for fixed j. Thus the above result states that if the length of time spent by each component in the "good" states is an IFRA random variable, then the corresponding length of time spent by the multistate system in the "good" states is also an IFRA random variable. In the binary case this represents the so-called IFRA closure (under formation of binary coherent systems) theorem.

The following definition is due to Ross [10].

<u>Definition 5.1</u>. The stochastic process $\{X(t),\ t \geq o\}$ is said to be an <u>IFRA process</u> if $T_a = \inf\{t : X(t) \leq a\}$ is an IFRA random variable for every $a \geq o$.

Having introduced this definition, Ross [10] then proves the following generalized IFRA closure theorem.

Theorem 5.1. Let $\{X_i(t), t \geq o\}$, $i = 1,\ldots,n$ be independent IFRA processes and ϕ a multistate structure function. Then $\{\phi(\underline{X}(t)), t \geq o\}$ is an IFRA process.

The crucial tool in proving the above theorem is theorem 4.4.

Ross [10] also defines an NBU process and proves a generalized NBU closure theorem (under formation of multistate structures).

Another definition of an NBU process is given by EPS [4], and then a simple characterization for this NBU process is derived. Using their characterization, they give a simple proof of a generalized NBU closure theorem. The EPS definition of an NBU process is as follows:

Definition 5.2. The stochastic process $\{X_i(t), t \geq o\}$ is an NBU process if $T_{i,j} = \inf\{t : X_i(t) \leq j\}$ is an NBU random variable for $j = o,\ldots,M$ and $i = 1,\ldots,n$.

Recall that the state space for the EPS [4] model is the set $\{o,\ldots,M\}$.

The following lemma gives a simple characterization of an NBU process.

Lemma 5.1. The stochastic process $\{X_i(t), t \geq o\}$ is NBU if and only if for all $s \geq o$, $t \geq o$,

$$X_i(s+t) \overset{st}{\leq} \min(X_i'(s), X_i'(t)),$$

where $X_i'(s)$ and $X_i'(t)$ are two independent random variables having the same distributions as $X_i(s)$, $X_i(t)$ respectively.

Using their lemma 5.1, EPS [4], prove the following generalized NBU closure theorem.

Theorem 5.2. Let ϕ be the structure function of an MCS having n components and $\{X_i(t), t \geq o\}$ be the $i^{\underline{th}}$ component performance process, $i = 1,\ldots,n$. Let $\{X_i(t), t \geq o\}$, $i = 1,\ldots,n$ be independent NBU processes. Then $\{\phi(\underline{X}(t), t \geq o\}$ is an NBU stochastic process.

The various generalizations that have been presented so far in this section have been obtained under the assumption that the components of the system are independent. However in many real life situations the components are subjected to common stresses which make them stochastically dependent. In recent papers by Block and Savits [3] and El-Neweihi [6], the authors introduce multivariate classes of stochastic processes that describe the joint performance of the n components of a system without insisting on the statistical independence of these components.

Now let $\{\underline{X}(t) = (X_1(t),\ldots,X_n(t)), t \geq o\}$ be a vector-valued stochastic process. Assume that $\underline{X}(t)$ is nonnegative, decreasing and right-continuous.

The following definition is due to Block and Savits [3].

Definition 5.3. $\{\underline{X}(t), t \geq 0\}$ is said to be a (vector-valued) IFRA process if and only if for every upper and open set $U \subset R^n$, the random variable

$$T_U = \inf\{t: \underline{X}(t) \not\in U\}$$

is IFRA.

Among several results, Block and Savits prove the following closure theorem

Theorem 5.3. If ϕ is a multistate monotone structure function and $\{\underline{X}(t), t \geq o\}$ is an IFRA process, then $\{\phi(\underline{X}(t)), t \geq o\}$ is an IFRA process.

The following definition is due to El-Neweihi [6].

Definition 5.4. The vector-valued stochastic process $\{\underline{X}(t), t \geq o\}$ is said to be MNBU process if and only if the random variable

$$T_C = \inf\{t: \underline{X}(t) \in C\}$$

is NBU, for every lower closed set $C \subset R^n$.

The following generalized NBU closure theorem is then proved by El-Neweihi [6].

Theorem 5.4. Let $\{\underline{X}(t), t \geq 0\}$ be MNBU process. Let ϕ be decreasing left-continuous and nonnegative function. Then $\{\phi(\underline{X}(t)), t \geq o\}$ is NBU process.

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