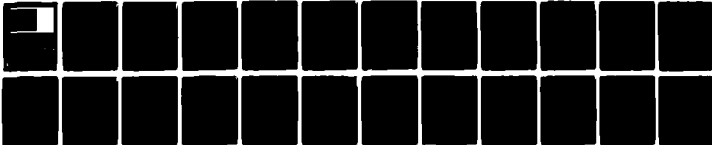


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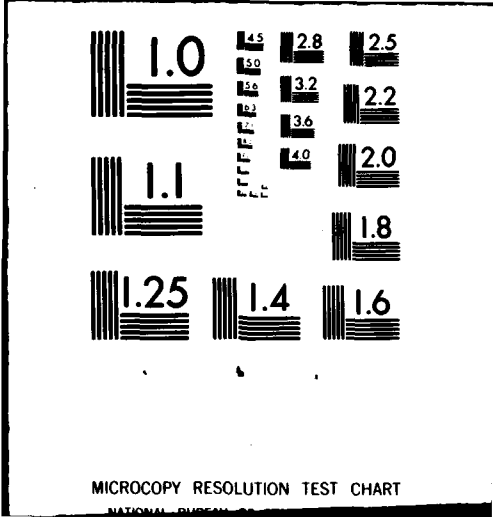
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A PSEUDO-PARABOLIC VARIATIONAL  
INEQUALITY AND STEFAN PROBLEM

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MATHEMATICS RESEARCH CENTER

A PSEUDO-PARABOLIC VARIATIONAL INEQUALITY  
AND STEFAN PROBLEM

E. DiBenedetto and R. E. Showalter<sup>(1)</sup>

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ABSTRACT

We study existence and uniqueness questions concerning non-linear variational inequalities of the type of (1.1a) - (1.1b) in the introduction. Applications to free boundary problems are also given.

AMS (MOS) Subject Classifications: 35A15, 35K22, 35K55, 35K70, 35R35.

Key Words: maximal monotone, convex of constraints, free-boundary problems,  
pseudo-parabolic

Work Unit Number 1 - Applied Analysis


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SIGNIFICANCE AND EXPLANATION

A variety of heat-transfer problems involving a change of phase, such as melting of ice, fusion of metals, solidification of alloys, involves basically two facts: a diffusion process (described for example by the heat equation) and a description of the progressive change of the region where the process takes place. Such phenomena can be formulated as free-boundary problems which are often treated mathematically using variational inequalities. The heat equation describes many heat transfer problems only in the crudest way.

In this paper we propose to study such processes by using a non-linear, more refined model of heat-conduction. We develop a mathematical theory for a convenient class of variational inequalities and apply it to study problems involving a change of phase (for example a Stefan problem) where the alternative model mentioned above is employed.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

E. DiBenedetto and R. E. Showalter<sup>(1)</sup>1. Introduction.

Let  $A$  and  $B$  be (possibly multi-valued) maximal monotone operators and let  $C$  be a non-empty closed convex set in the real Hilbert space  $V$ . We shall give existence and uniqueness results for evolution inequalities (formally) of the form

$$(1.1.a) \quad u(t) \in C: \left( \frac{d}{dt}(Au(t)) + Bu(t) - f(t), v - u(t) \right)_V \geq 0, \quad v \in C, \quad 0 \leq t \leq T,$$

$$(1.1.b) \quad (Au(0) - v_0, v - u(0))_V \geq 0, \quad v \in C,$$

where  $f \in L^2(0, T; V)$  and  $v_0 \in A(u_0)$  are given. In Section 2 we introduce a new notion of weak solution of (1.1) and verify uniqueness when  $A$  is linear self-adjoint and  $B$  is strictly monotone. Existence of a weak solution is proved in Section 3 when  $A$  is a (single-valued) function of the form "identity plus compact operator",  $B$  is bounded, and  $A$  or  $B$  is a subgradient.

Variational inequalities of the form (1.1) are of interest on their own as extensions of corresponding evolution equations of Sobolev type (where  $C = V$ ). Early work on such inequalities is described in [2]; we mention [6] specifically as a source of examples of initial-boundary value problems for the pseudo-parabolic partial differential equation

$$(1.2) \quad \frac{\partial}{\partial t}(u - a\Delta u) = k\Delta u$$

with  $a > 0$ ,  $k > 0$ . Such equations arise as models for diffusion, and they provide an interesting alternative to the classical diffusion equation wherein  $a = 0$ . In Section 4 we give an example of an initial-boundary-value problem consisting of a highly nonlinear partial differential equation of pseudo-parabolic type whose solution is subject to unilateral constraints. Existence and uniqueness results for weak solutions follow from our abstract results on (1.1).

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A one-phase free-boundary problem of Stefan type for the equation (1.2) is shown in Section 5 to lead to the variational inequality (1.1). This development is parallel to that of the classical case  $a=0$  which is described, e.g., in [7]. The existence of a classical solution of a Stefan problem for (1.2) in one spatial dimension was given in [9] by entirely different methods.

## 2. The Variational Inequality.

We denote by  $L^2(0,T;V)$  the Hilbert space of (Bochner) square-integrable functions on the interval  $(0,T)$  with values in the Hilbert space  $V$ . Let  $H^1(0,T;V)$  denote the absolutely continuous  $V$ -valued functions  $v$  whose derivatives  $\frac{dv}{dt}$  belong to  $L^2(0,T;V)$ . Denote the dual of  $V$  by  $V^*$  and recall the natural identification  $L^2(0,T;V) = L^2(0,T;V)^*$ ; thus we obtain the (dual) identification  $L^2(0,T;V) \hookrightarrow H^1(0,T;V)^*$  by restriction. The derivative  $\frac{d}{dt} : H^1(0,T;V) \hookrightarrow L^2(0,T;V)$  is a bounded linear operator which determines the dual operator  $L \equiv -\left(\frac{d}{dt}\right)^* : L^2(0,T;V) \rightarrow H^1(0,T;V)^*$  by the formula

$$\langle Lf, v \rangle \equiv -\int_0^T (f(t), v'(t))_V dt, \quad f \in L^2(0,T;V), \quad v \in H^1(0,T;V).$$

The restriction of  $Lf$  to  $V$ -valued test functions is the (distribution) derivative  $\frac{df}{dt}$ . Moreover, for  $f \in H^1(0,T;V)$  we have

$$\langle Lf, v \rangle = \left( \frac{df}{dt}, v \right)_{L^2(0,T;V)} + (f(0), v(0))_V - (f(T), v(T))_V, \quad v \in H^1(0,T;V).$$

Thus, we can regard " $Lf + f(T)$ " as formally equivalent to the Cauchy operator " $\frac{df}{dt} + f(0)$ ".

We shall use basic material on maximal monotone operators [1]. Specifically, recall  $A \subset V \times V$  is monotone if  $[x_j, y_j] \in A$  for  $j=1$  and  $2$  imply  $(x_1 - x_2, y_1 - y_2)_V \geq 0$ , and strictly monotone if in addition equality holds only if  $x_1 = x_2$ . If  $\varphi: V \rightarrow \mathbb{R} \cup \{+\infty\}$  is proper, convex and lower semicontinuous, its subgradient defined by

$$\partial\varphi(x) \equiv \{u \in V: (u, y-x)_V \leq \varphi(y) - \varphi(x) \text{ for all } y \in V\}$$

for  $x \in V$  is maximal monotone. More specifically, if  $C$  is a non-empty, convex and closed set in  $V$ , its indicator function

$$I_C(x) \equiv \begin{cases} 0 & , x \in C \\ +\infty & , x \notin C \end{cases}$$

is proper, convex and lower semicontinuous, and we have  $u \in \partial I_C(x)$  if and only if

$$x \in C: (u, y-x) \leq 0 \text{ for all } y \in C .$$

Thus, the subgradient of the indicator function provides a convenient method of expressing the variational inequality.

Suppose we are given the pair  $A, B$  of maximal monotone operators on the Hilbert space  $V$ , a closed convex subset  $C$  of  $V$ ,  $f \in L^2(0, T; V)$  and a pair  $[u_0, v_0] \in A$ . Then a function  $u$  is called a strong solution of (1.1) if there is a pair of functions  $v, w$  such that

$$u, v \in H^1(0, T; V) ; \quad w \in L^2(0, T; V) ,$$

$$(2.1.a) \quad u(t) \in C: \left( \frac{dv(t)}{dt} + w(t) - f(t), x - u(t) \right)_V \geq 0, \quad x \in C ,$$

$$v(t) \in A(u(t)) \text{ and } w(t) \in B(u(t)) \text{ for a.e. } t \in [0, T], \text{ and}$$

$$(2.1.b) \quad (v(0) - v_0, x - u(0))_V \geq 0, \quad x \in C .$$



Note that since  $u$  and  $v$  are continuous,  $C$  is closed in  $V$  and  $A$  is closed in  $V \times V$ , it follows that the inclusions  $u(t) \in C$  and  $v(t) \in A(u(t))$  hold for all  $t \in [0, T]$ . Also, (2.1) can be restated as

$$(2.2.a) \quad \frac{dv(t)}{dt} + w(t) + \partial I_C(u(t)) \ni f(t),$$

$v(t) \in A(u(t))$  and  $w(t) \in B(u(t))$  for a.e.  $t \in [0, T]$ , and

$$(2.2.b) \quad v(0) + \partial I_C(u(0)) \ni v_0$$

in terms of the indicator function.

We shall use a weak notion of solution in which it is not required that  $v \in H^1(0, T; V)$ . Set  $K \equiv \{u \in H^1(0, T; V) : u(t) \in C, 0 \leq t \leq T\}$ . Define a weak solution of (1.1) to be a function  $u$  for which there is a pair of functions  $v, w$  satisfying

$$u \in K; \quad v, w \in L^2(0, T; V),$$

$$v(t) \in A(u(t)), \quad w(t) \in B(u(t)), \quad \text{a.e. } t \in [0, T],$$

and for some  $\xi \in A(u(T))$  we have

$$(2.3) \quad \langle Lv + w - f, \eta - u \rangle + (\xi, \eta(T) - u(T))_V \geq (v_0, \eta(0) - u(0))_V, \quad \eta \in K.$$

Note that if  $u$  is a strong solution then it is a weak solution with  $\xi = v(T)$ .

Moreover we have the following elementary result.

Theorem 1. Let  $A$  be continuous, linear, self-adjoint and monotone; let  $B$  be strictly monotone. Then there is at most one weak solution.

Proof: Let  $u_1$  and  $u_2$  be weak solutions and let  $v_1, w_1$  and  $v_2, w_2$  be the corresponding selections from  $A(u_1), B(u_1)$ , etc. By our assumptions on  $A$  we have (after modification on a null set)  $v_j = A(u_j) \in H^1(0, T; V)$  and  $\xi_j = A(u_j(T))$  for  $j = 1, 2$ . Thus we have

$$\langle LAu_1 + w_1 - f, u_2 - u_1 \rangle + (Au_1(T), u_2(T) - u_1(T))_V \geq (v_0, u_2(0) - u_1(0))_V$$

$$\langle LAu_2 + w_2 - f, u_1 - u_2 \rangle + (Au_2(T), u_1(T) - u_2(T))_V \geq (v_0, u_1(0) - u_2(0))_V .$$

For any  $u \in H^1(0, T; V)$  we have

$$\langle LAu, u \rangle = 1/2((Au(0), u(0))_V - (Au(T), u(T))_V) ,$$

so adding the two inequalities and applying this identity with  $u = u_1 - u_2$  gives

$$(w_1 - w_2, u_1 - u_2)_{L^2(0, T; V)} + 1/2(Au(T), u(T))_V + 1/2(Au(0), u(0))_V \leq 0 .$$

Strict monotonicity of  $B$  shows  $u_1 = u_2$  .

Remarks. Without additional assumptions we should not expect uniqueness of the selections  $v, w$ . For example, in the extreme case  $C = \{0\}$ , (2.3) is vacuous and we need only choose  $v, w \in L^2(0, T; V)$  with  $v(t) \in A(0)$  and  $w(t) \in B(0)$  to obtain a weak solution. At the other extreme,  $C = V$ , any weak solution gives a strong solution of the equation  $\frac{dv}{dt} + w = f$  in  $L^2(0, T; V)$  with  $v(0) = v_0$ . Even for equations, the current uniqueness proofs require, e.g.,  $A$  or  $B$  to be linear self-adjoint. See [5].

### 3. Existence of a Weak Solution.

Our objective is to prove the following result on the existence of weak solutions of (1.1). Note that each of our hypotheses concerns only one of the three sources of nonlinearity in the problem; we have not placed any "compatibility" conditions on the operators  $A, B$  or the set  $C$ .

Theorem 2. Let  $C$  be a non-empty, closed and convex subset of the Hilbert space  $V$ . Let  $A$  and  $B$  be maximal monotone operators on  $V$  and assume the following:

- (i)  $A$  is a (single-valued) function which maps bounded sets in  $V$  into bounded sets in  $U$ , where  $U$  is a Banach space compactly imbedded in  $V$ .
- (ii)  $B$  maps bounded sets in  $V$  into bounded sets in  $V$ .
- (iii) Either  $A = \partial\varphi$  or  $B = \partial\varphi$ , where  $\varphi: V \rightarrow \mathbb{R}$  is a convex and lower-semicontinuous function.

Then for each  $u_0 \in C$  and  $f \in L^2(0, T; V)$  there is at least one pair  $u, w$  such that

$$(3.1.a) \quad u \in K, \quad w \in L^2(0, T; V), \quad w(t) \in B(u(t)), \quad 0 \leq t \leq T,$$

$$(3.1.b) \quad \langle L(u + A(u)) + w - f, \eta - u \rangle + (u(T) + A(u(T)), \eta(T) - u(T))_V \geq (u_0 + A(u_0), \eta(0) - u(0))_V,$$

for  $\eta \in K$ ,

and  $u(0) = u_0$ .

Remarks If in addition we had  $A(u) \in H^1(0, T; V)$ , then  $u$  would be a strong solution of

$$(3.2) \quad \frac{d}{dt}(u(t) + A(u(t))) + w(t) + \partial I_C(u(t)) \ni f(t), \quad \text{a.e. } t \in [0, T].$$

This is (2.2.a) with  $A$  replaced by  $A + I$ .

Since we do have  $u \in H^1(0, T; V)$  and  $u(0) = u_0$ , it follows that (3.1.b) is equivalent to

$$(3.3) \quad \left\langle \frac{du}{dt} + LA(u) + w - f, \eta - u \right\rangle + (A(u(T)), \eta(T) - u(T))_V \geq (A(u_0), \eta(0) - u(0))_V, \quad \eta \in K.$$

Proof: We shall prove Theorem 2 in the following steps. First we approximate (3.2) (and, hence, (3.1)) by replacing  $\partial I_C$  by its Lipschitz-continuous Yoshida approximation  $\partial I_C^\varepsilon$ ,  $\varepsilon > 0$ ; the resulting equation has a solution  $u_\varepsilon$  by [5]. Then we

establish estimates on  $\{u_\varepsilon\}$ , deduce the existence of a weak limit  $u \equiv \lim(u_\varepsilon)$ , and finally show  $u$  is a weak solution of (3.1).

The approximation. As an approximation of the indicator function  $I_C$  we take

$$I_C^\varepsilon(x) \equiv (2\varepsilon)^{-1} \|x - P_C(x)\|_V^2, \quad \varepsilon > 0, \quad x \in V.$$

Its Fréchet derivative is  $\partial I_C^\varepsilon(x) = \varepsilon^{-1}(x - P_C(x))$ , where  $P_C$  is the orthogonal projection onto  $C$ , and it is monotone and Lipschitz continuous. Thus  $B + \partial I_C^\varepsilon$  is maximal monotone and we obtain from [5] the existence of a pair  $u_\varepsilon \in H^1(0, T; V)$ ,  $w_\varepsilon \in L^2(0, T; V)$  for each  $\varepsilon > 0$  satisfying

$$(3.4) \quad \begin{aligned} \frac{d}{dt}(u_\varepsilon(t) + A(u_\varepsilon(t))) + w_\varepsilon(t) + \partial I_C^\varepsilon(u_\varepsilon(t)) &= f(t), \\ w_\varepsilon(t) \in B(u_\varepsilon(t)), \quad \text{a.e. } t \in [0, T], \end{aligned}$$

and  $u_\varepsilon(0) = u_0$ . This approximation (3.4) is strongly suggested by (3.2).

The estimates. Consider the two cases in (iii).

Case A =  $\partial\varphi$ : Take the scalar product of (3.4) with  $u_\varepsilon$  and integrate; this gives

$$\begin{aligned} 1/2 \|u_\varepsilon(t)\|_V^2 + \varphi^*(A(u_\varepsilon(t))) + \int_0^t (w_\varepsilon, u_\varepsilon)_V + \int_0^t (\partial I_C^\varepsilon(u_\varepsilon), u_\varepsilon)_V \\ = 1/2 \|u_0\|_V^2 + \varphi^*(A(u_0)) + \int_0^t (f, u_\varepsilon), \quad 0 \leq t \leq T, \end{aligned}$$

where  $\varphi^*(x) = \sup\{(x, y)_V - \varphi(y) : y \in V\}$  is the convex conjugate of  $\varphi$  [1, p. 41]. Since  $A$  is bounded, its domain is all of  $V$  so  $\varphi(0) < \infty$ . Thus, we may take  $\varphi(0) = 0$  and  $\varphi^*(x) \geq 0$ ,  $x \in V$ , with no loss of generality. Since  $B$  is monotone  $(w_\varepsilon(t), u_\varepsilon(t))_V \geq (B^0(0), u_\varepsilon(t))$  for  $t \in [0, T]$ , where, e.g.,  $B^0(0) \in B(0)$  is the minimal section of  $B$  at  $0$ . Finally we may assume  $0 \in C$  and thus

$$(\partial I_C^\epsilon(u_\epsilon(t)), u_\epsilon(t))_V \geq I_C^\epsilon(u_\epsilon(t)) \geq 0, \quad 0 \leq t \leq T,$$

from the definition of the subgradient. These observations and the preceding estimate give

$$(1/2) \|u_\epsilon(t)\|_V^2 \leq (1/2) \|u_0\|_V^2 + \varphi^*(Au_0) + (\|f\|_{L^1(0,T;V)} + T\|B^0(0)\|_V) \|u_\epsilon\|_{L^\infty(0,T;V)},$$

$$0 \leq t \leq T.$$

This implies that  $\|u_\epsilon\|_{L^\infty(0,T;V)}$  is bounded, and from (i) and (ii) it follows

$$\|w_\epsilon\|_{L^\infty(0,T;V)} \quad \text{and} \quad \|A(u_\epsilon)\|_{L^\infty(0,T;U)}$$

are bounded uniformly in  $\epsilon > 0$ . Next, we take the scalar product of (3.4) with  $u'_\epsilon(t)$  and integrate; this gives

$$\begin{aligned} & \int_0^t \|u'_\epsilon\|_V^2 + \int_0^t \left( \frac{d}{dt} A(u_\epsilon), \frac{du_\epsilon}{dt} \right)_V + I_C^\epsilon(u_\epsilon(t)) \\ & \leq I_C^\epsilon(u_0) + (\|w_\epsilon\|_{L^2(0,T;V)} + \|f\|_{L^2(0,T;V)}) \|u'_\epsilon\|_{L^2(0,T;V)}. \end{aligned}$$

The monotonicity of  $A$  implies the second term above is non-negative so we deduce that  $\|u'_\epsilon\|_{L^2(0,T;V)}$  and  $\|I_C^\epsilon(u_\epsilon)\|_{L^\infty}$  are bounded uniformly in  $\epsilon > 0$ .

Case B =  $\partial\varphi$ : Take the scalar product of (3.4) with  $u'_\epsilon(t)$ ; this gives

$$\|u'_\epsilon(t)\|_V^2 + \left( \frac{d}{dt} (Au_\epsilon(t)), \frac{d}{dt} u_\epsilon(t) \right)_V + (w_\epsilon(t) + \partial I_C^\epsilon(u_\epsilon(t)), u'_\epsilon(t))_V = (f(t), u'_\epsilon(t))_V.$$

The second term is non-negative because  $A$  is monotone. The third term is the derivative of  $\varphi(u_\epsilon(t)) + I_C^\epsilon(u_\epsilon(t))$  by the chain rule [1, p. 73]. Thus we integrate this identity and obtain

$$\int_0^t \|u'_\epsilon\|_V^2 + \varphi(u_\epsilon(t)) + I_C^\epsilon(u_\epsilon(t)) \leq \varphi(u_0) + \|f\|_{L^2(0,T;V)} \|u'_\epsilon\|_{L^2(0,T;V)}.$$

We can add to  $B = \partial\phi$  a constant, by adding the same to  $f(t)$ , so we may add an affine function to  $\phi$  with no loss of generality and thereby obtain  $\phi(x) \geq 0$  for all  $x \in V$ . The preceding estimate gives uniform bounds on  $\|u'_\epsilon\|_{L^2(0,T;V)}$  and  $\|I_C^\epsilon(u_\epsilon)\|_{L^\infty}$ . Similar bounds follow immediately on  $\|u_\epsilon\|_{L^\infty(0,T;V)}$  and by (i) and (ii) on  $\|A(u_\epsilon)\|_{L^\infty(0,T;U)}$  and  $\|w_\epsilon\|_{L^\infty(0,T;V)}$ , respectively.

The limit. From the estimates obtained above it follows there is a subnet of  $\{u_\epsilon\}$  (which we denote again by  $\{u_\epsilon\}$ ) for which

$$\begin{aligned} w\text{-}\lim(u_\epsilon) &= u \text{ in } H^1(0,T;V), \text{ and} \\ w\text{-}\lim(w_\epsilon) &= w \text{ in } L^2(0,T;V), \end{aligned}$$

where "w-lim" denotes the weak limit.

Lemma 1.  $u \in K \equiv \{v \in H^1(0,T;V) : v(t) \in C, \text{ all } t \in [0,T], u(0) = u_0 \text{ and } w\text{-}\lim(u_\epsilon(t)) = u(t) \text{ in } V \text{ for every } t \in [0,T]\}.$

Proof: Let  $t > 0$ . For each  $x \in V$  we have

$$(u_\epsilon(t) - u(t), x)_V = \int_0^t ((u_\epsilon - u)', x)_V + (u_0 - u(0), x)_V$$

convergent to  $(u_0 - u(0), x)_V$ . By bounded convergence

$$\lim \int_0^T (u_\epsilon - u, x)_V = \int_0^T (u_0 - u(0), x)_V = 0$$

so  $u(0) = u_0$  and  $w\text{-}\lim(u_\epsilon(t)) = u(t)$ . Next define  $z_\epsilon(t) = P_C(u_\epsilon(t))$ , the orthogonal projection onto  $C$ . Then  $\{z_\epsilon\}$  is bounded in  $L^2(0,T;V)$  so there is a subnet  $\{z_\epsilon\}$  which converges weakly to  $z$  in  $L^2(0,T;V)$ . Note that

$$\|u_\epsilon(t) - z_\epsilon(t)\|_V^2 = 2\epsilon I_C^\epsilon(u_\epsilon(t)) \leq (\text{const.})\epsilon,$$

so  $w\text{-}\lim(u_\xi) = u = z$ . Since the set  $\{v \in L^2(0, T; V) : v(t) \in C, \text{ a.e. } t \in C\}$  contains each  $z_\epsilon$  and is weakly closed, it also contains  $u = z$ . Finally  $u \in K$  follows since  $C$  is closed and  $u$  is continuous.

Lemma 2. We have the (strong) limits  $\lim A(u_\epsilon) = A(u)$  in  $L^2(0, T; V)$  and  $\lim A(u_\epsilon(t)) = A(u(t))$  in  $V$  for every  $t \in [0, T]$ .

Proof: Let  $t \in [0, T]$ . Since  $\{A(u_\epsilon(t))\}$  is in a compact set in  $V$  there is a subnet  $\{A(u_{\xi}(t))\}$  which converges (strongly) to  $v(t)$  in  $V$ . But  $w\text{-}\lim(u_\xi(t)) = u(t)$  in  $V$  and  $A$  is maximal monotone so  $v(t) = A(u(t))$ . The above applies as well to any subnet of  $\{A(u_\epsilon(t))\}$ , so the entire net converges to  $A(u(t))$ . The convergence in  $L^2(0, T; V)$  of  $\{A(u_\epsilon)\}$  to  $A(u)$  follows by the bounded convergence theorem.

Lemma 3.  $w \in B(u)$  and  $\lim (w_\epsilon, u_\epsilon)_{L^2(0, T; V)} = (w, u)_{L^2(0, T; V)}$ .

Proof: It suffices to show that [1, p. 27]

$$(3.5) \quad \limsup (w_\epsilon, u_\epsilon)_{L^2(0, T; V)} \leq (w, u)_{L^2(0, T; V)}$$

Take the scalar product of (3.4) with  $u_\epsilon - u$  and integrate. From the estimate

$$(\partial I_C^\epsilon(u_\epsilon), u_\epsilon - u)_{L^2(0, T; V)} \geq I_C^\epsilon(u_\epsilon) - I_C(u) = I_C^\epsilon(u_\epsilon) \geq 0$$

we obtain

$$(w_\epsilon, u_\epsilon)_{L^2} \leq (w_\epsilon, u)_{L^2} + \left( \frac{d}{dt}(u_\epsilon + A(u_\epsilon)), u - u_\epsilon \right)_{L^2} + (f, u_\epsilon - u)_{L^2}.$$

By taking the upper limit we have

$$\limsup (w_\epsilon, u_\epsilon)_{L^2} \leq (w, u)_{L^2} + \limsup \left( \frac{d}{dt}(u_\epsilon + A(u_\epsilon)), u - u_\epsilon \right)_{L^2}.$$

Thus it suffices to show the last term is non-positive. From the identity

$$\int_0^T (u'_\epsilon, u_\epsilon - u)_V = 1/2 (\|u_\epsilon(T)\|_V^2 - \|u_0\|_V^2) - \int_0^T (u'_\epsilon, u)_V$$

and Lemma 1 there follows

$$\liminf \int_0^T (u'_\epsilon, u_\epsilon - u)_V \geq 1/2 (\|u(T)\|_V^2 - \|u_0\|_V^2) - \int_0^T (u', u)_V = 0.$$

Similarly from Lemma 2 and

$$\int_0^T (A(u_\epsilon)', u_\epsilon - u)_V = (Au_\epsilon(T), u_\epsilon(T) - u(T))_V - \int_0^T (A(u_\epsilon), u'_\epsilon - u')_V$$

it follows that  $\lim \int_0^T (A(u_\epsilon)', u_\epsilon - u)_V = 0$ .

The solution. To show that  $u$  is a weak solution, it suffices by Lemma 1 to verify (3.1.b). For any  $\eta \in K$  it follows from (3.4) that

$$(3.6) \quad (u'_\epsilon, \eta - u_\epsilon)_{L^2(0, T; V)} + (A(u_\epsilon)', \eta - u_\epsilon)_{L^2(0, T; V)} + (w_\epsilon - f, \eta - u_\epsilon)_{L^2(0, T; V)} \\ = (\partial I_C^\epsilon(u_\epsilon), u_\epsilon - \eta)_{L^2(0, T; V)}.$$

From the definition of subgradient it follows that the right side is greater than  $I_C^\epsilon(u_\epsilon) - I_C^\epsilon(\eta) = I_C^\epsilon(u_\epsilon)$ , hence, non-negative. Consider the first term on the left side. We have from  $w\text{-}\lim u_\epsilon(T) = u(T)$  that



$$\begin{aligned} (u', u)_{L^2} &= (1/2) (\|u(T)\|_V^2 - \|u_0\|_V^2) \leq \liminf (1/2) (\|u_\epsilon(T)\|_V^2 - \|u_0\|_V^2) \\ &= \liminf (u'_\epsilon, u_\epsilon)_{L^2} \end{aligned}$$

so there follows

$$\langle Lu, \eta - u \rangle + \langle u(T), \eta(T) - u(T) \rangle_V - \langle u_0, \eta(0) - u_0 \rangle_V = (u', \eta - u)_{L^2} \geq \limsup (u'_\epsilon, \eta - u_\epsilon)_{L^2}.$$

Concerning the second term, we obtain from Lemma 2

$$\begin{aligned} &\langle L(Au), \eta - u \rangle + \langle A(u(T)), \eta(T) - u(T) \rangle_V - \langle A(u_0), \eta(0) - u_0 \rangle_V \\ &= \lim \{ \langle L(Au_\epsilon), \eta - u_\epsilon \rangle + \langle A(u_\epsilon(T)), \eta(T) - u_\epsilon(T) \rangle_V - \langle A(u_0), \eta(0) - u_0 \rangle_V \} \\ &= \lim (A(u_\epsilon)', \eta - u_\epsilon)_{L^2(0, T; V)} \end{aligned}$$

Finally, Lemma 3 identifies the limit of the third term, so by taking the "lim sup" in (3.6) we obtain (3.1.b).

#### 4. A Pseudo-Parabolic Inequality.

When our results from above are used to describe initial-boundary-value problems for partial differential equations or inequalities, it is usually more convenient to express them in terms of the equivalent notion of a maximal monotone operator  $\mathcal{G}$  from the Hilbert  $V$  to its dual  $V^*$ . Thus, letting  $\mathcal{R}: V \rightarrow V^*$  be the Riesz isomorphism given by the scalar product,

$$\mathcal{R}x(y) = (x, y)_V, \quad x, y \in V,$$

we say  $\mathcal{G} \subset V \times V^*$  is monotone if the composite operator  $A \equiv \mathcal{R}^{-1} \circ \mathcal{G}$  is monotone in  $V \times V$  and maximal monotone if, in addition,  $Rg(\mathcal{R} + \mathcal{G}) = V^*$ . We can easily state Theorem 2, for example, in this context. Thus, we are given a set  $C$  closed and convex in  $V$ ;  $\mathcal{G}$  and  $\mathcal{B}$  are maximal monotone operators from  $V$  to  $V^*$  satisfying

hypotheses corresponding to (i), (ii) and (iii). Then for each  $u_0 \in C$  and  $f \in L^2(0, T; V^*)$  there is a pair of functions  $u, w$  satisfying

$$(4.1.a) \quad u \in K \equiv \{v \in H^1(0, T; V) : v(t) \in C \text{ for } 0 \leq t \leq T\},$$

$$(4.1.b) \quad w \in L^2(0, T; V^*), \quad w(t) \in \mathcal{B}(u(t)) \text{ for a.e. } t \in [0, T],$$

$$(4.1.c) \quad \mathcal{A}(u) \in L^2(0, T; V^*) \text{ and } u(0) = u_0,$$

$$(4.1.d) \quad \langle L(\mathcal{R} + \mathcal{A})(u) + w - f, \eta - u \rangle + (\mathcal{R} + \mathcal{A})u(T)(\eta(T) - u(T)) \\ \geq (\mathcal{R} + \mathcal{A})u_0(\eta(0) - u(0)), \text{ for } \eta \in K.$$

In this setting the linear operator  $L: L^2(0, T; V^*) \rightarrow H^1(0, T; V)^*$  is given by

$$\langle Lg, u \rangle = -\int_0^T g(t)(u'(t))dt, \quad g \in L^2(0, T; V^*), \quad u \in H^1(0, T; V).$$

Since  $u \in H^1(0, T; V)$ , the inequality (4.1.d) is equivalent to

$$\left\langle \frac{d}{dt} \mathcal{R}u + L\mathcal{A}(u) + w - f, \eta - u \right\rangle + \mathcal{A}(u(T))(\eta(T) - u(T)) \geq \mathcal{A}(u_0)(\eta(0) - u(0)), \\ \text{for } \eta \in K.$$

We shall describe an example of a partial differential equation of pseudo-parabolic type which is to be resolved subject to unilateral boundary constraints.

(A similar equation with constraint over the entire region will be given in the next section.) Let  $G$  be a bounded open set in  $R^n$  which lies on one side of its boundary  $\partial G$ ; assume  $\partial G$  consists of two disjoint parts  $\Gamma_0$  and  $\Gamma$ , and let  $\vec{n}(s) = (n_1(s), \dots, n_n(s))$  be the unit outward normal at each point  $s \in \partial G$ .  $H^1(G)$  is the Sobolev space of those  $v \in L^2(G)$  for which all derivatives

$\frac{\partial v}{\partial x_j} = D_j v$ ,  $1 \leq j \leq n$ , belong to  $L^2(G)$ ; we set  $D_0 v = v$ . Let  $V = \{v \in H^1(G) : v|_{\Gamma_0} = 0\}$ ; by  $v|_{\Gamma_0}$  we mean the trace of  $v$  on  $\Gamma_0$  (see [8,10]). For  $v \in V$  we denote by  $\gamma(v) \in L^2(\Gamma)$  the trace of  $v$  on  $\Gamma$ . Let  $r_0 \in L^\infty(G)$  and  $r_1 \in L^\infty(\Gamma)$  be non-negative and define

$$(4.2) \quad \mathfrak{R}u(v) = (u, v)_V = \int_G \left( \sum_{j=1}^n D_j u D_j v + r_0 uv \right) + \int_\Gamma r_1 \gamma(u) \gamma(v), \quad u, v \in V.$$

It follows by a compactness argument that (4.2) is equivalent to the usual  $H^1(G)$  scalar-product if any one of  $\Gamma_0$  or  $\{x: r_0(x) > 0\}$  or  $\{s: r_1(s) > 0\}$  has strictly positive measure, and we assume this hereafter.

The operator  $\mathfrak{A}$  is given by a pair of continuous (maximal) monotone functions  $\alpha_0, \alpha_{-1}: \mathbb{R} \rightarrow \mathbb{R}$  which are linearly bounded:

$$|\alpha_j(z)| \leq Q(1 + |z|), \quad z \in \mathbb{R}, \quad j = 0, -1$$

for some constant  $Q > 0$ . We define

$$(4.3) \quad \mathfrak{A}(u)(v) = \int_G \alpha_0(u)v + \int_\Gamma \alpha_{-1}(\gamma u)\gamma v, \quad u, v \in V.$$

This operator is a subgradient (in fact, a Gateaux derivative) and is bounded from  $V$  to  $U \equiv L^2(G) \times L^2(\Gamma)$ . Since the imbeddings  $V \hookrightarrow U$  and  $U = U^* \hookrightarrow V^*$  are compact, the hypothesis (i) is fulfilled.

The operator  $\mathfrak{B}$  will be specified by a family of maximal monotone operators  $\beta_k: \mathbb{R} \rightarrow \mathbb{R}$  which are linearly bounded:

$$|w| \leq Q(1 + |z|) \quad \text{for } w \in \beta_j(z), \quad z \in \mathbb{R}, \quad -1 \leq k \leq n$$

for some  $Q > 0$ . Each  $\beta_k = \partial\varphi_k$  for a corresponding convex continuous  $\varphi_k: \mathbb{R} \rightarrow \mathbb{R}$ .

We then define  $\mathfrak{B} = \partial\varphi$  where

$$\varphi(v) \equiv \int_G \sum_{j=0}^n \varphi_j(D_j v) + \int_\Gamma \varphi_{-1}(\gamma v), \quad v \in V.$$

Thus  $\mathfrak{B}$  is given (formally) by

$$(4.4) \quad \mathfrak{B}(u) = \sum_{k=0}^n D_k^* \beta_k^*(D_k u) + \gamma^* \beta_{-1}^*(\gamma u), \quad u \in V.$$

To be precise, we have  $F \in \mathfrak{B}(u)$  if and only if there exist  $f_k \in \beta_k(D_k u)$  in  $L^2(G)$ ,  $0 \leq k \leq n$ , and  $f_{-1} \in \beta_{-1}(\gamma u)$  in  $L^2(\Gamma)$  for which

$$F(v) = \int_G \sum_{k=0}^n f_k D_k v + \int_\Gamma f_{-1}(\gamma v), \quad v \in V.$$

By restricting each of the functionals  $\mathfrak{R}(u)$ ,  $\mathfrak{Q}(u)$  and  $\mathfrak{B}(u)$  to test functions  $C_0^\infty(G)$  we obtain the corresponding distributions over  $G$

$$(4.5.a) \quad \mathfrak{R}_0(u) = -\Delta u + r_0 u,$$

$$(4.5.b) \quad \mathfrak{Q}_0(u) = \alpha_0(u),$$

$$(4.5.c) \quad \mathfrak{B}_0(u) = -\sum_{j=1}^n D_j \beta_j(D_j u) + \beta_0(u), \quad u \in V,$$

where the multi-valued  $\mathfrak{B}_0$  is interpreted as before. The respective differences are given (by Green's theorem for the first and last cases) for sufficiently regular  $u$  by

$$(4.6.a) \quad \mathfrak{R}u(v) - \int_G \mathfrak{R}_0(u)v = \int_{\Gamma} \left( \frac{\partial u}{\partial n} + r_1 \gamma u \right) \gamma v ,$$

$$(4.6.b) \quad \mathfrak{Q}(u)(v) - \int_G \mathfrak{Q}_0(u)v = \int_{\Gamma} \alpha_{-1}(\gamma u) \gamma v ,$$

$$(4.6.c) \quad \mathfrak{B}(u)(v) - \int_G \mathfrak{B}_0(u)v = \int_{\Gamma} \left[ \sum_{j=1}^n \beta_j (D_j u) n_j + \beta_{-1}(\gamma u) \right] \gamma v , \quad v \in V .$$

Thus we have realized the operators  $\mathfrak{R}$ ,  $\mathfrak{Q}$  and  $\mathfrak{B}$  as the sum of a distribution over  $G$  (4.5) and a boundary part over  $\Gamma$  (4.6). See [5] for details.

The remaining data is given as follows. Let  $C = \{v \in V : \gamma(v) \geq 0 \text{ a.e. on } \Gamma\}$  and let  $u_0 \in C$  be specified. Suppose  $F_0 \in L^2(G \times [0, T])$  and  $g_0 \in L^2(\Gamma \times [0, T])$  are given and define  $f \in L^2(0, T; V^*)$  by

$$f(t)(v) = \int_G F_0(\cdot, t)v + \int_{\Gamma} g_0(\cdot, t)\gamma v , \quad v \in V .$$

With the preceding data as given, the solution  $u, w$  of (4.1) is a generalized solution of the pseudo-parabolic problem

$$(4.7) \quad \left\{ \begin{array}{l} \frac{\partial}{\partial t} (\mathfrak{R}_0 u + \mathfrak{Q}_0(u)) + \mathfrak{B}_0(u) = F_0 \quad \text{in } G \times (0, T) \\ u(x, 0) = u_0(x) , \quad x \in G , \\ u = 0 \quad \text{on } \Gamma_0 \times [0, T] \\ u \geq 0 , \quad \Lambda(u) \geq 0 , \quad \Lambda(u)(u) = 0 \quad \text{on } \Gamma \times [0, T] \end{array} \right.$$

where  $\Lambda$  is the boundary operator obtained from (4.6) as

$$(4.8) \quad \Lambda(u) = \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial n} + r_1 \gamma(u) + \alpha_{-1}(\gamma u) \right) + \sum_{j=1}^n \beta_j (D_j u) n_j + \beta_{-1}(\gamma u) .$$

The other operators in (4.7) are given by (4.5). Note specifically that the multi-valued operators are to be interpreted precisely as was done above following (4.4).

### 5. A Stefan Problem.

We consider a problem of heat diffusion involving a solid-liquid phase change at a prescribed temperature. One application we have in mind is the melting of ice (initially at temperature zero) suspended in a reservoir or porous medium. The novelty in this treatment is that we assume the heat diffusion is governed by the pair of equations

$$\frac{\partial \theta}{\partial t} = k \Delta \varphi, \quad \theta = \varphi - a \Delta \varphi.$$

Chen and Gurtin [3] introduced such a model for heat conduction in non-simple materials where the energy, entropy, heat flux and thermodynamic temperature  $\theta(x,t)$  depend on the conductive temperature  $\varphi(x,t)$  and its first two spatial gradients. Here the heat flux is determined by the conductive temperature and the phase is determined by the thermodynamic temperature. Thus  $\theta > 0$  in the region occupied by water and  $\theta = 0$  corresponds to the frozen region.

We describe the geometry of the problem. Let the bounded domain  $G$  in  $R^n$  be the medium in which the ice/water is suspended and let its boundary  $\partial G$  consist of two disjoint pieces,  $\Gamma_0$  and  $\Gamma_1$ . Set  $\Omega = G \times (0, T)$ , where  $T > 0$ , and note that its lateral boundary is  $B_0 \cup B_1$ , where  $B_j = \Gamma_j \times (0, T)$  for  $j = 0, 1$ . The water-region  $\Omega_1 = \{(x, t) \in \Omega : \theta(x, t) > 0\}$  is separated from the ice-region  $\Omega_0 = \{(x, t) \in \Omega : \theta(x, t) = 0\}$  by an interface  $S$  which is the phase boundary. The unit outward normal on  $\partial \Omega_1$  is denoted by  $\vec{N} = (\vec{N}_x, N_t)$ ,  $\vec{N}_x \in R^n$ . If  $V(t)$  is the velocity in  $R^n$  of the interface at time  $t$ , then it follows by the chain

rule that  $V(t) \cdot \vec{N}_x + N_t = 0$  on  $S$ . Set  $\vec{n} = \vec{N}_x / \|\vec{N}_x\|$ , the unit outward normal in  $R^n$  of the lateral boundary of  $\Omega_1$ . Of course  $n = \vec{N}_x$  on  $B_1$ , and  $\vec{N}_x = 0$  where  $t = 0$  or  $t = T$ .

The problem is formulated as follows. We are given the conductivity  $k > 0$ , temperature discrepancy  $a > 0$ , and latent heat  $b > 0$ , of the material and a constant  $h \geq 0$  representing conductivity across the lateral boundary  $B_1$ . The initial thermodynamic temperature  $\theta_0(x)$ ,  $x \in G$ , and applied conductive temperature  $g(x, t)$ ,  $(x, t) \in B_1$ , are given with  $\theta_0 = 0$  on  $\Gamma_0$ ,  $\theta_0 > 0$  on  $\Gamma_1$ , and  $g \geq 0$ . The local form of the problem is to find a pair of non-negative functions  $\theta, \varphi$  on  $\Omega$  for which we have

$$(5.1.a) \quad \frac{\partial \theta}{\partial t} = k \Delta \varphi, \quad \theta = \varphi - a \Delta \varphi \quad \text{in } \Omega_1$$

$$(5.1.b) \quad k \frac{\partial \theta}{\partial n} + b V(t) \cdot \vec{n} = 0 \quad \text{on } S$$

$$(5.1.c) \quad k \frac{\partial \theta}{\partial n} + h(\varphi - g) = 0 \quad \text{on } \Gamma_1$$

$$(5.1.d) \quad \varphi = 0 \quad \text{on } \Gamma_0$$

$$(5.1.e) \quad \theta(\cdot, 0) = \theta_0 \quad \text{on } G.$$

Note that if  $\theta, \varphi$  is a solution of (5.1) and  $\theta_0 \geq 0$ , then

$$(5.2) \quad (a/k) \frac{\partial \theta}{\partial t} + \theta = \varphi \quad \text{in } \Omega$$

so it follows that  $\varphi = 0$  on  $\Omega_0 \cup S$ . Since  $g \geq 0$ , the maximum principle for the elliptic equation in (5.1.a) on the region  $G(t) \equiv \{x \in G: (x, t) \in \Omega_1\}$  shows that  $\varphi > 0$  in  $\Omega_1$  and  $\frac{\partial \theta}{\partial n} < 0$  on  $S$ . Thus  $N_t < 0$  on  $S$  and  $G(t)$  is increasing with  $t$ .

We shall show that the problem (5.1) leads to a variational inequality of the form (1.1). Define  $V = \{v \in H^1(G) : v|_{\Gamma_0} = 0\}$  as before. Regarding regularity of a solution, we assume  $\theta_0 \in V$ ,  $\theta : [0, T] \rightarrow V$  is absolutely continuous,  $\varphi \in L^1(0, T; V)$ , and (c.f. (5.2))

$$(5.3) \quad \frac{a}{k} \frac{d\theta(t)}{dt} + \theta(t) = \varphi(t), \quad \text{a.e. } t \in [0, T].$$

Define the continuous linear  $\mathcal{R} : V \rightarrow V^*$  by

$$\mathcal{R}u(v) = \int_G k(\vec{\nabla}u \cdot \vec{\nabla}v) dx + \int_{\Gamma_1} h(uv) ds, \quad u, v \in V.$$

For a test function  $v \in C_0^\infty((0, T), V)$  we obtain

$$\begin{aligned} \int_0^T \mathcal{R}\varphi(t)(v(t)) dt &= \int_{\Omega_1} k \vec{\nabla} \varphi \cdot \vec{\nabla} v \, dx dt + \int_{B_1} h\varphi v \, ds dt \\ &= \int_{\Omega_1} (-k\Delta\varphi)v \, dx dt + \int_{\partial\Omega_1} k \vec{\nabla} \varphi \cdot \vec{N}_x v \, ds dt + \int_{B_1} h\varphi v \, ds dt \\ &= -\int_{\Omega} \frac{\partial\theta}{\partial t} v + \int_{B_1} hgv + \int_S bN_t v \end{aligned}$$

from (5.1). Furthermore we have

$$\int_S N_t v = \int_{\Omega_1} \frac{\partial v}{\partial t} = \int_{\Omega} H(\theta) \frac{\partial v}{\partial t} = -\frac{\partial H(\theta)}{\partial t} (v)$$

in the sense of  $V^*$ -valued distributions, where  $H(s) = 1$  for  $s > 0$  and  $H(s) = 0$  for  $s \leq 0$  is the Heaviside function. We can summarize the above calculations as



$$(5.4) \quad \frac{d}{dt} (\theta + bH(\theta)) + \mathcal{R}\varphi = (gh)_{\Gamma_1} \quad \text{in } L^1(0, T; V^*)$$

where we define

$$(hg)_{\Gamma_1}(t)(v) = \int_{\Gamma_1} hg(s, t)v(s)ds, \quad v \in V, \quad t \in [0, T].$$

Combining (5.3) and (5.4) we find that the absolutely continuous function

$\theta: [0, T] \rightarrow V$  satisfies

$$(5.5.a) \quad \frac{d}{dt} (\theta + (a/k)\mathcal{R}(\theta) + bH(\theta)) + \mathcal{R}(\theta) = (gh)_{\Gamma_1} \quad \text{in } L^1(0, T; V^*),$$

$$(5.5.b) \quad \theta(0) = \theta_0, \quad \text{and}$$

$$(5.5.c) \quad \theta(x, t) \geq 0, \quad \text{a.e. } x \in G, \quad t \in [0, T].$$

If we integrate (5.5.a) and follow the suggestion in [7] to set

$$u(t) \equiv \int_0^t \theta(s)ds$$

$$f(t) \equiv (I + (a/k)\mathcal{R} + bH)\theta_0 - b + \int_0^t (hg)_{\Gamma_1}(s)ds,$$

there follows

$$\frac{d}{dt} (I + (a/k)\mathcal{R})u + \mathcal{R}u - f(t) = b(1-H(\theta)).$$

Finally we note that  $H(u) = H(\theta)$  since  $G(t)$  is increasing in  $t$ , hence,  $u(1-H(\theta)) = 0$  in  $\Omega$ .

The preceding computations show that  $u \in H^1(0, T; V)$  and it satisfies  $u(0) = 0$ ,

$$(5.6.a) \quad u(t) \geq 0 \text{ in } V$$

$$(5.6.b) \quad \frac{d}{dt} (\mathcal{R} + (k/a)I)u(t) + (k/a)\mathcal{R}u(t) \geq f(t) \text{ in } V^*, \text{ and}$$

$$(5.6.c) \quad \left( \frac{d}{dt} (\mathcal{R} + (k/a)I)u(t) + (k/a)\mathcal{R}u(t) - f(t) \right) (u(t)) = 0, \quad 0 \leq t \leq T.$$

Setting  $C = \{v \in V: v \geq 0 \text{ a.e. in } G\}$  we see that  $u$  is a strong solution of (4.1) with  $\mathcal{A} = (k/a)I$ ,  $\mathcal{B} = (k/a)\mathcal{R}$ , and  $u_0 = 0$ . Theorem 1 asserts the uniqueness of a solution of (5.1) under conditions considerably weaker than those leading to (5.6). Theorem 2 establishes the existence of a weak solution with certain additional regularity properties. In particular  $\mathcal{A}u \in H^1(0, T; V^*)$  since  $\mathcal{A}$  is continuous and linear, so (4.1.d) is equivalent to

$$\left\langle \frac{d}{dt} (\mathcal{R}u + \mathcal{A}u) + w - f, \eta - u \right\rangle \geq 0, \quad \text{for } \eta \in K.$$

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