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A NONLINEAR HYPERBOLIC VOLTERRA EQUATION IN VISCOELASTICITY.(U)
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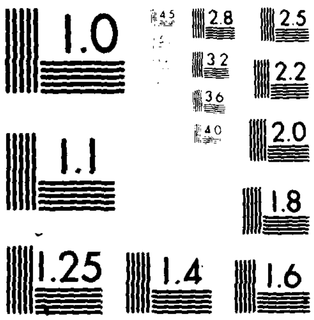

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A NONLINEAR HYPERBOLIC VOLTERRA EQUATION IN VISCOELASTICITY

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C. M. Dafermos* and J. A. Nohel**

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ABSTRACT

A general model for the nonlinear motion of a one dimensional, finite, homogeneous, viscoelastic body is developed and analysed by an energy method. It is shown that under physically reasonable conditions the nonlinear boundary, initial value problem has a unique, smooth solution (global in time), provided the given data are sufficiently "small" and smooth; moreover, the solution and its derivatives of first and second order decay to zero as $t \to \infty$. Various modifications and generalizations, including two and three dimensional problems, are also discussed.

AMS (MOS) Subject Classifications: 73F15, 73F99, 73H10, 45G10, 45K05, 45D05, 45M10, 45M99, 35L55, 35L67, 47H10, 47H15

Key Words: nonlinear viscoelastic motion, materials with memory, stress-strain relaxation functions, nonlinear Volterra equations, hyperbolic equations, dissipation, shocks, global smooth solutions, energy methods, asymptotic behaviour, resolvent kernels, frequency domain method

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SIGNIFICANCE AND EXPLANATION

In nonlinear systems of "hyperbolic" type, characteristic speeds are not constant so that weak waves are amplified and smooth solutions may blow up in finite time due to the formation of shock waves. It is interesting to consider situations where this destabilizing mechanism coexists (and thus competes) with dissipation.

In certain cases (e.g., viscosity of the rate type) dissipation is so powerful that waves cannot break and solutions remain globally smooth. A more interesting situation arises when the amplification and decay mechanisms have comparable power so that the outcome of their confrontation cannot be predicted at the outset. Elementary dimensional considerations indicate that breaking of waves develops on a time scale inversely proportional to wave amplitude while dissipation proceeds at a roughly constant time scale. It should thus be expected that dissipation prevails and waves do not break when the initial data are "small". Results of this type were obtained by T. Nishida for the quasilinear wave equation with first-order frictional damping for sufficiently smooth and small initial displacements and initial velocities.

In this paper we develop and study a general nonlinear model for the motion of a one dimensional, finite, homogeneous body. Here the dissipation mechanism which is induced by memory effects of the viscoelastic materials (stress-strain relaxation function - the stress is a nonlinear functional rather than a function of the strain) is different and more subtle. Using elementary energy methods, which are combined with frequency domain techniques for nonlinear Volterra equations, we show that under physically reasonable conditions on the stress-strain relaxation function, the known history of the displacement, the nonlinearities of the model, and on the assigned external body force, the boundary-history value problem (1.9), (1.18) in the text which describes the model has a unique, smooth solution (global in time), provided the given data (history and external body force) are sufficiently smooth and "small". Moreover, we also show that the solution and its spacial and time derivatives of first and second order decay to zero as $t \to \infty$. Various modifications and generalizations of the model, including two and three dimensional problems, are also considered.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A NONLINEAR HYPERBOLIC VOLTERRA EQUATION IN VISCOELASTICITY

C. M. Dafermos* and J. A. Nohel**

1. Introduction. In nonlinear systems of "hyperbolic" type, characteristic speeds are not constant so that weak waves are amplified and smooth solutions may blow up in finite time due to the formation of shock waves. It would be interesting to consider situations where this destabilizing mechanism coexists (and thus competes) with dissipation.

In certain cases (e.g., viscosity of the rate type) dissipation is so powerful that waves cannot break and solutions remain globally smooth. A more interesting situation arises when the amplification and decay mechanisms have comparable power so that the outcome of their confrontation cannot be predicted at the outset. Elementary dimensional considerations indicate that breaking of waves develops on a time scale inversely proportional to wave amplitude while dissipation proceeds at a roughly constant time scale. It should thus be expected that dissipation prevails and waves do not break when the initial data are "small". Results of this type for quasilinear wave equations with frictional damping were first obtained by Nishida [1] and, subsequently, by Matsumura [2], who uses methodology that goes back to Schauder [3]. The more delicate situation of thermal damping (one dimensional thermoelasticity) is discussed in Slemrod [4].

A different, subtler type of dissipation mechanism is induced by memory effects and arises in nonlinear viscoelasticity. A simple, one dimensional, model corresponds to the constitutive relation

$$\sigma(t,x) = \varphi(e(t,x)) + \int_{-\infty}^{t} a'(t-\tau)\psi(e(\tau,x))d\tau \ , \qquad (1.1)$$

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where σ is the stress, e the strain, a the relaxation function with $' = d/dt$, and φ, ψ assigned constitutive functions. We normalize the relaxation function so that $a(\infty) = 0$. When the reference configuration is a natural state, $\varphi(0) = \psi(0) = 0$. Experience indicates that $\varphi(e)$, $\psi(e)$, as well as the equilibrium stress

$$\chi(e) \overset{\text{def}}{=} \varphi(e) - a(0)\psi(e) \tag{1.2}$$

are increasing functions of e, at least near equilibrium (e small). Moreover, the effect of viscosity is dissipative. To express mathematically the above physical requirements, we impose upon $a(t)$, $\varphi(e)$, $\psi(e)$ and $\chi(e)$ the following assumptions:

(1.3) $a(t) \in W^{2,1}(0,\infty)$, $a(t)$ is strongly positive definite on $[0,\infty)$;

$$\varphi(e) \in C^3(-\infty,\infty), \quad \varphi(0) = 0, \quad \varphi'(0) > 0 \; ; \tag{1.4}$$

$$\psi(e) \in C^3(-\infty,\infty), \quad \psi(0) = 0, \quad \psi'(0) > 0 \; ; \tag{1.5}$$

$$\chi'(0) = \varphi'(0) - a(0)\psi'(0) > 0 \; . \tag{1.6}$$

Assumption (1.3), which requires that $a(t) - \alpha \exp(-t)$ be a positive definite kernel on $[0,\infty)$ for some $\alpha > 0$, expresses the dissipative character of viscosity. Smooth, integrable, nonincreasing, convex relaxation functions, e.g.,

$$a(t) = \sum_{k=1}^{K} \nu_k \exp(-\mu_k t), \quad \nu_k > 0, \quad \mu_k > 0 \; , \tag{1.7}$$

which are commonly employed in the applications of the theory of viscoelasticity, satisfy (1.3).

It is often convenient to express σ, given by (1.1), in terms of equilibrium stress, namely (integrate (1.1) by parts and use (1.2)),

$$\sigma(t,x) = \chi(e(t,x)) + \int_{-\infty}^{t} a(t-\tau)\psi(e(\tau,x))_\tau d\tau \; . \tag{1.8}$$

We now consider a homogeneous, one dimensional body (string or bar) with reference configuration $[0,1]$ of density $\rho = 1$ (for simplicity) and constitutive relation (1.1) which is moving under the action of an assigned body force $g(t,x)$,

$-\infty < t < \infty$, $0 \leq x \leq 1$. We let $u(t,x)$ denote the displacement of particle x at time t in which case the strain is $e(t,x) = u_x(t,x)$. Thus the equation of motion $\rho u_{tt} = \sigma_x + \rho g$ here takes the form

$$(1.9) \qquad u_{tt} = \varphi(u_x)_x + \int_{-\infty}^{t} a'(t-\tau)\psi(u_x)_x d\tau + g, \quad -\infty < t < \infty,\ 0 \leq x \leq 1 ,$$

or, if one uses representation (1.8) for the stress,

$$(1.10) \qquad u_{tt} = \chi(u_x)_x + \int_{-\infty}^{t} a(t-\tau)\psi(u_x)_{\tau x} d\tau + g, \quad -\infty < t < \infty, \quad 0 \leq x \leq 1 .$$

The history of the motion of the body up to time $t = 0$ is assumed known, i.e.,

$$(1.11) \qquad u(t,x) = v(t,x), \quad -\infty < t \leq 0, \quad 0 \leq x \leq 1 ,$$

where $v(t,x)$ is a given function which satisfies equation (1.9) together with appropriate boundary conditions, for $t \leq 0$. Our task is to determine a smooth extension $u(t,x)$ of $v(t,x)$ on $(-\infty,\infty) \times [0,1]$ which satisfies (1.9) together with assigned boundary conditions, for $-\infty < t < \infty$.

Upon setting

$$(1.12) \qquad h = \int_{-\infty}^{0} a'(t-\tau)\psi(v_x)_x d\tau + g, \quad t \geq 0, \quad 0 \leq x \leq 1 ,$$

$$(1.13) \qquad u_0(x) = v(0,x), \quad u_1(x) = v_t(0,x), \quad 0 \leq x \leq 1 ,$$

the history-value problem (1.9), (1.11) reduces to the initial-value problem

$$(1.14) \qquad u_{tt} = \varphi(u_x)_x + \int_{0}^{t} a'(t-\tau)\psi(u_x)_x d\tau + h, \quad 0 \leq t < \infty, \quad 0 \leq x \leq 1 ,$$

$$(1.15) \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad 0 \leq x \leq 1 .$$

Conversely, (1.14), (1.15) can be reduced to (1.9), (1.11) by constructing a function $v(t,x)$ on $(-\infty,0] \times [0,1]$ which satisfies $v(0,x) = u_0(x)$, $v_t(0,x) = u_1(x)$,

$$(1.16)\begin{cases} v_{tt}(0,x)=\varphi(u_{0x})_x+h(0,x)\ , & 0 < x < 1, \\ v_{ttt}(0,x)=\varphi''(u_{0x})u_{0xx}u_{1x}+\varphi'(u_{0x})u_{1xx}+a'(0)\psi(u_{0x})_x+h_t(0,x), & 0 < x < 1, \end{cases}$$

together with appropriate boundary conditions, for $t < 0$, and then defining $g(t,x)$ on $(-\infty,\infty) \times [0,1]$ by

$$(1.17)\qquad g(t,x) = \begin{cases} v_{tt} - \varphi(v_x)_x - \int_{-\infty}^{t} a'(t-\tau)\psi(v_x)_x d\tau, & t < 0,\ 0 < x < 1\ , \\ h - \int_{-\infty}^{0} a'(t-\tau)\psi(v_x)_x d\tau, & t > 0,\ 0 < x < 1\ . \end{cases}$$

The purpose of (1.16) is to ensure that $g(t,x)$, as defined by (1.17), has the smoothness properties, across $t = 0$, which will be required below in the existence theorem.

For the special case $\psi(e) \equiv \varphi(e)$ variants of existence theorems for (1.14), (1.15) were established by MacCamy [5], Dafermos and Nohel [6] and Staffans [7]. The assumption $\psi \equiv \varphi$ allows one to invert the linear Volterra integral operator on the right-hand side of (1.14) and thus express $\varphi(u_x)_x$ in terms of $u_{tt} - h$ through an inverse Volterra integral operator using the resolvent kernel associated with a'. One may then transfer time derivatives from u_{tt} to the resolvent kernel via integration by parts. This procedure reveals the instantaneous character of dissipation and, at the same time, renders the memory term linear and milder, thus simplifying the analysis considerably. On the other hand, the above approach is somewhat artificial: By inverting the right-hand side of (1.14), one loses sight of the original equation and of the physical interpretation of the derived a priori estimates. More importantly, the physical appropriateness of the restriction $\psi = \varphi$ is by no means clear.

<u>Remark</u>: The present normalization of the kernel a with $a(\infty) = 0$ is different from that in the existing literature (see [5], [6], [7]). The reader should note a', not a, enters the constitutive relation (1.1) as well as the equation of

motion (1.9). The present normalization is more convenient for technical reasons; for, the equivalent form (1.8) of the constitutive equation in which a (rather than a') enters, and the corresponding equation of motion (1.10), are extremely convenient for obtaining the crucial a priori estimates in our analysis, when a satisfies assumption (1.3). In the earlier literature in which only the special case $\psi \equiv \varphi$ was studied, the normalization

$$a(t) = a_\infty + A(t) \qquad 0 \leq t < \infty ,$$

$a(0) = 1$, $a_\infty > 0$, $A \in W^{2,1}(0,\infty)$, A strongly positive was used.

In another noteworthy special case, when $a(t) = \exp(-\mu t)$, (1.9) is equivalent to the third order partial differential equation

$$u_{ttt} + \mu u_{tt} = \varphi(u_x)_{tx} + \mu\chi(u_x)_x + g_t + \mu g$$

studied by Greenberg [8].

In this paper we show how one may deal with Equation (1.9) directly and establish existence of solutions without the assumption $\varphi \equiv \psi$. We will consider in detail the case where the boundary of the body is free of traction which leads to boundary conditions

(1.18) $$\sigma(t,0) = \sigma(t,1) = 0, \qquad -\infty < t < \infty .$$

Other types of boundary conditions will be discussed in Section 4. The change of variable (superposition of a rigid motion)

(1.19) $$u(t,x) = \bar{u}(t,x) + m_0 + m_1 t + \int_0^t \int_0^\tau \int_0^1 g(s,y)\,dy\,ds\,d\tau$$

shows that without loss of generality we may assume

(1.20) $$\int_0^1 g(t,x)dx = 0, \qquad -\infty < t < \infty ,$$

(1.21) $$\int_0^1 u(t,x)dx = 0, \qquad -\infty < t < \infty .$$

Of the body force g we require

$$(1.22)\quad \begin{cases} g(t,\cdot), g_t(t,\cdot), g_x(t,\cdot) \quad \text{in} \quad C((-\infty,\infty);\, L^2(0,1)) \cap L^2((-\infty,\infty);\, L^2(0,1)) \\ g(t,x)=g_1(t,x)+g_2(t,x) \text{ with } g_{1tt}(t,\cdot), g_{2tx}(t,\cdot) \text{ in } L^2((-\infty,\infty);L^2(0,1)). \end{cases}$$

As noted above, despite the presence of viscous dissipation, it is not to be expected that a global smooth solution to (1.9), (1.11), (1.18) will exist unless the amplitude of waves remains small. Consequently, one may only hope to obtain global existence results under the restriction that $g(t,x)$ be appropriately "small". We "measure" $g(t,x)$ by

$$(1.23)\quad G \stackrel{\text{def}}{=} \sup_{(-\infty,\infty)} \int_0^1 \{g^2+g_t^2+g_x^2\}(t,x)dx + \int_{-\infty}^{\infty}\int_0^1 \{g^2+g_t^2+g_x^2+g_{1tt}^2+g_{2tx}^2\}dxdt .$$

Our main result is

__Theorem 1.1. Under assumptions__ (1.3), (1.4), (1.5), (1.6), __there exists a constant__ $\mu > 0$ __with the following property: For every__ $g(t,x)$ __on__ $(-\infty,\infty) \times [0,1]$ __which satisfies__ (1.20) __and__ (1.22) __with__

$$(1.24)\qquad G < \mu^2 ,$$

__and for any__ $v(t,x)$ __on__ $(-\infty,0] \times [0,1]$, __with__ $v(t,\cdot)$, $v_t(t,\cdot)$, $v_x(t,\cdot)$, $v_{tt}(t,\cdot)$, $v_{tx}(t,\cdot)$, $v_{xx}(t,\cdot)$, $v_{ttt}(t,\cdot)$, $v_{ttx}(t,\cdot)$, $v_{txx}(t,\cdot)$, $v_{xxx}(t,\cdot)$ __in__ $C((-\infty,0];\, L^2(0,1)) \cap L^2((-\infty,0];\, L^2(0,1))$, __which satisfies Equation__ (1.9) __together with the boundary conditions__ (1.18) __for__ $t < 0$, __there exists a unique__ $u(t,x)$ __on__ $(-\infty,\infty) \times [0,1]$, __with__ $u(t,\cdot)$, $u_t(t,\cdot)$, $u_x(t,\cdot)$, $u_{tt}(t,\cdot)$, $u_{tx}(t,\cdot)$, $u_{xx}(t,\cdot)$, $u_{ttt}(t,\cdot)$, $u_{ttx}(t,\cdot)$, $u_{txx}(t,\cdot)$, $u_{xxx}(t,\cdot)$ __in__ $C((-\infty,\infty);\, L^2(0,1)) \cap L^2((-\infty,\infty);\, L^2(0,1))$, __which satisfies__ (1.9), (1.11), (1.18), __as well as__ (1.21). __Furthermore,__

$$(1.25)\quad u(t,\cdot),u_t(t,\cdot),u_x(t,\cdot),u_{tt}(t,\cdot),u_{tx}(t,\cdot),u_{xx}(t,\cdot) \xrightarrow[{[0,1]}]{\text{unif.}} 0,\ t \to \infty.$$

The proof of the above theorem employs the general strategy developed in [2,6,7]. We first establish, in Section 2, the existence of a local solution, defined on a maximal interval $(-\infty, T_0)$, with the property that when $T_0 < \infty$ a certain norm blows up as $t \uparrow T_0$. Then, in Section 3, we show that, due to viscous dissipation, the afforementioned norm remains uniformly bounded on the maximal interval, provided that (1.24) holds with μ sufficiently small. In particular, $T_0 = \infty$ and the smooth solution exists *globally*.

In the final Section 4, we have collected information on various extensions of the above results. We show how one can handle boundary conditions other than (1.18). We indicate how alternative sets of assumptions on $v(t,x)$ and $g(t,x)$ lead to variants of Theorem 1.1 rendering information on the smoothness of solutions. Finally, we explain how the present techniques may be used to establish existence theorems for the equations of multidimensional viscoelasticity as well as abstract integrodifferential equation in Hilbert space.

2. Local Solutions. In this section we establish a local existence theorem on a maximal interval. It is more convenient to work with Equation (1.14) to which, as we have seen, (1.9) may be reduced. Also we shall impose here boundary conditions

(2.1) $$u_x(t,0) = u_x(t,1) = 0, \qquad t \geq 0 ,$$

which, though apparently stronger than (1.18), are actually equivalent to (1.18), as will be shown in Section 3. Finally, we temporarily strengthen assumption (1.4) into

(2.2) $$\varphi(e) \in C^3(-\infty,\infty), \quad \varphi(0) = 0, \quad \varphi'(e) \geq \kappa > 0, \qquad -\infty < e < \infty .$$

On the other hand, assumptions $\psi'(0) > 0$, $\chi'(0) > 0$ and the positivity of the kernel $a(t)$ *will not play any role in this section.*

Theorem 2.1. Let $u_0(x)$, $u_{0x}(x)$, $u_{0xx}(x)$, $u_{0xxx}(x)$, $u_1(x)$, $u_{1x}(x)$, $u_{1xx}(x)$ be in $L^2(0,1)$ and assume

(2.3) $$u_{0x}(0) = u_{0x}(1) = 0, \quad u_{1x}(0) = u_{1x}(1) = 0 .$$

Moreover, let $h(t,x)$ be defined on $[0,\infty) \times [0,1]$ with $h(t,\cdot)$, $h_t(t,\cdot)$, $h_x(t,\cdot)$ in $C([0,\infty); L^2(0,1))$ and $h(t,x) = h_1(t,x) + h_2(t,x)$, h_{1tt}, h_{2tx} in

$L^2([0,\infty); L^2(0,1))$. *Then there is* T_0, $0 < T_0 \leq \infty$, *and a unique function* $u(t,x) \in C^2([0,T_0) \times [0,1])$, *with* $u_{ttt}(t,\cdot)$, $u_{ttx}(t,\cdot)$, $u_{txx}(t,\cdot)$, $u_{xxx}(t,\cdot)$ *in* $C([0,T]; L^2(0,1))$, *for every* $0 < T < T_0$, *such that* u *satisfies* (1.14) *on* $[0,T_0) \times [0,1]$ *together with initial conditions* (1.15) *and boundary conditions* (2.1) *on* $[0,T_0)$. *Furthermore, if* $T_0 < \infty$,

$$(2.4) \qquad \int_0^1 \{u^2(t,x)+u_t^2(t,x)+u_x^2(t,x)+u_{tt}^2(t,x)+u_{tx}^2(t,x)+u_{xx}^2(t,x)+u_{ttt}^2(t,x) + u_{ttx}^2(t,x)+u_{txx}^2(t,x)+u_{xxx}^2(t,x)\}dx \to \infty, \quad \text{as} \quad t \uparrow T_0 .$$

We note that $h(t,x)$ and $u_0(x)$, $u_1(x)$, defined by (1.12), (1.13) with $v(t,x)$ and $g(t,x)$ as in Theorem 1.1, do satisfy the assumptions of Theorem 2.1.

The proof of Theorem 2.1 which is a variant of the local result in [6] will be based upon the Banach fixed point theorem. We begin with some preparation. For $M,T > 0$, we let $X(M,T)$ denote the set of functions $w(t,x)$ on $[0,T] \times [0,1]$, with $w(t,\cdot)$, $w_t(t,\cdot)$, $w_x(t,\cdot)$, $w_{tt}(t,\cdot)$, $w_{tx}(t,\cdot)$, $w_{xx}(t,\cdot)$, $w_{ttt}(t,\cdot)$, $w_{ttx}(t,\cdot)$, $w_{txx}(t,\cdot)$, $w_{xxx}(t,\cdot)$ in $L^\infty([0,T]; L^2(0,1))$ which assume initial data $w(0,x) = u_0(x)$, $w_t(0,x) = u_1(x)$ and boundary conditions $w_x(t,0) = w_x(t,1) = 0$, $t \in [0,T]$, and satisfy

$$(2.5) \qquad \underset{[0,T]}{\text{ess-sup}} \int_0^1 \{w_{ttt}^2(t,x) + w_{ttx}^2(t,x) + w_{txx}^2(t,x) + w_{xxx}^2(t,x)\}dx \leq M^2 .$$

For $w(t,x) \in X(M,T)$, (2.5) and the Poincaré inequality yield

$$(2.6) \qquad w_x^2(t,x) + w_{tx}^2(t,x) + w_{xx}^2(t,x) \leq M^2, \quad 0 \leq t \leq T, \quad 0 \leq x \leq 1 .$$

We now consider the map $S : X(M,T) \to C^2([0,T] \times [0,1])$ which carries $w(t,x) \in X(M,T)$ into the solution $u(t,x)$ of the linear equation

$$(2.7) \qquad u_{tt} - \varphi'(w_x)u_{xx} = \int_0^t a'(t-\tau)\psi(w_x)_x d\tau + h$$

satisfying initial conditions (1.15) and boundary conditions (2.1). We note that $\varphi'(w_x(t,x))$ is C^1 smooth and $\varphi'(w_x)_{tt}$, $\varphi'(w_x)_{tx}$ are in $L^\infty([0,T]; L^2(0,1))$. Furthermore, if $f(t,x)$ denotes the right-hand side of (2.7), then $f(t,\cdot)$, $f_t(t,\cdot)$, $f_x(t,\cdot)$ are in $C([0,T]; L^2(0,1))$ and $f(t,x) = f_1(t,x) + f_2(t,x)$ with f_{1tt}, f_{2tx} in $L^2([0,T]; L^2(0,1))$. It then follows by standard theory that $u_{ttt}(t,\cdot)$, $u_{ttx}(t,\cdot)$, $u_{txx}(t,\cdot)$ and $u_{xxx}(t,\cdot)$ are in $C([0,T]; L^2(0,1))$. Our strategy is to show that, under proper conditions, S has a unique fixed point in $X(M,T)$ which will obviously be the solution to (1.14), (1.15), (2.1) with the desired properties.

<u>Lemma 2.1</u>. <u>When</u> M <u>is sufficiently large and</u> T <u>is sufficiently small</u>, S <u>maps</u> $X(M,T)$ <u>into itself</u>.

<u>Proof</u>. We fix $\eta > 0$ and apply to (2.7) the forward difference operator Δ, $(\Delta\omega)(t) \overset{\text{def}}{=} \omega(t + \eta) - \omega(t)$, thus obtaining

$$\begin{aligned}\Delta u_{tt} - \varphi'(w_x)\Delta u_{xx} &= \Delta\varphi'(w_x)u_{xx} + \Delta\varphi'(w_x)\Delta u_{xx} \\ &\quad + \Delta \int_0^t a'(t-\tau)\psi(w_x)_x d\tau + \Delta h \ .\end{aligned} \tag{2.8}$$

We multiply (2.8) by Δu_{txx} and integrate over $[0,s] \times [0,1]$, $0 < s < T$. After appropriate integrations by parts, we divide through by η^2 and we let $\eta \downarrow 0$. We give the details of the computation of one term:

$$\begin{aligned}\int_0^s\int_0^1 \Delta u_{tt}\Delta u_{txx}dxdt &= - \int_0^s\int_0^1 \Delta u_{ttx}\Delta u_{tx}dxdt \\ &= -\frac{1}{2}\int_0^1 (\Delta u_{tx})^2(s,x)dx + \frac{1}{2}\int_0^1 (\Delta u_{tx})^2(0,x)dx \ ,\end{aligned} \tag{2.9}$$

whence

$$\lim_{\eta\downarrow 0} \frac{1}{\eta^2}\int_0^s\int_0^1 \Delta u_{tt}\Delta u_{txx}dxdt = -\frac{1}{2}\int_0^1 u_{ttx}^2(s,x)dx + \frac{1}{2}\int_0^1 u_{ttx}^2(0,x)dx \tag{2.10}$$

where

(2.11) $$u_{ttx}(0,x) = \varphi(u_{0x}(x))_{xx} + h_x(0,x) .$$

We apply the same procedure to the remaining terms of (2.8) thus obtaining

(2.12) $$\frac{1}{2}\int_0^1 u_{ttx}^2(s,x)dx + \frac{1}{2}\int_0^1 \varphi'(w_x(s,x))u_{txx}^2(s,x)dx = \frac{1}{2}\int_0^1 u_{ttx}^2(0,x)dx$$

$$+ \frac{1}{2}\int_0^1 \varphi'(u_{0x}(x))u_{1xx}^2(x)dx + \frac{1}{2}\int_0^s\int_0^1 \varphi''(w_x)w_{tx}u_{txx}^2 dxdt$$

$$+ \int_0^s\int_0^1 \varphi'''(w_x)w_{tx}w_{xx}u_{xx}u_{ttx}dxdt + \int_0^s\int_0^1 \varphi''(w_x)w_{txx}u_{xx}u_{ttx}dxdt$$

$$+ \int_0^s\int_0^1 \varphi''(w_x)w_{tx}u_{xxx}u_{ttx}dxdt - \int_0^1 a'(s)\psi(u_{0x}(x))_x u_{txx}(s,x)dx$$

$$+ \int_0^1 a'(0)\psi(u_{0x}(x))_x u_{1xx}(x)dx + \int_0^s\int_0^1 a''(t)\psi(u_{0x})_x u_{txx}dxdt$$

$$- \int_0^1 u_{txx}(s,x)\int_0^s a'(s-t)\psi(w_x(t,x))_x dtdx + \int_0^s\int_0^1 a'(0)\psi(w_x)_x u_{txx}dxdt$$

$$+ \int_0^s\int_0^1 u_{txx}\int_0^t a''(t-\tau)\psi(w_x)_x d\tau dxdt + \int_0^s\int_0^1 h_{2tx}u_{ttx}dxdt$$

$$- \int_0^1 h_{1t}(s,x)u_{txx}(s,x)dx + \int_0^1 h_{1t}(0,x)u_{1xx}(x)dx + \int_0^s\int_0^1 h_{1tt}u_{txx}dxdt .$$

We now differentiate (2.7) with respect to t and x to obtain

(2.13) $$u_{ttt} - \varphi'(w_x)u_{txx} - \varphi''(u_{0x})u_{0xx}u_{1x} - \int_0^t \{\varphi''(w_x)w_{\tau x}u_{xx}\}_\tau d\tau$$

$$= a'(t)\psi(u_{0x})_x + \int_0^t a'(t-\tau)\psi(w_x)_{\tau x}d\tau + h_t ,$$

$$(2.14)\qquad u_{ttx} - \varphi'(w_x)u_{xxx} - \varphi''(u_{0x})u_{0xx}^2 - \int_0^t \{\varphi''(w_x)w_{xx}u_{xx}\}_\tau d\tau$$

$$= \int_0^t a'(t-\tau)\psi(w_x)_{xx}d\tau + h_x ,$$

from which we easily get the estimates

$$(2.15)\qquad \int_0^1 u_{ttt}^2(s,x)dx - 6\int_0^1 \varphi'(w_x(s,x))^2 u_{txx}^2(s,x)dx$$

$$\leq 6\int_0^1 \varphi''(u_{0x}(x))^2 u_{0xx}^2(x)u_{1x}^2(x)dx$$

$$+ 6\int_0^1 \{\int_0^s [\varphi''(w_x)w_{tx}u_{xx}]_t dt\}^2 dx + 6\int_0^1 a'(t)^2\psi(u_{0x}(x))_x^2 dx$$

$$+ 6\int_0^1 \{\int_0^s a'(s-t)\psi(w_x)_{tx}dt\}^2 dx + 6\int_0^1\int_0^s h_t^2 dxdt ,$$

$$(2.16)\qquad \int_0^1 \varphi'(w_x(s,x))^2 u_{xxx}^2(s,x)dx - 5\int_0^1 u_{ttx}^2(s,x)dx$$

$$\leq 5\int_0^1 \varphi''(u_{0x}(x))^2 u_{0xx}^4 dx + 5\int_0^1 \{\int_0^s [\varphi''(w_x)w_{xx}u_{xx}]_t dt\}^2 dx$$

$$+ 5\int_0^1 \{\int_0^s a'(s-t)\psi(w_x)_{xx}dt\}^2 dx + 5\int_0^1\int_0^s h_x^2 dxdt .$$

Let us set

$$N^2 = \int_0^1 \{u_0^2 + u_{0x}^2 + u_{0xx}^2 + u_{0xxx}^2 + u_1^2 + u_{1x}^2 + u_{1xx}^2\}dx$$

$$(2.17)\qquad + \sup_{[0,\infty)} \int_0^1 \{h^2(t,x)+h_t^2(t,x)+h_x^2(t,x)\}dx + \int_0^\infty\int_0^1 \{h_{1tt}^2+h_{2tx}^2\}dxdt ,$$

$$(2.18)\qquad V^2 = \sup_{[0,T]} \int_0^1 \{u_{ttt}^2(t,x) + u_{ttx}^2(t,x) + u_{txx}^2(t,x) + u_{xxx}^2(t,x)\}dx .$$

Then, by virtue of (2.5), (2.6), the Poincaré inequality and Schwarz's inequality, every term on the right-hand side of (2.12), (2.15), and (2.16) can be majorized by one of $p(N)$, $Tq(M)V^2$, $p(N)V$, $Tq(M)V$, $T^2q(M)V^2$, $T^{1/2}p(N)V$, $T^2q(M)$, $Tp(N)$, where $p(\cdot)$ and $q(\cdot)$ are locally bounded functions on $[0,\infty)$. Thus, combining (2.12), (2.15) and (2.16) and using (2.2), we arrive at an estimate of the form

$$(2.19)\qquad V^2 \le c\{p(N) + Tq(M)V^2 + p(N)V + Tq(M)V + T^2q(M)V^2 + T^{1/2}p(N)V + T^2q(M) + Tp(N)\} .$$

Applying the Cauchy-Schwarz inequality,

$$(2.20)\qquad \{1 - cTq(M) - 2cT^2q(M) - cTp(N) - 1/2\}V^2 \le c\{2p(N) + cp^2(N) + q(M) + T^2q(M) + Tp(N)\} .$$

Thus, if one fixes $M^2 > 8c\{2p(N) + cp^2(N) + q(M)\}$ and then selects T so small that, at the same time, $cTq(M) + 2cT^2q(M) + cTp(N) < 1/4$ and $cT^2q(M) + cTp(N) \le M^2/8$, (2.20) yields $V^2 \le M^2$ and $u(t,x) \in X(M,T)$. The proof of the lemma is complete.

We now equip $X(M,T)$ with the metric

$$(2.21)\qquad \rho(u,\bar{u}) = \max_{[0,T]} \{\int_0^1 [(u_{tt}-\bar{u}_{tt})^2 + (u_{tx}-\bar{u}_{tx})^2 + (u_{xx}-\bar{u}_{xx})^2](t,x)dx\}^{1/2} ,$$

where $u, \bar{u} \in X(M,T)$. On account of the lower semicontinuity property of norms in Banach space, $X(M,T)$ is complete under ρ.

<u>Lemma 2.2</u>. <u>For</u> M <u>sufficiently large and</u> T <u>sufficiently small, the map</u> $S : X(M,T) \to X(M,T)$ <u>is a strict contraction with respect to the metric</u> ρ.

<u>Proof</u>. Let $w(t,x)$, $\bar{w}(t,x) \in X(M,T)$. We set $u = Sw$, $\bar{u} = S\bar{w}$, $W = w - \bar{w}$, $U = u - \bar{u}$. Then $U(t,x)$ is the solution of the problem:

$$U_{tt} - \varphi'(w_x)U_{xx} = A(t,x)\bar{u}_{xx}\int_0^t W_{\tau x}d\tau \tag{2.22}$$

$$+ \int_0^t a'(t-\tau)[\psi'(w_x)W_{xx} + B(\tau,x)\bar{w}_{xx}W_x]d\tau ,$$

$$U(0,x) = 0, \quad U_t(0,x) = 0, \qquad 0 \le x \le 1 , \tag{2.23}$$

$$U_x(t,0) = U_x(t,1) = 0, \qquad 0 \le t \le T , \tag{2.24}$$

where

$$A(t,x) = \begin{cases} \dfrac{\varphi'(w_x(t,x)) - \varphi'(\bar{w}_x(t,x))}{w_x(t,x) - \bar{w}_x(t,x)} , & w_x(t,x) \ne \bar{w}_x(t,x) \\ \varphi''(w_x(t,x)), & w_x(t,x) = \bar{w}_x(t,x) , \end{cases} \tag{2.25}$$

$$B(t,x) = \begin{cases} \dfrac{\psi'(w_x(t,x)) - \psi'(\bar{w}_x(t,x))}{w_x(t,x) - \bar{w}_x(t,x)} , & w_x(t,x) \ne \bar{w}_x(t,x) \\ \psi''(w_x(t,x)) , & w_x(t,x) = \bar{w}_x(t,x) . \end{cases} \tag{2.26}$$

Furthermore,

$$U_{ttt} - \varphi'(w_x)U_{txx} = A(t,x)\bar{u}_{txx}W_x + \varphi''(w_x)w_{tx}U_{xx} + \varphi''(w_x)\bar{u}_{xx}W_{tx} \tag{2.27}$$

$$- C(t,x)\bar{w}_{tx}W_x + a'(0)\psi'(w_x)W_{xx} + a'(0)B(t,x)\bar{w}_{xx}W_x$$

$$+ \int_0^t a''(t-\tau)\psi'(w_x)W_{xx}d\tau + \int_0^t a''(t-\tau)B(\tau,x)\bar{w}_{xx}W_x d\tau$$

where

$$(2.28)\qquad C(t,x) = \begin{cases} \dfrac{\varphi''(w_x(t,x)) - \varphi''(\bar{w}_x(t,x))}{w_x(t,x) - \bar{w}_x(t,x)}, & w_x(t,x) \neq \bar{w}_x(t,x), \\ \varphi'''(w_x(t,x)), & w_x(t,x) = \bar{w}_x(t,x). \end{cases}$$

Multiplying Equation (2.27) by U_{tt} and integrating over $[0,1] \times [0,s]$, $0 < s \leq T$, we obtain, after certain integrations by parts,

$$(2.29)\qquad \frac{1}{2}\int_0^1 U_{tt}^2(s,x)dx + \frac{1}{2}\int_0^1 \varphi'(w_x(s,x))U_{tx}^2(s,x)dx$$

$$= \frac{1}{2}\int_0^s\int_0^1 \varphi''(w_x)w_{tx}U_{tx}^2 dxdt - \int_0^s\int_0^1 \varphi''(w_x)w_{xx}U_{tt}U_{tx}dxdt$$

$$+ \int_0^s\int_0^1 A\bar{u}_{txx}W_xU_{tt}dxdt + \int_0^s\int_0^1 \varphi''(w_x)w_{tx}U_{tt}U_{xx}dxdt$$

$$+ \int_0^s\int_0^1 \varphi''(w_x)\bar{u}_{xx}W_{tx}U_{tt}dxdt - \int_0^s\int_0^1 C\bar{w}_{tx}W_xU_{tt}dxdt$$

$$+ a'(0)\int_0^s\int_0^1 \psi'(w_x)W_{xx}U_{tt}dxdt + a'(0)\int_0^s\int_0^1 B\bar{w}_{xx}W_xU_{tt}dxdt$$

$$+ \int_0^s\int_0^1 U_{tt}\int_0^t a''(t-\tau)\psi'(w_x)W_{xx}d\tau dxdt$$

$$+ \int_0^s\int_0^1 U_{tt}\int_0^t a''(t-\tau)B\bar{w}_{xx}W_x d\tau dxdt\,.$$

Moreover, from (2.22) we get

$$(2.30)\qquad \int_0^1 \varphi'(w_x(s,x))^2U_{xx}^2(s,x)dx \leq 3\int_0^1 U_{tt}^2(s,x)dx + 3\int_0^1 A^2\bar{u}_{xx}^2\left(\int_0^s W_{\tau x}d\tau\right)^2dx$$

$$+ 3\int_0^1 \left\{\int_0^s a'(s-t)[\psi'(w_x)W_{xx} + B\bar{w}_{xx}W_x]dt\right\}^2dx\,.$$

Combining (2.29) with (2.30) and using (2.2), (2.5), (2.6), the Poincaré inequality and the Cauchy-Schwarz inequality, we arrive, after a long computation, at an estimate of the form

$$(2.31)\qquad \int_0^1 \{U_{tt}^2(s,x) + U_{tx}^2(s,x) + U_{xx}^2(s,x)\}dx$$

$$\leq (T + T^2) \max_{[0,T]} \int_0^1 \{W_{tt}^2(t,x) + W_{tx}^2(t,x) + W_{xx}^2(t,x)\}dx$$

$$+ m \int_0^s \int_0^1 \{U_{tt}^2(t,x) + U_{tx}^2(t,x) + U_{xx}^2(t,x)\}dxdt ,$$

where m depends solely upon $a(t)$, M, and bounds of φ, ψ and their derivatives on the interval $[-M,M]$. In order to assist the reader to see how (2.31) is derived from (2.29), (2.30), we give the details of the estimation of one of the most complicated terms on the right-hand side of (2.29):

$$(2.32)\qquad \int_0^s \int_0^1 U_{tt} \int_0^t a''(t-\tau)\psi'(w_x)W_{xx}d\tau dxdt$$

$$\leq \frac{2}{\varepsilon} \int_0^s \int_0^1 U_{tt}^2 dxdt + 2\varepsilon \int_0^s \int_0^1 \{\int_0^t a''(t-\tau)\psi'(w_x)W_{xx}d\tau\}^2 dxdt$$

$$\leq \frac{2}{\varepsilon} \int_0^s \int_0^1 U_{tt}^2 dxdt$$

$$+ 2\varepsilon \int_0^s \{\int_0^t |a''(t-\tau)|d\tau\}\{\int_0^t |a''(t-\tau)| \int_0^1 \psi'(w_x)^2 W_{xx}^2 d\tau\}dxdt$$

$$\leq \frac{2}{\varepsilon} \int_0^s \int_0^1 U_{tt}^2 dxdt$$

$$+ 2\varepsilon s\{\int_0^\infty |a''(\tau)|d\tau\}^2\{\max_{[-M,M]} \psi'(e)^2\}\{\sup_{[0,T]} \int_0^1 W_{xx}^2(t,x)dx\} .$$

From (2.31) and Gronwall's inequality we deduce

$$\text{(2.33)} \qquad \max_{[0,T]} \int_0^1 \{U_{tt}^2(t,x) + U_{tx}^2(t,x) + U_{xx}^2(t,x)\}dx$$

$$\leq (T + T^2)e^{mT} \max_{[0,T]} \int_0^1 \{W_{tt}^2(t,x) + W_{tx}^2(t,x) + W_{xx}^2(t,x)\}dx .$$

Thus, when T is so small that $(T + T^2)\exp(mT) < \frac{1}{4}$, (2.33) yields

$$\text{(2.34)} \qquad \rho(Sw,S\bar{w}) \leq \frac{1}{2}\rho(w,\bar{w}), \quad \text{for} \quad w,\bar{w} \in X(M,T)$$

and the proof of the lemma is complete.

Proof of Theorem 2.1. From Lemma 2.2 and the Banach fixed point theorem we deduce the existence of a unique fixed point of S in $X(M,T)$, for conveniently large M and appropriately small T, which will be the unique solution of (1.14), (1.15), (2.1) on $[0,T] \times [0,1]$. Let $T_0 \leq \infty$ be the maximal interval of existence of a solution $u(t,x)$ to (1.14), (1.15), (2.1) with $u_t(t,\cdot)$, $u_x(t,\cdot)$, $u_{tt}(t,\cdot)$, $u_{tx}(t,\cdot)$, $u_{xx}(t,\cdot)$, $u_{ttt}(t,\cdot)$, $u_{ttx}(t,\cdot)$, $u_{txx}(t,\cdot)$, $u_{xxx}(t,\cdot)$ in $L^\infty([0,T]; L^2(0,1))$ for every $0 < T < T_0$. If $T_0 < \infty$ and (2.4) is not satisfied, we can extend $u(t,x)$ up to $t = T_0$ so that $u(t,x) \quad C^2([0,T_0] \times [0,1])$. Moreover, by weak convergence in $L^2(0,1)$, $u(T_0,x)$, $u_x(T_0,x)$, $u_{xx}(T_0,x)$, $u_{xxx}(T_0,x)$, $u_t(T_0,x)$, $u_{tx}(T_0,x)$, $u_{txx}(T_0,x)$ are all in $L^2(0,1)$. But then, using $u(T_0,x)$, $u_t(T_0,x)$ as new initial data, we may extend $u(t,x)$ to some interval $[T_0,T_0 + \varepsilon]$, beyond T_0, and this is a contradiction since $[0,T_0)$ is assumed maximal. The function $u(t,x)$ will be a solution of (2.7), with $w(t,x) \equiv u(t,x)$, and thus, as noted above, $u_{ttt}(t,\cdot)$, $u_{ttx}(t,\cdot)$, $u_{txx}(t,\cdot)$, $u_{xxx}(t,\cdot)$ are all in $C([0,T]; L^2(0,1))$, for every T in $(0,T_0)$. The proof is complete.

3. Global Solutions.

Our objective in this section is to show that when the body force is "small" the maximal interval of existence of solution to (1.9), (1.11), (1.18) is $(-\infty,\infty)$ and solutions decay as $t \to \infty$. For that purpose, the dissipative character of viscosity, embodied in assumptions (1.3) on the relaxation function a, plays the crucial role. Assumption (1.3) will be exploited here through its consequences recorded in the following

Lemma 3.1. *There exist positive constants* β, γ *such that*

$$(3.1) \qquad \int_{-\infty}^{s} [\int_{-\infty}^{t} a(t-\tau)w(\tau)d\tau]^2 dt \le \beta \int_{-\infty}^{s} w(t) \int_{-\infty}^{t} a(t-\tau)w(\tau)d\tau dt ,$$

$$(3.2) \qquad \int_{-\infty}^{s} [\int_{-\infty}^{t} a'(t-\tau)w(\tau)d\tau]^2 dt \le \gamma \int_{-\infty}^{s} w(t) \int_{-\infty}^{t} a(t-\tau)w(\tau)d\tau dt ,$$

for any $s \in (-\infty,\infty)$ *and every* $w(t) \in L^2(-\infty,s)$.

The proof can be read off, for example, from Lemma 4.2 of [7], recalling that $a(t), a'(t), a''(t)$ are in $L^1(0,\infty)$, and that (by assumption (1.3)) $a(t) - \alpha \exp(-t)$ is a positive definite kernel on $[0,\infty)$ for some $\alpha > 0$. As a matter of fact, we may use

$$(3.3) \qquad \beta = \frac{1}{\alpha} \{\int_0^\infty |a(t)|dt\}^2 + \frac{4}{\alpha} \{\int_0^\infty |a'(t)|dt\}^2 ,$$

$$(3.4) \qquad \gamma = \frac{1}{\alpha} \{\int_0^\infty |a'(t)|dt\}^2 + \frac{4}{\alpha} \{\int_0^\infty |a''(t)|dt\}^2 .$$

Another important implication of a combination of (1.3), (1.5) and (1.6) is the property:

Lemma 3.2. *Let* $k(t)$ *be the resolvent kernel of the operator*

$$(3.5) \qquad \varphi'(0)\omega(t) + \int_{-\infty}^{t} a'(t-\tau)\psi'(0)\omega(\tau)d\tau ;$$

that is, k *is the unique solution of the linear Volterra equation*

$$(3.6) \qquad \varphi'(0)k(t) + \int_0^t a'(t-\tau)\psi'(0)k(\tau)d\tau = -\psi'(0)a'(t) .$$

Then $k(t) \in L^1(0,\infty)$.

The proof of Lemma 3.2 follows by a standard argument: Since $a'(t) \in L^1(0,\infty)$, the Paley-Wiener theorem states that $k(t) \in L^1(0,\infty)$ if and only if

$$P(z) \overset{\text{def}}{=} \varphi'(0) + \psi'(0)\hat{a}'(z) = \chi'(0) + \psi'(0)z\hat{a}(z) \tag{3.7}$$

does not vanish on the half plane $\operatorname{Re} z \geq 0$. (In (3.7) $z = \xi + i\zeta$ and $\hat{}$ denotes the Laplace transform).

A simple calculation yields

$$\operatorname{Re}P(z) = \chi'(0) + \psi'(0)\xi\operatorname{Re}\hat{a}(z) - \psi'(0)\zeta\operatorname{Im}\hat{a}(z) \ , \tag{3.8}$$

$$\operatorname{Im}P(z) = \psi'(0)\zeta\operatorname{Re}\hat{a}(z) + \psi'(0)\xi\operatorname{Im}\hat{a}(z) \ . \tag{3.9}$$

On account of (1.3), (1.5) and (1.6),

$$\operatorname{Re}P(\xi + i0) = \chi'(0) + \psi'(0)\xi\hat{a}(\xi) > 0, \quad 0 \leq \xi < \infty \ . \tag{3.10}$$

As regards $\operatorname{Im}P(z)$, since by the strong positivity of $a(t)$, $\operatorname{Re}\hat{a}(i\zeta) > 0$, we have $\operatorname{Im}P(0 + i\zeta) = \psi'(0)\zeta\operatorname{Re}\hat{a}(i\zeta)$ is positive for $\zeta > 0$ and negative for $\zeta < 0$. On the other hand, $\operatorname{Im}P(\xi + i0) = 0$, $0 \leq \xi < \infty$. Furthermore, since $a'(t) \in L^1(0,\infty)$, we deduce by the Riemann-Lebesgue lemma that $\lim_{|z|\to\infty} \operatorname{Im}P(z) = \psi'(0) \lim_{|z|\to\infty} \operatorname{Im}\hat{a}'(z) = 0$, uniformly on $\operatorname{Re} z \geq 0$. But $\operatorname{Im}P(z)$ is harmonic on $\operatorname{Re} z \geq 0$ so that, by the maximum principle, we conclude that $\operatorname{Im}P(z) > 0$ on $\{z = \xi + i\zeta \mid \xi \geq 0, \zeta > 0\}$ and $\operatorname{Im}P(z) < 0$ on $\{z = \xi + i\zeta \mid \xi \geq 0, \zeta < 0\}$. In conjunction with (3.10) this yields $P(z) \neq 0$ on $\operatorname{Re} z \geq 0$ and the proof of the lemma is complete.

Before proceeding to the proof of Theorem 1.1, let us show that the boundary conditions (1.18) are equivalent to

$$u_x(t,0) = u_x(t,1) = 0, \quad -\infty < t < \infty \ . \tag{3.11}$$

We multiply (1.8) by $\psi(e(t,x))_t$, integrate over $(-\infty,s)$, $-\infty < s < \infty$, and use the positivity of $a(t)$ to get

$$\int_{-\infty}^{s} \chi(e(t,x))\psi(e(t,x))_t dt \leq 0, \quad -\infty < s < \infty, \ x = 0,1 \ , \tag{3.12}$$

or

$$(3.13) \qquad \Psi(e(s,x)) \leq 0, \quad -\infty < s < \infty, \quad x = 0,1 ,$$

where

$$(3.14) \qquad \Psi(e) \overset{\text{def}}{=} \int_0^e \chi(\eta)\psi'(\eta)d\eta .$$

On account of (1.5), (1.6), $\Psi(e) > 0$ on $(-\delta,\delta)\setminus\{0\}$, δ positive small. Thus (3.13) yields $e(s,x) = 0$, $-\infty < s < \infty$, $x = 0,1$, and (3.11) has been established.

Proof of Theorem 1.1. By virtue of (1.4), (1.5), (1.6), there are positive δ and κ such that

$$(3.15) \qquad \varphi'(e) > \kappa, \ \psi'(e) > \kappa, \ \chi'(e) > \kappa, \ |e| \leq \delta .$$

We modify $\varphi(e)$ outside the interval $[-\delta,\delta]$ so that (2.2) be satisfied and we let $u(t,x)$ be the solution to (1.9), (1.11), (3.11) on a maximal time interval $(-\infty,T_0)$.

For $T \in (-\infty,T_0)$, we set

$$(3.16) \qquad U(T) = \sup_{(-\infty,T]} \int_0^1 \{u^2(t,x)+u_t^2(t,x)+u_x^2(t,x)+u_{tt}^2(t,x)+u_{tx}^2(t,x)$$
$$+ u_{xx}^2(t,x)+u_{ttt}^2(t,x)+u_{ttx}^2(t,x)+u_{txx}^2(t,x)+u_{xxx}^2(t,x)\}dx$$
$$+ \int_{-\infty}^T \int_0^1 \{u^2+u_t^2+u_x^2+u_{tt}^2+u_{tx}^2+u_{xx}^2+u_{ttt}^2+u_{ttx}^2+u_{txx}^2+u_{xxx}^2\}dxdt .$$

Our strategy is to show that there are positive constants ν, K, $\nu \leq \delta$, such that, if

$$(3.17) \qquad |u_x(t,x)|^2 + |u_{tx}(t,x)|^2 + |u_{xx}(t,x)|^2 \leq \nu^2, \ -\infty < t \leq T, \ 0 \leq x \leq 1,$$

then

$$(3.18) \qquad U(T) \leq KG$$

where G is defined by (1.23). Once this claim has been established, we may complete the proof of the theorem by the following line of argument similar to that previously used in [6]: First we note that, by virtue of our assumptions on $v(t,x)$, (3.17) is automatically satisfied, as a strict inequality, when t is sufficiently small. Next we observe that, in view of the Poincaré inequality

$$(3.19)\qquad \begin{aligned} &|u_x(t,x)|^2 + |u_{tx}(t,x)|^2 + |u_{xx}(t,x)|^2 \\ &\le \int_0^1 \{|u_{xx}(t,y)|^2 + |u_{txx}(t,y)|^2 + |u_{xxx}(t,y)|^2\}dy , \end{aligned}$$

and when $G \le \mu^2$ with $\mu^2 < \nu^2/K$, (3.18) implies (3.17) (as a strict inequality). Thus, for $G \le \mu^2 < \nu^2/K$, (3.17) and (3.18) will hold for every T on the maximal interval of existence in which case Theorem 2.1 (in particular (2.4)) implies $T_0 = \infty$. From (3.16), (3.18) we have $u(t,\cdot)$, $u_t(t,\cdot)$, $u_x(t,\cdot)$, $u_{tt}(t,\cdot)$, $u_{tx}(t,\cdot)$, $u_{xx}(t,\cdot)$, $u_{ttt}(t,\cdot)$, $u_{ttx}(t,\cdot)$, $u_{txx}(t,\cdot)$, $u_{xxx}(t,\cdot)$ in $L^2((-\infty,\infty); L^2(0,1)) \cap L^\infty((-\infty,\infty); L^2(0,1))$. Now $u(t,\cdot)$, $u_t(t,\cdot)$, $u_x(t,\cdot)$, $u_{tt}(t,\cdot)$, $u_{tx}(t,\cdot)$, $u_{xx}(t,\cdot)$, $u_{ttt}(t,\cdot)$, $u_{ttx}(t,\cdot)$, $u_{txx}(t,\cdot)$ in $L^2((-\infty,\infty); L^2(0,1))$ implies

$$(3.20)\qquad u(t,\cdot),u_t(t,\cdot),u_x(t,\cdot),u_{tt}(t,\cdot),u_{tx}(t,\cdot),u_{xx}(t,\cdot) \xrightarrow{L^2(0,1)} 0, t \to \infty,$$

which, in conjunction with $u_x(t,\cdot)$, $u_{tx}(t,\cdot)$, $u_{xx}(t,\cdot)$, $u_{ttx}(t,\cdot)$, $u_{txx}(t,\cdot)$, $u_{xxx}(t,\cdot)$ in $L^\infty((-\infty,\infty); L^2(0,1))$, yields (1.25).

It thus remains to verify (3.18) under the assumption (3.17). We fix s in $(-\infty,T]$. The first estimate is obtained by multiplying (1.10) by $\psi(u_x)_{tx}$, integrating over $(-\infty,s] \times [0,1]$ and integrating by parts. The reader should be cautioned that in these and the many integrations by parts which follow, there are several possible ways to carry out such integrations. The ones selected in this section are chosen for the purpose of using the same estimates when considering the boundary conditions (4.1), (4.2) below (see Theorem 4.1). The result of this calculation is

$$(3.21)\quad \frac{1}{2}\int_0^1 \psi'(u_x(s,x))u_{tx}^2(s,x)dx + \frac{1}{2}\int_0^1 \chi'(u_x(s,x))\psi'(u_x(s,x))u_{xx}^2(s,x)dx$$

$$+ \int_{-\infty}^{s}\int_0^1 \psi(u_x)_{tx}\int_{-\infty}^{t} a(t-\tau)\psi(u_x)_{\tau x}d\tau dxdt$$

$$= \frac{1}{2}\int_{-\infty}^{s}\int_0^1 \psi''(u_x)u_{tx}^3 dxdt - \frac{1}{2}\int_{-\infty}^{s}\int_0^1 \{\chi'(u_x)\psi''(u_x)$$

$$- \chi''(u_x)\psi'(u_x)\}u_{tx}u_{xx}^2 dxdt - \int_0^1 g(s,x)\psi(u_x(s,x))_x dx$$

$$+ \int_{-\infty}^{s}\int_0^1 g_t\psi(u_x)_x dxdt .$$

To motivate our next estimate, we differentiate (1.10) with respect to t and then integrate formally by parts to get

$$(3.22)\qquad u_{ttt} = \chi(u_x)_{tx} + \int_{-\infty}^{t} a(t-\tau)\psi(u_x)_{\tau\tau x}d\tau + g_t .$$

We would like to multiply (3.22) by $\psi(u_x)_{ttx}$ and then integrate over $(-\infty,s] \times [0,1]$ in order to arrive at an estimate analogous to (3.21). Unfortunately, this operation is not legitimate since $\psi(u_x)_{ttx}$ does not necessarily exist as a function. Consequently, same as with the derivation of (2.12) in Section 2, we shall have to work first with a discrete analog of (3.22) and then pass to the limit. To this end we apply to (1.10) the forward difference operator Δ, of step $\eta > 0$, thus arriving at

$$(3.23)\qquad \Delta u_{tt} = \Delta\chi(u_x)_x + \int_{-\infty}^{t} a(t-\tau)\Delta\psi(u_x)_{\tau x}d\tau + \Delta g .$$

We now multiply (3.23) by $\Delta\psi(u_x)_{tx}$, we integrate over $(-\infty,s] \times [0,1]$, we perform a number of integrations by parts, we divide through by η^2, and we pass to the limit as $\eta \to 0$. The outcome of this tedious but straightforward calculation is

$$(3.24)\quad \frac{1}{2}\int_0^1 \psi'(u_x(s,x))u_{ttx}^2(s,x)dx + \frac{1}{2}\int_0^1 \chi'(u_x(s,x))\psi'(u_x(s,x))u_{txx}^2(s,x)dx$$

$$+ \lim_{\eta\downarrow 0}\frac{1}{\eta^2}\int_{-\infty}^{s}\int_0^1 \Delta\psi(u_x)_{tx}\int_{-\infty}^{t} a(t-\tau)\Delta\psi(u_x)_{\tau x}d\tau dxdt$$

$$= \frac{5}{2}\int_{-\infty}^{s}\int_0^1 \psi''(u_x)u_{tx}u_{ttx}^2 dxdt - \int_0^1 \psi''(u_x(s,x))u_{tx}^2(s,x)u_{ttx}(s,x)dx$$

$$+ \int_{-\infty}^{s}\int_0^1 \psi'''(u_x)u_{tx}^3u_{ttx}dxdt + \frac{3}{2}\int_{-\infty}^{s}\int_0^1 \{\chi''(u_x)\psi'(u_x)$$

$$- \chi'(u_x)\psi''(u_x)\}u_{tx}u_{txx}^2 dxdt$$

$$+ \int_{-\infty}^{s}\int_0^1 \{\chi''(u_x)\psi'(u_x) - \chi'(u_x)\psi''(u_x)\}u_{xx}u_{ttx}u_{txx}dxdt$$

$$+ \int_{-\infty}^{s}\int_0^1 \{\chi'''(u_x)\psi'(u_x)-\chi''(u_x)\psi''(u_x)-\chi'(u_x)\psi'''(u_x)\}u_{tx}^2u_{xx}u_{txx}dxdt$$

$$- \int_{-\infty}^{s}\int_0^1 \chi''(u_x)\psi''(u_x)u_{tx}u_{xx}^2u_{ttx}dxdt - \int_{-\infty}^{s}\int_0^1 \chi''(u_x)\psi'''(u_x)u_{tx}^3u_{xx}^2dxdt$$

$$- \int_0^1 \chi''(u_x(s,x))\psi'(u_x(s,x))u_{tx}(s,x)u_{xx}(s,x)u_{txx}(s,x)dx$$

$$- \int_0^1 g_{1t}(s,x)\psi(u_x(s,x))_{tx}dx + \int_{-\infty}^{s}\int_0^1 g_{1tt}\psi(u_x)_{tx}dxdt$$

$$+ \int_{-\infty}^{s}\int_0^1 g_{2tx}\psi(u_x)_{tt}dxdt\ .$$

The reader should note that because $\psi(u_x)_{ttx}$ does not exist as a function, one cannot, after dividing (3.23) by η, pass to the limit as $\eta \to 0$ under the integral on the right hand side of (3.23). However, the limit of every other term in (3.23) exists as $\eta \to 0$, and therefore,

$$\lim_{\eta\to 0} \int_{-\infty}^{t} a(t-\tau)\Delta\psi(u_x)_{\tau x}(\tau,x)d\tau$$

exists for $t \in [0,T]$, $0 \leq x \leq 1$. The same comment (arrived at by the same reasoning) applies to the limit as $\eta \to 0$ of the multiple (quadratic form) integral on the left hand side of (3.24). It is important for the subsequent estimates (in particular (3.27), (3.28)) to know that this limit exists and is finite (in fact positive). This is important for the concluding part of the proof of Theorem 1.1 (see argument preceding (3.37) below).

To get our next estimate we multiply by u_{txx} the identity

$$a(0)\Delta\psi(u_x)_x = -\int_{-\infty}^{t} a'(t-\tau)\Delta\psi(u_x)_x d\tau + \int_{-\infty}^{t} a(t-\tau)\Delta\psi(u_x)_{\tau x} d\tau \tag{3.25}$$

and we integrate over $(-\infty,s] \times [0,1]$. We majorize the right-hand side of the resulting equation by first applying Schwarz's inequality and then using (3.1) and (3.2). The result is

$$a(0)\int_{-\infty}^{s}\int_{0}^{1} u_{txx}\Delta\psi(u_x)_x dxdt \tag{3.26}$$

$$\leq \{\int_{-\infty}^{s}\int_{0}^{1} u_{txx}^2 dxdt\}^{1/2}\{\gamma\int_{-\infty}^{s}\int_{0}^{1} \Delta\psi(u_x)_x \int_{-\infty}^{t} a(t-\tau)\Delta\psi(u_x)_x d\tau dxdt\}^{1/2}$$

$$+ \{\int_{-\infty}^{s}\int_{0}^{1} u_{txx}^2 dxdt\}^{1/2}\{\beta\int_{-\infty}^{s}\int_{0}^{1} \Delta\psi(u_x)_{tx} \int_{-\infty}^{t} a(t-\tau)\Delta\psi(u_x)_{\tau x} d\tau dxdt\}^{1/2}.$$

Dividing through by η (the step of the forward difference operator Δ), letting $\eta \to 0$, and using (3.15) and the Cauchy-Schwarz inequality, we end up with the estimate

$$(3.27)\quad \frac{1}{2}\kappa a(0)\int_{-\infty}^{s}\int_{0}^{1} u_{txx}^2\,dxdt - \frac{\gamma}{\kappa a(0)}\int_{-\infty}^{s}\int_{0}^{1} \psi(u_x)_{tx}\int_{-\infty}^{t} a(t-\tau)\psi(u_x)_{\tau x}\,d\tau dxdt$$

$$- \frac{\beta}{\kappa a(0)}\lim_{\eta\downarrow 0}\frac{1}{\eta^2}\int_{-\infty}^{s}\int_{0}^{1} \Delta\psi(u_x)_{tx}\int_{-\infty}^{t} a(t-\tau)\Delta\psi(u_x)_{\tau x}\,d\tau dxdt$$

$$\le - a(0)\int_{-\infty}^{s}\int_{0}^{1} \psi''(u_x)u_{tx}u_{xx}u_{txx}\,dxdt\ .$$

Next we integrate over $(-\infty,s] \times [0,1]$ the square of (3.23), we use (3.1), then we divide through by η^2 and we let $\eta \downarrow 0$ to get

$$(3.28)\quad \int_{-\infty}^{s}\int_{0}^{1} u_{ttt}^2\,dxdt - 4\int_{-\infty}^{s}\int_{0}^{1} \chi'(u_x)^2 u_{txx}^2\,dxdt$$

$$- 4\beta\lim_{\eta\downarrow 0}\frac{1}{\eta^2}\int_{-\infty}^{s}\int_{0}^{1} \Delta\psi(u_x)_{tx}\int_{-\infty}^{t} a(t-\tau)\Delta\psi(u_x)_{\tau x}\,d\tau dxdt$$

$$\le 4\int_{-\infty}^{s}\int_{0}^{1} \chi''(u_x)^2 u_{tx}^2 u_{xx}^2\,dxdt + 4\int_{-\infty}^{s}\int_{0}^{1} g_t^2\,dxdt\ .$$

To the above estimate we append

$$(3.29)\quad \int_{-\infty}^{s}\int_{0}^{1} u_{ttx}^2\,dxdt - \int_{-\infty}^{s}\int_{0}^{1} u_{ttt}u_{txx}\,dxdt + \int_{0}^{1} u_{tt}(s,x)u_{txx}(s,x)\,dx = 0$$

which can be derived from

$$(3.30)\quad \int_{-\infty}^{s}\int_{0}^{1} \Delta u_{tx}^2\,dxdt = \int_{-\infty}^{s}\int_{0}^{1} \Delta u_{tt}\Delta u_{xx}\,dxdt - \int_{0}^{1} \Delta u_t(s,x)\Delta u_{xx}(s,x)\,dx$$

by passing to the limit. In turn, (3.30) can be easily verified via integrations by parts.

We now differentiate (1.9) with respect to t,

$$u_{ttt} = \varphi(u_x)_{tx} + \int_{-\infty}^{t} a'(t - \tau)\psi(u_x)_{\tau x}d\tau + g_t \,, \tag{3.31}$$

and we easily get the estimate

$$\begin{aligned}
&\int_0^1 u_{ttt}^2(s,x)dx - 5\int_0^1 \varphi'(u_x(s,x))u_{txx}^2(s,x)dx \\
&\quad - 5\{\int_0^\infty |a'(t)|dt\}^2 \sup_{(-\infty,s]} \int_0^1 \psi'(u_x)^2 u_{txx}^2 dx \\
&\quad < 5\int_0^1 \varphi''(u_x(s,x))u_{tx}^2(s,x)u_{xx}^2(s,x)dx \\
&\quad + 5\{\int_0^\infty |a'(t)|dt\}^2 \sup_{(-\infty,s]} \int_0^1 \psi''(u_x)^2 u_{tx}^2 u_{xx}^2 dx + \int_0^1 g_t^2(s,x)dx \,.
\end{aligned} \tag{3.32}$$

The final set of estimates is derived by the following procedure: We differentiate Equation (1.9) with respect to x and then add and subtract appropriate terms to arrive at

$$\begin{aligned}
\varphi'(0)u_{xxx} + \int_{-\infty}^{t} a'(t - \tau)\psi'(0)u_{xxx}d\tau &= u_{ttx} - [\varphi'(u_x) - \varphi'(0)]u_{xxx} \\
&\quad - \varphi''(u_x)u_{xx}^2 - \int_{-\infty}^{t} a'(t - \tau)[\psi'(u_x) - \psi'(0)]u_{xxx}d\tau \\
&\quad - \int_{-\infty}^{t} a'(t - \tau)\psi''(u_x)u_{xx}^2 d\tau - g_x \overset{\text{def}}{=} X(t,x) \,.
\end{aligned} \tag{3.33}$$

Thus, if $k(t)$ is the resolvent kernel of the operator (3.5),

$$\varphi'(0)u_{xxx}(t,x) = X(t,x) + \int_{-\infty}^{t} k(t - \tau)X(\tau,x)d\tau \,. \tag{3.34}$$

By *Lemma* 3.2, $k(t) \in L^1(0,\infty)$ so that we have estimates

$$(3.35) \qquad \varphi'(0)^2 \int_0^1 u_{xxx}^2(s,x)\,dx \le 2 \int_0^1 X^2(s,x)\,dx$$

$$+ 2\{\int_0^\infty |k(t)|dt\}^2 \sup_{(-\infty,s]} \int_0^1 X^2(t,x)\,dx ,$$

$$(3.36) \qquad \varphi'(0)^2 \int_{-\infty}^{s} \int_0^1 u_{xxx}^2\,dxdt \le 2\{1 + [\int_0^\infty |k(t)|dt]^2\} \int_{-\infty}^{s} \int_0^1 X^2(t,x)\,dxdt .$$

We are now ready to prove (3.18) under assumption (3.17). First we note that, on account of (3.15), $U(T)$ can be majorized, with the help of the Poincaré inequality, by the supremum over $(-\infty,T]$ of an appropriate linear combination of the left-hand sides of the estimates (3.21), (3.24), (3.27), (3.28), (3.29), (3.32), (3.35) and (3.36). On the other hand, each term on the right-hand sides of these estimates can be majorized, by means of the Cauchy-Schwarz inequality and (3.17), by either cG, or $O(\nu)U(T)$, or $\varepsilon U(T) + c(\varepsilon)G$ for any $\varepsilon > 0$. We thus arrive at an estimate of the form

$$(3.37) \qquad U(T) \le \{O(\nu) + O(\varepsilon)\}U(T) + c(\varepsilon)G$$

from which one can get (3.18) by fixing ν and ε sufficiently small. The proof of Theorem 1.1 is complete.

4. Remarks and Extensions.

When the endpoints of the body are pinned, in the place of (1.18) we have boundary conditions

$$(4.1) \qquad u(t,0) = u(t,1) = 0, \qquad -\infty < t < \infty .$$

Similarly, when one endpoint (say $x = 0$) is pinned and the other is free,

$$(4.2) \qquad u(t,0) = 0,\ \sigma(t,1) = 0, \qquad -\infty < t < \infty .$$

In these cases no rigid motions are possible so we don't have to assume (1.20) nor should we expect that (1.21) will generally hold.

When the body force satisfies (1.22) with $g_2(t,x) \equiv 0$, all estimates derived in Section 3 for the case (3.11) are also valid under (4.1) or (4.2). We thus have

Theorem 4.1. *There is* $\mu > 0$ *with the property that for every* $g(t,x)$ *on* $(-\infty,\infty) \times [0,1]$, *which satisfies* (1.22) *with* $g_2(t,x) \equiv 0$ *and* (1.24), *and any* $v(t,x)$ *on* $(-\infty,0] \times [0,1]$, *with* $v(t,\cdot)$, $v_t(t,\cdot)$, $v_x(t,\cdot)$, $v_{tt}(t,\cdot)$, $v_{tx}(t,\cdot)$, $v_{xx}(t,\cdot)$, $v_{ttt}(t,\cdot)$, $v_{ttx}(t,\cdot)$, $v_{txx}(t,\cdot)$, $v_{xxx}(t,\cdot)$ *in* $C((-\infty,0]; L^2(0,1)) \cap L^2((-\infty,0]; L^2(0,1))$, *which satisfies* (1.9) *together with* (4.1) (*or* (4.2)) *for* $t \leq 0$, *there exists a unique* $u(t,x)$ *on* $(-\infty,\infty) \times [0,1]$, *with* $u(t,\cdot)$, $u_t(t,\cdot)$, $u_x(t,\cdot)$, $u_{tt}(t,\cdot)$, $u_{tx}(t,\cdot)$, $u_{xx}(t,\cdot)$, $u_{ttt}(t,\cdot)$, $u_{ttx}(t,\cdot)$, $u_{txx}(t,\cdot)$, $u_{xxx}(t,\cdot)$ *in* $C((-\infty,\infty); L^2(0,1)) \cap L^2((-\infty,\infty); L^2(0,1)$, *which satisfies* (1.9), (1.11) *and* (4.1) (*or* (4.2)). *Moreover*, (1.25) *holds*.

As history and body force get smoother, solutions become smoother. Regularity results can be obtained by establishing a priori estimates for derivatives of u of order 4, 5, etc. Such estimates fall into three categories: those derived by differentiating (1.10) a number of times with respect to t and/or x and then multiplying by the appropriate multiplier (recall the derivation of (3.24)); those derived by expressing certain derivatives in terms of other derivatives through the equation itself (compare with (3.32) or (3.35)); those obtained through interpolation (such as (3.29)). The program is feasible because, since the problem is autonomous (kernel of convolution type), differentiations with respect to x or t essentially preserve the form of the equation (compare, for example, (3.22) with (1.10)). Time derivatives of u satisfy the same homogeneous boundary conditions as u, at $x = 0,1$, so that differentiating the equation with respect to t is generally a better prospect than differentiating with respect to x. In any event there are so many possible combinations of differentiations, integrations by parts, etc., that one may establish several variants of regularity theorems. Here is a typical one:

Theorem 4.2. *Suppose the assumptions of Theorem* 1.1 *hold and, in addition,* φ *and* ψ *are* C^4 *smooth,* $v_{tttt}(t,\cdot)$, $v_{tttx}(t,\cdot)$, $v_{ttxx}(t,\cdot)$, $v_{txxx}(t,\cdot)$ $v_{xxxx}(t,\cdot)$ *are in* $C((-\infty,0]; L^2(0,1)) \cap L^2((-\infty,0]; L^2(0,1))$ *and*

$$
(4.3) \quad \begin{cases} g_{tt}(t,\cdot), g_{tx}(t,\cdot), g_{xx}(t,\cdot) \text{ in } C((-\infty,\infty); L^2(0,1)) \cap L^2((-\infty,\infty); L^2(0,1)) \\ g(t,x) = g_1(t,x) + g_2(t,x) \text{ with } g_{1ttt}(t,\cdot), g_{2ttx}(t,\cdot) \text{ in } L^2((-\infty,\infty); L^2(0,1)) . \end{cases}
$$

Then, when (1.24) is satisfied with μ sufficiently small, the solution $u(t,x)$ of (1.9), (1.11), (1.18) possesses $u_{tttt}(t,\cdot)$, $u_{tttx}(t,\cdot)$, $u_{ttxx}(t,\cdot)$, $u_{txxx}(t,\cdot)$, $u_{xxxx}(t,\cdot)$ in $C((-\infty,\infty); L^2(0,1)) \cap L^2((-\infty,\infty); L^2(0,1))$ and

$$
(4.4) \qquad u_{ttt}(t,\cdot), u_{ttx}(t,\cdot), u_{txx}(t,\cdot), u_{xxx}(t,\cdot) \xrightarrow[{[0,1]}]{\text{unif.}} 0, \quad t \to \infty .
$$

It is noteworthy that the extra derivatives (4.3) of g that are required in order to guarantee smoothness of the solution need not be "small". This is due to the fact that all energy integrals are quadratic forms in the higher order derivatives of u, with coefficients that are solely controlled by ν of (3.17). As we have seen in Section 3, ν is controlled by U which, in turn, is controlled by G.

We close with remarks on the multidimensional situation. The configuration of the body is now a bounded set $\Omega \subset R^n$ with smooth boundary $\partial\Omega$ and the displacement is an n-dimensional vector field $\underset{\sim}{u}$. A typical problem is to determine $\underset{\sim}{u}(t,\underset{\sim}{x})$, $-\infty < t < \infty$, $\underset{\sim}{x} \in \Omega$, such that

$$
(4.5) \qquad \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{n} \left\{ \frac{\partial \Phi_{ij}}{\partial x_j} + \int_{-\infty}^{t} a'(t-\tau) \frac{\partial \Psi_{ij}}{\partial x_j} d\tau \right\} + g_i, \quad i = 1,\dots,n , \quad -\infty < t < \infty, \quad \underset{\sim}{x} \in \Omega ,
$$

$$
(4.6) \qquad \underset{\sim}{u}(t,\underset{\sim}{x}) = \underset{\sim}{v}(t,\underset{\sim}{x}), \qquad -\infty < t \leq 0, \quad \underset{\sim}{x} \in \Omega ,
$$

$$
(4.7) \qquad \underset{\sim}{u}(t,\underset{\sim}{x}) = \underset{\sim}{0}, \qquad -\infty < t < \infty, \quad \underset{\sim}{x} \in \partial\Omega ,
$$

where $\underset{\sim}{\Phi}$ and $\underset{\sim}{\Psi}$ are known, smooth, matrix valued functions of the matrix $\underset{\sim}{e} = \nabla \underset{\sim}{u}$ (strain), $\underset{\sim}{g}(t,\underset{\sim}{x})$ is an assigned body force and $\underset{\sim}{v}(t,\underset{\sim}{x})$ is the given history.

We assume $\underset{\sim}{\Phi}(\underset{\sim}{0}) = \underset{\sim}{\Psi}(\underset{\sim}{0}) = \underset{\sim}{0}$, set

(4.8) $$C_{ijk\ell}(\underset{\sim}{e}) = \frac{\partial\Phi_{ij}(\underset{\sim}{e})}{\partial e_{k\ell}} , \quad D_{ijk\ell}(\underset{\sim}{e}) = \frac{\partial\Psi_{ij}(\underset{\sim}{e})}{\partial e_{k\ell}} ,$$

(4.9) $$E_{ijk\ell}(\underset{\sim}{e}) = C_{ijk\ell}(\underset{\sim}{e}) - a(0)D_{ijk\ell}(\underset{\sim}{e}) ,$$

and impose the symmetry restrictions

(4.10) $$C_{ijk\ell} = C_{k\ell ij}, \quad D_{ijk\ell} = D_{k\ell ij} .$$

Assumptions (1.4), (1.5), (1.6) will here turn into coercivity conditions for the partial differential operators associated with $C_{ijk\ell}(\underset{\sim}{0})$, $D_{ijk\ell}(\underset{\sim}{0})$ and $E_{ijk\ell}(\underset{\sim}{0})$. Under boundary conditions (4.7), coercivity is equivalent to strong ellipticity

(4.11) $$\sum_{i,j,k,\ell} C_{ijk\ell}(\underset{\sim}{0})\xi_i\xi_k\zeta_j\zeta_\ell > 0, \quad |\underset{\sim}{\xi}| = |\underset{\sim}{\zeta}| = 1 ,$$

(4.12) $$\sum_{i,j,k,\ell} D_{ijk\ell}(\underset{\sim}{0})\xi_i\xi_k\zeta_j\zeta_\ell > 0, \quad |\underset{\sim}{\xi}| = |\underset{\sim}{\zeta}| = 1 ,$$

(4.13) $$\sum_{i,j,k,\ell} E_{ijk\ell}(\underset{\sim}{0})\xi_i\xi_k\zeta_j\zeta_\ell > 0, \quad |\underset{\sim}{\xi}| = |\underset{\sim}{\zeta}| = 1 .$$

Assumptions (4.10), (4.11), (4.12) and (4.13) can be motivated by Mechanics. However, in order to carry through the analysis, we require an additional condition whose physical interpretation is less clear. We define

(4.14) $$F_{jk\ell pqr}(\underset{\sim}{e}) = \sum_i D_{ijk\ell}(\underset{\sim}{e})E_{ipqr}(\underset{\sim}{e})$$

and assume that $\underset{\sim}{F}$ is symmetric,

(4.15) $$F_{jk\ell pqr} = F_{pqrjk\ell} ,$$

and that its value at $\underset{\sim}{e} = \underset{\sim}{0}$ corresponds to a coercive operator. We note that in the special case $\underset{\sim}{\Phi} \equiv \underset{\sim}{\Psi}$ the resulting $\underset{\sim}{F}$ automatically satifies the above conditions.

Under the above assumptions it is possible to establish the existence of globally defined smooth solutions to (4.5), (4.6), (4.7) by the procedure followed here in the one-dimensional case. The strategy is to establish a priori energy estimates for the $L^2(\Omega)$ norms of derivatives of $\underset{\sim}{u}$ of sufficiently high order (depending upon n) that would guarantee, via Sobolev's lemma, pointwise bounds analogous to (3.17). The calculations, however, are very long.

It is easy to discern the essential ingredients in the proofs and it thus seems feasible to develop an existence theory for the history-value problem in a class of abstract nonlinear integrodifferential equations

$$\frac{d^2u}{dt^2} = A(u(t)) + \int_{-\infty}^{t} a'(t - \tau)B(u(\tau))d\tau + g(t) \tag{4.16}$$

on a Hilbert space H, where A and B are nonlinear operators defined on a scale of Hilbert spaces (abstracting the scale of Sobolev spaces $[W^{k,2}(\Omega)]^n$) and satisfying appropriate symmetry and coercivity conditions. We remark also that the general and physically interesting case in which the stress-strain relaxation function is a $n \times n$ matrix $\underset{\sim}{a}$ (in (4.5) $\underset{\sim}{a} = aI$) is considerably more complicated than the situation considered here.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A general model for the nonlinear motion of a one dimensional, finite homogeneous, viscoelastic body is developed and analysed by an energy method. It is shown that under physically reasonable conditions the nonlinear boundary, initial value problem has a unique, smooth solution (global in time), provided the given data are sufficiently "small" and smooth; moreover, the solution and its derivatives of first and second order decay to zero as $t \to \infty$. Various modifications and generalizations, including two and three dimensional problems, are also discussed.

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