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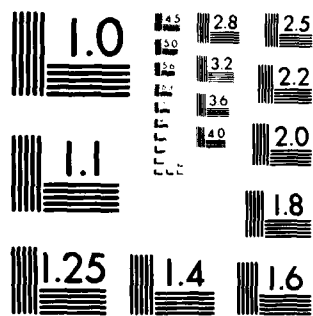
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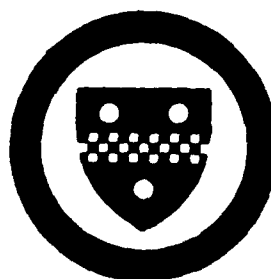
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A MULTIVARIATE CORRELATION RATIO

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May 1980

Technical Report No. 80-6

Institute for Statistics and Applications
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where A is a given positive definite matrix. The properties of η_A are discussed, with particular attention paid to a 'correlation-maximizing' property. A number of examples illustrating the application of η_A are given; these examples include the multivariate normal, the elliptically symmetric distributions, the Farlie-Morgenstern-Gumbel family, and the multinomial. The problem of maximizing $\eta_A(BY; X)$ over suitable matrices B is considered and the results that are obtained are related to canonical correlations for the multivariate normal.

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A Multivariate Correlation Ratio

by

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ABSTRACT

A brief review of the historical background and certain known results concerning the univariate correlation ratio are given. A multivariate correlation ratio of a random vector \underline{Y} upon a random vector \underline{X} is defined by

$$\eta_A(\underline{Y};\underline{X}) = \{\text{tr}(\underline{A}^{-1} \text{Cov } E(\underline{Y}|\underline{X}))\}^{\frac{1}{2}} \{\text{tr}(\underline{A}^{-1} \underline{\Sigma}_Y)\}^{-\frac{1}{2}},$$

where \underline{A} is a given positive definite matrix. The properties of η_A are discussed, with particular attention paid to a "correlation-maximizing" property. A number of examples illustrating the application of η_A are given; these examples include the multivariate normal, the elliptically symmetric distributions, the Farlie-Morgenstern-Gumbel family, and the multinomial. The problem of maximizing $\eta_A(\underline{B}\underline{Y};\underline{X})$ over suitable matrices \underline{B} is considered and the results that are obtained are related to canonical correlations for the multivariate normal.

AMS Classification: Primary 62H20, Secondary 62J05

Key Words: Correlation ratio, multivariate correlation ratio, vector correlation, canonical correlation, sup-correlation, elliptically symmetric.

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A Multivariate Correlation Ratio

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1. INTRODUCTION AND HISTORICAL BACKGROUND.

The correlation ratio, $\eta(Y;X)$, of a random variable Y upon a random variable X , defined by (with suitable assumptions)

$$\eta(Y;X) = \{\text{Var}(E(Y|X))/\text{Var } Y\}^{1/2}, \quad (1.1)$$

was first introduced by K. Pearson in (1903, p. 304), who wrote " η is a useful constant which ought always to be given for non-linear systems...it measures the approach of the system not only to linearity but to single valued relationship, i.e., to a causal nexus". Pearson further discussed η in his papers of (1905; or see (1948, pp. 477-528)) and (1909). In his 1905 paper, he wrote "the correlation ratio...is an excellent measure of the stringency of correlation always lying numerically between the values 0 and 1, which mark absolute independence and complete causation respectively". He further noted, based on his considerations of non-normal bivariate data, that "the ease with which η can be calculated suggests that in many cases it should accompany, if not replace the determination of the correlation coefficient".

Blakeman (1905) also introduced a criteria based on η to test for linearity of regression. Fisher (1925; pp. 257-260 of 14th Ed. (1970)),

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seemingly less enthusiastic about η , wrote concerning the sample analogue of η in the regression model that "as a descriptive statistic the utility of the correlation ratio is extremely limited". It appears that much of his concerns were based on certain distributional properties. A more recent discussion concerning η can be found in Lancaster (1969, pp. 201-202).

Various other properties of η have been considered within the literature which focuses on measures of association and measures of dependence, most of this literature having been written within the last approximately 20 years. Kruskal (1958) in his survey on ordinal measures of association discussed η , and Rényi (1959) in his axiomatic development of measures of dependence examined properties of η . More recently, Hall (1970) defined the dependence characteristic function as $\eta(e^{itY}; X)$, where a suitable extension of η to complex-valued random variables was given; the relative merits of the dependence characteristic function versus the correlation ratio were considered. Kotz and Soong (1977) further reviewed some of the probabilistic properties of η . Hall (1970) also noted that when X is vector valued; the correlation ratio $\eta(Y; X)$, defined by (1.1) now with $E(Y|X)$ in the numerator, has essentially the same properties as when X is a scalar random variable. Within a specific multivariate normal setting, Johnson and Kotz (1972, p. 186) noted that a certain multivariate beta random variable could be viewed as a multivariate generalization of η .

The correlation ratio is in some ways connected to the sup-correlation coefficient between random variables X and Y , defined by

$$\rho'(X,Y) = \sup \rho(f(X),g(Y)), \quad (1.2)$$

where the supremum is over suitable functions f and g , and ρ is the Pearson correlation coefficient. This measure of dependence was introduced by Gebelein (1941) and developed further by Sarmanov (1958A), (1958B), Rényi (1959) and Lancaster (whose work is summarized in Lancaster (1969)). Sarmanov and Zaharov (1960) extended this concept to the multivariate case, defining $\rho'(X,Y) = \sup \rho(f(X),g(Y))$, where the supremum is over suitable f,g , which map R^p and R^q , respectively, into R^1 . We note that except in very special cases, it is difficult to obtain an explicit evaluation of $\rho'(X,Y)$.

In this paper, we consider defining the correlation ratio for the case when both Y and X are vector random variables. This extension would, for example, accomodate the situation when we are studying the relationship of X_{r+1}, \dots, X_{r+s} to X_1, \dots, X_r , or when we are relating jointly a time series Y_1, \dots, Y_t to a time series X_1, \dots, X_s . The properties of this multivariate correlation ratio are explored in light of the properties of $\eta(Y;X)$. We also examine maximizing the multivariate correlation ratio over certain linear combinations, and study the relationship of this concept to other multivariate notions, including the sup-correlation. A number of specific multivariate distributional examples are considered including the normal, elliptically symmetric and Farlie-Morgenstern-Gumbel.

2. A REVIEW OF RESULTS PERTAINING TO $\eta(Y;X)$.

In this section, we survey some results concerning $\eta(Y;X)$ and discuss briefly some of the implications of these results.

Theorem 2.1. Let Y be a random variable with $0 < \text{Var } Y < \infty$.

Let \underline{X} be a p -dimensional random vector, jointly distributed with Y .

(a) Then $\min E(Y - g(\underline{X}))^2$ occurs at $g(\underline{x}) = E(Y | \underline{X} = \underline{x})$, where the minimum is over all measurable $g: R^p \rightarrow R^1$, for which $E(Y - g(\underline{X}))^2 < \infty$.

(b) Furthermore, the minimum value is $(1 - \eta^2(Y; \underline{X})) \text{Var } Y$.

A proof of essentially Theorem 2.1 can be found, for example, in Parzen (1960) or Hall (1970).

Theorem 2.2. Let Y be a random variable with $0 < \text{Var } Y < \infty$.

Let \underline{X} be a p -dimensional random vector jointly distributed with Y .

(a) Then $\max |\rho(Y, g(\underline{X}))|$ occurs at $g(\underline{x}) = E(Y | \underline{X} = \underline{x})$, where the maximum is over all measurable functions $g: R^p \rightarrow R^1$ for which the correlation is defined. (b) Furthermore, the maximum value is $\eta(Y; \underline{X})$.

Note that in Theorem 2.2 if $g(\underline{x})$ maximizes, then $\alpha g(\underline{x}) + \beta$, $\alpha \neq 0$, maximizes. A proof of Theorem 2.2 in the case X is a scalar can be found in Kotz and Soong (1977); the proof when \underline{X} is a vector is identical. This very interesting interpretation of $\eta(Y; \underline{X})$, according to Kruskal (1958), was first noted by Fréchet (1933), (1934). Earlier, Pearson (1905) had proved $\eta(Y; X) \geq |\rho(Y, X)|$ and Fisher (1925) had shown $\eta(Y; X) \geq \max |\rho(Y, g(X))|$.

An immediate result of Theorem 2.2 is that $0 \leq \eta \leq 1$. From Theorem 2.1, we observe that Y being predicted by $g(\underline{X})$ with an expected squared error of zero is equivalent to $\eta(Y; \underline{X}) = 1$. A further consequence of Theorem 2.2 is that $\eta(Y; \underline{X}) = 0$ is equivalent to $\rho(Y, h(\underline{X})) = 0$ for all measurable functions h , with $0 < \text{Var } h(\underline{X}) < \infty$. This also implies that $E(Y | \underline{X} = \underline{x}) = E(Y)$ a.e. for all \underline{x} .

One commonly perceived deficiency of η as a measure of dependence

(e.g., Renyi (1959)) is that $\eta = 0$ does not imply Y and \underline{X} are independent. The correlation ratio being zero may be interpreted as being "between" independence and uncorrelatedness (in terms of multiple correlation) in the following sense. Independence of Y and \underline{X} is equivalent to $\rho(f(Y), g(\underline{X})) = 0$ for all suitable f, g ; and uncorrelatedness is equivalent to $\rho(\alpha_1 Y + \beta_1, \alpha_2' \underline{X} + \beta_2) = 0$ for all $\alpha_1, \beta_1, \alpha_2, \beta_2$; while zero correlation ratio, as noted, is equivalent to $\rho(\alpha_1 Y + \beta_1, g(\underline{X})) = 0$, for all α_1, β_1 , and suitable g .

When $(Y, \underline{X})'$ have a joint multivariate normal distribution, the multiple correlation is defined (e.g., Anderson (1958)) by $\max_{\alpha} \rho(Y, \alpha' \underline{X})$. For the multivariate normal $E(Y|\underline{X})$ is linear in \underline{X} , so that it follows from Theorems 2.1 and 2.2 that $\eta(Y; \underline{X})$ is the same as the multiple correlation of Y with \underline{X} . For the linear prediction problem where g in Theorems 2.1 and 2.2 is restricted to be linear, it is more appropriate (e.g., Parzen (1960)) to use $\max_{\alpha} \rho(Y, \alpha' \underline{X})$ than $\eta(Y; \underline{X})$ for measuring dependence. This suggests, perhaps, that when \underline{X} is an arbitrary random vector, one might adopt the nomenclature, linear multiple correlation generalized multiple correlation, and sup-multiple-correlation, respectively for $\max_{\alpha} \rho(Y, \alpha' \underline{X})$, $\eta(Y; \underline{X})$, and $\sup_{f, g} \rho(f(Y), g(\underline{X}))$. For the multivariate normal these three measures are equal.

3. A MULTIVARIATE CORRELATION RATIO.

In this section, we consider the definition of a correlation ratio of Y upon \underline{X} , where Y is a $q \times 1$ random vector and \underline{X} is a $p \times 1$ random vector. Even though Y is now a vector, it seems suitable to have the multivariate correlation ratio take scalar values, rather than

attempt a multiple-valued version. To motivate our definition of the multivariate correlation ratio we first consider the multivariate prediction problem and proceed from there.

In referring to the covariance matrix of a random vector \underline{W} , we use the notation $\text{Cov}(\underline{W})$ and $\underline{\Sigma}_W$ interchangeably. The cross-covariance between random vectors $\underline{S}, \underline{T}$ is denoted by $\text{Cov}(\underline{S}, \underline{T}) = E[(\underline{S} - E(\underline{S}))(\underline{T} - E(\underline{T}))']$. For (random) vectors, the notation $||\underline{X}||_A^2 = \underline{X}' \underline{A}^{-1} \underline{X}$ is employed, where \underline{A} is a positive definite matrix.

Theorem 3.1. Let $\underline{Y}: q \times 1$, $\underline{X}: p \times 1$ be jointly distributed random vectors with $0 < \text{tr } \underline{\Sigma}_Y < \infty$; and let $\underline{A}: q \times q$ be positive definite. Then $\min E ||\underline{Y} - g(\underline{X})||_A^2$ occurs at $g(\underline{x}) = E(\underline{Y} | \underline{X} = \underline{x})$, where the minimum is over all measurable $g: R^p \rightarrow R^q$, for which $E ||\underline{Y} - g(\underline{X})||_A^2 < \infty$.

Proof. The result is immediate from the well-known identity

$$E ||\underline{Y} - g(\underline{X})||_A^2 = E ||\underline{Y} - E(\underline{Y} | \underline{X})||_A^2 + E ||g(\underline{X}) - E(\underline{Y} | \underline{X})||_A^2.$$

Theorem 3.2. Let $\underline{Y}: q \times 1$, $\underline{X}: p \times 1$ be jointly distributed random vectors with $0 < \text{tr } \underline{\Sigma}_Y < \infty$; and let $\underline{A}: q \times q$ be positive definite. Then $E ||\underline{Y} - E(\underline{Y} | \underline{X})||_A^2 = \text{tr}(\underline{A}^{-1} \underline{\Sigma}_Y) (1 - \{[\text{tr}(\underline{A}^{-1} \text{Cov } E(\underline{Y} | \underline{X}))] / [\text{tr}(\underline{A}^{-1} \underline{\Sigma}_Y)]\})$.

Proof. Note $E ||\underline{Y} - E(\underline{Y} | \underline{X})||_A^2 = E ||\underline{Y}||_A^2 - E ||E(\underline{Y} | \underline{X})||_A^2$. Without loss of generality, assume $E \underline{Y} = 0$. It is readily shown that

$E\|E(\underline{Y}|\underline{X})\|_{\underline{A}}^2 = \text{tr}[\underline{A}^{-1}\text{Cov}(E(\underline{Y}|\underline{X}))]$, and $E\|\underline{Y}\|_{\underline{A}}^2 = \text{tr}(\underline{A}^{-1}\underline{\Sigma}_Y)$, so that the result follows.

Observe that without any knowledge of \underline{X} the best expected minimum normed predictor of \underline{Y} is $E(\underline{Y})$ in the sense that $\min E\|\underline{Y}-\underline{c}\|_{\underline{A}}^2 = \text{tr}(\underline{A}^{-1}\underline{\Sigma}_Y)$ and occurs at $\underline{c} = E(\underline{Y})$. Thus, Theorem 3.2, like Theorem 2.1b, allows the measurement of the reduction in mean normed error due to knowledge of \underline{X} . This then suggests a suitable definition of a multivariate correlation ratio.

Definition 3.1. Let $\underline{Y}: q \times 1$ and \underline{X} be jointly distributed random vectors with $0 < \text{tr } \underline{\Sigma}_Y < \infty$. The correlation ratio of \underline{Y} upon \underline{X} is given by

$$\eta_{\underline{A}}(\underline{Y};\underline{X}) = \left(\frac{\text{tr}(\underline{A}^{-1}\text{Cov } E(\underline{Y}|\underline{X}))}{\text{tr}(\underline{A}^{-1}\underline{\Sigma}_Y)} \right)^{1/2},$$

where \underline{A} is a positive definite $q \times q$ matrix.

In the case when \underline{Y} is a scalar this definition clearly reduces to the previous definition (1.1) of correlation ratio. Obvious possible choices of \underline{A} are $\underline{A} = \underline{I}$ and $\underline{A} = \underline{\Sigma}_Y$, when $\underline{\Sigma}_Y$ is nonsingular. Note that $\eta_{\underline{I}}^2(\underline{Y};\underline{X}) = (\text{tr } \text{Cov } E(\underline{Y}|\underline{X})) / (\text{tr } \underline{\Sigma}_Y)$ and that $\eta_{\underline{\Sigma}_Y}^2(\underline{Y};\underline{X}) = q^{-1} \text{tr}(\underline{\Sigma}_Y^{-1/2} \text{Cov } E(\underline{Y}|\underline{X}) \underline{\Sigma}_Y^{-1/2})$. In practice, the actual choice of \underline{A} might depend on how difficult \underline{A} would be to estimate from the data for a particular multivariate model. It may be the case that $\text{tr } \underline{\Sigma}_Y$ is easier to estimate than the entire matrix $\underline{\Sigma}_Y$, so that $\underline{A} = \underline{I}$ should be chosen.

Recall that when \underline{Y} is a scalar, we have the important result that $\eta(\underline{Y};\underline{X}) = \max_{\underline{g}} \rho(\underline{Y};\underline{g}(\underline{X}))$. When \underline{Y} is a vector there is an analogous

result; however, we must introduce a suitable version of "correlation" between two vectors.

Definition 3.2. Let $\underline{S}: p \times 1$ and $\underline{T}: p \times 1$ be two jointly distributed random vectors with $0 < \text{tr } \underline{\Sigma}_S < \infty$, and $0 < \text{tr } \underline{\Sigma}_T < \infty$. The vector correlation between \underline{S} and \underline{T} is defined by

$$\rho_A(\underline{S}, \underline{T}) = \frac{\text{tr}(\underline{A}^{-1} \text{Cov}(\underline{S}, \underline{T}))}{(\text{tr}(\underline{A}^{-1} \underline{\Sigma}_S))^{1/2} (\text{tr}(\underline{A}^{-1} \underline{\Sigma}_T))^{1/2}},$$

where \underline{A} is a $p \times p$ positive definite matrix.

Note that $\rho_A(\underline{S}, \underline{T}) = \rho_A(\underline{T}, \underline{S})$, $|\rho_A(\underline{S}, \underline{T})| \leq 1$, and $\rho_A(\underline{S}, -\underline{T}) = -\rho_A(\underline{S}, \underline{T})$. For S, T scalars, $\rho_A(S, T) = \rho(S, T)$.

Theorem 3.3. Let $\underline{Y}: q \times 1$, $\underline{X}: p \times 1$ be jointly distributed random vectors with $0 < \text{tr } \underline{\Sigma}_Y < \infty$; and let $\underline{A}: q \times q$ be positive definite. Then $\max |\rho_A(\underline{Y}, g(\underline{X}))| = \eta_A(\underline{Y}; \underline{X})$, where the maximum is taken over all measurable functions $g: R^p \rightarrow R^q$, for which the vector correlation is defined.

Proof. Without loss of generality, assume $E(\underline{Y}) = \underline{0}$ and $Eg(\underline{X}) = \underline{0}$. From Theorem 3.1, it follows that

$$E \|\underline{Y} - E(\underline{Y}|\underline{X})\|_{\underline{A}}^2 \leq E \|\underline{Y} - c_g g(\underline{X})\|_{\underline{A}}^2, \quad (3.1)$$

for all g and constants c_g , which can depend on g . Expand (3.1) and choose $c_g = \{[\text{tr}(\underline{A}^{-1} \text{Cov } E(\underline{Y}|\underline{X}))]/[\text{tr}(\underline{A}^{-1} \text{Cov } g(\underline{X}))]\}^{1/2}$ to obtain

$$\text{Cov}(\underline{A}^{-1} \underline{Y}, E(\underline{Y}|\underline{X})) \geq c_g \text{Cov}(\underline{A}^{-1} \underline{Y}, g(\underline{X})). \quad (3.2)$$

Divide both sides of (3.2) by $[(\text{tr}(\underline{A}^{-1}\underline{\Sigma}_Y))(\text{tr}(\underline{A}^{-1}\text{Cov } E(\underline{Y}|\underline{X})))]^{\frac{1}{2}}$ to yield

$$\rho_A(\underline{Y}, E(\underline{Y}|\underline{X})) \geq \rho_A(\underline{Y}, g(\underline{X})), \quad (3.3)$$

for all measurable g . Because (3.3) holds for g and $-g$, and because $\rho_A(\underline{S}, -\underline{T}) = -\rho_A(\underline{S}, \underline{T})$, (3.3) holds with the right hand side of (3.3) in absolute values. Now

$$\begin{aligned} (\text{tr}(\underline{A}^{-1}\underline{\Sigma}_Y))^{\frac{1}{2}} \eta_A(\underline{Y}; \underline{X}) &= [\text{tr}(\underline{A}^{-1}\text{Cov } E(\underline{Y}|\underline{X}))]^{\frac{1}{2}} \\ &= [E\|E(\underline{Y}|\underline{X})\|_A^2]^{\frac{1}{2}} \\ &= \frac{\text{tr}(\underline{A}^{-1}\text{Cov}(\underline{Y}; E(\underline{Y}|\underline{X})))}{[\text{tr}(\underline{A}^{-1}\text{Cov } E(\underline{Y}|\underline{X}))]^{\frac{1}{2}}} \end{aligned} \quad (3.4)$$

The result now follows from (3.3) and (3.4).

Corollary 3.1. Let $\underline{Y}: q \times 1$, $\underline{X}: p \times 1$ be jointly distributed random vectors with $0 < \text{tr } \underline{\Sigma}_Y < \infty$; and let $\underline{A}: q \times q$ be positive definite. Then $\eta_A(\underline{Y}; \underline{X}) = \rho_A(\underline{Y}, E(\underline{Y}|\underline{X}))$.

Proof. This is the result of (3.4).

Observe that if $E(\underline{Y}|\underline{X})$ is linear in \underline{X} , then $\eta_A(\underline{Y}; \underline{X}) = \max |\rho_A(\underline{Y}, \underline{B}\underline{X})|$, where the maximum is taken over all matrices $\underline{B}: q \times p$.

Before considering a number of examples, we briefly discuss further properties of the vector correlation coefficient when $\underline{A} = \underline{I}$. It is easily shown that

$$\rho_I(\underline{S}, \underline{T}) = \sum_{i=1}^p (\alpha_i \beta_i)^{1/2} \rho(S_i, T_i),$$

where $\underline{S} = (S_1, \dots, S_p)'$, $\underline{T} = (T_1, \dots, T_p)'$; and $\alpha_i = \text{Var } S_i / (\sum \text{Var } S_i)$, $\beta_i = \text{Var } T_i / (\sum \text{Var } T_i)$. Note $0 \leq \alpha_i, \beta_i \leq 1$ and $\sum \alpha_i = \sum \beta_i = 1$; thus, $\rho_I(\underline{S}, \underline{T})$ can almost be viewed as a weighted average of the pair-wise correlations. If $\text{Var } S_1 = \dots = \text{Var } S_p$ and $\text{Var } T_1 = \dots = \text{Var } T_p$, then $\rho_I(\underline{S}, \underline{T}) = (\sum_{i=1}^p \rho(S_i, T_i)) / p$. If $\text{Var } S_1 \gg \text{Var } S_2 = \dots = \text{Var } S_p$ and $\text{Var } T_1 \gg \text{Var } T_2 = \dots = \text{Var } T_p$, then $\rho_I(\underline{S}, \underline{T}) = \rho(S_1, T_1)$. Let $\underline{S}^* = \underline{\Gamma} \underline{S} + \underline{a}$, $\underline{T}^* = \underline{\Gamma} \underline{T} + \underline{b}$, where $\underline{\Gamma}$ is a $p \times p$ orthonormal matrix, and $\underline{a}, \underline{b}$ are arbitrary vectors; then $\rho_I(\underline{S}^*, \underline{T}^*) = \rho_I(\underline{S}, \underline{T})$.

Example 3.1. Multivariate Normal.

Let $(\underline{X}', \underline{Y}')' \sim N(\underline{0}, \underline{\Sigma})$, where \underline{X} is $q \times 1$, \underline{Y} is $p \times 1$ and $\underline{\Sigma}$ is partitioned similarly, i.e.,

$$\underline{\Sigma} = \begin{bmatrix} \underline{\Sigma}_X & \underline{\Sigma}_{XY} \\ \underline{\Sigma}_{XY}' & \underline{\Sigma}_Y \end{bmatrix}. \quad (3.5)$$

Then $E(\underline{X}|\underline{Y}) = \underline{\Sigma}_{XY} \underline{\Sigma}_Y^{-1} \underline{Y}$ and $\text{Cov}(E(\underline{X}|\underline{Y})) = \underline{\Sigma}_{XY} \underline{\Sigma}_Y^{-1} \underline{\Sigma}_{XY}'$; hence,

$$\eta_I(\underline{X}; \underline{Y}) = \{\text{tr}(\underline{\Sigma}_{XY} \underline{\Sigma}_Y^{-1} \underline{\Sigma}_{XY}')\}^{1/2} (\text{tr} \underline{\Sigma}_X)^{-1/2}$$

and

$$\eta_{\underline{\Sigma}_X}(\underline{X}; \underline{Y}) = (q^{-1} \text{tr}(\underline{\Sigma}_X^{-1/2} \underline{\Sigma}_{XY} \underline{\Sigma}_Y^{-1} \underline{\Sigma}_{XY}' \underline{\Sigma}_X^{-1/2}))^{1/2}.$$

Observe that $\eta_{\underline{\Sigma}_X}(\underline{X}; \underline{Y}) = (q^{-1} \sum_{i=1}^q C_i^2(\underline{X}, \underline{Y}))^{1/2}$, where $C_i(\underline{X}, \underline{Y})$ is the i^{th} canonical correlation between \underline{X} , \underline{Y} , $i = 1, \dots, q$. If $p = q$, then

$$\rho_I(\underline{X}, \underline{Y}) = (\text{tr } \underline{\Sigma}_{XY}) \{ (\text{tr } \underline{\Sigma}_X) (\text{tr } \underline{\Sigma}_Y) \}^{-\frac{1}{2}} \quad \text{and} \quad \rho_{\underline{\Sigma}_X} = \text{tr}(\underline{\Sigma}_X^{-1} \underline{\Sigma}_{XY}) \{ p \text{tr}(\underline{\Sigma}_X^{-1} \underline{\Sigma}_Y) \}^{-\frac{1}{2}}.$$

Example 3.2. Elliptically Symmetric Distributions.

Let $\underline{Z} = (\underline{X}', \underline{Y}')'$ have p.d.f. $c|\underline{\Psi}|^{-\frac{1}{2}} \phi(\underline{z}' \underline{\Psi}^{-1} \underline{z})$. (Such distributions are called elliptically symmetric, e.g., Kelker (1970).) Let \underline{X} be $q \times 1$, \underline{Y} be $p \times 1$ and suppose $\underline{\Psi}$ is partitioned as in (3.5), where we assume $0 < \text{tr Cov}(\underline{X}) < \infty$. Then $E(\underline{X}|\underline{Y}) = \underline{\Psi}_{XY} \underline{\Psi}_Y^{-1} \underline{Y}$ and $\text{Cov}(E(\underline{X}|\underline{Y})) = \alpha \underline{\Psi}_{XY} \underline{\Psi}_Y^{-1} \underline{\Psi}_{XY}'$; also $\text{Cov}(\underline{X}) = \alpha \underline{\Psi}_X$, so that $\eta_I(\underline{X}; \underline{Y}) = \{ \text{tr}(\underline{\Psi}_{XY} \underline{\Psi}_Y^{-1} \underline{\Psi}_{XY}') \}^{\frac{1}{2}} (\text{tr } \underline{\Psi}_X)^{-\frac{1}{2}}$ and $\eta_{\underline{\Psi}_X}(\underline{X}; \underline{Y}) = \{ q^{-1} \text{tr}(\underline{\Psi}_X^{-\frac{1}{2}} \underline{\Psi}_{XY} \underline{\Psi}_Y^{-1} \underline{\Psi}_{XY}' \underline{\Psi}_X^{-\frac{1}{2}}) \}^{\frac{1}{2}}$.

Example 3.3. Multivariate Farlie-Gumbel-Morgenstern (FGM) Distribution.

In order to have tractable results, we consider a special four-dimensional FGM distribution with uniform marginals. (See Johnson and Kotz (1975) for the general multivariate distribution.) Let

$$\begin{aligned} f_{X_1, X_2, X_3, X_4}(x_1, x_2, x_3, x_4) &= 1 + \alpha_{12} v_{12} + \alpha_{13} v_{13} + \alpha_{14} v_{14} \\ &+ \alpha_{23} v_{23} + \alpha_{24} v_{24} + \alpha_{123} v_{123} \\ &+ \alpha_{124} v_{124} + \alpha_{1234} v_{1234}, \end{aligned} \quad (3.6)$$

where $v_{ij} = (1-2x_i)(1-2x_j)$, $v_{ijk} = (1-2x_i)(1-2x_j)(1-2x_k)$, and $v_{ijkl} = (1-2x_i)(1-2x_j)(1-2x_k)(1-2x_l)$. Direct calculation yields that $E(X_1|X_3, X_4) = (\frac{1}{2} - \alpha_{13}/6 - \alpha_{14}/6) + (\alpha_{13}/3)X_3 + (\alpha_{14}/3)X_4$, and $E(X_2|X_3, X_4) = (\frac{1}{2} - \alpha_{23}/6 - \alpha_{24}/6) + (\alpha_{23}/3)X_3 + (\alpha_{24}/3)X_4$. Further calculations yield that

$$\text{Cov}(E(X_1, X_2 | X_3, X_4))' = \begin{bmatrix} \frac{\alpha_{13}^2 + \alpha_{14}^2}{9} & \frac{\alpha_{13}\alpha_{23} + \alpha_{14}\alpha_{24}}{9} \\ \frac{\alpha_{13}\alpha_{23} + \alpha_{14}\alpha_{24}}{9} & \frac{\alpha_{23}^2 + \alpha_{24}^2}{9} \end{bmatrix} \quad (3.7)$$

and

$$\text{Cov}(X_1, X_2)' = \begin{bmatrix} 1/12 & \frac{\alpha_{12}}{36} \\ \frac{\alpha_{12}}{36} & 1/12 \end{bmatrix}. \quad (3.8)$$

Hence, for this example $\eta_I((X_1, X_2)'; (X_3, X_4)) = \{2/3(\alpha_{13}^2 + \alpha_{14}^2 + \alpha_{23}^2 + \alpha_{24}^2)\}^{1/2}$ and $\eta_{\Sigma_{12}}((X_1, X_2)'; (X_3, X_4)) = \{(2/9)(1 - 4\alpha_{12}^2)^{-1}[3(\alpha_{13}^2 + \alpha_{14}^2 + \alpha_{23}^2 + \alpha_{24}^2) - 2(\alpha_{12}\alpha_{13}\alpha_{23} + \alpha_{12}\alpha_{14}\alpha_{24})]\}^{1/2}$, where Σ_{12} is given by (3.8).

Suppose that f is given by (3.6), where $\alpha_{123} = \alpha_{124} = \alpha_{12} = 0$. By symmetry $\eta_I((X_3, X_4)'; (X_1, X_2)) = \{2/3(\alpha_{13}^2 + \alpha_{14}^2 + \alpha_{23}^2 + \alpha_{24}^2)\}^{1/2}$. Thus, if we set $\alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = 0$, we have $(X_1, X_2)'$ and $(X_3, X_4)'$ mutually have zero correlation ratio, each upon the other; yet $(X_1, X_2)'$ and $(X_2, X_4)'$ are not independent vectors. Very roughly speaking, α_{1234} measures a "higher order" level of dependence that is not measured in this example by η_I and $\eta_{\Sigma_{12}}$, where Σ_{12} is given by (3.8).

Example 3.4. Multinomial.

Let $\underline{X} = (X_1, \dots, X_I)'$ have p.m.f.

$$f_{\underline{X}}(\underline{x}) = \frac{n!}{\prod_{i=1}^I x_i! (n - \sum x_i)!} \prod_{i=1}^I p_i^{x_i} (1 - \sum p_i)^{n - \sum x_i}.$$

Then $E((X_1, \dots, X_J) | (X_{J+1}, \dots, X_I)) = n^*(p_1^*, \dots, p_J^*)$, where $n^* = n - (X_{J+1} + \dots + X_I)$.

and $p_i^* = p_i / (1 - (p_{J+1} + \dots + p_I))$, $i = 1, \dots, J$. It follows that $\text{Cov } E((X_1, \dots, X_J)' | (X_{J+1}, \dots, X_I)) = \Sigma_{1.2}$, where

$$\Sigma_{1.2} = \{n^*(p_i^* \delta_{ij} - p_i^* p_j^*)\}, \quad (3.9)$$

where $\delta_{ij} = 1$, if $i = j$, and $= 0$, otherwise. Since $\text{Cov}(X_1, \dots, X_J)' \equiv \Sigma_1$ is of the same form as (3.9), with n^* , p_i^* replaced by n , p_i , it follows that $\eta_1((X_1, \dots, X_J)'; (X_{J+1}, \dots, X_I)') = (n^*/n)(\Sigma p_i^*(1-p_i^*)) / (\Sigma p_i(1-p_i))$. Observe that $(\text{Cov}(X_1, \dots, X_J))^{-1} = \{n^{-1}(p_i^{-1} \delta_{ij} + (1 - \sum_{k=1}^J p_k)^{-1})\}$, so that $\eta_{\Sigma_1}((X_1, \dots, X_J)'; (X_{J+1}, \dots, X_I)') = (J)^{-1} [\sum_{i=1}^{J+1} ((1-p_i^*)(p_i^*/p_i))]$, where for $i = J+1$, $p_{J+1}^* = 1 - (p_1^* + \dots + p_J^*)$ and $p_{J+1} = 1 - (p_1 + \dots + p_J)$.

4. MOST PREDICTABLE LINEAR FUNCTIONS.

The multivariate correlation ratio $\eta_A(Y; X)$ measures the amount of predictability X has for Y for any jointly distributed random vectors X and Y . This relational notion is, of course, directional in that we are using the information in X in order to predict Y . To further understand this predictive relationship, it is natural to attempt to find what information in Y , that is, what function of Y , is most predictable from X . If we allow the consideration of all suitable measurable functions of Y , and follow the minimum norm approach with $A = I$, we are in essence, evaluating the sup-correlation $\rho'(X, Y)$ and finding the appropriate maximizing functions. As noted previously, this often is a difficult task mathematically. Consequently, we proceed analogously to the theory of principal components and canonical correlations, and restrict attention to linear functions of Y . The goal then is to find

which linear functions of Y are most predictable from X and to measure this degree of predictability.

For simplicity, we assume in the following that Σ_Y is non-singular; however, this could be avoided by examining determinantal equations.

Theorem 4.1. Let $Y: q \times 1$, $X: p \times 1$ be jointly distributed random vectors with $\text{tr } \Sigma_Y < \infty$ and Σ_Y nonsingular. Then

$$\max_{\beta} \eta^2(\beta'Y|X) = \lambda_{\text{MAX}}(\Sigma_Y^{-1/2} \text{Cov } E(Y|X) \Sigma_Y^{-1/2}), \quad (4.1)$$

where λ_{MAX} denotes the largest eigenvalue. Furthermore, this max occurs at $\beta = c \Sigma_Y^{-1/2} e_{\text{MAX}}$, where c is any nonzero constant, and e_{MAX} is an eigenvector corresponding to λ_{MAX} .

Proof.

$$\begin{aligned} \max_{\beta} \eta^2(\beta'Y|X) &= \max_{\beta} \frac{\text{Var } E(\beta'Y|X)}{\text{Var}(\beta'Y)} \\ &= \max_{\beta} \frac{\beta' \text{Cov } E(Y|X) \beta}{\beta' \Sigma_Y \beta} \\ &= \max_Y \frac{Y' \Sigma_Y^{-1/2} \text{Cov } E(Y|X) \Sigma_Y^{-1/2} Y}{Y' Y}, \end{aligned}$$

where $Y = \Sigma_Y^{-1/2} \beta$, so that the result is immediate (e.g., Rao (1973, p. 62)).

An immediate implication of Theorem 4.1 is the following corollary, whose proof is obvious.

Corollary 4.1. Let $\underline{Y}: q \times 1$, $\underline{X}: p \times 1$ be jointly distributed random vectors with $\text{tr } \Sigma_Y < \infty$ and Σ_Y nonsingular. Then

$$\begin{aligned} \text{Max}_{\substack{\beta: q \times 1 \\ g: R^p \rightarrow R^1}} \rho^2(\beta' \underline{Y}, g(\underline{X})) &= \lambda_{\text{MAX}}(\Sigma_Y^{-1/2} \text{Cov } E(\underline{Y}|\underline{X}) \Sigma_Y^{-1/2}), \end{aligned}$$

and occurs at $\beta = c \Sigma_Y^{-1/2} e_{\text{MAX}}$ and $g(x) = c e_{\text{MAX}}' \Sigma_Y^{-1/2} E(\underline{Y}|\underline{X})$.

The result of Corollary 4.1 is in some sense "half way" between linear canonical correlation for arbitrary random vectors and the sup-correlation of Sarmanov and Zaharov (1960).

For convenience, we introduce the following definition:

Definition 4.1. $L_1(\underline{Y}; \underline{X}) = \{\lambda_1(\Sigma_Y^{-1/2} \text{Cov } E(\underline{Y}|\underline{X}) \Sigma_Y^{-1/2})\}^{1/2}$, where λ_1 denotes the 1th largest eigenvalue.

When $\underline{Y}: q \times 1$ and $\underline{X}: p \times 1$ have a joint multivariate normal distribution $L_1(\underline{Y}; \underline{X}) = C_1(\underline{Y}, \underline{X})$, the first canonical correlation. For Example 3.3, $L_1^2((X_1, X_2)'; (X_3, X_4)')$ is the largest root of the determinantal equation $|\Sigma_{12 \cdot 34} - \lambda \Sigma_{12}| = 0$ where $\Sigma_{12 \cdot 34}$ is given by (3.7) and Σ_{12} by (3.8).

As in canonical correlation theory, multivariate versions of Theorem 4.1 could be obtained by examining uncorrelated iterations of (4.1). Another approach which we follow is to employ η_A . Suppose \underline{Y} is $q \times 1$ and \underline{X} is $p \times 1$; let B be an $r \times q$ matrix and A be an $r \times r$ positive definite matrix. Consider maximizing $\eta_A(B\underline{Y}; \underline{X})$ over B satisfying $\Sigma_{BY} = A^{-1}$. When $A = I$, this is equivalent to uncorrelatedness among the entries of $B \underline{Y}$.

Theorem 4.2. Let $\underline{Y}: q \times 1$, $\underline{X}: p \times 1$ be jointly distributed random vectors with $\text{tr } \underline{\Sigma}_Y < \infty$ and $\underline{\Sigma}_Y$ nonsingular; and let $\underline{A}: r \times r$ be positive definite. Then

$$\max_{\substack{\underline{B}: r \times q \\ \text{Cov}(\underline{B}\underline{Y}) = \underline{A}^{-1}}} \eta_A^2(\underline{B}\underline{Y}; \underline{X}) = \left\{ r^{-1} \sum_{i=1}^r L_1^2(\underline{Y}; \underline{X}) \right\}^{\frac{1}{2}}.$$

Proof. Observe that

$$\begin{aligned} \max_{\underline{B}} \eta_A^2 &= \max_{\underline{B}} \text{tr}(\underline{A}^{-1} \text{Cov } E(\underline{B}\underline{Y}|\underline{X})) / (\text{tr } \underline{A}^{-1} \underline{\Sigma}_{BY}) \\ &= \max_{\underline{B}} \frac{\text{tr}(\underline{A}^{-\frac{1}{2}} \underline{B} \text{Cov } E(\underline{Y}|\underline{X}) \underline{B}' \underline{A}^{-\frac{1}{2}})}{\text{tr}(\underline{A}^{-\frac{1}{2}} \underline{B} \underline{\Sigma}_Y \underline{B}' \underline{A}^{-\frac{1}{2}})} \\ &= r^{-1} \max_{\substack{\underline{C}: \underline{C}\underline{C}' = \underline{I}}} \text{tr } \underline{C} \underline{\Sigma}_Y^{-\frac{1}{2}} \text{Cov } E(\underline{Y}|\underline{X}) \underline{\Sigma}_Y^{-\frac{1}{2}} \underline{C}', \end{aligned}$$

where $\underline{C} = \underline{A}^{-\frac{1}{2}} \underline{B} \underline{\Sigma}_Y^{\frac{1}{2}}$. The result follows immediately (e.g., Rao (1973, pp. 63)).

For $\underline{X}, \underline{Y}$ having a multivariate normal distribution $\max_{\underline{B}} \eta_A^2(\underline{B}\underline{Y}; \underline{X}) = r^{-1} \sum_{i=1}^r C_i^2(\underline{Y}, \underline{X})$ i.e., the average of the squares of the first r canonical correlations.

An actual maximizing \underline{B} in Theorem 4.2 is given by

$$\underline{B} = \underline{A}^{\frac{1}{2}} (\underline{e}_1 : \dots : \underline{e}_r)' \underline{\Sigma}_Y^{-\frac{1}{2}},$$

where \underline{e}_i is an eigenvector corresponding to $L_i(\underline{Y}; \underline{X})$ and $\underline{e}_1, \dots, \underline{e}_r$ are orthogonal vectors.

The q values, $\{ \sum_{i=1}^r L_i^2(\underline{Y}; \underline{X}) \}^{\frac{1}{2}}$, $j = 1, \dots, q$, could themselves be viewed as measures of dependences of \underline{Y} upon \underline{X} and their properties explored. Note that knowing these q values is equivalent to knowing

the q eigenvalues, $L_i(\underline{Y}; \underline{X})$, $i = 1, \dots, q$.

For a fixed A , the estimation of $\eta_A(\underline{Y}; \underline{X})$ based upon n observations would be of interest, as would the estimation of $L_i(\underline{Y}; \underline{X})$. Both the actual estimation techniques employed and the resultant distribution theory would be dependent upon the underlying model assumptions.

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where A is a given positive definite matrix. The properties of η_A are discussed, with particular attention paid to a "correlation-maximizing" property. A number of examples illustrating the application of η_A are given; these examples include the multivariate normal, the elliptically symmetric distributions, the Farlie-Morgenstern-Gumbel family, and the multinomial. The problem of maximizing $\eta_A(BY; X)$ over suitable matrices B is considered and the results that are obtained are related to canonical correlations for the multivariate normal.

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