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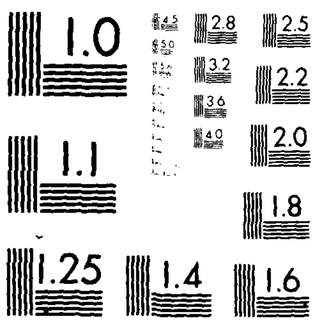

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ON A SEMI-COERCIVE QUASI-VARIATIONAL INEQUALITY

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ON A SEMI-COERCIVE QUASI-VARIATIONAL INEQUALITY

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ABSTRACT

We prove the existence of solutions for a nonlinear semi-coercive quasi-variational inequality with obstacle on the boundary.

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Key Words: quasi-variational inequality, Signorini problem, semi-coercive, nonlinear heat flow

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SIGNIFICANCE AND EXPLANATION

One feature of the so-called quasi-variational problems is that the constraints are not given in advance. The problem considered in this paper is related to the description of a *stationary* temperature distribution inside a material with thermally semi-permeable boundary (here are the constraints) in the case where the exterior temperature varies proportionally to some average of the heat flux crossing the boundary (here is the dependence of the constraints on the solution). Some existence results were obtained in a previous work by the authors, assuming that the heat balance equation is coercive, a condition which eventually yields solutions for any forcing term. Here we deal with a weakened form of this condition, the semi-coercive case, which, in some respects, is physically more natural. A sufficient and almost necessary condition on the forcing term is obtained for the existence of solutions.

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ON A SEMI-COERCIVE QUASI-VARIATIONAL INEQUALITY

Maria Giovanna Garroni[1] and Jean-Pierre Gossez[2]

The existence of solutions for the following nonlinear implicit Signorini type problem was studied in [3]: find $u(x)$, $x \in \Omega$, satisfying

$$(1) \qquad -\sum_{i=1}^{N} D^i A_i(x,u,\nabla u) + A_0(x,u,\nabla u) = f \quad \text{in } \Omega$$

together with

$$(2) \qquad u \geq \Psi(u) \quad \text{on } \Gamma ,$$

$$(3) \qquad \gamma_a u \geq 0 \quad \text{on } \Gamma ,$$

$$(4) \qquad \gamma_a u \cdot (u - \Psi(u)) = 0 \quad \text{on } \Gamma ,$$

where $\Psi(u)$, the obstacle on Γ, is defined by

$$\Psi(u)(x) = h(x) - \int_\Gamma \gamma_a u(y)\varphi(y)\,d\Gamma_y .$$

Here Ω is a bounded open set in R^N with smooth boundary Γ, f is given in Ω, h and φ on Γ, and γ_a denotes the conormal derivative associated to (1). It was proved that under some suitable growth, monotonicity and coercivity assumptions on the coefficients A_i and A_0, problem (1)-(4) always has a solution provided the negative part φ^- of the averaging factor φ is small (a condition which can be released when the coefficients in (1) grow slower than linearly). Earlier results in the linear case can be found in [5,8,1].

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As indicated in [3], this problem is related to the description of a stationary temperature distribution inside a material with thermally semi-permeable boundary in the case where the exterior temperature varies proportionally to some average of the heat flux crossing the boundary. From this point of view it is of interest to consider the situation where no lower order coefficient appears in (1), i.e. where the heat balance equation reads

$$Lu \equiv - \sum_{i=1}^{N} D^i A_i(x,u,\nabla u) = f \quad \text{in} \quad \Omega \; . \tag{5}$$

Solutions may then fail to exist for some forcing terms f, which is physically understandable. Mathematically, there is a lack of coercivity.

It is our purpose in this paper to show how the method of [3] can be combined with the techniques of semi-coercive problems [2,7,9,4] in order to deal with this situation. For simplicity we limit ourselves to the problem stated above. Its variants considered in [3] could be treated by similar arguments.

The precise assumptions imposed on the coefficients $A_i(x,\eta,\zeta)$ of L are the following (compare with the standard Leray-Lions conditions):

(6) the functions $A_i(x,\eta,\zeta)$ satisfy the Caratheodory conditions;

(7) there exist $1 < p < \infty$, $k_1(x) \in L^{p'}(\Omega)$ and a constant c_1 such that

$$|A_i(x,\eta,\zeta)| \leq c_1|\zeta|^{p-1} + k_1(x)$$

for a.e. x, all η, ζ, all i;

(8) for a.e. x, all η,

$$\sum_{i=1}^{N} (A_i(x,\eta,\zeta) - A_i(x,\eta,\zeta'))(\zeta_i - \zeta_i') > 0$$

if $\zeta \neq \zeta'$;

(9) there exist $d_1 > 0$ and $\ell_1(x) \in L^1(\Omega)$ such that

$$\sum_{i=1}^{N} A_i(x,\eta,\zeta)\zeta_i \geq d_1|\zeta|^p - \ell_1(x)$$

for a.e. x, all η, ζ.

We are also given $f \in L^{p'}(\Omega)$, h and $\varphi \in W^{1-1/p,p}(\Gamma)$.

Using Green's formula, one can define the conormal derivative $\gamma_a u \in W^{-(1-1/p),p'}(\Gamma)$ for any $u \in W^{1,p}(\Omega)$ such that the distribution $Lu \in L^{p'}(\Omega)$:

(10) $$a(u,v) = \langle Lu,v\rangle + \langle \gamma_a u,\gamma_0 v\rangle \quad \text{for} \quad v \in W^{1,p}(\Omega) .$$

Here $a(u,v)$ is the usual Dirichlet form associated to L, γ_0 is the trace operator, and $\langle \ , \ \rangle$ denotes either the pairing in the distribution sense on Ω or the duality pairing between $W^{-(1-1/p),p'}(\Gamma)$ and $W^{1-1/p,p}(\Gamma)$. The obstacle

$$\Psi(u) = h - \langle \gamma_a u,\varphi\rangle$$

is thus defined for $u \in W^{1,p}(\Omega)$ with $Lu \in L^{p'}(\Omega)$. Let

$$Q(u) = \{v \in W^{1,p}(\Omega);\ \gamma_0 v \geq \Psi(u) \ \text{ a.e. on } \Gamma\}$$

be the corresponding closed convex set. If we interpret, for $u \in W^{1,p}(\Omega)$, equation (5) in the distribution sense in Ω, condition (2) as $\gamma_0 u \geq \Psi(u)$ a.e. on Γ, condition (3) in the sense of the dual of $W^{1,1/p,p}(\Gamma)$, and condition (4) as $\langle\gamma_a u,\gamma_0 u - \Psi(u)\rangle = 0$, then the problem of finding $u \in W^{1,p}(\Omega)$ satisfying (5), (2), (3), (4) is easily seen to be equivalent to that of solving the quasi-variational inequality

(11) $$\begin{cases} u \in W^{1,p}(\Omega) \ \text{ with } \ Lu \in L^{p'}(\Omega) , \\ u \in Q(u) , \\ a(u,u-v) \leq \langle f,u-v\rangle \ \text{ for all } \ v \in Q(u) . \end{cases}$$

THEOREM. _Assume_ (6)-(9). _If_ $\int_\Omega f < 0$, _then problem_ (11) _has a solution_.

We remark that $\int_\Omega f \leq 0$ is necessary for the existence of a solution. Indeed if u solves (11), then $Lu = f$, and it follows from (10) that for $w = \text{cst} > 0$,

$$0 = w\int_\Omega f + \langle \gamma_a u,w\rangle .$$

This implies $\int_\Omega f \leq 0$ since $\gamma_a u$ is a positive element of the dual of $W^{1-1/p,p}(\Gamma)$ as is easily verified by taking in (11) $v = u + \tilde{z}$ with $z \in W^{1-1/p,p}(\Gamma)$, $z \geq 0$ a.e. on Γ. (Here and below, $z \to \tilde{z}$ denotes a fixed right inverse of the trace operator).

We also remark that no restriction is imposed on the averaging factor φ. Such restrictions are however needed when treating some of the variants considered in [3], for instance the one mentioned at the end of the present paper.

The following result of [4] will be used in the proof of the theorem. Let T be a pseudo-monotone mapping from a reflexive Banach space X to its dual X', K a closed convex subset of X containing the origin, and consider the variational inequality

$$(12) \qquad \begin{cases} u \in K , \\ (Tu, u - v) \leq (g, u - v) \quad \text{for all} \quad v \in K , \end{cases}$$

where g is given in X' and $(\, , \,)$ denotes the pairing between X' and X. Let Y be a second Banach space with X compactly imbedded in Y and let q be a continuous semi-norm on X such that $\|\cdot\|_X$ and $q(\cdot) + \|\cdot\|_Y$ are equivalent on X. Assume that for some constants $c_2 > 0$, $p > 1$ and d_2,

$$(13) \qquad (Tv, v) \geq c_2 q(v)^p - d_2 \quad \text{for} \quad v \in X .$$

Then (12) is solvable if either $K \cap \{v \in X;\ q(v) = 0\}$ is bounded in X or $(g, v) < 0$ for all nonzero v in $K \cap \{v \in X;\ q(v) = 0\}$.

To prove the theorem let us write for $\lambda \in \mathbb{R}$,

$$Q_\lambda = \{v \in W^{1,p}(\Omega);\ \gamma_0 v \geq h - \lambda \ \text{ a.e. on } \Gamma\} ,$$

and for $w \in W^{1,p}(\Omega)$,

$$L_w(u) \equiv - \sum_{i=1}^{N} D^i A_i(x, w, \nabla u) ,$$

and let a_w and γ_{a_w} be the Dirichlet form and the conormal derivative corresponding to L_w. Consider the variational inequality

$$(14) \qquad \begin{cases} u \in Q_\lambda , \\ a_w(u,u-v) \le \langle f,u-v\rangle \quad \text{for all} \quad v \in Q_\lambda . \end{cases}$$

Its solvability follows from the above mentioned result after proceeding to a translation in order to bring the origin inside the convex set (note in this respect that (13) is slightly weaker than the semi-coercivity condition as given in [4]). Moreover (8) implies that two solutions of (14) differ by a constant. But if u solves (14), then $u - \delta \notin Q_\lambda$ for any constant $\delta > 0$. Indeed, in the contrary case, then, given $\eta \in C^\infty(\bar{\Omega})$, $u + \varepsilon\eta \in Q_\lambda$ for ε sufficiently small, and replacing in (14), we eventually derive that u solves the Neumann problem

$$a_w(u,v) = \langle f,v\rangle \quad \text{for} \quad v \in W^{1,p}(\Omega) ,$$

which is impossible since $\int_\Omega f \neq 0$. Consequently (14) has an unique solution $u = u_{\lambda,w}$. (Uniqueness is not really needed here for continuing the argument, see [3]). Defining now

$$\theta(\lambda,w) = (\langle\gamma_{a_w}(u_{\lambda,w}),\varphi\rangle,\ u_{\lambda,w}) \in R \times W^{1,p}(\Omega) ,$$

we are reduced to finding a fixed point for the mapping $(\lambda,w) \to \theta(\lambda,w)$ in $R \times W^{1,p}(\Omega)$.

We claim that the following estimates hold:

$$(15) \qquad \|\nabla u_{\lambda,w}\|_p \le c|\lambda|^{1/p} + c ,$$

$$(16) \qquad \|u_{\lambda,w}\|_p \le c|\lambda| + c ,$$

where c is (here and below) a constant independent of λ and w and $\|\ \|_p$ denotes the $L^p(\Omega)$ norm. Indeed if (15) does not hold, then, for some sequence (λ_n,w_n), one has

$$(17) \qquad \|\nabla u_n\|_p > n|\lambda_n|^{1/p} + n$$

where u_n stands for u_{λ_n,w_n}. Note that $\|u_n\|_X$, the norm of u_n in $X = W^{1,p}(\Omega)$, goes to $+\infty$. Taking $v = \tilde{h} - \lambda_n$ in (14) and using (7), (9), we obtain

(18) $$\|\nabla u_n\|_p^p \leq d + \langle f,u_n\rangle + \langle f,\lambda_n\rangle$$

where d is a constant independent of n. Write $y_n = u_n/\|u_n\|_X$. Clearly $\|y_n\|_X = 1$, and for a subsequence, $y_n \to y$ weakly in X. Also, dividing (18) by $\|u_n\|_X^p$ and using (17), we deduce that $\|\nabla y_n\|_p \to 0$. So, finally, $y_n \to y$ strongly in X and y is a nonzero constant. Now we divide (18) by $\|u_n\|_X$ and use the fact that $\|\nabla u_n\|_p^p - \langle f,\lambda_n\rangle$ is ≥ 0 for n large (a consequence of (17)) in order to obtain at the limit that $\langle f,y\rangle \geq 0$. This will contradict our assumption $\int_\Omega f < 0$ if y can be shown to be ≥ 0. For that purpose recall that $u_n \in Q_{\lambda_n}$ so that

$$y_n \geq h/\|u_n\|_X - \lambda_n/\|u_n\|_X \quad \text{a.e. on } \Gamma .$$

One has $h/\|u_n\|_X \to 0$ a.e. on Γ and, for a subsequence, $y_n \to y$ a.e. on Γ and $\lambda_n/\|u_n\|_X \to \beta \in [-\infty,+\infty]$. If $\beta \leq 0$, we are done. If not, then $\lambda_n/\|u_n\|_X \geq \alpha > 0$ for some α and all n sufficiently large. In particular $\lambda_n > 0$ and so $\langle f,\lambda_n\rangle < 0$. Using this information in (18), we get

$$\|\nabla u_n\|_p^p/\|u_n\|_X \leq d/\|u_n\|_X + \langle f,y_n\rangle .$$

Here the right-hand side converges to $\langle f,y\rangle$ and the left-hand side is greater than $\alpha\|\nabla u_n\|_p^p/\lambda_n$. But the latter converges to $+\infty$ by (17), a contradiction. The proof of the second estimate (16) follows the same lines and is simpler.

Since the growth condition (7) implies that

$$\|\gamma_{a_w} u_{\lambda,w}\|_{\Gamma'} \leq c\|\nabla u_{\lambda,w}\|_p^{p-1} + c$$

where $\| \ \|_{\Gamma'}$ denotes the norm in $W^{-(1-1/p),p'}(\Gamma)$, we deduce from (15) that

$$|\langle \gamma_{a_w} u_{\lambda,w},\varphi\rangle| \leq c|\lambda|^{(p-1)/p} + c .$$

This estimate combined with (16) implies the existence of $r > 0$ such that $[-r,+r] \times B_r$ is mapped into itself by θ. Here B_r denotes the closed ball of radius r centered at 0 in $W^{1,p}(\Omega)$. The continuity of θ and the fact that θ transforms a bounded set into a relatively compact set can be verified by exactly the same arguments as in [3], using the convergence theorems for nonlinear elliptic operators given there in order to deal with the dependence on w. Hence Schauder's fixed point theorem applies, and the proof is complete.

We remark that in the case where the coefficients A_i in (5) do not depend on u, i.e. $A_i = A_i(x,\nabla u)$, then the conclusion of the theorem still holds when equality is allowed in (8). The proof is simpler since no freezing procedure involving w is needed.

The above method also applies to the situation where Γ is composed of two parts Γ_1 and Γ_2 separated by a third part Γ_3 and one requires (2)-(4) on Γ_1, (2)-(4) with reverse inequality sign on Γ_2 and the Neumann boundary condition on Γ_3. In this situation, existence results valid for any forcing term f can be proved. Variational inequalities of this type were considered recently in [6].

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