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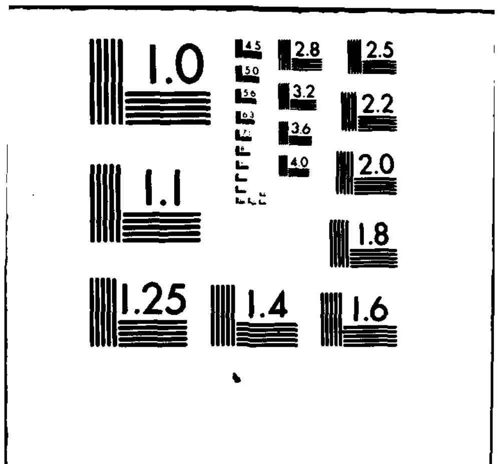

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ON THE GENERALIZED EULER-FROBENIUS POLYNOMIAL

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June 1980

(Received April 28, 1980)

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ABSTRACT

In this paper the properties of the generalized Euler-Frobenius polynomial $\cancel{\Pi_n(\lambda,q)}$ are studied. It is proved that its zeroes are separated by a factor q and their asymptotic behavior as $q \to \infty$ (q approaches infinity) is obtained. As a consequence it is shown that least squares spline approximation on a biinfinite geometric mesh is boundable independently of the (local) mesh ratio q and that the norm of the inverse of the corresponding B-spline Gram matrix decreases monotonly to $2k - 1$ for large q, as $q \to \infty$.

AMS (MOS) Subject Classification: 41A15

Key Words: Euler-Frobenius polynomial, Splines, Interpolation, Least-squares approximation

Work Unit Number 3 - Numerical Analysis and Computer Science

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+Supported by Fulbright Grant No. 79-045-A.

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SIGNIFICANCE AND EXPLANATION

The Euler-Frobenius polynomial Π_n plays an important role in the analysis of cardinal polynomial splines. It has simple, negative zeroes, and the fact

$$\Pi_{2k-1}(-1) \neq 0$$

allows us to conclude that there exists a unique bounded cardinal polynomial spline $f \in S_{2k,\mathbb{Z}}$ that interpolates prescribed data $\underline{b} \in \ell_\infty$ at integers.

The essential properties of cardinal polynomial splines have been later extended to the more general case of cardinal $\mathcal{L}$-splines, and thereby, by an appropriate change of variables, to polynomial splines on the biinfinite geometric mesh

$$(q^i)_{-\infty}^{\infty}, \qquad 1 < q < \infty .$$

The generalized Euler-Frobenius polynomial is in the latter case given by

$$\Pi_n(\lambda;q) := \frac{1}{n!t^n} \sum_{i=0}^{n} (-)^{n-i}\binom{n}{i} \prod_{\substack{j=0 \\ j\neq i}}^{n} (q^j - \lambda), \quad t := \ell n q .$$

In the report some new characteristics of $\Pi_n(\cdot;q)$ are outlined. A simple but far reaching property is proved: the zeroes are separated by a factor q. This fact helps us to analyse spline interpolation on the biinfinite geometric mesh. In particular, it is proved that the least squares spline approximation is boundable independently of the local mesh ratio q, and that the norm of the inverse of the corresponding B-spline Gram matrix decreases monotonly to $2k - 1$ for large q, as $q \to \infty$.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON THE GENERALIZED EULER-FROBENIUS POLYNOMIAL

Y. Y. Feng* and J. Kozak+

1. Introduction

The exponential Euler polynomial $A_n(x;t)$ played an important role in the analysis of cardinal polynomial splines. This is much due to the fact that the spline defined by the functional relation

$$\phi_n(x) := A_n(x;\lambda), \quad x \in [0,1[,$$
$$\phi_n(x+1) := \lambda\phi_n(x) \quad \text{otherwise} ,$$

vanishes at all integers for particular values of λ, the zeroes of the Euler-Frobenius polynomial $\Pi_n(\lambda) := (1-\lambda)^n A_n(0;\lambda)$. A nice survey of cardinal polynomial splines can be found in [7]. Micchelli [6] showed that the essential properties of cardinal polynomial splines can be extended to the more general case of cardinal $\mathcal{L}$-splines. By applying his results to the particular differential operator,

$$\mathcal{L}_t(D) := \prod_{i=0}^{n} (D - it), \qquad D := \frac{d}{dx}, \qquad t \in R$$

and to the corresponding generalized exponential Euler polynomial

$$A_n(x;\lambda,q) := \frac{1}{n!t^n} \sum_{i=0}^{n} (-)^{n-i} \binom{n}{i} \frac{q^{ix}}{q^i - \lambda}, \qquad q := e^t \tag{1.1}$$

he analyzed spline interpolation at knots on the biinfinite geometric mesh

$$(q^i)_{-\infty}^{+\infty} . \tag{1.2}$$

In this case, the generalized Euler-Frobenius polynomial is given by

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+Supported by Fulbright Grant No. 79-045-A.

$$\Pi_n(\lambda;q) := \prod_{i=0}^{n} (q^i - \lambda)A_n(0;\lambda,q) = \frac{1}{n!t^n} \sum_{i=0}^{n} (-)^{n-i}\binom{n}{i} \prod_{\substack{j=0\\j\neq i}}^{n} (q^j - \lambda) , \qquad (1.3)$$

and satisfies a "difference-delay" equation [6]

$$\Pi_0(\lambda;q) := 1 , \qquad (1.4)$$

$$\Pi_{n+1}(\lambda;q) = \frac{1}{(n+1)t} ((1-\lambda)q^n\Pi_n(q^{-1}\lambda;q) - (q^{n+1} - \lambda)\Pi_n(\lambda;q)), \quad n = 0,1,\ldots .$$

A recent paper by Höllig [5] shows that more general spline interpolation problems on a biinfinite geometric mesh can be understood in terms of properties of $\Pi_n(\lambda;q)$.

The main part of the present paper is an outline of some new characteristics of $\Pi_n(\lambda;q)$. A simple but far reaching property is the: zeroes $\mu_{n,i}(q)$ are separated by a factor q. This produces the bounds

$$-\text{const}_1\, q^{n-i} < \mu_{n,i}(q) < -\text{const}_2\, q^{n-i}$$

for some property chosen const_1, const_2.

In Section 3, the properties developed are used in an analysis of spline interpolation Pf to f defined by the conditions

$$\int_I M_{i,r} Pf = \int_I M_{i,r} f, \quad \text{all } i ,$$

on a biinfinite geometric mesh. In this way, some of the results in [5] are obtained by a different approach.

2. The Zeroes of $\Pi_n(\lambda;q)$

We start the section with the symmetries of the generalized Euler-Frobenius polynomial. In addition to the description (1.3), we shall use

$$\sum_{i=0}^{n-1} a_{n,i}(q)\lambda^i := \frac{1}{\gamma_n(q-1)^n}\Pi_n(\lambda;q), \quad \gamma_n := \frac{1}{n!t^n},$$

to emphasize its polynomial character in λ.

<u>Theorem 2.1</u>. The polynomial $\Pi_n(\lambda;q)$ satisfies

$$\Pi_n(\lambda;q) = \lambda^{n-1}q^{-n(n-1)/2}\Pi_n(q^n\lambda^{-1};q). \tag{2.1}$$

The coefficients $a_{n,i}(q)$ can be recurrently computed by

$$a_{n+1,i}(q) = (q-1)^{-1}((q^{n+1}-q^{n-i})a_{n,i}(q) + (q^{n+1-i}-1)a_{n,i-1}(q)), \tag{2.2}$$

where

$$a_{n,0}(q) := 1, \quad a_{n,-1}(q) = a_{n,n}(q) := 0.$$

<u>Proof</u>. For $n = 1$ or $\lambda = 0$, (2.1) obviously holds. Assume $\lambda \neq 0$, $n \geq 2$. Then

$$\Pi_n(\lambda;q) = \gamma_n \sum_{i=0}^{n} (-)^{n-i}\binom{n}{i} \prod_{\substack{j=0\\ j\neq i}}^{n} (q^j - \lambda)$$

$$= \gamma_n \sum_{i=0}^{n} (-)^{n-i}\binom{n}{i}q^i\lambda^{-1} \prod_{\substack{j=0\\ j\neq i}}^{n} (q^j - \lambda),$$

since the n-th order finite difference of a constant vanishes. But

$$q^i\lambda^{-1}\prod_{\substack{j=0\\ j\neq i}}^{n} (q^j-\lambda) = (-)^n\lambda^{n-1}q^{-n(n-1)/2}\prod_{\substack{j=0\\ j\neq n-i}}^{n} (q^j - q^n\lambda^{-1}),$$

which completes the proof of (2.1).

In terms of the $a_{n,i}(q)$, the recurrence relation (1.4) reads

$$\sum_{i=0}^{n} a_{n+1,i}(q)\lambda^i = -(q-1)^{-1}((1-\lambda)q^n \sum_{i=0}^{n-1} a_{n,i}(q)q^{-i}\lambda^i - (q^{n+1}-\lambda) \sum_{i=0}^{n-1} a_{n,i}(q)\lambda^i)$$

$$= (q-1)^{-1} \sum_{i=0}^{n} ((q^{n+1}-q^{n-i})a_{n,i}(q) + (q^{n+1-i}-1)a_{n,i-1}(q))\lambda^i ,$$

if we define $a_{n,-1}(q) = a_{n,n}(q) := 0$, and this confirms (2.2). ∎

<u>Corollary 2.1</u>. The coefficients $a_{n,i}(q)$ satisfy

$$a_{n,i}(q) = q^{n(n-2i-1)/2} a_{n,n-1-i}(q) , \tag{2.3}$$

and for $n \geq 2$

$$a_{n,i}(q) = q^{(n-i)(n-1-i)/2} \sum_{j=0}^{i(n-1-i)} a_{n,i}^{(j)} q^j . \tag{2.4}$$

The integer coefficients $a_{n,i}^{(j)}$ are symmetric

$$a_{n,i}^{(j)} = a_{n,i}^{(i(n-1-i)-j)}, \quad \text{all } j . \tag{2.5}$$

In particular,

$$\begin{aligned} a_{n,i}^{(0)} &= \binom{n-1}{i} , \\ a_{n,i}^{(1)} &= (n-2)\binom{n-1}{i} - \binom{n-2}{i+1} - \binom{n-2}{i-2} . \end{aligned} \tag{2.6}$$

It is easy to prove (2.3)-(2.6) by using (2.2) and mathematical induction. We shall omit this step.

From now on we think of the zeroes of $\Pi_n(\cdot;q)$ as functions of q. It is proved in [6] that the $n - 1$ zeroes $\mu_{n,i}(q)$, $i = 1,\ldots,n-1$, of $\Pi_n(\cdot;q)$ are all simple and real, in fact negative. They satisfy

$$\mu_{n,i}(q) < 0, \quad \frac{d}{dq}\mu_{n,i}(q) < 0 , \tag{2.7}$$

$$\lim_{q\to 0+} \mu_{n,i}(q) = 0, \quad \lim_{q\to\infty} \mu_{n,i}(q) = -\infty, \quad \text{all } i , \tag{2.8}$$

and

$$\mu_{n,i}(q^{-1}) = \mu_{n,n-i}^{-1}(q), \quad \text{all} \quad i \ . \tag{2.9}$$

We shall think of the $\mu_{n,i}(q)$ as ordered,

$$\mu_{n,1}(q) < \mu_{n,2}(q) < \cdots < \mu_{n,n-1}(q) < 0 \ . \tag{2.10}$$

Then, additionally, by [4] and (2.15)

$$\frac{d}{dq}\left(\frac{\mu_{n,n-1}(q)}{q}\right) < 0, \quad \frac{d}{dq}\left(\frac{\mu_{n,1}(q)}{q^{n-1}}\right) > 0 \ . \tag{2.11}$$

The symmetry (2.9) tells us that we can restrict our discussion to the case $q > 1$.

<u>Lemma 2.1</u>. Let $q > 1$. Then

$$\mu_{n,i-1}(q) < \mu_{n+1,i}(q) < q\mu_{n,i}(q), \quad i = 2,3,\ldots,n-1;\ n = 2,3,\ldots \ . \tag{2.12}$$

<u>Proof</u>. Suppose $\mu_{n,i-1}(q) < q\mu_{n,i}(q)$ holds for some n. By hypothesis then

$$\operatorname{sign}(\Pi_n(q^{-1}\lambda;q)) \cdot \operatorname{sign}(\Pi_n(\lambda;q)) < 0, \quad \lambda \in [q\mu_{n,i}(q), \mu_{n,i}(q)] \ ,$$

and from (1.4)

$$\Pi_{n+1}(\lambda;q) \neq 0, \quad \lambda \in [q\mu_{n,i}(q), \mu_{n,i}(q)] \ . \tag{2.13}$$

But $\mu_{n,i}(q)$ is a zero of $\Pi_n(\cdot;q)$, thus another look at (1.4) tells us

$$\operatorname{sign}(\Pi_{n+1}(q\mu_{n,i}(q);q)) \cdot \operatorname{sign}(\Pi_{n+1}(\mu_{n,i-1}(q);q)) < 0 \ ,$$

and there is at least one zero of $\Pi_{n+1}(\cdot;q)$ in each of the intervals

$$]\mu_{n,i-1}(q), q\mu_{n,i}(q)[\ , \quad \text{all} \quad i \ . \tag{2.14}$$

Also by (1.4)

$$\operatorname{sign}(\Pi_{n+1}(0+;q)) \cdot \operatorname{sign}(\Pi_{n+1}(\mu_{n,n-1}(q);q)) < 0 \ ,$$

$$\operatorname{sign}(\Pi_{n+1}(\mu_{n,1}(q);q)) \cdot \operatorname{sign}(\Pi_{n+1}(-\infty;q)) < 0 \ ,$$

and this reveals the position of the smallest and the largest zero of $\Pi_{n+1}(\cdot;q)$. However, $\Pi_{n+1}(\cdot;q)$ is a polynomial of degree $< n+1$, and in each of the intervals (2.14) there is exactly one zero, $\mu_{n+1,i}(q)$.

Now (2.15) brings the induction hypothesis to the next level and (2.12) is proved since it obviously holds for $n = 2$. ∎

It is easy to deduce the following interesting properties of $\mu_{n,i}(q)$.

__Corollary 2.2__. The zeroes $\mu_{n,i}(q)$ of $\Pi_n(\cdot;q)$ have the following properties

$$\mu_{n,i}(q) \cdot \mu_{n,n-i}(q) = q^n, \quad \text{all } i . \tag{2.15}$$

In particular

$$\mu_{2k,k}(q) = -q^k , \tag{2.16}$$

and for $i < \lfloor\frac{n-1}{2}\rfloor$

$$\mu_{n,i}(q) < -q^{n-i}, \; \mu_{n,n-i}(q) > -q^i , \tag{2.17}$$

as well as

$$\frac{d}{dq}\left(\frac{\mu_{n,i}(q)}{q^n}\right) > 0, \quad \text{all } i . \tag{2.18}$$

__Proof__. By (2.1)

$$\Pi_n(\lambda;q) = 0 \quad \text{iff} \quad \Pi_n(q^n/\lambda;q) = 0 .$$

Since we have ordered $\mu_{n,i}(q)$ as in (2.10), (2.15) follows. (2.16) is a special case of (2.15). From (2.15) and (2.12) we find

$$q^n = \mu_{n,i}(q)\mu_{n,n-i}(q) > q^{n-2i}\mu^2_{n,n-i}(q) ,$$

which implies (2.17). Finally, combining (2.15) and (2.7) we obtain (2.18). ∎

__Theorem 2.2__. Let $q > 1$. Then for $i = 1,2,\ldots, \lfloor\frac{n-1}{2}\rfloor$

$$-c_1q^{n-i} < \mu_{n,i}(q) < -c_2q^{n-i} , \tag{2.19}$$

$$-\frac{1}{c_2}q^i < \mu_{n,n-i}(q) < -\frac{1}{c_1}q^i , \tag{2.20}$$

the constants c_1, c_2 do not depend on q and i, and

$$c_1 = |\mu_{n,1}(1)| , \tag{2.21}$$

$$1 < c_2 < \begin{cases} \frac{n+1}{n-1}, & n \text{ odd} \\ \frac{n+2}{n-2}, & n \text{ even} \end{cases} . \tag{2.22}$$

__Proof__. It is enough to prove (2.19), since (2.20) follows from it by (2.15). Observe from (2.11) that $\mu_{n,1}(q) > \mu_{n,1}(1)q^{n-1}$. Then by Lemma 2.1

$$\mu_{n,i}(q) > \frac{\mu_{n,1}(q)}{q^{i-1}} > \mu_{n,1}(1)q^{n-i} = -c_1 q^{n-i} ,$$

and the left inequality is proved.

Since $\mu_{n,i}(q)/q^{n-i}$ is a *continuous* function on $[1,\infty[$ and satisfies (2.17), while by Theorem 2.4

$$\lim_{q\to\infty} \frac{\mu_{n,i}}{q^{n-i}} = -\frac{n-i}{i} ,$$

there obviously exists a *constant* $1 < c_2 < \min_i \frac{n-i}{i}$ independent q such that the right inequality of (2.19) holds. In particular note that $1 < c_2 < \frac{k}{k-1}$ for $n = 2k-1$. ∎

Theorem 2.2 bounds $\mu_{n,i}(q)$ as functions of q. However, it is of interest also to ask the opposite question: suppose $\mu_{n,i}(\tilde{q}) = \mu_{n,i-1}(q)$. What can we say about q, $\tilde{q}$? We believe its answer is beautiful enough to deserve its place in the paper.

<u>Theorem 2.3</u>. There exists *a constant*, *const* < 1 *so that, for any* q, $\tilde{q}$ *or* i,

$$\mu_{n,i}(\tilde{q}) = \mu_{n,i-1}(q) , \tag{2.23}$$

implies

$$\frac{q}{\tilde{q}} < \text{const} < 1 .$$

<u>Proof</u>. Let q, $\tilde{q}$ satisfy (2.23) for some i, Then (2.18) gives us

$$\left(\frac{q}{\tilde{q}}\right)^n < \left(\frac{q}{\approx q}\right)^n , \quad \overset{\approx}{q} := \text{a solution of} \quad \left(\frac{q}{\approx q}\right)^n = \frac{\mu_{n,i}(q)}{\mu_{n,i-1}(q)} =: \rho_i(q) .$$

The function $\rho_i(q)$ is a continuous function of q, and by Lemma 2.1 and (2.9)

$$q > 1 : \rho_i(q) < \frac{1}{q} ,$$

$$q < 1 : \rho_i(q) = \frac{\mu_{n,n-i+1}(1/q)}{\mu_{n,n-i}(1/q)} < q .$$

Thus $\rho_i(0+) = \rho_i(\infty) = 0$. Clearly we find

$$\text{const} = \max_i \max_q \rho_i(q) < 1 .$$

∎

The last part of this section we devote to the asymptotic behavior of $\mu_{n,i}(q)$ as $q \to \infty$.

<u>Theorem 2.4</u>. For $i = 1,2,\ldots, \lceil\frac{n-1}{2}\rceil$,

$$\mu_{n,i}(q) = -\frac{n-i}{i}\, q^{n-i} - c_{n,i} q^{n-i-1} + O(q^{n-2-i}) , \tag{2.24}$$

$$\mu_{n,n-i}(q) = -\frac{i}{n-i}\, q^{i} + \left(\frac{i}{n-i}\right)^2 c_{n,i} q^{i-1} + O(q^{i-2}) . \tag{2.25}$$

Here

$$\begin{aligned} 0 < c_{n,i} := \frac{1}{i^2(i+1)(n-1)(n-i+1)} & [(n-2i)^4 + (6i-1)(n-2i)^3 \\ & + 4i(3i-1)(n-2i)^2 + 4i^2(2i-1)(n-2i)] . \end{aligned} \tag{2.26}$$

In particular,

$$c_{2k-1,k-1} = \frac{(2k-1)^2}{k(k-1)^2(k+1)} . \tag{2.27}$$

<u>Proof</u>. By (2.15) it is enough to prove (2.24). Let $\lambda = \mu_{n,i}(q)$. By (2.8) and (2.15)

$$\lim_{q\to\infty} \frac{\mu_{n,i}}{q^n} = 0, \quad \text{all } i .$$

Thus for some $\alpha \neq 0$ and some $r > 0$

$$\lambda = \alpha q^{n-r} + \beta q^{n-r-1} + O(q^{n-r-2}) . \tag{2.28}$$

Since the coefficients $a_{n,i}(q)$ are polynomials in q, and after a proper normalization in $1/q$, r is an integer. From Corollary 2.1 we conclude that as $q \to \infty$

$$\sum_{i=0}^{n-1} a_{n,i}(q)\lambda^i \approx \sum_{i=0}^{n-1} q^{(n+i)(n-1-i)/2+(n-r)i}(a_{n,i}^{(0)}\alpha^i + q^{-1}(a_{n,i}^{(1)}\alpha^i + a_{n,i}^{(0)} i\alpha^{i-1}\beta)) . \tag{2.29}$$

An inspection of the exponent

$$\psi(i,r) := (n + i)(n - 1 - i)/2 + (n - r)i$$

shows

$$\psi\left(\frac{2(n - r) - 1}{2} + i, r\right) = \psi\left(\frac{2(n - r) - 1}{2} - i, r\right) ,$$

$$\Delta_1 \psi(i,r) := \psi(i + 1, r) - \psi(i,r) = n - r - i - 1 .$$

Since $\Delta_1 \psi(n - r - 1, r) = 0$, the leading power of q occurs in the terms $i = n - r - 1$, $n - r$. Thus (2.29) can vanish precisely for $r = 1,2,\ldots,n - 1$ as $q \to \infty$, and we conclude from (2.10): $\mu_{n,i}(q) = O(q^{n-i})$. By using (2.6) it is now straightforward to complete the proof. ∎

3. Polynomial Splines on a Biinfinite Geometric Mesh

To start more generally, let $\underline{t} := (t_i)_{-\infty}^{+\infty}$ be a strictly increasing biinfinite sequence with $t_{\pm\infty} := \lim_{i\to\pm\infty} t_i$, $I :=]t_{-\infty}, t_{+\infty}[$. Let further

$$mS_{n,\underline{t}}(I) := \{f \in C^{n-2}(I) \cap L_\infty(I) \,\big|\, f|_{]t_i,t_{i+1}[} \text{ is a polynomial of degree } < n\}$$

be the normed linear space of polynomial splines of order n with the breakpoint sequence $\underline{t}$ and the norm $\|f\| := \sup_{x\in I} |f(x)|$. Let $r,k \in \mathbb{N}$ be given integers, $0 \le r < 2k$, $0 < k$. Consider the map

$$R_r : mS_{2k-r,\underline{t}}(I) \to \ell_\infty : f \mapsto (\phi_{i,r} f)_{-\infty}^{+\infty} \tag{3.1}$$

associated with interpolation conditions

$$\phi_{i,0} f := f(t_i), \quad \phi_{i,r} f := \int_I M_{i,r} f, \quad r > 0 .$$

Here, as usual the B-splines of order k with knots $\underline{t}$ are defined by

$$M_{ik}(x) := k[t_i, t_{i+1}, \ldots, t_{i+k}](\cdot - x)_+^{k-1} ,$$

$$N_{ik} := \frac{1}{k}(t_{i+k} - t_i) M_{ik} .$$

The interpolation problem: for given $\underline{b} := (b_i)_{i=-\infty}^{+\infty} \in \ell_\infty$, find $f \in S_{2k-r,\underline{t}}(I)$ such that

$$R_r f = \underline{b}$$

is by [2] correct, if R_r is invertible, i.e. the Gramian (totally positive) matrix

$$G_r := (\phi_{i,r} N_{j,2k-r})_{i,j=-\infty}^{+\infty}$$

is boundedly invertible.

Let us restrict ourselves now to a particular geometric knot sequence $\underline{t} := (q^i)_{-\infty}^{+\infty}$ for some $q \in]0,\infty[$. In this case the matrix is a Toeplitz matrix and is boundedly invertible iff the characteristic polynomial

$$\sum_j \lambda^j \phi_{i,r} N_{j,2k-r} \tag{3.2}$$

has no zero on the unit circle $|\lambda| = 1$, or since G_r is totally positive, at $\lambda = -1$. The case $r = 0$ is treated in [6], where it is proved that

$$\Pi_{2k-1}(\lambda;q) = \sum_{j=0}^{2k-2} \lambda^j \phi_{2k-1,0} N_{j,2k} = \lambda^{2k-i-1} \sum_j \lambda^j \phi_{i,0} N_{j,2k} ,$$

and from properties of the generalized Euler-Frobenius polynomial determined when R_0 is invertible. A nice argument shown to us by de Boor [3] leads to the conclusion: The characteristic polynomial (3.2) has -1 as a zero iff

$$\Pi_{2k-1}(-q^{k-r};q) = 0$$

for any r, $0 \le r \le 2k-1$. A recent result of Höllig [5] states

$$\|G_r^{-1}\|_\infty = h_r(q) := \left| \frac{\Pi_{2k-1}(q^r;q)}{\Pi_{2k-1}(-q^r;q)} \right| .$$

He proves that $h_r(q)$ is bounded independently of q and G_r, $r \ne k-1,k$ is not boundedly invertible for at least one $q \in [1,\infty[$. We give here an alternative proof by simply rereading Theorem 2.2. By Theorem 2.1 we can restrict to the case $0 \le r \le k-1$.

The equation

$$\eta_i(q) := \eta_{i,k,r}(q) := \mu_{2k-1,i}(q)/q^r = -1 \tag{3.5}$$

has (at least one) solution $q \in]0,\infty[$ exactly for $r+1 \le i \le 2k-2-r$. Put

$$Q_r := \{q | q \text{ is a solution of } (3.5)\}$$

and $|Q_r|$:= number of elements in Q_r. Choose $r+1 \le i \le k-1$. Then

$$\begin{aligned} -c_1 q^{2k-1-r-i} &\le \eta_i(q) \le -c_2 q^{2k-1-r-i}, \qquad q \ge 1 , \\ -c_2^{-1} q^{-r+i} &\le \eta_i(q) \le -c_1^{-1} q^{-r+i}, \qquad q \le 1 . \end{aligned} \tag{3.6}$$

If $q \ge 1$ obviously there is no solution to (3.5) since this would imply $i \ge 2k-r$. In the case $q \le 1$ there is $q \in Q_r$ exactly for $i \ge r+1$. Since

$$\eta_i(q) = -1 \text{ iff } \eta_{2k-1-i}(1/q) = -1$$

our claim is confirmed.

We note that the case of a finite-partition [1] suggests that R_r, $r \ne k-1,k$ is not invertible for all q, but as already pointed out in [2] the same proof can not be applied since the quotients

$$\min_{i+1-r\le j\le i} \frac{q^{j+r} - q^j}{q^{i+1} - q^i} \le r$$

are bounded independently of q.

Let now $q > 1$. From (2.15) we get

$$h_k(q) = h_{k-1}(q) = \prod_{i=1}^{2k-2} \left| \frac{q^k - \mu_{2k-1,i}(q)}{q^k + \mu_{2k-1,i}(q)} \right| = \prod_{i=1}^{k-1} \frac{w_i(q) + 1}{w_i(q) - 1} \tag{3.7}$$

with

$$w_i(q) := -(\mu_{2k-1,i}(q) + \mu_{2k-1,2k-1-i}(q))/(q^k + q^{k-1}) . \tag{3.8}$$

From Theorem 2.2 we conclude

$$\bar{w}_i(q) := -\mu_{2k-1,i}(q)/q^k > w_i(q) > (c_2 q^{k-1-i} + c_2^{-1} q^{-k+1+i})/2 =: \underline{w}_i(q)$$

and

$$\bar{h}_{k-1}(q) := \prod_{i=1}^{k-1} \frac{\bar{w}_i(q) + 1}{\bar{w}_i(q) - 1} < h_{k-1}(q) < \underline{h}_{k-1}(q) := \prod_{i=1}^{k-1} \frac{\underline{w}_i(q) + 1}{\underline{w}_i(q) - 1} .$$

Since $\underline{h}_{k-1}(q)$ is decreasing as a function of q, this suggests that $h_{k-1}(q)$ is too. However, we succeeded in proving this only as $q \to \infty$, as a consequence of Theorem 2.4 and (3.7), (3.8).

<u>Theorem 3.5</u>. For $0 \le r \le 2k - 1$, the Gramian matrix G_r is not boundedly invertible for $q \in Q_r$, and $|Q_r| = |Q_{2k-1-r}| \ge 2(k - 1 - r)$, $0 \le r \le k - 1$. In particular, $|Q_k| = |Q_{k-1}| = 0$, *and the norm* $h_k(q)$ *for* $q \in [1,\infty[$ *satisfies*

$$\bar{h}_k(q) \le h_k(q) < \underline{h}_k(q) < \left(\frac{c_2 + 1}{c_2 - 1}\right)^{2(k-1)} \tag{3.9}$$

also as $q \to \infty$

$$h_k(q) = (2k - 1)(1 + 4(k-2)/(k+1)q^{-1} + O(q^{-2})) . \tag{3.10}$$

ACKNOWLEDGEMENT

We thank Professor Carl de Boor for many friendly and fertile discussions that helped us in understanding of spline functions, and also lead to this paper. We are also grateful for his careful reading and questioning of the manuscript.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2088	2. GOVT ACCESSION NO. AD-A089 661	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE GENERALIZED EULER-FROBENIUS POLYNOMIAL		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Y. Y. Feng and J. Kozak		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Computer Science
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE June 1980
		13. NUMBER OF PAGES 13
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Euler-Frobenius polynomial
Splines
Interpolation
Least-squares approximation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper the properties of the generalized Euler-Frobenius polynomial $\Pi_n(\cdot;q)$ are studied. It is proved that its zeroes are separated by a factor q and their asymptotic behavior as $q \to \infty$ is obtained. As a consequence it is shown that least squares spline approximation on a biinfinite geometric mesh is boundable independently of the (local) mesh ratio q and that the norm of the inverse of the corresponding B-spline Gram matrix decreases monotonly to $2k - 1$ for large q, as $q \to \infty$.