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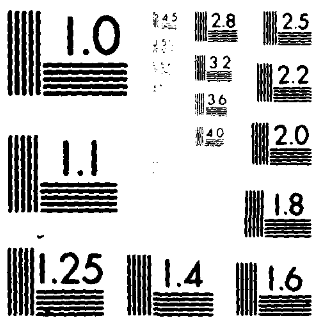


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# A COUNTEREXAMPLE FOR THE TROTTER PRODUCT FORMULA

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# A COUNTEREXAMPLE FOR THE TROTTER PRODUCT FORMULA

Thomas G. Kurtz [redacted] Michel Pierre

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ABSTRACT

We exhibit here two linear m-accretive operators $A_1$ and $A_2$ whose sum is m-accretive but for which the associated product formulas

$\left[S^{A_1}(\frac{t}{n})S^{A_2}(\frac{t}{n})\right]^n$ and $\left[(I + \frac{t}{n} A_1)^{-1}(I + \frac{t}{n} A_2)^{-1}\right]^n$ do not converge.

AMS (MOS) Subject Classifications: Primary 47H15; Secondary 34G05, 35K55

Key Words: m-accretive operators, semigroups of contraction, approximation

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## SIGNIFICANCE AND EXPLANATION

A wide variety of partial differential equations as well as other equations can be written as ordinary differential equations of the form $u'(t) + Au(t) = 0$, where $u$ takes values in a linear space $X$ and $A$ is an operator on $X$. The solution is given by $u(t) = S(t)u(0)$ where $S(t)$ is a semigroup of operators on $X$. In many cases the operator $A$ can be written as the sum $A_1 + A_2$ of (possibly simpler) operators where $A_1$ and $A_2$ correspond to semigroups $S_1(t)$ and $S_2(t)$. Under appropriate conditions, the Trotter product formula $S(t)f = \lim_{n\to\infty}\left[S_1(\frac{t}{n})S_2(\frac{t}{n})\right]^n f$ relates $S(t)$ to $S_1(t)$ and $S_2(t)$ and provides one approach to the study of $S(t)$.

While various sufficient conditions for the validity of this limit are known, no satisfactory necessary conditions are known even when $A_1$ and $A_2$ are linear.

As part of the effort to understand the limitations on the validity of the product formula, we give an example in which $A_1$, $A_2$ and $A_1 + A_2$ are all m-accretive but the corresponding semigroups do not satisfy the product formula.



The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A COUNTEREXAMPLE FOR THE TROTTER PRODUCT FORMULA

Thomas G. Kurtz[1] and Michel Pierre[1,2]

In [10], Trotter proved the following result: given $-A_1$, $-A_2$ the infinitesimal generators of two strongly continuous semigroups $S_1(t)$, $S_2(t)$ of linear contractions on a Banach space $X$, if $-\overline{(A_1 + A_2)}$ (the closure of $-(A_1 + A_2)$) is also the generator of such a semigroup, say $S_3(t)$, then, for any $f \in X$:

$$\forall t \in [0,\infty), \quad \lim_{n\to\infty} \left[S_1\left(\frac{t}{n}\right) S_2\left(\frac{t}{n}\right)\right]^n f = S_3(t) f \; . \tag{1}$$

Many attempts arose in the literature to extend this result to the case of nonlinear semigroups of contractions. In this context a natural question is: given $A_1$, $A_2$ two m-accretive operators on $X$ such that $A_3 = A_1 + A_2$ is also m-accretive, is (1) true for the semigroups of contractions "generated" (in the sense of Crandall-Liggett [5]) by $-A_1$, $-A_2$ and $-A_3$ and for any $f \in \overline{D(A_3)}$ (assuming the product makes sense)?

A positive answer to this question has been provided with extra assumptions on $A_1$, $A_2$ or (and) on the space $X$, for instance the following:

* $A_1$ and $A_2$ are continuous on $X$.

* $-A_1$ is the generator of a linear contraction semigroup and $A_2$ is continuous on $X$.

* $X$ is a Hilbert space and $A_1$, $A_2$, $A_1 + A_2$ are single-valued maximal monotone operators (see Brézis-Pazy [2] or Brézis [1]).

* $X$ is a Hilbert space and $A_1$, $A_2$ are the subdifferentials of lower semi-continuous convex functions from $X$ into $]-\infty,\infty]$ (see Masuda-Kato [7]).

Other results are also mentioned in Kato [6]. It is interesting to notice that all the results above are (more or less easy) consequences of the nonlinear version

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of Chernoff's lemma (see [3]) given by Brézis-Pazy in [2] which says: given $(U(t))_{t\geq 0}$, a family of contractions from a closed convex subset $C$ of $X$ into itself, if there exists $A_3$ m-accretive such that $\overline{D(A_3)} = C$ and

$$\forall f \in C,\ \forall \lambda > 0,\ \lim_{t\to 0^+} \left[I + \frac{\lambda}{t}(I - U(t))\right]^{-1} f = (I + \lambda A_3)^{-1} f ,$$

then

$$\forall f \in C,\ \forall t \in [0,\infty[,\ \lim_{n\to\infty} \left[U(\tfrac{t}{n})\right]^n f = S_3(t) f .$$

The purpose of this paper is to give a counterexample showing that the question above has a negative answer in that general setting. Moreover we exhibit here two linear m-accretive operators $A_1$, $A_2$ whose sum $A_3 = A_1 + A_2$ is also m-accretive and for which (1) fails for some $f \in \overline{D(A_3)}$ as well as

$$\forall t \in [0,\infty),\ \lim_{n\to\infty} \left[\left(I + \frac{t}{n} A_1\right)^{-1} \left(I + \frac{t}{n} A_2\right)^{-1}\right]^n f = S_3(t) f .$$

To understand this counterexample with respect to Trotter's result, it is necessary to remember that an operator $A$ on a Banach space $X$ is said to be m-accretive if, for any $\lambda > 0$, $(I + \lambda A)^{-1}$ is a nonexpansive mapping defined on the whole space $X$ (see e.g. [2] for more details). Consequently, by the well-known Hille-Yosida theorem, if $A$ is a linear m-accretive operator, $-A$ is the (infinitesimal) generator of a strongly continuous semigroup of contractions if and only if its domain $D(A)$ is dense. Obviously this property fails in our examples below. Therefore, if these operators generate semigroups in the "nonlinear sense" (see Crandall-Liggett [5]), that is

$$\text{(2)} \qquad \forall f \in \overline{D(A)},\ \forall t \in [0,\infty),\ S(t) f = \lim_{n\to\infty} \left(I + \frac{t}{n} A\right)^{-n} f ,$$

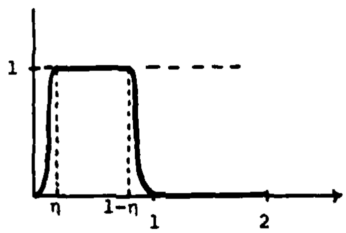
they are not strong generators of these semigroups.

Let $C_b(\mathbb{R})$ (resp. $C(K)$) denote the Banach space of the bounded continuous functions on $\mathbb{R}$ (resp. on the compact set $K$ of $\mathbb{R}$) with the norm

$$\forall u \in C_b(\mathbb{R}), \qquad \|u\| = \sup_{x\in\mathbb{R}} |u(x)|$$

$$(\text{resp.} \quad \forall u \in C(K), \quad \|u\| = \sup_{x\in K} |u(x)|) \ .$$

Let $\rho \in C^{\infty}(\mathbb{R})$ be a periodic function with period 2 whose graph on $[0,2]$ is:




On $C_b(\mathbb{R})$, we define the following operators (the derivative is taken in the sense of distributions).

(i) $D(A_1) = \{u \in C_b(\mathbb{R}); \quad \rho x^3 u' \in C_b(\mathbb{R})\}$

$A_1 u = \rho x^3 u'$ .

(ii) $D(A_2) = \{u \in C_b(\mathbb{R}); \quad (1-\rho)x^3 u' \in C_b(\mathbb{R})\}$

$A_2 u = (1-\rho)x^3 u'$ .

(iii) $D(A_3) = \{u \in C_b(\mathbb{R}); \quad x^3 u' \in C_b(\mathbb{R})\}$

$A_3 u = x^3 u'$ .

For any compact set $K$ of $\mathbb{R}$, symmetric with respect to $0$, we define on $C(K)$:

$$\forall i = 1,2,3, \quad D(A_i^K) = \{u \in C(K);\ \alpha_i x^3 u' \in C(K)\}$$

$$A_i^K u = \alpha_i x^3 u' \ ,$$

where $\alpha_1 = \rho|_K$, $\alpha_2 = (1-\rho)|_K$, $\alpha_3 = 1_K$. Here the derivative is taken in $\mathcal{D}'(\mathring{K})$ and "$\alpha_i x^3 u' \in C(K)$" means that $\alpha_i x^3 u'$ is continuous on $\mathring{K}$ and can be continuously extended to $K$.

PROPOSITION 1.

(i) For $i = 1,2,3$, $-A_i^K$ is the generator of a strongly continuous contraction semigroup $S_i^K$ on $C(K)$ and $A_1^K + A_2^K = A_3^K$.

(ii) For $i = 1,2,3$, $A_i$ is m-accretive on $C_b(\mathbb{R})$ and $A_1 + A_2 = A_3$.

(iii) For $i = 1,2,3$,

$$\forall f \in C_b(\mathbb{R}), \quad \forall \lambda > 0, \quad [(I + \lambda A_i)^{-1} f]|_K = (I + \lambda A_i^K)^{-1}(f|_K) .$$

(iv) If $S_i(t) : \overline{D(A_i)} \to \overline{D(A_i)}$ is defined by

$$\forall f \in \overline{D(A_i)}, \quad \forall t \geq 0, \quad S_i(t)f = \lim_{n\to\infty} \left[I + \frac{t}{n} A_i\right]^{-n} f ,$$

then:

$$\forall f \in \overline{D(A_i)}, \quad \forall t \geq 0, \quad [S_i(t)f]|_K = S_i^K(t)(f|_K) .$$

Remark 1. If $u \in D(A_3)$, $x^3 u'$ is bounded. Hence $\lim_{x\to+\infty} u(x)$ and $\lim_{x\to-\infty} u(x)$ exist. Therefore $D(A_3)$ is not dense in $C_b(\mathbb{R})$.

Note also that, if $x_n, y_n \in [2n + \eta, 2n + 1 - \eta]$ and if $u \in D(A_1)$, then:

$$|u(x_n) - u(y_n)| \leq \frac{1}{2} \|\rho x^3 u'\| \left[\frac{1}{x_n^2} + \frac{1}{y_n^2}\right] .$$

This also proves that $D(A_1)$ is not dense in $C_b(\mathbb{R})$.

PROPOSITION 2.

(i) $S_1(t)$ and $S_2(t)$ leave $\overline{D(A_3)}$ invariant and for all $f \in \overline{D(A_3)}$ and all $t \in [0,\infty)$, $\left[S_1(\frac{t}{n}) S_2(\frac{t}{n})\right]^n f$ converges to $S_3(t)f$ uniformly on compact subsets of $\mathbb{R}$.

(ii) For all $f \in C_b(\mathbb{R})$ and all $t > 0$, $\left[\left(I + \frac{t}{n} A_1\right)^{-1} \left(I + \frac{t}{n} A_2\right)^{-1}\right]^n f$ converges to $S_3(t)f$ uniformly on compact subsets of $\mathbb{R}$.

But:

(iii) For any $f \in C_b(\mathbb{R})$ with compact support and $f \neq 0$, there exists $t \in (0,\infty)$ such that $\left[S_1(\frac{t}{n}) S_2(\frac{t}{n})\right]^n f$ does not converge in $C_b(\mathbb{R})$.

For all $t \in ]0,\infty)$, there exists $f \in C_b(\mathbb{R})$ such that $\left[S_1(\frac{t}{n}) S_2(\frac{t}{n})\right]^n f$ does not converge in $C_b(\mathbb{R})$.

(iv) For any $f \in C_b(\mathbb{R})$ with compact support and $f \neq 0$, there exists $t$ such that $\left[\left(I + \frac{t}{n} A_1\right)^{-1}\left(I + \frac{t}{n} A_2\right)^{-1}\right]^n f$ does not converge in $C_b(\mathbb{R})$.

<u>Proof of Proposition 1.</u>

The equalities $A_1^K + A_2^K = A_3^K$, $A_1 + A_2 = A_3$ follow directly from the definition.

For each $i = 1,2,3$, the proposition is a consequence of the following lemma.

<u>Lemma</u>. Let $\alpha$ be a nonnegative function of $C^\infty(\mathbb{R}) \cap C_b(\mathbb{R})$. Let $A$ (resp. $A^K$) be defined on $C_b(\mathbb{R})$ (resp. $C(K)$) by

$$D(A) = \{u \in C_b(\mathbb{R});\ \ \alpha x^3 u' \in C_b(\mathbb{R})\},\quad Au = \alpha x^3 u'$$

$$(\text{resp. } D(A^K) = \{u \in C(K);\ \ \alpha x^3 u' \in C(K)\},\quad A^K u = \alpha x^3 u')\ .$$

Then:

(i) $-A^K$ is the generator of the strongly continuous semigroup of contractions $S^K(t)$ on $C(K)$ defined by

$$\forall f \in C(K),\quad S^K(t)f(x) = f(X(t,x))\ , \tag{3}$$

where $X(\cdot,x)$ is the solution of

$$\frac{d}{dt} X(t,x) = -\alpha(X(t,x))X^3(t,x),\quad X(0,x) = x\ . \tag{4}$$

Moreover, for all $\lambda > 0$

$$\forall f \in C(K),\quad \forall x \in K,\quad (I + \lambda A^K)^{-1} f(x) = \frac{1}{\lambda}\int_0^\infty e^{-\frac{t}{\lambda}} f(X(t,x))dt\ . \tag{5}$$

(ii) $A$ is m-accretive on $C_b(\mathbb{R})$ and

$$\forall f \in C_b(\mathbb{R}),\quad \forall x \in \mathbb{R},\quad (I + \lambda A)^{-1} f(x) = \frac{1}{\lambda}\int_0^\infty e^{-\frac{t}{\lambda}} f(X(t,x))dt\ ,$$

$$\forall f \in \overline{D(A)},\quad \forall x \in \mathbb{R},\quad S(t)f(x) = f(X(t,x))\ ,$$

where $S(t)$ is defined by (2).

<u>Proof of the Lemma.</u>

The proof of (i) is similar to the proof of Theorem (1.1) in [8].

Since $K$ is symmetric and since $[X \mapsto -\alpha(X)X^3]$ is Lipschitz continuous on $K$ and has the same sign as $-X$, (4) has a unique solution which stays in $K$ for $x \in K$

and satisfies

$$\forall t \geq 0 \quad |X(t,x)| \leq |x|$$

$$(t,x) \in [0,\infty[ \times K \to X(t,x) \text{ is continuous .}$$

It follows that (3) defines a strongly continuous semigroup of contractions $S^K(t)$ on $C(K)$ whose generator $L$ is given by

$$Lu(x) = \lim_{t\to 0^+} \frac{u(X(t,x)) - u(x)}{t} ,$$

when the limit exists uniformly in $x \in K$. Proceeding as in [8], we prove that $L$ is the closure of its restriction $L_0$ to $C^1(K)$. Indeed let $\mathcal{L}$ denote the Lipschitz continuous functions on $K$. Then, if $u \in D(L) \cap \mathcal{L}$

$$Lu(x) = -\alpha(x)x^3u'(x) ,$$

and $[u,Lu]$ is the limit in $C(K) \times C(K)$ of some $[u_n, L_0 u_n]$ with $u_n \in C^1(K)$. This proves that $\overline{L}_0$ contains the restriction of $L$ to $D(L) \cap \mathcal{L}$. But one can show that $L$ is the closure of this restriction by using the fact that $S(t)$ leaves $D(L) \cap \mathcal{L}$ invariant.

Now let us show $-\overline{L}_0 = A^K$. If $[u_n, \alpha x^3 u_n'] \in -L_0$ converges to $[u,v]$ in $C(K) \times C(K)$, then $\alpha x^3 u_n'$ converges to $\alpha x^3 u'$ in the sense of distributions; hence $\alpha x^3 u' = v \in C(K)$ which proves $-\overline{L}_0 \subset A^K$.

For the converse, as $I - \overline{L}_0$ is onto on $C(K)$, it is sufficient to remark that $I + A^K$ is one-one, that is:

(6) $$(u \in C(K),\ u + \alpha x^3 u' = 0 \text{ in } \mathcal{D}'(\overset{\circ}{K})) \Longrightarrow (u = 0 \text{ on } K) .$$

This achieves the proof of (i), the property (5) being well-known.

To prove that $A$ is m-accretive, let us consider for $f \in C_b(\mathbb{R})$ and $\lambda > 0$:

(7) $$u_\lambda(x) = \frac{1}{\lambda}\int_0^\infty e^{-\frac{t}{\lambda}} f(X(t,x))dt .$$

For any $K$ as above, we have

$$\forall x \in K,\ u_\lambda(x) = (I + \lambda A^K)^{-1}(f_{|_K})(x) .$$

As $K$ is arbitrary, this proves that $u_\lambda$ and $\alpha x^3 u'_\lambda$ are continuous on $\mathbb{R}$ and verify

$$u_\lambda + \lambda\alpha x^3 u'_\lambda = f \quad \text{in} \quad \mathcal{D}'(\mathbb{R}) .$$

Since $\|u_\lambda\| \leq \|f\|$ by definition, $u_\lambda$ and $\alpha x^3 u'_\lambda \in C_b(\mathbb{R})$. Hence $u_\lambda \in D(A)$ and $u_\lambda + \lambda A u_\lambda = f$.

This proves that $A$ is an extension of an m-accretive operator. Since $I + A$ is one-one (see (6)), $A$ is m-accretive.

The relations (5) and (7) give

$$\forall f \in C_b(\mathbb{R}), \quad [(I + \lambda A)^{-1} f]\big|_K = (I + \lambda A^K)^{-1}(f\big|_K) .$$

Hence, by the definition (2):

$$\forall f \in \overline{D(A)}, \quad S(t)f\big|_K = \lim_{n\to\infty} \left[I + \frac{t}{n} A^K\right]^{-n}(f\big|_K) = S^K(t)(f\big|_K) .$$

(The last equality is well-known for the linear generators.) Finally

$$\forall f \in \overline{D(A)} \quad S(t)f(x) = f(X(t,x)) .$$

<u>Remark 2</u>. If $\alpha \equiv 1$ (i.e. $A = A_3$), we obtain that

$$X(t,x) = \frac{\operatorname{sgn} x}{\sqrt{2t + \frac{1}{x^2}}} .$$

Then, $\tilde{S}(t)f(x) = f(X(t,x))$ defines a semigroup of contractions <u>on $C_b(\mathbb{R})$</u>, but one can directly verify that $t \mapsto \tilde{S}(t)f$ is continuous at $0$ if and only if $f \in C(\overline{\mathbb{R}}) = \{g \in C_b(\mathbb{R});\ \lim_{x\to+\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$ exist$\}$. Since $\tilde{S}(t)$ leaves $C(\overline{\mathbb{R}})$ invariant and since $\overline{D(A_3)} \subset C(\overline{\mathbb{R}})$ by the remark 1, $S_3(t)$ is exactly the restriction of $\tilde{S}(t)$ to $C(\overline{\mathbb{R}})$ and $C(\overline{\mathbb{R}}) = \overline{D(A_3)}$.

<u>Proof of Proposition 2</u>.

Observe that, by the definition of $\rho$, for $i = 1,2$:

$$\text{(8)} \qquad \begin{cases} \forall x > 0, & x - 1 - \eta \leq X_i(t,x) \leq x \\ \forall x < 0, & x \leq X_i(t,x) \leq x + 1 + \eta . \end{cases}$$

$(X_i$, $i = 1,2$, is the solution of (4) with $[illegible]_1 = [illegible]$, $[illegible]_2 = 1 - [illegible]$). [illegible]

(which is the set $\{g \in C_b(\mathbb{R});\ \lim_{x\to+\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$ exist$\}$ by [illegible]) [illegible]

invariant under $S_1(t)$ *and* $S_2(t)$. *Hence* $\left[S_1(\frac{t}{n})S_2(\frac{t}{n})\right]^n f$ is defined for all $f \in$ [illegible].

*Then,* using (i), (iii) and (iv) in proposition 1, parts (i) and (ii) are consequences

*of* Trotter and Chernoff's results (see [10], [3]).

Now by (8), if $f \in C_b(\mathbb{R})$ has compact support in $[-R,R]$, $S_1(t)f$ *and* $S_2(t)f$ also have compact support in $[-R - 1 - n, R + 1 + n]$ for any $t > 0$ and so do $(I + tA_1)^{-1}f$ and $(I + tA_2)^{-1}f$ by (ii) in the lemma.

So let $f \in C_b(\mathbb{R})$ have compact support and assume that $\left[S_1(\frac{t}{n})S_2(\frac{t}{n})\right]^n f$ or $\left[\left(I + \frac{t}{n} A_2\right)^{-1}\left(I + \frac{t}{n} A_2\right)^{-1}\right]^n f$ converge <u>uniformly on $\mathbb{R}$</u>. The limit is *necessarily* $S_3(t)f$ which is given by:

$$\forall t > 0,\ \forall x \neq 0,\ S_3(t)f(x) = f\left(\frac{\operatorname{sgn} x}{\sqrt{2t + \frac{1}{x^2}}}\right) .$$

Then we have

$$0 = S_3(t)f(+\infty) = f\left(\frac{1}{\sqrt{2t}}\right) , \quad 0 = S_3(t)f(-\infty) = f\left(\frac{-1}{\sqrt{2t}}\right) .$$

If $f \not\equiv 0$, this is false for some $t \in (0,\infty)$.

For the last statement of (iii), given $t > 0$, let $f \in C_b(\mathbb{R})$ *have compact* support and $f = 1$ on $\left[-\frac{1}{\sqrt{2t}}, \frac{1}{\sqrt{2t}}\right]$. *Then*

$$S_3(t)f \equiv 1 .$$

Clearly $\left[S_1(\frac{t}{n})S_2(\frac{t}{n})\right]^n f$, which has compact support, cannot converge uniformly to 1.

<u>Remark 3</u>. If $\hat{C}(\mathbb{R})$ denotes the continuous functions on $\mathbb{R}$ which vanish at $\pm\infty$, let $\hat{A}_i = A_i \cap \hat{C}(\mathbb{R}) \times \hat{C}(\mathbb{R})$. *Then we can show that* $-\hat{A}_1, -\hat{A}_2$ are the (strong) generators of continuous semigroups of contractions $\hat{S}_1(t), \hat{S}_2(t)$. The same remarks as above prove that $\left[\hat{S}_1(\frac{t}{n})\hat{S}_2(\frac{t}{n})\right]^n f$ do not always converge in $\hat{C}(\mathbb{R})$. (Obviously

$-\hat{A}_3$ does not generate any semigroup even in the nonlinear sense.) Trotter also noted in [10] that the convergence of this product may fail for the sum of two generators.

Let us finally recall the example given by Pitt [9] showing that, if $-A_1$, $-A_2$ are two generators, the above product may converge even if $D(A_1) \cap D(A_2) = \{0\}$. See also Chernoff [4] for more pathological cases.

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~~We exhibit here~~ This report describes two linear m-accretive operators $A_1$ and $A_2$ whose sum is m-accretive but for which the associated product formulas

$$\left[S^{A_1}\left(\frac{t}{n}\right)S^{A_2}\left(\frac{t}{n}\right)\right]^n \text{ and } \left[\left(I+\frac{t}{n}A_1\right)^{-1}\left(I+\frac{t}{n}A_2\right)^{-1}\right]^n$$

do not converge.


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