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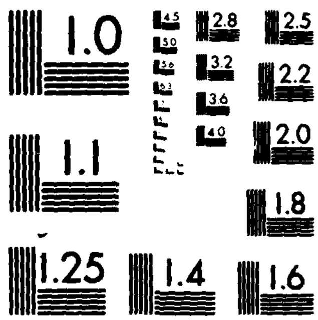

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STOCHASTIC APPROXIMATION WITH DISCONTINUOUS DYNAMICS and STATE DEPENDENT NOISE: w.p.1 CONVERGENCE*

by

Harold J. Kushner
Divisions of Applied Mathematics & Engineering
Lefschetz Center for Dynamical Systems
Brown University
Providence, R. I. 02912

July 10, 1980

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Stochastic approximations of the form $X_{n+1} = X_N + a_n h(X_n, \xi_n)$ are treated where $h(\cdot,\cdot)$ might not be continuous and the noise sequence $\{\xi_n\}$ might depend on $\{X_n\}$. An "averaging" and an "ordinary differential equation" method are combined to get w.p.1 convergence for both the above algorithm and for the case where the iterates are projected back onto a bounded set G if they ever leave it. Two examples are developed, the first being an

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automata problem where the dynamics are not smooth and the noise is state dependent, and the second a Robbins-Monro process with observation averaging (which causes the noise to be state dependent). Each examples is typical of a larger class.

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ABSTRACT

Stochastic approximations of the form $X_{n+1} = X_n + a_n h(X_n, \xi_n)$ are treated where $h(\cdot,\cdot)$ might not be continuous and the noise sequence $\{\xi_n\}$ might depend on $\{X_n\}$. An 'averaging' and an 'ordinary differential equation' method are combined to get w.p.1 convergence for both the above algorithm and for the case where the interates are projected back onto a bounded set G if they ever leave it. Two examples are developed, the first being an automata problem where the dynamics are not smooth and the noise is state dependent, and the second a Robbins-Monro process with observation averaging (which causes the noise to be state dependent). Each example is typical of a larger class.

1. Introduction.

References [1], [2] present a collection of fairly general methods for proving w.p.1 and weak convergence results for stochastic approximations of the type

$$(1.1) \qquad X_{n+1} = X_n + a_n h(X_n,\xi_n), \quad X_n \in R^r, \text{ Euclidean r-space,}$$

where $\{\xi_n\}$ is a sequence of random variables and $0 < a_n \to 0$, $\Sigma a_n = \infty$. Also, several stochastic approximation schemes for sequential monte carlo function minimization or equation solving under equality and inequality constraints were dealt with. One, among others, is the projection method. Let $q_1,\ldots,q_m$ denote continuously differentiable functions, define $G = \{x : q_i(x) \leq 0, i=1,\ldots,m\}$, then the algorithm is

$$(1.2) \qquad X_{n+1} = \pi_G[X_n + a_n h(X_n,\xi_n)]$$

where $\pi_G(y)$ denotes the closest point on G to y. Both weak convergence and w.p.1 results were proved for this and several other 'constrained' algorithms.

If $h(x,\xi)$ is not additive in ξ, then the methods in [1] (and also in [3], which deals with related algorithms, at least for the unconstrained case) require that $h(\cdot,\cdot)$ be continuous. In many applications, $h(\cdot,\cdot)$ is not continuous (e.g., $h(\cdot,\cdot)$ might be an indicator function). Here, we combine some of the basic ideas from [1] together with the averaging methods of [4],

[5] to develop an alternative method which is more convenient when $h(\cdot,\cdot)$ is not smooth, and which is often quite advantageous if $\{\xi_n\}$ is state dependent. We rely on the assumption that even if $h(\cdot,\cdot)$ is not smooth, expectations or conditional expectations of the types $Eh(x,\xi_n)$, $E[h(x,\xi_n)|\xi_{n-1},\xi_{n-2},\ldots]$ are smooth functions of x. This situation occurs in many examples. Reference [6] also makes such an assumption for non-smooth $h(\cdot,\cdot)$, but deals with $a_n \equiv a > 0$, and a finite time interval $[n:an \le T]$.

In Sections 2,3, respectively, we treat the case (1.1), (1.2), respectively, and where $\{\xi_n\}$ is bounded and not state dependent. Section 4 deals with the case of state dependent $\{\xi_n\}$ and the 'unbounded' noise case is briefly discussed. The convergence is w.p.1 in all cases. Two interesting classes of examples appear in Sections 5 and 6.

2. The algorithm (1.1).

Assumptions. E_n denotes expection conditioned on $\{\xi_j, j < n\}$. K denotes a constant whose value might change from usage to usage and δX_n denotes $X_{n+1} - X_n$.

A1. $\Sigma a_n^2 < \infty$, $\Sigma a_n = \infty$, $\{a_{n+1}/a_n\}$ is bounded, $h(\cdot,\cdot)$ is measurable and $h(x,\cdot)$ is bounded uniformly on bounded x-sets. $\{\xi_n\}$ is uniformly bounded.

A2. There is a twice continuously differentiable Liapunov function $0 \le V(x)$ such that $|V_{xx}(\cdot)|$ is bounded, $V(x) \to \infty$ as $|x| \to \infty$ and for some $\varepsilon_0 > 0$ and compact set Q_0 of the form $\{x:V(x) \le \lambda_0\}$, $V_x'(x)\bar{h}(x) < -\varepsilon_0$ for $x \notin Q_0$, where $\bar{h}(\cdot)$ is defined in (A3).

A3. There is a continuously differentiable function $\bar{h}(\cdot)$ and a null set N_0 such that for each n and x and $\omega \notin N_0$, the function defined by

$$V_0(x,n) \equiv \sum_{j=n}^{\infty} a_j V_x'(x) E_n[h(x, \xi_j) - \bar{h}(x)],$$

is bounded by $Ka_n(1+|V_x'(x)\bar{h}(x)|)$ where the convergence for $V_0(x,n)$ and for all infinite sums of the sequel is in the sense $\lim_N \sum_n^N a_j[\;]$ for each x, and where the sequence of partial sums is bounded uniformly on compact x-sets.

A4. $E_n|h(x,\xi_j)|^2 \le K(1+|V_x'(x)\bar{h}(x)|)$, $j \ge n$

A5. $|V_x'(x)\bar{h}(x)| \le K(1+V(x))$

A6. Let $[\;]_x$ denote the gradient here. Then

$$\Big|\sum_{j=n+1}^{\infty} a_j [V_x'(x)E_{n+1}(h(x,\xi_j)-\bar{h}(x))]_x\Big| \le Ka_n(1+|V_x'(x)\bar{h}(x)|^{1/2})$$

A7. For $0 \le s \le 1$

$$E_n|V_x'(x+sa_nh(x,\xi_n))\bar{h}(x+sa_nh(x,\xi_n))| \le K(1+|V_x'(x)\bar{h}(x)|).$$

The examples show that the assumptions are often not restrictive.

Let $X^0(\cdot)$ denote the continuous piecewise linear function which equals X_0 on $[-\infty,0]$, X_n, $n \ge 0$, at $t_n \equiv \sum_{i=0}^{n-1} a_i$ and in each (t_n,t_{n+1}) is a linear interpolation of X_n and X_{n+1}.

Define $X^n(\cdot)$ by $X^n(t) = X^0(t+t_n)$. Note that $X^n(0) = X^0(t_n) = X_n$, and define $m(t) = \max\{n : t_n \leq t\}$ for $t \geq 0$ and $m(t) = 0$ for $t < 0$.

Theorem 1. Assume (A1)-(A7). Then $\{X_n\}$ is bounded w.p.1. If $V_x'(x)\bar{h}(x) \leq 0$ for all x, then $X_n \to \{x : V_x'(x)\bar{h}(x) = 0\}$ w.p.1. In general, $\{X_n\}$ converges w.p.1 to the largest bounded invariant set of

$$\dot{x} = \bar{h}(x). \tag{2.1}$$

If $x_0 \equiv x(t)$ is an asymptotically stable solution of (2.1) (in the sense of Liapunov) with domain of attraction $DA(x_0)$, and if $X_n \in$ compact $A \subset DA(x_0)$ infinitely often, then (except for ω in a null set) $X_n \to x_0$ as $n \to \infty$.

Proof. We have

$$E_n V(X_{n+1}) - V(X_n) = a_n V_x'(X_n) E_n h(X_n, \xi_n) + \frac{a_n^2}{2} \int_0^1 E_n h'(X_n, \xi_n) V_{xx}(X_n + s\delta X_n) h(X_n, \xi_n) ds. \tag{2.2}$$

Also

$$E_n V_0(X_{n+1},n+1) - V_0(X_n,n) =$$

$$(2.3)\qquad E_n \sum_{n+1}^{\infty} a_j V_x'(X_{n+1}) E_{n+1}[h(X_{n+1},\xi_j) - \bar{h}(X_{n+1})]$$

$$- \sum_{n+1}^{\infty} a_j V_x'(X_n) E_n[h(X_n,\xi_j) - \bar{h}(X_n)]$$

$$- a_n V_x'(X_n)[E_n h(X_n,\xi_n) - \bar{h}(X_n)],$$

which equals

(2.4) last line of (2.3) +

$$a_n E_n h'(X_n,\xi_n) \int_0^1 \sum_{j=n+1}^{\infty} a_j [E_{n+1} V_x'(X_n+s\delta X_n)(h(X_n+s\delta X_n,\xi_j) - \bar{h}(X_n+s\delta X_n))]_x .$$

The last term in (2.4) is bounded by $O(a_n^2)O(1+|V_x'(X_n)\bar{h}(X_n)|)$. Define $\tilde{V}(n) = V(X_n) + V_0(X_n,n)$. Then, by the above calculations,

$$(2.5)\qquad E_n\tilde{V}(n+1) - \tilde{V}(n) = a_n(1+a_n\varepsilon_n)V_x'(X_n)\bar{h}(X_n) + \tilde{\varepsilon}_n a_n^2,$$

where $\{\varepsilon_n\},\{\tilde{\varepsilon}_n\}$ are sequences of uniformly bounded random variables. Thus we can write

$$(2.6)\qquad \tilde{V}(n) - \sum_{i=0}^{n-1} a_i(1+a_i\varepsilon_i)V_x'(X_i)\bar{h}(X_i) - \sum_{i=0}^{n-1}\tilde{\varepsilon}_i a_i^2 \equiv \sum_{i=0}^{n-1} m_i \equiv M_n,$$

where (2.6) defines m_i, M_n, and $\{M_n\}$ is a martingale. Note that

$$m_n = \tilde{V}(n+1) - \tilde{V}(n) - a_n(1+a_n\varepsilon_n)V_x'(X_n)\bar{h}(X_n) - \tilde{\varepsilon}_n a_n^2.$$

Define $W(n) = \tilde{V}(n) + E_n \sum_{j=n}^{\infty} \tilde{\varepsilon}_i a_i^2$ and note that $W(n) \geq -O(a_n)$ for large n by (A3), (A5).

Let n_0 be a stopping time such that $X_{n_0} \notin Q_0$ and define $n_1 = \min\{n : n > n_0, X_{n_1} \in Q_0\}$. Then $\{\tilde{W}(n) = W(n \cap n_1), n \geq n_0\}$ is a super martingale bounded below by $-O(a_n)$, and $E_n W(n+1) - W(n) \leq -\varepsilon_0 a_n/2$ if $X_n \notin Q_0$ and n is large. This implies that Q_0 is a recurrence set; i.e., $X_n \in Q_0$ for infinitely many n w.p.1. Let $\lambda_1 > \lambda_0$ and define $Q_1 = \{x: V(x) \leq \lambda_1\}$. For each such Q_1 there is a real $K(Q_1)$ such that $|m_n|^2 \leq K(Q_1)a_n^2$ if $X_n \in Q_1$. Define $n_2 = \min\{n: X_n \notin Q_1, n \geq n_0\}$. Then

$$\text{(2.7)} \qquad P\{\sup_{n_0 \leq n < n_2} |\sum_{i=n_0}^{n} m_i| \geq \varepsilon\} \leq K(Q_1) E_{n_0} \sum_{i=n_0}^{n_2-1} a_i^2/\varepsilon^2.$$

From the above part of this paragraph and the fact that $V_x'(x)\bar{h}(x) \leq -\varepsilon_0$ for $x \notin Q_0$ and the boundedness of $|h(x,\xi)|$, $x \in Q_1$, we conclude that eventually (w.p.1) X_n stays in Q_1 (for any $\lambda_1 > \lambda_0$). Also,

$$\text{(2.8a)} \quad \sup_{m \geq n} \left|V(X_m) - V(X_n) - \sum_{i=n}^{m-1} a_i(1+a_i\varepsilon_i)V_x'(X_i)\bar{h}(X_i)\right| \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty$$

or, equivalently, using $m(t_n) = n$,

$$\text{(2.8b)} \quad \sup_{s \geq 0} \left|V(X^n(s)) - V(X^n(o)) - \sum_{i=n}^{m(t_n+s)-1} a_i(1+a_i\varepsilon_i)V_x'(X_i)\bar{h}(X_i)\right| \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

Let Ω_1 = {set of non-recurrence of Q_0} ∪ {set of non-convergence of Σm_n}. By the w.p.1 boundedness of $\{X_n\}$, $X^0(\cdot)$ is uniformly continuous for $\omega \notin$ a null set Ω_2. Fix $\omega \notin \Omega_1 \cup \Omega_2 = \Omega_0$. Via the Arzelà-Ascoli Theorem, pick a convergent subsequence (converging uniformly on bounded intervals) of $\{X^n(\cdot)\}$, with limit $X(\cdot)$. Then

$$(2.9) \qquad V(X(t)) = V(X(0)) + \int_0^t V_x'(X(s))\bar{h}(X(s))ds.$$

Equation (2.8) implies that if $V_x'(x)\bar{h}(x) \leq 0$ for all x, then $X_n \to S_0 = \{x: V_x'(x)\bar{h}(x) = 0\}$ w.p.1 as $n \to \infty$.

Next, let $f(\cdot)$ be a real valued function on R^r with compact support and continuous second derivatives. With $f(\cdot)$ replacing $V(\cdot)$, define $f_0(x,n), \tilde{f}(n)$ as $V_0(x,n), \tilde{v}(n)$ were defined. Then (2.8) holds for $f(x)$ replacing $V(\cdot)$. By choosing $f(\cdot)$ such that $f(x) = x^i$, $i=1,\ldots,r$, in the set Q_1, where x^i is the i^{th} component of x, we see there is a bounded sequence $\{\hat{\varepsilon}_n\}$ such that

$$(2.10) \qquad \sup_{s \geq 0} \left| X^n(s) - X^n(0) - \sum_{i=n}^{m(t_n+s)-1} a_i(1+a_i\hat{\varepsilon}_i)\bar{h}(X_i) \right| \to 0 \text{ w.p.1. as } n \to \infty.$$

Thus any limit $X(\cdot)$ of $\{X^n(\cdot)\}$ must satisfy (2.1) and the possible limit points of $\{X_n\}$ are contained w.p.1 in the largest bounded invariant set of (2.1). The assertion concerning asymptotically stable $x(t) \equiv x_0$ is now readily proved (see, e.g., proof of Theorem (2.3.1) of [1]), and the details are omitted. Q.E.D.

3. The Projection Method.

Let G be as defined in Section 1. For the continuous vector field $\bar{h}(\cdot)$ define $\bar{\pi}(\bar{h}(x))$ = projection of $\bar{h}(x)$ onto G; i.e., $\bar{\pi}(\bar{h}(x)) = \lim_{\Delta \downarrow 0} [\pi_G(x+\Delta\bar{h}(x)) - x]/\Delta$. The limit need not be unique. We will need

(A8). (A3) and (A6) hold, but with V_x dropped and the right sides $O(a_n)$.

(A9) $q_i(\cdot)$, $i = 1,\ldots,m$, are continuously differentiable, G is bounded and is the closure of its interior $G^0 = G - \partial G = \{x: q_i(x) < 0, i = 1,\ldots,m\}$, at each $x \in \partial G$, the gradients of the active constraints are linearly independent.

Theorem 2. Assume (A1), (A8), (A9). Then $\{X^0(\cdot)\}$ is uniformly continuous on $[0,\infty]$. There is a null set Ω_0 such that for $\omega \notin \Omega_0$ any limit $X(\cdot)$ of a convergent (uniformly on bounded intervals) subsequence of $\{X^n(\cdot)\}$ satisfies

$$\dot{x} = \bar{\pi}(\bar{h}(x)). \tag{3.1}$$

If $\{X_n\} \subset$ compact $A \subset DA(x_0)$ infinitely often and $\omega \notin \Omega_0$, and $x_0 = x(t)$ is an asymptotically stable point of (3.1), then $X_n \to x_0$ w.p.1. Let $H(\cdot) \geq 0$ be a real valued function whose second mixed partial derivatives are continuous and $\bar{h}(x) = -H_x(x)$. Define KT = set of points where $\bar{h}'(x)\bar{\pi}(\bar{h}(x)) = 0$, and suppose that $KT = \bigcup_{i=1}^{\ell} S_i$, where the S_i are disjoint, closed and such that $H(x)$ is constant on each S_i. Then $X_n \to KT$ w.p.1 as $n \to \infty$.

Proof. The proof is very similar to that of Theorem 1. Let $f(\cdot)$ be an arbitrary real valued function on R^r with continuous second partial derivatives. Then

$$E_n f(X_{n+1}) - f(X_n) = a_n f_x'(X_n) E_n h(X_n,\xi_n) + a_n f_x'(X_n) E_n \tau_n$$
$$+ \frac{a_n^2}{2} \int_0^1 E_n (\delta X_n/a_n)' f_{xx}(X_n + s\delta X_n)(\delta X_n/a_n) ds,$$

where $\tau_n = [\pi_G(X_n + a_n h(X_n,\xi_n)) - (X_n + a_n h(X_n,\xi_n))]/a_n = O(1)$. Note that there is a K such that $\tau_n = 0$ if distance$(X_n, \partial G) \geq Ka_n$ and that τ_n lies in the cone $C(X_{n+1}) = \{y: q_{i,x}'(X_{n+1})y \leq 0$ for $i: q_i(X_{n+1}) = 0\}$.

Define $f_0(x,n)$ by

$$f_0(x,n) = \sum_{j=n}^{\infty} a_j f_x'(x) E_n [h(x,\xi_j) - \bar{h}(x)]$$

and set $\tilde{f}(n) = f(X_n) + f_0(X_n,n)$. There is a bounded sequence ε_i such that

$$E_n \tilde{f}(n+1) - \tilde{f}(n) - \varepsilon_n a_n^2 - a_n f_x'(X_n)\bar{h}(X_n) - a_n f_x'(X_n) E_n \tau_n = 0,$$

$$\tilde{f}(n) - \tilde{f}(0) - \sum_{i=0}^{n-1} \varepsilon_i a_i^2 - \sum_{i=0}^{n-1} a_i f_x'(X_i)\bar{h}(X_i) - \sum_{i=0}^{n-1} a_i f_x'(X_i)\tau_i \equiv \sum_{i=0}^{n-1} m_i \equiv M_n,$$

where $\{M_n\}$ is a martingale and $|m_i|^2 \leq Ka_i^2$. As in Theorem 1,

$$(3.2) \quad \sup_{s \geq 0} \left| f(X^n(s)) - f(X^n(0)) - \sum_{i=n}^{m(t_n+s)-1} a_i f_x'(X_i)\bar{h}(X_i) - \sum_{i=n}^{m(t_n+s)-1} a_i f_x'(X_i)\tau_i \right| \to 0$$

w.p.1 as $n \to \infty$,

from which follows

$$\text{(3.3)} \qquad \sup_{s\geq 0}\left|X^n(s) - X^n(0) - \sum_{i=n}^{m(t_n+s)-1} a_i\bar{h}(X_i) - \sum_{i=n}^{m(t_n+s)-1} a_i\tau_i\right| \to 0$$

w.p.1 as $n \to \infty$.

Also, $\{X^n(\cdot)\}$ is equicontinuous, since $h(\cdot,\cdot)$ is bounded.

Let Ω_0 denote the set of nonconvergence in (3.3) and for fixed $\omega \notin \Omega_0$, extract a convergent subsequence of $\{X^n(\cdot)\}$ (uniformly on bounded intervals) with limit denoted by $X(\cdot)$. Define $\bar{h}_0(x) = \bar{\pi}(\bar{h}(x))$ and $\bar{h}_1(x) = \bar{h}(x) - \bar{h}_0(x)$. Then, by (3.3) there is a bounded R^r-valued measurable function $\tau(\cdot)$ such that $\tau(s) = 0$ unless $X(s) \in \partial G$, and if $X(s) \in \partial G$ then $\tau(s)$ is in the cone $C(X(s))$ and (3.4) holds.

$$\text{(3.4)} \qquad \begin{aligned} X(t) &= X(0) + \int_0^t \bar{h}(X(s))ds + \int_0^t \tau(s)ds \\ &= X(0) + \int_0^t \bar{h}_0(X(s))ds + \int_0^t \bar{h}_1(X(s))ds + \int_0^t \tau(s)ds, \end{aligned}$$

The last two integrals on the right of (3.4) must cancel if $X(t)$ is to remain in G for all t. Thus (3.1) holds w.p.1.

If $\bar{h}(x) = -H_x(x)$, then use $H(\cdot)$ as a Liapunov function for (3.1) to get

$$\text{(3.5)} \qquad \dot{H}(x) = H_x(x)\bar{\pi}(-H_x(x)) \leq 0,$$

from which we see that $X(t) \to KT$ as $t \to \infty$. Thus, for each $\varepsilon > 0$, $\{X_n\}$ is in an ε neighborhood $N_\varepsilon(KT)$ of KT infinitely often w.p.1. Fix $\varepsilon > 0$. Define $H_1 = \underline{\lim}_n H(X_n)$. Suppose that S_1 and $\hat{\Omega}_1$ are such that H_1 = value of $H(x)$ on S_1 if $\omega \in \hat{\Omega}_1$ and $P\{\hat{\Omega}_1\} > 0$, and for some $\varepsilon_1 > \varepsilon > 0$, $\{X_n\}$ leaves the ε_1-neighborhood $N_{\varepsilon_1}(S_1)$ infinitely often for $\omega \in \hat{\Omega}_1$. Then for (almost all) $\omega \in \hat{\Omega}_1$, there are real numbers $\ell_n \to \infty$ and $k_n \geq K_0 > 0$ with $k_n \to T \leq \infty$ and a solution $X(\cdot)$ to (3.1) which is a limit of the sequence $\{X^0(\ell_n+s),\ s \leq k_n,\ n = 1,2,\ldots\}$ and where $X(0) \in \partial N_\varepsilon(S_1)$ and either $X(t) \in \partial N_{\varepsilon_1}(S_1)$ if $T < \infty$ or else $X(t) \to \partial N_{\varepsilon_1}(S_1)$ as $t \to \infty$. Using an argument like that used in [1], Theorem 2.3.5, the last sentence and (3.5) imply that $H_1 \neq \underline{\lim}_n H(X_n)$ almost everywhere on $\hat{\Omega}_1$, a contradiction. The next to the last assertion of the theorem is proved in a similar way. Q.E.D.

4. State Dependent and Unbounded Noise

State Dependent and Bounded Noise

There are several ways in which the state dependent and bounded noise case can be treated. The noise can be parameterized as in [4], Section 9. Here, we choose a Markovian representation. Suppose that $\{\xi_{n-1}, X_n\}$ is a Markov process. In applications, this might require an augmentation of the state space of the 'original' $\{\xi_n\}$ and a redefinition of the 'original' $h(\cdot,\cdot)$. Let E_n denote conditioning on ξ_j, $j < n$, X_j, $j \leq n$, and define the 'partial' transition function

$$P(\xi,\alpha,\Gamma|x) = P\{\xi_{n+\alpha-1} \in \Gamma | X_n = x, \xi_{n-1} = \xi\}.$$

It is supposed that P does not depend on n, for notational simplicity only.

Write $V_0(x,n)$ in the form

$$(4.1)\quad V_0(x,n) = V_x'(x)\sum_{j=n}^{\infty} a_j[\int h(x,\xi)P(\xi_{n-1},j-n+1,d\xi|x) - \bar{h}(x)].$$

Note that $E_nP(\xi_n,j-n,\Gamma|X_n) = P(\xi_{n-1},j-n+1,\Gamma|X_n)$ by the Markov property. Assume that the sum in (4.1) is continuously differentiable in x, and that the derivatives can be taken termwise and that (replacing A6))

$$(4.2)\quad |\sum_{j=n+1}^{\infty} a_j\,[V_x'(x)\ \{\int h(x,\xi)P(\xi_n,j-n,d\xi|x) - \bar{h}(x)\}]_x| \le Ka_n(1+|V_x'(x)\bar{h}(x)|^{1/2}).$$

<u>Theorem 3</u>. <u>Assume</u> (A1)-(A7) <u>but with</u> (4.1), (4.2) <u>replacing</u> (A3), (A6), <u>resp</u>. <u>and</u> (A4) <u>replaced by</u>

$$\int |h(x,\xi)|^2 P(\xi_{n-1},j-n,d\xi|x) \le K(1+|V_x'(x)\bar{h}(x)|),\ j > n.$$

<u>Then the conclusions of Theorem 1 hold</u>.

<u>Assume</u> (A1), (A8), (A9) <u>but with the modifications of</u> (A3), (A6) <u>stated above</u>. <u>Then the conclusions of Theorem 2 continue to hold</u>.

Remark on the proof. In the proof the difference (4.3) occurs,

$$(4.3)\qquad \sum_{j=n+1}^{\infty} E_n a_j V_x'(X_{n+1})\left[\int h(X_{n+1},\xi)P(\xi_n,j-n,d\xi\,|\,X_{n+1}) - \bar{h}(X_{n+1})\right]$$

$$- \sum_{j=n+1}^{\infty} a_j V_x'(X_n)\left[\int h(X_n,\xi)P(\xi_{n-1},j-n+1,d\xi\,|\,X_n) - \bar{h}(X_n)\right].$$

Using the differentiability and the equality below (4.1) and the bounds from (A1) - (A7) (modified for Theorem 3), (4.3) can be seen to be of the order of $a_n^2(1+|V_x'(X_n)\bar{h}(X_n)|)$.

The proof of Theorem 3 is the same as those of Theorems 1 and 2.

Unbounded noise

We state a generalization of Theorem 1 for the case where $\{\xi_n\}$ is unbounded. First, make the following alterations in the assumptions. Drop the boundedness of $\{\xi_n\}$ in (A1) and suppose that there are $K_0 < \infty$ and $\beta_n \geq 0$, $\gamma_n \geq 0$ such that $\sup_n(E\beta_n+E\gamma_n) < \infty$, $a_n\beta_n + a_n\gamma_n \to 0$ w.p.1 as $n \to \infty$ and $|h(x,\xi_n)| \leq K_0\beta_n$ for $x \in Q_0$ and A3,4 hold with K replaced by $K\gamma_n$. An additional assumption is required. (A6) and (A7) were used in Theorem 1 to get the bound (below (2.4)) on (2.4). We require that the bound hold with $O(a_n^2)$ replaced by $\gamma_n O(a_n^2)$. This is, perhaps, an awkward way of stating the assumption, but it can be verified in many standard examples. For an alternative condition see the remark after the example. We now have

Theorem 4. Under the conditions of Theorem 1, altered as above, the conclusions of Theorem 1 continue tc hold.

The proof is very similar to that of Theorem 1; with only a few changes requires; e.g., $a_n \varepsilon_n$ is replaced by a $\delta_n \to 0$ w.p.1 and $\tilde{W}(n) \geq -\tilde{\delta}_n \to 0$ w.p.1 as $n \to \infty$. There is an analogous result for the cases of Theorem 2.

Example. Let $\{\xi_n\}$ be stationary and Markov and $h(x,\xi) = \bar{h}(x) + h_0'(x)g(\xi)$, where $Eg(\xi_n) = 0$, $Eg^2(\xi_n) < \infty$. Here γ_n is a function of ξ_{n-1} and β_n is a function of ξ_n. Such a form occurs in applications to the identification and adaptive control of linear systems, where $\bar{h}$ and h_0 are affine functions of x. Then, Theorem 1 holds under a simple stability condition on $\dot{x} = \bar{h}(x)$, and on reasonable conditions on $\{\xi_n\}$. A standard and important special case occurs in the identification problem for linear systems where we use $\xi_n = L_1\hat{\xi}_n$, $y_n = L_2\hat{\xi}_n$, $\{\hat{\xi}_n\}$ Markov and

$$X_{n+1} = X_n - a_n\psi_n[\psi_n' X_n - y_n],$$

$\psi_n \in R^n$, $y_n \in R$.

Remark on Theorem 3. The 'unbounded noise' analog of Theorem 3 also holds under the conditions of Theorem 3, modified as follows.

(A4) is replaced by the expression in the statement of Theorem 3, but with K replaced by $K\gamma_n$. (4.1) is used for $V_0(x,n)$ and the K there is replaced by $K\gamma_n$. As an alternative to (A6), (A7), assume that

$$(4.4) \quad E_n|\text{left hand side of } (4.2)|^2 \le \gamma_n K(1+|V_x'(x)\bar{h}(x)|),$$

where x is replaced by $x + sa_n h(x,\xi_n)$, $s \in [0,1]$, in evaluating (4.4). Then under the conditions on β_n, γ_n, K_0, $h(x,\xi_n)$ in the paragraph above Theorem 4, the conclusions of the first paragraph of Theorem 3 continue to hold. There is a similar extension of the second paragraph of Theorem 3.

The following two classes of examples have state dependent noise and they illustrate two different ways of using Theorem 3.

5. A Learning Automata Example.

This example is a modification of one in [5], where $a_n \equiv \varepsilon > 0$ and an extensive development of the asymptotic distributional properties is given. Here we are concerned with w.p.1 convergence only for the case where $a_n \to 0$. A relatively simple case is treated. Clearly, more complicated arrival and adaptive processes and systems can be treated.

The problem. Calls arrive at a switching terminal at random at time instants $n = 0,1,2,\ldots$, with P{one call arrives at n^{th} instant} = $\mu \in (0,1)$, P{>1 call arrives at n^{th} instant} = 0. There are two possible routings to the destination, routes i, $i = 1,2$, where route i has N_i independent lines - and can handle up to N_i calls simultaneously. Let $[n,n+1)$ denote the n^{th} interval of time. The duration of each call has the distribution: P{call completed in the $(n+1)^{st}$ interval|uncompleted at end of n^{th} interval, route i used} = $\lambda_i \in (0,1)$. The members of the sequence of interarrival times and call durations are mutually independent. The use of an adaptive automaton for adjusting the routing comes from [7].

The routing automaton operates as follows. Let $\{X_n\}$ denote a sequence of random variables - with values in $[0,1]$. In order to have an unambiguous sequencing of events, let the calls ending in the n^{th} interval actually end at time $n + \frac{1}{2}$, and let both arrivals and route assignments be at the ends of the intervals; i.e., at the instants $0,1,2,\ldots$ precisely. Thus the state of the route occupancy at time $(n+1)^-$ does not include the calls just terminated or calls arriving at $(n+1)$. Define the "route occupancy process" $Z_n = (Z_n^1, Z_n^2)$, where Z_n^i is the number of lines of route i occupied at time n^+. Thus, $Z_n^i \le N_i$. If a call arrives at instant $n + 1$, the automaton chooses route 1 with probability X_n and route 2 with probability $1 - X_n$. If all lines of the chosen route i are occupied at instant $(n+1)^-$, then the call is switched to route j $(j \neq i)$. If all lines of route j are also occupied at instant $(n+1)^-$, then the call is rejected. The choice probabilities $\{X_n\}$ are to be adjusted or adapted according to the 'experience' of the system.

The specific adjustment scheme for $\{X_n\}$ is the following "linear-reward" algorithm [7]. Let J_{in} denote the indicator of the event {call arrives at $n+1$, is assigned first to route i and is accepted by route i}. For practical as well as theoretical purposes, it is important to bound X_n away from the points 0 and 1. Let $0 < x_\ell < x_u < 1$. We use the (projected) algorithm (5.1), where $\Big|_{x_\ell}^{x_u}$ denotes truncation at x_u or x_ℓ, and $\alpha(x) = 1 - x$, $\beta(x) = -x$.

$$(5.1) \qquad X_{n+1} = [X_n + a_n\alpha(X_n)J_{1n} + a_n\beta(X_n)J_{2n}]\Big|_{x_\ell}^{x_u} .$$

<u>Some definitions</u>. If the choice probabilities X_n are held fixed at some value x for all n, then the route choice <u>automaton</u> still is well defined. For fixed route selection probability $x \in (0,1)$, let $\{Z_n(x)\} = \{(Z_n^1(x), Z_n^2(x)),\ 0 \le n < \infty\}$ denote the corresponding route occupancy process. For the process $\{Z_n(x)\}$, the state space $Z = \{(i,j): i \le N_1,\ j \le N_2\}$ (whose points are ordered in some fixed way) is a single ergodic class, and the probability transition matrix, denoted by $A'(x)$, has infinitely differentiable components. With given initial condition, define $P_n(\alpha|x) = P\{Z_n(x) = \alpha\}$ and define the vector $P_n(x) = \{P_n(\alpha|x),\ \alpha \in Z\}$. Then $P_{n+1}(x) = A(x)P_n(x)$.

The pair $\{Z_n, X_n\}, n \geq 0\}$ is a Markov process on $Z \times [x_\ell, x_u]$ and the marginal transition probability $P\{Z_{n+1} = (k,\ell) | Z_n = (i,j), X_n\}$ is just the $((i,j)$-column, (k,ℓ)-row) entry of $A(X_n)$. Define the vector $P_n = \{P_n(\alpha), \alpha \in Z\}$ where $P_n(\alpha) = P\{Z_n = \alpha | X_\ell, \ell < n, Z_0\}$. Then $P_{n+1} = A(X_n)P_n$. Also, let $P(x) = \{P(\alpha|x), \alpha \in Z\}$ denote the unique invariant measure for $\{Z_n(x)\}$, with marginal defined by $P^1(N_1|x)$ = asymptotic probability that $Z_n^1 = N_1$, and similarly for route 2. Finally, define the transition probability $P(\alpha,j,\alpha_1|x) = P\{Z_j(x) = \alpha_1 | Z_0(x) = \alpha\}$, and define the marginal transition probability

$$P^i(\alpha,j,N_i|x) = P\{Z_j^i(x) = N_i | Z_0(x) = \alpha\}.$$

Define E_n to be the expectation conditioned on $\{Z_\ell, X_\ell, \ell \leq n\}$ and set $\nu_i = (1-\lambda_i)^{N_i}$.

Application of Theorem 3.

We have $h(X_n,\xi_n) = \alpha(X_n)J_{1n} + \beta(X_n)J_{2n}$ and, with $I\{\cdot\}$ denoting the indicator function,

$$E_n h(X_n,\xi_n) = \mu\alpha(X_n)X_n[1-\nu_1 I\{Z_n^1 = N_1\}]$$
$$+ \mu\beta(X_n)(1-X_n)[1-\nu_2 I\{Z_n^2 = N_2\}],$$

which can be written in the form

$$(5.2)\qquad = \mu X_n(1-X_n)[\nu_2 P^2(Z_n,0,N_2|X_n) - \nu_1 P^1(Z_n,0,N_1|X_n)].$$

Define $\bar{h}(\cdot)$ to be the limit

$$\bar{h}(x) = \mu x(1-x)\lim_{n\to\infty} E[\nu_2 P^2(Z_n,n,N_2|x) - \nu_1 P^1(Z_n,n,N_2|x)]$$

$$(5.3)\qquad = \mu x(1-x)[\nu_2 P^2(N_2|x) - \nu_1 P^1(N_1|x)].$$

The sum (A3) is replaced by (since the second part of Theorem 3 is to be used, the $V_x(x)$ component can be dropped)

$$V_0(x,n) = \mu x(1-x)V_x'(x)\sum_{j=n}^{\infty} a_j[\nu_2(P^2(x,j-n,N_2|x) - P^2(N_2|x))$$

$$(5.4)\qquad - \nu_1(P^1(x,j-n,N_1|x) - P^1(N_1|x))].$$

The sum (A6) is replaced by the analogous sum of the derivatives (again drop the $V_x(x)$ component). There is a unique $\bar{x} \in (0,1)$ such that $\bar{h}(\bar{x}) = 0$ and $\bar{h}(x) > 0$ for $x \in (0,\bar{x})$ and $\bar{h}(x) < 0$ for $x \in (\bar{x},1)$. The $P_n(x)$ and $P_{n,x}(x)$ converge [5] to the limits $P(x)$, $P_x(x)$ geometrically with a rate uniform in $x \in [x_\ell,x_u]$ and in $P_0(x)$ ($P_{0,x}(x) = 0$ is the appropriate initial condition to get the limit for the derivative sequence in (A6)). This result implies that (A3), (A6) exist and converge absolutely and uniformly in (n,X_n) at a geometric rate. See [5] for the details of the convergences.

Part 2 of Theorem 3 now yields Theorem 4 below. Theorem 4 can also be proved directly, via the method of Theorem 2 (here the boundary is only $\{x_\ell, x_u\}$) with the 'corrected' test function (5.4) used in lieu of the sum in (A3).

Theorem 5. Let $\Sigma a_i^2 < \infty$, $\Sigma a_i = \infty$. Then if $\bar{x} \in [x_\ell, x_u]$, we have $\{X_n\} \to \bar{x}$ w.p.1. Otherwise $\{X_n\}$ converges w.p.1 to the point x_ℓ or x_u which is nearest to $\bar{x}$.

6. Observation Averaging for Stochastic Approximations.

The general method of Theorems 1 and 2 can be easily used to prove w.p.1 convergence for stochastic approximations of the Robbins-Monro or Kiefer-Wolfowitz type but with averaged observations. The main difficulty is due to the fact that the quantity which plays the role of the noise is always state dependent. The idea will be illustrated via a very simple example. We use a Robbins-Monro scheme to estimate the root of $Kx = 0$, x = scalar, $K > 0$ (but the method is applicable to the general problem).

Define the estimates by

$$X_{n+1} = (X_n + a_n \xi_n)\Big|_{x_\ell}^{x_u}$$

(6.1)

$$\xi_n = \alpha \xi_{n-1} - \beta [K X_n + \psi_n],$$

where $\alpha \in (0,1)$, $\beta > 0$ and $\{\psi_n\}$ is a bounded sequence of mutually independent random variables with zero mean value. If $\alpha = 0$, then (6.1) is the usual Robbins-Monroe method, truncated at values x_u, x_ℓ. If $\alpha \in (0,1)$, then the observations are exponentially weighted. Theorem 3 requires truncation to some finite interval $[x_\ell, x_u]$. Such truncation is usually done in practice anyway. Define $\bar{h}(x) = -\beta Kx/(1-\alpha)$ and $h(x,\xi) = \xi$. Instead of writing $V_0(x,n)$ in the form (4.1), it is more convenient to do the following. For each x,n, define the auxiliary processes $\{\xi_j(x), j \geq n\}$ where the initial condition $\xi_{n-1}(x)$ is to be defined and $\xi_j(x) = \alpha\xi_{j-1}(x) - (\beta Kx + \psi_j)$, $j \geq n$. Write $V_0(x,n)$ as

$$(6.2) \qquad V_0(x,n) = \sum_{j=n}^{\infty} a_j V_x'(x) E_n[h(x,\xi_j(x)) - \bar{h}(x)],$$

where $\xi_{n-1}(X_n) \equiv \xi_{n-1}$, and E_n denotes expectation conditioned on X_i, $i \leq n$, ψ_i, $i < n$. Note that $\xi_n(X_n) = \xi_n$.

Now Theorem 3 yields

<u>Theorem 6</u>. <u>Let</u> $\Sigma a_i^2 < \infty$, $\Sigma a_i = \infty$. <u>If</u> $0 \in [x_\ell, x_u]$, <u>then</u> $\{X_n\} \to 0$ w.p.1. <u>Otherwise</u> $\{X_n\}$ <u>converges</u> w.p.1 <u>to the point</u> x_ℓ, x_u <u>which is closest to zero</u>.

In [4] there is an analysis of the asymptotic properties of (6.1) when $a_n \equiv \varepsilon > 0$.

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