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IMPLICIT DEGENERATE EVOLUTION EQUATIONS AND APPLICATIONS. (U)

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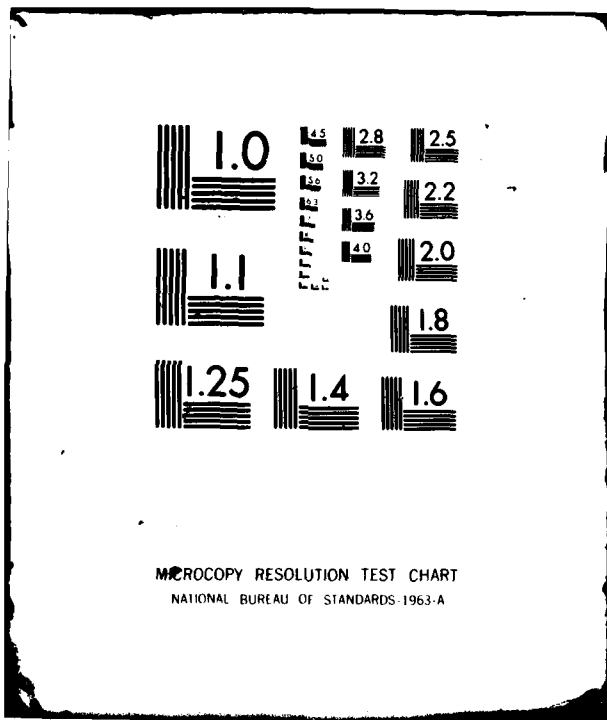
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IMPLICIT DEGENERATE EVOLUTION EQUATIONS AND APPLICATIONS

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IMPLICIT DEGENERATE EVOLUTION EQUATIONS AND APPLICATIONS

Emmanuele DiBenedetto and R. E. Showalter\*

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ABSTRACT

The initial-value problem is studied for evolution equations in Hilbert space of the general form

$$\frac{d}{dt} A(u) + B(u) \ni f$$

where  $A$  and  $B$  are maximal monotone operators. Existence of a solution is proved when  $A$  is a subgradient and either  $A$  is strongly-monotone or  $B$  is coercive; existence is established also in the case where  $A$  is strongly-monotone and  $B$  is subgradient. Uniqueness is proved when one of  $A$  or  $B$  is continuous self-adjoint and the sum is strictly-monotone; examples of non-uniqueness are given. Applications are indicated for various classes of degenerate nonlinear partial differential equations or systems of mixed elliptic-parabolic-pseudoparabolic types and problems with non-local nonlinearity.

AMS(MOS) Subject Classification: Primary 34G20, 35A05, 35D05  
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Key Words: Maximal monotone, Evolution equations, Degenerate parabolic, Free boundary

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SIGNIFICANCE AND EXPLANATION

A variety of models of physical phenomena can be written in a unified way as

$$\frac{d}{dt} Au + Bu = f$$

where  $A$  and  $B$  are monotone operators between Hilbert spaces. Therefore it seems convenient to study questions of existence and uniqueness for abstract equations as the above in view of their range of applications.

We will divide the applications in two categories: problems with local non-linearities and problems with global non-linearities. The former occur when at some point of the region where the process occur, the function describing the process either jumps or degenerates. Typical are the Stefan problem, diffusion in porous media, diffusion in partially saturated porous media. They include also certain kinds of diffusion in a medium with a singularity due to a crack or fissure.

Problems with global non-linearities are typical of processes where a threshold (which separates the different aspects of the phenomenon, say for example elliptic-parabolic) is given through the global measure of the energy (or the variation of the energy).

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

# IMPLICIT DEGENERATE EVOLUTION EQUATIONS AND APPLICATIONS

Emmanuele DiBenedetto and R. E. Showalter\*

## 1. Introduction.

Let  $A$  and  $B$  be maximal monotone operators from a Hilbert space  $V$  to its dual  $V^*$ . Such operators are in general multi-valued and their basic properties will be recalled below. We shall consider initial-value problems of the form

$$(1.1) \quad \frac{d}{dt} A(u) + B(u) \ni f, \quad u(0) \ni v_0$$

where  $f \in L^2(0, T; V^*)$  and  $v_0 \in V^*$  are given. It is assumed throughout our work that  $A$  is a compact operator from  $V$  to  $V^*$ . In applications to partial differential equations this assumption limits the order of the operator  $A$  to be strictly lower than that of  $B$ . Both operators will be required to satisfy boundedness conditions and one or the other is assumed to be a subgradient.

The objective of this work is to prove existence of a solution of (1.1) when  $A$  and  $B$  are possibly degenerate. Observe that we must in general assume some condition of coercivity on the pair of operators. To see this, we note that if one of them is identically zero then (1.1) is equivalent to a one-parameter family of "stationary" problems of the form  $M(u(t)) \ni F(t)$  where  $M$  is maximal monotone. But if  $M$  is, e.g., a subgradient in a space of finite dimension, it is surjective only if it is coercive. Thus it is appropriate to assume at least one of  $A$  or  $B$  is coercive. In accord with this remark our work will proceed as follows. First we replace  $A$  by the coercive operator

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$A + \epsilon R$ , where  $\epsilon > 0$  and  $R : V \rightarrow V^*$  is the Riesz isomorphism determined by the scalar product on  $V$ , and we solve the initial-value problem for the "regularized" equation

$$(1.2) \quad \frac{d}{dt} (A + \epsilon R)(u_\epsilon) + B(u_\epsilon) = f.$$

Here we may take  $\epsilon = 1$  with no loss of generality and we make no coercivity assumptions on either  $A$  or  $B$ . Next we assume  $B$  is coercive and let  $\epsilon \rightarrow 0^+$  in order to recover (1.1) with (possibly) degenerate  $A$ . Since  $R$  is of the same order as  $B$  this regularization is analogous to the Yoshida approximation. The operator  $A$  is assumed to be a subgradient in the above. Finally, we show the initial value problem can be solved for (1.2) when  $B$  (but not necessarily  $A$ ) is a subgradient.

We mention some related work on equations of the form in (1.1). The theory of such implicit evolution equations divides historically into three cases. The first and certainly the easiest is where  $B \circ A^{-1}$  is Lipschitz or monotone in some space [6, 23]. The second is that one of the operators is (linear) self-adjoint, and this case includes the majority of the applications to problems where singular or degenerate behavior arises due to spatial coefficients or geometry [2, 25]. These situations are described in the book [9] to which we refer for details and a very extensive bibliography. The third case is that wherein both operators are possibly nonlinear. This considerably more difficult case has been investigated by Grange and Mignot [12] and more recently by Barbu [4]. In both of these studies a compactness assumption similar to ours is made. Our boundedness assumptions are more restrictive than those in the papers above, but they assume  $f$  is smooth and that both operators are subgradients. By not requiring that  $B$  be a subgradient in (1.1) we obtain a significantly larger class of applications to partial differential equations, especially to systems.

Our work is organized as follows. In Section 2 we recall certain information on maximal monotone operators and then state our results on the existence of solutions of the initial-value problems (1.1) and for (1.2). The proofs are given in Sections 3 and 4. Section 5 contains elementary examples of how non-uniqueness occurs, and we show there that uniqueness holds in the situation where one of the operators is self-adjoint. Section 6 is concerned with the structure and construction of maximal monotone operators between Hilbert

spaces which characterize certain partial differential equations and associated boundary conditions. These operators are used to present in Section 7 a collection of initial-boundary-value problems for partial differential equations which illustrate the applications of our results to the existence theory of such problems.



## 2. Preliminaries and Main Results.

We begin by reviewing information on maximal monotone operators. Refer to [1, 3, 11] for additional related material and proofs. Then we shall state our existence theorems for the Cauchy problem (1.1).

Let  $V$  be real Hilbert space and  $A$  a subset of the product  $V \times V$ . We regard  $A$  as a function from  $V$  to  $2^V$ , the set of subsets of  $V$ , or as a multi-valued mapping or operator from  $V$  into  $V$ ; thus,  $f \in A(u)$  means  $[u, f] \in A$ . We define the domain  $D(A) = \{u \in V: Au \text{ non-empty}\}$ , range  $R_g(A) = \cup \{Au: u \in V\}$  and inverse  $A^{-1}(u) = \{v \in V: u \in A(v)\}$  of  $A$  as indicated. The operator  $A$  is monotone if  $(f_1 - f_2, u_1 - u_2)_V > 0$  whenever  $[u_j, f_j] \in A$  for  $j = 1, 2$ . This is equivalent to  $(I + \lambda A)^{-1}$  being a contraction for every  $\lambda > 0$ . We call  $A$  maximal monotone if it is maximal in the sense of inclusion of graphs. Then we have a monotone  $A$  maximal monotone if and only if  $R_g(I + \lambda A) = V$  for some (hence, all)  $\lambda > 0$ . If  $A$  is maximal monotone we can define its resolvent  $J_\lambda \equiv (I + \lambda A)^{-1}$ , a contraction defined on all  $V$ , and its Yoshida approximation  $A_\lambda = \lambda^{-1}(I - J_\lambda)$ , a monotone Lipschitz function defined on all  $V$ . For  $u \in V$  we have  $A_\lambda(u) \in A(J_\lambda(u))$ . We denote weak convergence of  $x_n$  to  $x$  by  $x_n \rightharpoonup x$ .

**Lemma 2.1.** Let  $A$  be maximal monotone,  $[x_n, y_n] \in A$  for  $n > 1$ ,  $x_n \rightharpoonup x$ ,  $y_n \rightharpoonup y$  and  $\liminf (y_n, x_n)_V < (y, x)_V$ . Then  $[x, y] \in A$ . If in addition  $\limsup (y_n, x_n)_V < (y, x)_V$ , then  $(y_n, x_n)_V \rightarrow (y, x)_V$ . We observe that  $A$  induces on  $L^2(0, T; V)$  a maximal monotone operator (denoted also by  $A$ ) defined by  $v \in A(u)$  if and only if  $v(t) \in A(u(t))$  for a.e.  $t \in [0, T]$ .

A special class of maximal monotone operators arises as follows. If  $\varphi: V \rightarrow (-\infty, \infty]$  is a proper, convex and lower semicontinuous function we define the subgradient  $\partial\varphi \subset V \times V$  by

$$\partial\varphi(x) = \{z \in V: \varphi(y) - \varphi(x) \geq (z, y-x) \text{ for all } y \in V\}.$$

The operator  $\partial\varphi$  is maximal monotone. Furthermore it is useful to consider the convex conjugate of  $\varphi$  defined by

$$\varphi^*(z) \equiv \sup\{(z, y)_V - \varphi(y), y \in V\}.$$

The following are equivalent:  $z \in \partial\varphi(x)$ ,  $x \in \partial\varphi^*(z)$ , and  $\varphi(x) + \varphi^*(z) = (x, z)_V$ ; thus  $\varphi\partial^*$  is the inverse of  $\partial\varphi$ . We mention the following chain rule. [1]. Let  $H^1(0, T; V)$  denote the space of absolutely continuous  $V$ -valued functions on  $[0, T]$  whose derivatives belong to  $L^2(0, T; V)$ .

Lemma 2.2. If  $u \in H^1(0, T; V)$ ,  $v \in L^2(0, T; V)$  and  $[u(t), v(t)] \in \partial\varphi$  for a.e.  $t \in [0, T]$ , then the function  $t \rightarrow \varphi(u(t))$  is absolutely continuous on  $[0, T]$  and

$$\frac{d}{dt} \varphi(u(t)) = (w, u'(t)), \quad \text{all } w \in \partial\varphi(u(t)),$$

for a.e.  $t \in [0, T]$ .

There is a version of monotone operator from  $V$  to its dual space  $V^*$  which is equivalent to the above through the Riesz map  $R: V \rightarrow V^*$ . Thus,  $A \subset V \times V^*$  is monotone if and only if  $A \equiv R^{-1} \circ \tilde{A}$  is monotone in  $V \times V$  and maximal monotone if and only if  $R_g(R + A) = V^*$  in addition. We shall use these two equivalent notions interchangeably. Our applications to partial differential equations will lead to operators on  $V \times V^*$ . Also the subgradient is naturally constructed in the  $W - W^*$  duality of a Banach (or topological vector) space  $W$ . Finally we cite the following chain rule.

Lemma 2.3. Let  $V$  and  $W$  be locally convex spaces with duals  $V^*$  and  $W^*$ . Let  $\Lambda: V \rightarrow W$  be continuous and linear with dual  $\Lambda^*: W^* \rightarrow V^*$ . If  $\varphi: W \rightarrow (-\infty, \infty]$  is proper, convex and lower semicontinuous then so also is  $\varphi \circ \Lambda: V \rightarrow (-\infty, \infty]$ , and if  $\varphi$  is continuous at some point of  $R_g(\Lambda)$  we have [11]

$$\partial(\varphi \circ \Lambda) = \Lambda^* \circ \partial\varphi \circ \Lambda.$$

Our results on the existence of solutions of the Cauchy problem (1.1) are stated as follows.

Theorem 1. Let  $W$  be a reflexive Banach space and  $V$  a Hilbert space which is dense and imbedded compactly in  $W$ . Denote the injection by  $i: V \rightarrow W$  and the dual (restriction) operator by  $i^*: W^* \rightarrow V^*$ . Assume the following:

[A<sub>1</sub>] The real-valued  $\varphi$  is proper, convex and lower semicontinuous on  $W$ , continuous at some point of  $V$ , and  $\partial\varphi \circ i: V \rightarrow W^*$  is bounded.

[B<sub>1</sub>] The operator  $B : V \rightarrow V^*$  is maximal monotone and bounded.

Define  $A \equiv i^* \circ \partial\varphi \circ i$ . Then for each given  $f \in L^2(0, T; V^*)$  and  $[u_0, v_0] \in A$  there exists a triple  $u \in H^1(0, T; V)$ ,  $v \in H^1(0, T; V^*)$ , and  $w \in L^2(0, T; V^*)$  such that

$$(2.1.a) \quad \frac{d}{dt} (Ru(t) + v(t)) + w(t) = f(t) ,$$

$$(2.1.b) \quad v(t) \in A(u(t)), w(t) \in B(u(t)), \text{ a.e. } t \in [0, T] ,$$

$$(2.1.c) \quad Ru(0) + v(0) = Ru_0 + v_0 .$$

Theorem 2. In addition to the above, assume

$$[A_2] \quad \partial\varphi \circ i : L^2(0, T; V) \rightarrow L^2(0, T; W^*) \text{ is bounded}$$

$$[B_2] \quad B : L^2(0, T; V) \rightarrow L^2(0, T; V^*) \text{ is bounded and coercive, i.e.,}$$

$$\lim_{\|u\|_{L^2(0, T; V)} \rightarrow +\infty} \frac{\int_0^T v(t)(u(t)) dt}{\|u\|_{L^2(0, T; V)}} = +\infty .$$

$[u, v] \in B$

Then for each given  $f \in L^2(0, T; V^*)$  and  $v_0 \in R_g(A)$  there exists a triple  $u \in L^2(0, T; V)$ ,  $v \in H^1(0, T; V^*)$ ,  $w \in L^2(0, T; V^*)$  such that

$$(2.2.a) \quad \frac{d}{dt} v(t) + w(t) = f(t) ,$$

$$(2.2.b) \quad v(t) \in A(u(t)), w(t) \in B(u(t)), \text{ a.e. } t \in [0, T] ,$$

$$(2.2.c) \quad v(0) = v_0 .$$

Remarks. From Lemma 2.3 it follows that  $A = \partial(\varphi|_V)$  where  $\varphi|_V = \varphi \circ i$  is the restriction of  $\varphi$  to  $V$ . Since  $A : V \rightarrow V^*$  is bounded it follows  $D(A) = V$ , hence,

$$V \subset D(\partial\varphi) \subset \text{dom}(\varphi) \subset W$$

and  $\varphi$  is continuous on the space  $V$ . Also, since  $\varphi(0) < \infty$  we may assume with no loss of generality that  $\varphi(0) < 0$  and thus  $\varphi^*(z) > 0$  for all  $z \in V$ .

From the compactness of  $i^* : W^* \rightarrow V^*$  it follows  $A : V \rightarrow V^*$  is compact, i.e., maps bounded sets into relatively compact sets.

Since  $B$  is bounded and maximal monotone we have  $D(B) = V$ . It is important for our applications that we have made no assumptions which directly relate  $A$  and  $B$ . Specifically, we do not compare  $A(x)$  and  $B(x)$  in angle or in norm.

Finally, we give a variation on Theorem 1 in which only the second operator  $\delta$  is a subgradient. The compactness assumption on  $A$  is retained.

Theorem 3. Let the spaces  $V$  and  $W$  be given as before. Assume the following:

[A<sub>3</sub>] The operator  $A: V \rightarrow V^*$  is maximal monotone with  $R_g(A) \subset W^*$  and  $A: V \rightarrow W^*$  is bounded.

[B<sub>3</sub>] The real-valued  $\psi$  is proper, convex and lower semicontinuous on  $V$  and  $\delta \equiv \partial\psi: V \rightarrow V^*$  is bounded.

Then for given  $f \in L^2(0, T; V^*)$  and  $\{u_0, v_0\} \in A$  there exists a triple  $u \in H^1(0, T; V)$ ,  $v \in H^1(0, T; V^*)$ , and  $w \in L^2(0, T; V^*)$  satisfying (2.1).

**3. Proofs of Theorem 1 and Theorem 3.**

These proofs are very similar; let us consider first the Theorem 1. We formulate (2.1) in the space  $V$ . Set  $A = R^{-1} \circ A$ ,  $B = R^{-1} \circ B$ , etc, and consider the equivalent equation

$$(3.1.a) \quad \frac{d}{dt} (u(t) + v(t)) + w(t) = f(t) ,$$

$$(3.1.b) \quad v(t) \in A(u(t)), w(t) \in B(u(t)), \text{ a.e. } t \in [0, T] .$$

Let  $\lambda > 0$  and consider the approximation of (3.1) by

$$(3.2.a) \quad \frac{d}{dt} (u_\lambda(t) + v_\lambda(t)) + B_\lambda(u_\lambda(t)) = f(t) ,$$

$$(3.2.b) \quad v_\lambda(t) \in A(u_\lambda(t)), \quad t \in [0, T] .$$

Since  $(I + A)^{-1}$  and  $B_\lambda$  are both Lipschitz continuous from  $V$  to  $V$ , (3.2) has a unique absolutely continuous solution  $u_\lambda$  with  $u_\lambda(0) + v_\lambda(0) = u_0 + v_0$ . Since  $(I + A)^{-1}$  is a function, we have  $u_\lambda(0) = u_0$  and  $v_\lambda(0) = v_0$ .

We derive a priori estimates on  $u_\lambda$ . Take the scalar product in  $V$  of (3.2.a) with  $u_\lambda(t)$  and note

$$(v'_\lambda(t), u_\lambda(t))_V = \frac{d}{dt} \varphi^*(v_\lambda)$$

by Lemma 2.2 where  $\varphi^*$  is the conjugate of  $\varphi|_V$  in  $V$ . Integrating the resulting identity gives

$$\begin{aligned} \frac{1}{2} \|u_\lambda(t)\|_V^2 + \varphi^*(v_\lambda(t)) &< \frac{1}{2} \|u_0\|_V^2 + \varphi^*(v_0) \\ &+ \int_0^t (\|f(s)\|_V + \|B_\lambda(0)\|_V) \|u_\lambda(s)\|_V ds, \quad 0 \leq t \leq T . \end{aligned}$$

Since  $\{B_\lambda(0)\}$  is bounded by the fact that  $0 \in D(B)$ ,  $\varphi^* > 0$  and  $f \in L^2(0, T; V)$ , we have proved the first part of the following.

**Lemma 3.1.** The following are bounded independent of  $\lambda > 0$ :

$$(a) \quad \begin{aligned} \|u_\lambda\|_{L^\infty(0, T; V)} , \quad \|Rv_\lambda\|_{L^\infty(0, T; W^*)} , \\ \|J_\lambda(u_\lambda)\|_{L^\infty(0, T; V)} , \quad \|B_\lambda(u_\lambda)\|_{L^\infty(0, T; V)} , \end{aligned}$$

$$(b) \quad \|u'_\lambda\|_{L^2(0, T; V)} , \quad \|v'_\lambda\|_{L^2(0, T; V)} .$$

Proof: The second and third terms of (a) are bounded because the operators  $A : V \rightarrow W^*$  and  $J_\lambda \equiv (I + \lambda B)^{-1} : V \rightarrow V$  are bounded. Since  $B_\lambda(u_\lambda) \in B(J_\lambda(u_\lambda))$  and  $B$  is bounded, the last term in (a) is bounded.

To obtain (b) we take the scalar product of (3.2.a) by  $u'_\lambda(t)$ , note that  $(v'_\lambda(t), u'_\lambda(t))_V > 0$  by (3.2.b) and monotonicity of  $A$ , and thereby obtain

$$\|u'_\lambda(t)\|_V^2 \leq (\|f(t)\|_V + \|B_\lambda(u_\lambda(t))\|_V) \|u'_\lambda(t)\|_V,$$

so we bound the first term in (b). The second follows from (3.2.a).

Note that we have  $\{Rv_\lambda\}$  bounded in  $L^2(0, T; W^*)$  and  $\{Rv'_\lambda\}$  bounded in  $L^2(0, T; V^*)$ . Since  $W^*$  is compact in  $V^*$  it follows from [17, p. 58] that  $\{Rv_\lambda\}$  is (strongly) relatively compact in  $L^2(0, T; V^*)$ . From this observation and Lemma 3.1 it follows we may pass to a subsequence, again denoted by  $u_\lambda, v_\lambda$ , for which we have

$$(3.3.a) \quad u_\lambda \rightarrow u, \quad B_\lambda(u_\lambda) \rightarrow w, \quad u'_\lambda \rightarrow u',$$

$$(3.3.b) \quad v_\lambda \rightarrow v \text{ (strongly)}, \quad v'_\lambda \rightarrow v' \text{ in } L^2(0, T; V),$$

$$(3.3.c) \quad u_\lambda(t) \rightarrow u(t) \text{ and } v_\lambda(t) \rightarrow v(t), \text{ all } t \in [0, T].$$

Since  $u_\lambda - J_\lambda(u_\lambda) = \lambda B_\lambda(u_\lambda) \rightarrow 0$  there follows

$$(3.3.d) \quad J_\lambda(u_\lambda) \rightarrow u \text{ in } L^2(0, T; V).$$

It remains to show that  $u, v, w$  satisfy (3.1) and the initial condition. First we use (3.3.a) and (3.3.b) and Lemma 2.1 to obtain  $v \in A(u)$ . Next take the scalar product of (3.2.a) with any  $x \in V$  and integrate to get

$$(u_\lambda(t) + v_\lambda(t), x)_V + \int_0^t (B_\lambda(u_\lambda(s)), x)_V ds = \int_0^t (f(s), x)_V ds + (u_0 + v_0, x)_V.$$

Taking the limit as  $\lambda \rightarrow 0$  gives (since  $x$  is arbitrary)

$$u(t) + v(t) + \int_0^t (w - f) ds = u_0 + v_0, \quad 0 \leq t \leq T.$$

From this identity we obtain (3.1.a) and  $u(0) + v(0) = u_0 + v_0$ ; since  $v(0) \in A(u(0))$  and  $(I + A)^{-1}$  is a function we have  $u(0) = u_0$ . In order to show  $w \in B(u)$ , and thereby finish the proof of Theorem 1, it suffices by Lemma 2.1 to show

$$\limsup_{\lambda \rightarrow 0} (B_\lambda(u_\lambda), J_\lambda(u_\lambda) - u)_{L^2(0, T; V)} < 0.$$

We note further that

$$\begin{aligned} (B_\lambda(u_\lambda), J_\lambda(u_\lambda)) &= (B_\lambda(u_\lambda), J_\lambda(u_\lambda) - u_\lambda) + (B_\lambda(u_\lambda), u_\lambda) \\ &= -\lambda(B_\lambda(u_\lambda), B_\lambda(u_\lambda)) + (B_\lambda(u_\lambda), u_\lambda) \end{aligned}$$

so it suffices to show

$$(3.4) \quad \limsup_{\lambda \rightarrow 0} (B_\lambda(u_\lambda), u_\lambda - u)_{L^2(0, T; V)} < 0.$$

By (3.2.a) it follows (3.4) is equivalent to

$$(3.5) \quad \liminf_{\lambda \rightarrow 0} (u'_\lambda + v'_\lambda, u_\lambda - u)_{L^2(0, T; V)} > 0.$$

Define  $\psi(x) = \frac{1}{2} \|x\|_V^2 + \varphi(x)$ ,  $x \in V$  so that  $\partial\psi = I + \partial\varphi$ . From (3.2.b) and Lemma 2.2 we obtain

$$(u'_\lambda(t) + v'_\lambda(t), u_\lambda(t))_V = \frac{d}{dt} \psi^*(u_\lambda(t) + v_\lambda(t)),$$

and by integrating there follows

$$(u'_\lambda + v'_\lambda, u_\lambda)_{L^2(0, T; V)} = \psi^*(u_\lambda(T) + v_\lambda(T)) - \psi^*(u_0 + v_0).$$

Similarly we have from (3.1.a)

$$(u' + v', u)_{L^2(0, T; V)} = \psi^*(u(T) + v(T)) - \psi^*(u_0 + v_0).$$

By (3.3.c) and weak lower semicontinuity of  $\psi^*$  we have

$$\psi^*(u(T) + v(T)) < \liminf_{\lambda \rightarrow 0} \psi^*(u_\lambda(T) + v_\lambda(T)),$$

and our preceding calculations show this is equivalent to (3.5).

Remark 3.1. From Lemma 2.1 we find that

$$(B_\lambda(u_\lambda), J_\lambda(u_\lambda))_{L^2(0, T; V)} \rightarrow (w, u)_{L^2(0, T; V)}.$$

If we also have  $B$  (or  $B$ ) strongly monotone then we can take the limit in the estimate

$$\|B_\lambda(u_\lambda) - w, J_\lambda(u_\lambda) - u\|_{L^2(0,T;V)} > c \|J_\lambda(u_\lambda) - u\|_{L^2(0,T;V)}^2$$

to conclude  $\{J_\lambda(u_\lambda)\}$  and  $\{u_\lambda\}$  converge strongly to  $u$  in  $L^2(0,T;V)$ .

**Remark 3.2.** It is clear that we actually have  $v(t) \in A(u(t))$  for every  $t \in [0,T]$ .

The proof of Theorem 3 closely follows the preceding pattern. That is, formulate (2.1) as the equivalent initial value problem for (3.1) and approximate this by (3.2) with  $u_\lambda(0) + v_\lambda(0) = u_0 + v_0$  for each  $\lambda > 0$ .

To derive a priori bounds we take the scalar product of (3.2.a) with  $u'_\lambda(t)$  and integrate to obtain

$$(3.6) \quad \int_0^T \|u'_\lambda\|_V^2 + \psi_\lambda(u_\lambda(T)) \leq \psi_\lambda(u_0) + \int_0^T (f(t), u'_\lambda(t))_V dt.$$

Here  $\psi_\lambda$  is the Yoshida approximation of  $\psi$ . We may assume  $\psi$  is non-negative and the same holds for  $\psi_\lambda$ , so we have the first part of the following.

**Lemma 5.1.** The following are bounded independent of  $\lambda > 0$ :

$$(a) \quad \|u_\lambda\|_{L^\infty(0,T;V)}, \|u'_\lambda\|_{L^2(0,T;V)}, \|Rv_\lambda\|_{L^\infty(0,T;W^*)},$$

$$\|J_\lambda(u_\lambda)\|_{L^\infty(0,T;V)}, \|B_\lambda(u_\lambda)\|_{L^\infty(0,T;V)},$$

$$(b) \quad \|v_\lambda\|_{L^\infty(0,T;V)}, \|v'_\lambda\|_{L^2(0,T;V)}.$$

**Proof:** The bound on the first two terms in (a) follow from (3.6) and the remaining terms in (a) are bounded by  $[A_3]$  and  $[B_3]$ . Next take the scalar product of (3.2.a) with  $v'_\lambda(t)$  and obtain (b) as was done in Lemma 3.1.

We may pass to a subsequence satisfying (3.3) and we obtain as in Theorem 1 the triple  $u, v, w$  satisfying the equation (3.1.a) and initial condition and  $v(t) \in Au(t)$ ,  $t \in [0,T]$ . It remains to show  $w \in B(u)$  and this is equivalent to showing (c.f. (3.5))

$$(3.7) \quad \liminf_{\lambda \rightarrow 0} (u'_\lambda + v'_\lambda, u_\lambda)_{L^2(0,T;V)} \geq (v', u)_{L^2(0,T;V)}.$$



Since  $u'_\lambda \in L^2(0, T; V)$  we may integrate by parts to compute

$$(3.8.a) \quad (u'_\lambda + v'_\lambda, u'_\lambda)_{L^2(0, T; V)} = \frac{1}{2} (\|u_\lambda(T)\|_V^2 - \|u_0\|_V^2) \\ - (v_\lambda, u'_\lambda)_{L^2(0, T; V)} + (v_\lambda(T), u_\lambda(T))_V - (v_0, u_0)$$

and similarly, since  $u' \in L^2(0, T; V)$ ,

$$(3.8.b) \quad (u' + v', u')_{L^2(0, T; V)} = \frac{1}{2} (\|u(T)\|_V^2 - \|u_0\|_V^2) \\ - (v, u')_{L^2(0, T; V)} + (v(T), u(T))_V - (v_0, u_0)_V.$$

Finally we observe that (3.7) follows immediately from (3.3) and (3.8).

4. Proof of Theorem 2.

Choose  $u_0 \in A^{-1}(v_0)$ . For each  $\lambda > 0$  let  $u_\lambda, v_\lambda \in H^1(0, T; V)$ ,  $w_\lambda \in L^2(0, T; V)$

satisfy

$$(4.1.a) \quad \lambda u_\lambda'(t) + v_\lambda'(t) + w_\lambda(t) = f(t) ,$$

$$(4.1.b) \quad v_\lambda(t) \in A(u_\lambda(t)), w_\lambda(t) \in B(u_\lambda(t)), \text{ a.e. } t \in [0, T] ,$$

$$(4.1.c) \quad \lambda u_\lambda(0) + v_\lambda(0) = \lambda u_0 + v_0 .$$

The problem (4.1) has such a solution by Theorem 1, and our plan is to show that we may take the limit as  $\lambda \rightarrow 0$  in (4.1) to obtain a solution  $u, w \in L^2(0, T; V)$ ,  $v \in H^1(0, T; V)$  of

$$(4.2.a) \quad v'(t) + w(t) = f(t) ,$$

$$(4.2.b) \quad v(t) \in A(u(t)), w(t) \in B(u(t)), \text{ a.e. } t \in [0, T] ,$$

$$(4.2.c) \quad v(0) = v_0 .$$

With our notation  $A = R^{-1} \circ A$ , etc., (4.2) is equivalent to (2.2).

We proceed to derive a priori estimates. Consider first the initial condition. Since

$(\lambda I + A)^{-1}$  is a function it follows from (4.1.c) that

$$(4.3) \quad u_\lambda(0) = u_0, v_\lambda(0) = v_0, \lambda > 0 .$$

Lemma 4.1. The following are bounded independent of  $\lambda > 0$  :

$$(a) \quad \|u_\lambda\|_{L^2(0, T; V)}, \lambda^{1/2} \|u_\lambda\|_{L^\infty(0, T; V)},$$

$$(b) \quad \|w_\lambda\|_{L^2(0, T; V)}, \|Rv_\lambda\|_{L^2(0, T; W^*)} .$$

Proof: Take the scalar product of (4.1.a) with  $u_\lambda(t)$  and integrate to obtain

$$(4.4) \quad (\lambda/2) \|u_\lambda(t)\|_V^2 + \varphi^*(v_\lambda(t)) + \int_0^t (w_\lambda, u_\lambda)_V \\ < (\lambda/2) \|u_0\|_V^2 + \varphi^*(v_0) + \int_0^t (f, u_\lambda)_V, \quad 0 \leq t \leq T .$$

We drop the second (non-negative) term in (4.4) and note by monotonicity of  $B$  that

$$(w_\lambda, u_\lambda)_V \geq (\xi, u_\lambda)_V \text{ for some } \xi \in B(0). \text{ Thus (4.4) gives}$$

$$\int_0^T (w_\lambda, u_\lambda)_V < \|f\|_{L^2(0,T;V)} \|u_\lambda\|_{L^2(0,T;V)} + C,$$

and the coercivity of  $B$  implies the boundedness of the first term in (a). The second now follows from (4.4) and now part (b) follows from our assumptions  $A_1$  and  $B_1$ .

Lemma 4.2. The following are bounded independent of  $\lambda > 0$ :

$$\|v_\lambda'\|_{L^2(0,T;V)}, \quad \|\lambda u_\lambda'\|_{L^2(0,T;V)}.$$

*Proof:* Take the scalar product of (4.1.a) with  $v_\lambda'(t)$ . Since  $(u_\lambda'(t), v_\lambda'(t))_V > 0$  by monotonicity of  $A$  we obtain

$$\|v_\lambda'(t)\|_V^2 < (\|f(t)\|_V + \|w_\lambda(t)\|_V) \|v_\lambda'(t)\|_V$$

from which the first bound is immediate. To obtain the second we take the scalar-product of (4.1.a) with  $u_\lambda'(t)$  and drop the non-negative term  $(u_\lambda'(t), v_\lambda'(t))_V$ . This gives

$$\lambda \|u_\lambda'(t)\|_V^2 < (\|f(t)\|_V + \|w_\lambda(t)\|_V) \|u_\lambda'(t)\|_V$$

and hence the desired bound.

We have now shown that  $\{Rv_\lambda\}$  is bounded in  $L^2(0,T;W^*)$  and that  $\{Rv_\lambda'\}$  is bounded in  $L^2(0,T;V^*)$ . Since  $W^*$  is compact in  $V^*$  it follows that  $\{Rv_\lambda\}$  is strongly compact in  $L^2(0,T;V^*)$ . From this observation, Lemma 4.1, and Lemma 4.2 it follows we may pass to a subsequence (which we denote again by  $\{u_\lambda\}, \{v_\lambda\}, \{w_\lambda\}$ ) for which in  $L^2(0,T;V)$  we have

$$\begin{aligned} u_\lambda &\rightarrow u, & w_\lambda &\rightarrow w, \\ v_\lambda &\rightarrow v, & v_\lambda' &\rightarrow v'. \end{aligned}$$

Note that  $\lambda u_\lambda \rightarrow 0$  and it follows  $\lambda u_\lambda' \rightarrow 0$  by standard arguments. Furthermore we may assume  $v_\lambda(t) \rightarrow v(t)$  in  $V$  for all  $t \in [0,T]$  by equi-continuity of  $\{v_\lambda\}$ , and similarly  $\lambda u_\lambda(t) \rightarrow 0$  in  $V$  for all  $t \in [0,T]$ .

It remains to show that the triple  $u, v, w$  obtained above constitutes a solution of (4.2). Let  $x \in V$ , take the scalar product of  $x$  with (4.1.a) and integrate to obtain

$$(\lambda u_\lambda(t) + v_\lambda(t), x)_V + \int_0^t (w_\lambda(s), x)_V ds = \int_0^t (f(s), x)_V ds + (\lambda u_0 + v_0, x)_V$$

Since weak convergence in  $L^2(0, T; V)$  implies weak convergence in  $L^2(0, t; V)$ , it follows by letting  $\lambda \rightarrow 0$  that

$$(v(t), x)_V + \int_0^t (w(s), x)_V ds = \int_0^t (f(s), x)_V ds + v_0, \quad x \in V, \quad t \in [0, T].$$

That is,

$$v(t) + \int_0^t w(s) ds = \int_0^t f(s) ds + v_0, \quad \text{a.e. } t \in [0, T],$$

and this implies (4.2.a) and (4.2.c). From Lemma 2.1 there follows  $v \in A(u)$  so it remains only to establish  $w \in B(u)$ . For this it suffices by Lemma 2.1 to show

$$(4.4) \quad \limsup_{\lambda \rightarrow 0} (w_\lambda, u_\lambda)_{L^2(0, T; V)} \leq (w, u)_{L^2(0, T; V)}.$$

In order to prove (4.4) we first note by (4.1.a) and (4.2.a) that it is equivalent to

$$(4.5) \quad \liminf_{\lambda \rightarrow 0} (\lambda u_\lambda' + v_\lambda', u_\lambda)_{L^2(0, T; V)} \geq (v', u)_{L^2(0, T; V)}.$$

Since  $u_\lambda(t) \in A^{-1}(v_\lambda(t)) = \partial \varphi^*(v_\lambda(t))$  a.e. on  $[0, T]$ , where  $\varphi^*$  is the conjugate of  $\varphi|_V$ , we obtain from Lemma 2.2

$$(\lambda u_\lambda'(t) + v_\lambda'(t), u_\lambda(t))_V = \frac{d}{dt} \left\{ (\lambda/2) \|u_\lambda(t)\|_V^2 + \varphi^*(v_\lambda(t)) \right\}$$

so we integrate and obtain

$$\begin{aligned} (\lambda u_\lambda' + v_\lambda', u_\lambda)_{L^2(0, T; V)} &= (\lambda/2) \|u_\lambda(T)\|_V^2 + \varphi^*(v_\lambda(T)) - (\lambda/2) \|u_0\|_V^2 - \varphi^*(v_0) \\ &> \varphi^*(v_\lambda(T)) - (\lambda/2) \|u_0\|_V^2 - \varphi^*(v_0). \end{aligned}$$

Similarly we compute

$$(\nu', u)_{L^2(0, T; V)} = \varphi^*(\nu(T)) - \varphi^*(\nu_0) .$$

Since  $\{\nu_\lambda\}$  are equi-uniformly-continuous we have  $\nu_\lambda(t) \rightarrow \nu(t)$  at every  $t \in [0, T]$ , so the lower semicontinuity of  $\varphi^*$  gives

$$\liminf_{\lambda \rightarrow 0} \varphi^*(\nu_\lambda(T)) \geq \varphi^*(\nu(T)) .$$

In view of the preceding computations this is exactly (4.5).

5. Remarks on Uniqueness.

We first present an example which shows that gross non-uniqueness of solutions of (1.1) can occur, even if both operators are strongly monotone subgradients. Moreover the non-uniqueness occurs in each term of the triple,  $u, v, w$ , not just in the latter two terms selected, respectively, from  $A(u)$  and  $B(u)$ . Next we shall show that uniqueness does hold for (1.1) when at least one of the operators is continuous, linear and symmetric and the sum of the operators is strictly monotone. Our last example shows that symmetry of the linear operator is essential.

Example 1. Let  $V = W = \mathbb{R}$ , the space of real numbers, and define

$$A(s) = B(s) = s + H(s-1),$$

where

$$H(r) = \begin{cases} 1 & , \quad r > 1 \\ [0,1] & , \quad r = 0 , \\ 0 & , \quad r < 1 \end{cases}$$

denotes the Heaviside function and  $f \equiv 0$ . Consider the initial-value problem (1.1) which takes the form

$$(5.1) \quad \begin{cases} v'(t) + w(t) = 0, & v(0) = 2 \\ v(t) - u(t) \in H(u(t) - 1), & w(t) - u(t) \in H(u(t) - 1) . \end{cases}$$

Let  $g$  be any maximal monotone graph or continuous function from  $\mathbb{R}$  to  $\mathbb{R}$  such that  $g(s) = s$  for  $s \notin [1,2]$  and  $g(s) \subset [1,2]$  for  $s \in [1,2]$ . Then, if  $v$  is a solution of

$$(5.2) \quad v'(t) + g(v(t)) = 0, \quad t > 0, \quad v(0) = 2,$$

it follows that with  $u(t) \equiv A^{-1}(v(t))$  and  $w(t) \equiv -v'(t)$  we have a solution of (5.1). This procedure yields an abundance of solutions.

We display some special cases of the above. Pick  $c \in [1/2, 1]$  and define  $g_c$  to be that maximal monotone graph such that  $g_c(t) = \{c^{-1}\}$ ,  $t \in (1,2)$ , and  $g_c(t) = \{t\}$ ,  $t \notin [1,2]$ . The corresponding solution  $v_c$  of (5.2) is given by

$$v_c(t) = 2 - t/c, \quad 0 \leq t < c, \quad v_c(t) = e^{c-t}, \quad t > c.$$

With the two functions  $u_c$  and  $w_c$  given by

$$\begin{aligned} u_c(t) &= 1, \quad w_c(t) = 1/c \quad \text{for } 0 < t < c \\ u_c(t) &= w_c(t) = e^{-t/c}, \quad t > c, \end{aligned}$$

this provides a continuum of solutions of (5.1).

We can give the following elementary sufficient conditions for uniqueness to hold for (1.1) or, equivalently, for (4.2).

**Theorem 4.** Let  $A$  and  $B$  be monotone operators on a Hilbert space  $V$ . Suppose  $A + B$  is strictly monotone and that one of  $A$  or  $B$  is continuous, linear and symmetric. Then for each function  $f : [0, T] \rightarrow V$  and  $v_0 \in V$  there is at most one solution  $u, v, w$  of (4.2).

**Proof:** Suppose  $A$  is continuous, linear and symmetric. For  $j = 1, 2$  let  $u_j, v_j, w_j$  be a solution of (4.2). Take the scalar product of the difference of (4.2.a) with  $u_1 - u_2$  to obtain

$$\left(\frac{1}{2}\right) \frac{d}{dt} (A(u_1(t) - u_2(t)), u_1(t) - u_2(t))_V + (w_1(t) - w_2(t), u_1(t) - u_2(t))_V = 0.$$

Integrating this identity and using (4.2.c) gives

$$\left(\frac{1}{2}\right) (A(u_1(t) - u_2(t)), u_1(t) - u_2(t))_V + \int_0^t (w_1 - w_2, u_1 - u_2)_V ds = 0, \quad 0 < t < T$$

and this implies

$$Au_1(t) = Au_2(t), \quad (w_1(t) - w_2(t), u_1(t) - u_2(t))_V = 0 \quad \text{a.e. } t \in [0, T].$$

Since  $A + B$  is strictly monotone we have  $u_1(t) = u_2(t)$ , hence

$$v_1(t) = Au_1(t) = Au_2(t) = v_2(t), \quad \text{and, by (4.1.a), } w_1(t) = w_2(t) \quad \text{a.e. on } [0, T].$$

Suppose now  $B$  is continuous linear and symmetric. Starting with two solutions as above we integrate the corresponding equations (4.2.a) to obtain

$$(5.3) \quad v_j(t) + B(\theta_j(t)) = v_0 + \int_0^t f, \quad j = 1, 2$$

where  $\theta_j(t) \equiv \int_0^t u_j$ . Taking the difference of (5.3) for  $j = 1, 2$ , then scalar-product with  $\theta_1' - \theta_2'$  and integrating gives us

$$(5.4) \quad \int_0^t (v_1 - v_2, \theta_1' - \theta_2')_V + (1/2)(B(\theta_1(t) - \theta_2(t)), \theta_1(t) - \theta_2(t))_V = 0.$$

Since  $v_j(t) \in A(\theta_j'(t))$  a.e., each term is non-negative. It follows that

$B(\theta_1(t) - \theta_2(t)) = 0$  on  $[0, T]$  and thus from (5.4) that

$$(v_1(t) - v_2(t), u_1(t) - u_2(t))_V = 0 \text{ a.e. } t \in [0, T],$$

so the desired results follows by strict monotonicity of  $A + B$ .

Finally we cite an example to show that the symmetry condition cannot be eliminated from Theorem 4.

Example 2. Let  $H^1(0, 1)$  be the Sobolev space of those absolutely continuous functions on the interval  $(0, 1)$  whose first derivatives belong to  $L^2(0, 1)$ ; set

$V = \{v \in H^1(0, 1) : v(1) = 0\}$  and note that  $V \subset L^2(0, 1) \subset V^*$ . Define  $A : V \rightarrow V^*$  by  $A(v) = -v'$ . Clearly  $A$  is linear and we have

$$(v)(v) = -\int_0^1 v'v \, ds = (1/2)|v(0)|^2 > 0$$

so  $A$  is monotone. Let  $\beta$  be given by

$$\beta(r) = \begin{cases} r/2, & r < 0 \text{ or } r > 1, \\ r^2/2, & 0 \leq r \leq 1, \end{cases}$$

and define  $\tilde{B} : V \rightarrow V^*$  by

$$\tilde{B}(u)(v) = \int_0^1 \beta(u'(s))v'(s) \, ds, \quad u, v \in V.$$

It is easy to check that  $\tilde{B}$  is a strictly monotone subgradient on  $V$ .



Consider the Cauchy problem

$$(5.5) \quad \frac{d}{dt} (A(u)) + B(u) = 0, \quad Au(0) = -1$$

with the above operators. A solution  $u$  of (5.5) is a weak solution of the initial-boundary-value problem

$$(5.6.a) \quad (-u_x)_t - (\beta(u_x))_x = 0, \quad 0 < x < 1, \quad 0 < t,$$

$$(5.6.b) \quad u_x(0,t) = u_x(1,t) = 0,$$

$$(5.6.c) \quad -u_x(x,0) = -1$$

where the subscripts denote partial derivatives. Consider the following two functions:

$$u^{(1)}(x,t) = \begin{cases} (x^2 + t^2)/2t - 1, & 0 < x < t < 1, \\ x - 1, & 0 < t < x < 1, \end{cases}$$

$$u^{(2)}(x,t) = \begin{cases} t/2 - 1, & 0 < x < t/2 < 1/2, \\ x - 1, & 0 < t/2 < x < 1, \quad t < 1. \end{cases}$$

It is a straightforward computation to check that both  $u^{(1)}$  and  $u^{(2)}$  are solutions of (5.6), hence, both are solutions of (5.5). Note that the only condition of Theorem 4 not met in this example is the symmetry of  $A$ . It shows also that  $B$  being a subgradient is not a satisfactory substitute for  $B$  to be continuous and self-adjoint.

## 6. Construction of Differential Operators.

We have been discussing evolution equations which contain a pair of nonlinear operators from a Hilbert space  $V$  to its dual  $V^*$ . In our applications the generalized solutions obtained in our theorems may satisfy natural or variational boundary conditions (e.g., of Neumann type) which are implicit in the functional identity

$$(6.1) \quad \frac{d}{dt} A(u(t)) + B(u(t)) = f(t)$$

in  $V^*$ . Such boundary conditions are classically recovered by Green's formula so we shall describe an appropriate extension of this formula which requires a minimum of regularity of the generalized solution. The objective is to resolve each term in (6.1) into two parts, a differential operator in distributions over a region  $\Omega$ , the formal operator, and a constraint on the boundary  $\Gamma$ , the boundary operator. Then we briefly recall basic facts on Sobolev spaces and construct a rather general nonlinear operator  $B$  which will be used in the next section to illustrate our Theorems in some examples of initial-boundary-value problems.

Assume we are given a linear surjection  $\gamma : V \rightarrow T$ , called a "trace" operator, which is a strict homomorphism onto its range  $T$ , called "boundary values" of  $V$ . Let  $V_0$  be the kernel of  $\gamma$  and note that the dual operator,  $\gamma^*(g) = g \circ \gamma$ , is an isomorphism of the dual space  $T^*$  onto the annihilator  $V_0$  in  $V^*$ . Suppose there is given a continuous seminorm  $|\cdot|$  on  $V$  for which  $V_0$  is dense in the seminorm space  $U \equiv \{V, |\cdot|\}$ . Then we naturally identify  $U^*$  simultaneously as a subspace of  $V^*$  and of  $V_0^*$ .

We resolve the operator  $A : V \rightarrow 2^{V^*}$  into a formal part in  $V_0^*$  and a boundary part in  $T^*$ . For each  $u \in D[A]$  set  $A_0(u) = \{F|_{V_0} : F \in A(u)\}$ , the set of restrictions to  $V_0$  of functionals in  $A(u)$ . Then set  $D[A_0] \equiv \{u \in V : A_0(u) \cap U^* \neq \emptyset\}$  and define

$A_0 : V \rightarrow 2^{U^*}$  by  $A_0(u) = A_0(u) \cap U^*$ . That is,  $A_0$  is the set of those functionals in  $A_0(u)$  which have (unique) continuous extensions in  $U^* \subset V^*$ . Now let  $u \in D[A_0]$  and  $F \in A(u)$  with  $F_0 = F|_{V_0} \in U^*$ , hence,  $F_0 \in A_0(u)$ . Then in  $V_0$  we have

$F - F_0 = \gamma^*(g)$  for a unique  $g \in T^*$ , so we can define  $\partial_A(u) \subset T^*$  to be the set of all such  $g$ . Thus, for each  $F \in A(u)$  for which  $F_0 = F|_{V_0} \in U^*$ , there is a unique  $g \in T^*$  for which

$$F(v) = F_0(v) + g(\gamma v), \quad v \in V$$

and we indicate this by

$$(6.2) \quad A(u) = A_0(u) + \gamma^*(\partial_A(u)), \quad u \in D[A_0].$$

In our applications  $V_0^*$  is a space of distributions over  $\Omega$  and  $T$  is the space of boundary values of the Sobolev space  $V$ , so (6.2) is the abstract Green's formula for the operator  $A$ .

In many examples the solutions of (6.1) will have the additional regularity properties described below.

**Lemma 6.1.** Let  $v \in H^1(0, T; V^*)$  with  $v(t) \in A(u(t))$  a.e. on  $[0, T]$ , and set  $v_0(t) = v(t)|_{V_0}$  for each  $t \in [0, T]$ . Let  $v_0(t) \in U^*$  and define  $g(t) \in T^*$  by  $v(t) = v_0(t) + \gamma^*(g(t))$  for  $t \in [0, T]$ . If  $v_0'(t) \in U^*$  a.e. on  $[0, T]$  then  $g \in H^1(0, T; T^*)$  and

$$v'(t) = v_0'(t) + \gamma^*(g'(t)), \quad \text{a.e. } t \in [0, T].$$

The preceding situation occurs, for example, in the case of linear symmetric  $A$  and in certain other special cases [2, 9, 17, 25].

Suppose the operator  $A$  is given as above and let a second operator  $B: V + 2V^*$  be given. Resolve it likewise into two parts,

$$(6.3) \quad B(u) = B_0(u) + \gamma^*(\partial_B(u)), \quad u \in D[B_0].$$

Let there be given  $f_0 \in L^2(0, T; U^*)$ ,  $g_0 \in L^2(0, T; T^*)$ ,  $v_0 \in R_q[A_0]$  and  $g_0 \in T^*$  with  $v_0 + \gamma^*(g_0) \in R_q[A_0]$ . Consider a solution of the Cauchy problem

$$\begin{aligned} \frac{d}{dt} A(u(t)) + B(u(t)) &= f_0(t) + \gamma^*(g_0(t)), \quad \text{a.e. } t \in [0, T], \\ A(u(0)) &= v_0 + \gamma^*(g_0), \end{aligned}$$

that is, a triple  $u, v, w$  for which

$$(6.4) \quad \begin{cases} v(t) \in A(u(t)), \quad w(t) \in B(u(t)), \\ v'(t) + w(t) = f_0(t) + \gamma^*(g_0(t)), \quad \text{a.e. } t \in [0, T], \\ v(0) = v_0 + \gamma^*(g_0). \end{cases}$$

By restricting the above functionals to  $V_0$  we obtain

$$(6.5) \quad \begin{cases} v_0(t) \in A_0(u(t)), & w_0(t) \in B_0(u(t)), \\ v_0'(t) + w_0(t) = f_0(t) & \text{in } V_0^*, \text{ a.e. } t \in [0, T], \\ v_0(0) = v_0. \end{cases}$$

If Lemma 6.1 applies then we obtain  $w_0(t) \in U^*$  and the identities (6.2) and (6.3) give

$$(6.6) \quad \begin{cases} g_A(t) \in \partial_A(u(t)), & g_B(t) \in \partial_B(u(t)) \\ g_A'(t) + g_B(t) = g_0(t) & \text{in } T^*, \text{ a.e. } t \in [0, T], \\ g_A(0) = g_0. \end{cases}$$

Thus (6.4) implies (6.5) and, in the situation of Lemma 6.1, also (6.6), so we call a solution of (6.4) a weak solution of the pair (6.5), (6.6). The first will give a partial differential equation and the second yields variational boundary conditions in our examples.

Let  $\Omega$  be a bounded open set in  $R^n$  which lies locally on one side of its smooth boundary  $\Gamma$ .  $H^1(\Omega)$  is the space of functions  $\varphi$  in  $L^2(\Omega)$  for which each of the partial derivatives  $D_j \varphi = \frac{\partial \varphi}{\partial x_j}$  belongs to  $L^2(\Omega)$ ,  $1 \leq j \leq n$ . Letting  $D_0$  denote the identity on  $L^2(\Omega)$ , we can express the norm on  $H^1(\Omega)$  by

$$\|\varphi\|_{H^1(\Omega)} = \left( \sum_{j=0}^n \|D_j \varphi\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

We shall let  $V$  be a closed subspace of  $H^1(\Omega)$  containing  $C_0^\infty(\Omega)$  and let  $\gamma : V \rightarrow L^2(\Gamma)$  be the indicated restriction to  $V$  of the trace map [19]. We let  $T$  be the range of  $\gamma$  (a subspace of  $H^{1/2}(\Gamma)$ ) and denote the kernel by  $V_0 \equiv H_0^1(\Omega)$ . Since  $\Gamma$  is smooth there is a unit outward normal vector  $n(s) = [n_1(s), \dots, n_n(s)]$  at each point  $s \in \Gamma$ . Note that the test functions  $C_0^\infty(\Omega)$  are dense in  $V_0$  so the dual  $V_0^*$  is the space of (first order) distributions on  $\Omega$ . We refer to [19] for information on these Sobolev spaces.

Specifically, we shall use the trace operator between Sobolev spaces of fractional order.

We shall construct an operator  $B : V \rightarrow 2^{V^*}$  which will occur in many of our examples. For each integer  $k$ ,  $-1 \leq k \leq n$ , let there be given a continuous, convex function  $\psi_k : R \rightarrow R$  whose subgradient,  $\beta_k \equiv \partial \psi_k$ , satisfies

$$(6.7) \quad |w| < C(|s| + 1) \text{ if } w \in \beta_k(s), \quad s \in \mathbb{R}, \quad -1 \leq k \leq n,$$

where  $C$  is some large constant. Then define  $\psi : V \rightarrow \mathbb{R}$  by

$$\psi(u) = \sum_{k=0}^n \int_{\Omega} \psi_k(D_k u(x)) dx + \int_{\Gamma} \psi_{-1}(\gamma(u(s))) ds, \quad u \in V.$$

From the estimates (6.7) it follows that  $\psi$  is a sum of continuous convex functions so we can compute its subgradient term-by-term. Recall that the subgradient  $F$  of the convex function  $v \rightarrow \int_{\Omega} \varphi_k(v) dx$  at  $w \in L^2(\Omega)$  is determined by  $F(x) \in \beta_k(w(x))$ , a.e.  $x \in \Omega$ . Since  $D_k : V \rightarrow L^2(\Omega)$  is continuous linear, the subgradient of the convex function  $v \rightarrow \int_{\Omega} \varphi_k(D_k v) dx$  at  $u \in V$  is given by  $\{D_k^* F : F \in \beta_k(D_k u) \text{ a.e.}\}$ . See [11, pp. 26-28] and [1, p. 47] for proofs of these facts. These observations show that the subgradient of  $\psi$  is

$$(6.8) \quad \mathcal{B}(u) = \partial\psi(u) = \sum_{k=0}^n D_k^* \beta_k(D_k u) + \gamma^* \beta_{-1}(\gamma u), \quad u \in V.$$

To be precise, we have  $F \in \mathcal{B}(u)$  if and only if there exists  $f_k \in \beta_k(D_k u)$  in  $L^2(\Omega)$ ,  $0 \leq k \leq n$ , and  $f_{-1} \in \beta_{-1}(\gamma u)$  in  $L^2(\Gamma)$  for which

$$F(v) = \int_{\Omega} \sum_{k=0}^n f_k(x) D_k v(x) dx + \int_{\Gamma} f_{-1}(s) v(s) ds, \quad v \in V.$$

By restricting the above to  $v \in V_0 = H_0^1(\Omega)$  we see the formal part is the distribution

$$F|_{V_0} = - \sum_{k=1}^n D_k f_k + f_0 \in V_0^*.$$

We denote this by the equality (of sets)

$$(6.9) \quad \mathcal{B}_0(u) = - \sum_{k=1}^n D_k \beta_k(D_k u) + \beta_0(u).$$

Let's interpret (6.3) with  $U^* = L^2(\Omega)$ . First, if  $D_k f_k \in U^*$  for  $1 \leq k \leq n$ , then by the classical Green's theorem we have from above

$$F(v) - F|_{V_0}(v) = \int_{\Gamma} \left\{ \sum_{k=1}^n f_k(s) n_k(s) + f_{-1}(s) \right\} v(s) ds, \quad v \in V.$$

Thus  $u \in D(B)$  and we have shown

$$\sum_{k=1}^n f_k n_k + f_{-1} \in \partial(u) \text{ with } f_k \in \beta_k(D_k u).$$

That is, when the terms are as regular as indicated we have

$$(6.10) \quad \partial_B(u) = \sum_{k=1}^n \beta_k(D_k u) n_k + \beta_{-1}(u).$$

Furthermore,  $\partial_B(u)$  is defined without these regularity assumptions on the individual terms; it is sufficient to have  $F|_{V_0} \in U^*$ . Finally, we note that from (6.7) it follows that  $\delta$  satisfies the assumptions  $[B_1]$  of Theorem 1 and  $[B_2]$  of Theorem 3. It is also bounded from  $L^2(0, T; V)$  to  $L^2(0, T; V^*)$  and it will satisfy  $[B_2]$  of Theorem 2 if, in addition there is a pair of numbers  $K, c > 0$  such that

$$(6.11) \quad \left\{ \begin{array}{l} \psi_k(s) > c|s|^2 - K, \quad s \in \mathbb{R}, \quad 1 \leq k \leq n \\ \text{and one of the following:} \\ \text{(a) the estimate holds for } k = 0, \text{ or} \\ \text{(b) the estimate holds for } k = -1, \text{ or} \\ \text{(c) } v \in V \text{ and } v = \text{constant imply } v \equiv 0. \end{array} \right.$$

From (6.11) we can show that

$$\psi(v) > c_1 \|v\|_V^2 - K_1, \quad v \in V,$$

and this implies the coercivity condition in  $[B_2]$ .

7. Examples of Partial Differential Equations.

We shall describe some examples of initial-boundary-value problems for partial differential equations to illustrate the applications of our results. These examples were chosen merely to suggest a variety of problems that can be resolved by our Theorems, and they are not intended to be best possible in any sense.

(a). Elliptic-Parabolic Equations. For  $k = 0$  and  $-1$ , let  $\varphi_k : \mathbb{R} \rightarrow \mathbb{R}$  be convex and continuous with subgradient,  $\alpha_k \equiv \partial\varphi_k$ , satisfying

$$|w| < C(|s| + 1) \text{ if } w \in \alpha_k(s), \quad s \in \mathbb{R}.$$

Set  $W = H^r(\Omega)$ ,  $1/2 < r < 1$ ,  $V = H^1(\Omega)$ , and note that  $V \rightarrow W$  is compact and

$\gamma : W \rightarrow L^2(\Gamma)$  is continuous [19]. Thus we can define by

$$(\nu) \equiv \int_{\Omega} \varphi_0(\nu(x)) dx + \int_{\Gamma} \varphi_{-1}(\gamma(\nu(s))) ds, \quad \nu \in W$$

a continuous and convex function  $\varphi : W \rightarrow \mathbb{R}$  with subgradient

$$(u) = \partial\varphi(u) = \alpha_0(u) + \gamma^*(\alpha_{-1}(\gamma u))$$

bounded from  $W$  to  $W^*$ . That is,  $F \in (u)$  if and only if there exist  $f_0 \in \alpha_0(u)$  in  $L^2(\Omega)$  and  $f_{-1} \in \alpha_{-1}(\gamma(u))$  in  $L^2(\Gamma)$  for which

$$(7.1) \quad F(\nu) = \int_{\Omega} f_0(x)\nu(x) dx + \int_{\Gamma} f_{-1}(s)\nu(s) ds, \quad \nu \in V,$$

so the formal and boundary parts of  $A$  are given, respectively, by

$$(7.2) \quad A_0(u) = \alpha_0(u), \quad \partial_A(u) = \alpha_{-1}(\gamma u).$$

From Theorem 2 we obtain the existence of a weak solution of the initial-boundary-value problem

$$(7.3) \quad \begin{cases} \frac{\partial}{\partial t} A_0(u) + B_0(u) \ni f_0 & \text{in } L^2(0, T; H^{-1}(\Omega)), \\ A_0 u(0) \ni v_0, \\ \frac{\partial}{\partial t} \partial_A(u) + \partial(u) \ni g_0 & \text{in } L^2(0, T; H^{-1/2}(\Gamma)), \\ \partial_A u(0) \ni b_0. \end{cases}$$

This is made precise in the form (6.5) and (6.6) where the operators are specified in (6.9), (6.10) and (7.2).

Remarks. By our choice of  $V = H^1(\Omega)$ , all boundary conditions in (7.3) are of variational type. Dirichlet-type constraints are obtained by taking subspaces of  $H^1(\Omega)$ .

We require that  $f_0$  and  $g_0$  be square-summable with values in  $H^{-1}(\Omega)$  and  $H^{-1/2}(\Gamma)$ , respectively, and we assume (6.11) to obtain coercivity of  $B$ . The boundedness assumptions on  $\alpha_k$  ( $k = 0, -1$ ) can be relaxed somewhat by using embedding theorems, e.g., of  $W$  into  $L^p(\Omega)$ .

There is no bound on the degeneracy permitted in the operator  $A$ ; we include even the (uninteresting) elliptic case  $A \equiv 0$ . The case of  $A_0 = 0$  leads to an evolution on the boundary subject to an elliptic equation in the interior; such problems arise from diffusion in a medium bounded by material of markedly lower diffusivity [25].

The classical porous-media-equation and the weak form of the two-phase Stefan free-boundary problem are included in (7.3). In the latter, the enthalpy is given by  $\alpha_0(s) = (1 + cH(s))s + LH(s)$  where  $L > 0$  is the latent heat of fusion and  $H(\cdot)$  is the Heaviside function [14, 16]. Such problems arise in welding with the nonlinear term  $\beta_0(u)$  representing a source of heat due to electrical resistance.

Note that each solution of (5.1) is also a (spatially independent) solution of (7.3) so there is much non-uniqueness in (7.3).

(b) Pseudo-parabolic Equations. Here we set  $V = H_0^1(\Omega)$ , so  $T = \{0\}$  and all boundary conditions are of Dirichlet type. The operator  $A$  is given as above by (7.2); the operator  $B$  is also given as before but we shall only assume (6.7), not (6.11). On the space  $V$  we take the (equivalent) scalar product and corresponding Riesz map

$$Ru(v) = \int_{\Omega} \sum_{k=1}^n D_k u(x) D_k v(x) dx, \quad u, v \in V,$$

so we have  $R = -\Delta_n \equiv -\sum_{k=1}^n D_k^2$ . Assume  $f_0 \in L^2(0, T; H^{-1}(\Omega))$  and  $v_0 \in \alpha_0(u_0)$ ,  $u_0 \in H_0^1(\Omega)$  are given. Then from either Theorem 1 or from Theorem 3 we obtain existence of a solution of the problem



$$(7.4) \quad \left\{ \begin{array}{l} u \in H^1(0, T; H_0^1(\Omega)), \quad u(0) = u_0, \\ v \in H^1(0, T; H^{-1}(\Omega)), \quad v(0) = v_0, \\ w \in L^2(0, T; H^{-1}(\Omega)), \\ \frac{\partial}{\partial t} (v(t) - \Delta_n u(t)) + w(t) = f_0(t), \\ v(t) \in A_0(u(t)), \quad w(t) \in B_0(u(t)). \end{array} \right.$$

The operators  $A_0$  and  $B_0$  are given by (7.2) and (6.9) respectively.

Remarks. The partial differential equation in (7.4) is of the form of a nonlinear parabolic plus the term  $\frac{\partial}{\partial t} \Delta_n u(x, t)$ . Such equations are known to arise in various diffusion problems and are called pseudoparabolic [9, 15, 28]. Similar problems with variational boundary conditions can be considered; we obtain weak solutions in the form (6.4). However, since  $R_g(A_0 + R) = H^{-1}(\Omega)$ , we cannot use Lemma 6.1, in general, to deduce (6.6). This situation occurs even in the linear case [26].

The operator  $-\Delta_n$  in (7.4) can be replaced by the Riesz operator of any equivalent scalar product on  $H_0^1(\Omega)$ . This trivial observation is useful in introducing elliptic linear operators in its place.

We have not made use of the fact that only one of the operators  $A, B$  need be a subgradient. In particular, we are free to add to one of  $A$  or  $B$  any linear combination of first order derivatives. (See Example (d) below.)

Non-uniqueness of solutions of (7.4) follows from that of solutions of (5.1).

In the preceding examples the nonlinearity arises from the local dependence on the solution, e.g., from nonlinear functions of the values of  $u$  or  $\nabla u$  at each point of  $\Omega$ . We next display examples of global nonlinearity arising from the "total energy" or the "total flux" in the system. The following preliminary result will be useful.

Lemma 7.1. Let  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$  be continuous, bilinear, symmetric and nonnegative real-valued functions on the Hilbert space  $V$ . Then for  $\alpha, \beta \in \mathbb{R}$ , the function

$$\varphi(u) \equiv \frac{1}{2} \max(a(u, u) + \alpha, b(u, u) + \beta), \quad u \in V$$

is convex, continuous and its subgradient is given by

$$\partial\varphi(u) = \begin{cases} \{A(u)\} & \text{if } a(u,u) + \alpha > b(u,u) + \beta \\ \{(\lambda A + (1-\lambda)B)(u), 0 \leq \lambda \leq 1\} & \text{if } a(u,u) + \alpha = b(u,u) + \beta \\ \{B(u)\}, & \text{if } a(u,u) + \alpha < b(u,u) + \beta \end{cases}$$

where  $Au(v) = a(u,v)$ ,  $Bu(v) = b(u,v)$ ,  $v \in V$ .

Proof: We need only to compute  $\partial\varphi(u)$ . For the first and last cases we compute the Gateaux derivative  $\lim_{t \rightarrow 0} \{\varphi(u + tv) - \varphi(u)\}/t$  to obtain the desired results. Now assume  $a(u,u) + \alpha = b(u,u) + \beta$ . An easy computation gives

$$t^{-1}(\varphi(u+tv) - \varphi(u)) = \max\{a(u,v) + \frac{t}{2} a(v,v), b(u,v) + \frac{t}{2} b(v,v)\}$$

so we have the equivalence of  $f \in \partial\varphi(u)$ ,

$$f(v) \leq t^{-1}(\varphi(u+tv) - \varphi(u)), \quad v \in V, \quad t > 0,$$

and of

$$f(v) \leq \max\{a(u,v), b(u,v)\}, \quad v \in V.$$

This is equivalent to  $f = \lambda Au + (1 - \lambda)Bu$  for some  $\lambda$ ,  $0 \leq \lambda \leq 1$ .

(c) Energy-Dependent Elliptic-Parabolic Equation. We shall use Theorem 2 with the operator  $B$  given by (6.8), so we assume (6.7) and (6.11). Choose  $V = H^1(\Omega)$  so the space of boundary values is  $T = H^{1/2}(\Gamma)$ . Define on  $W \equiv L^2(\Omega)$  the function

$$\varphi(u) = \frac{1}{2} \max\{1, \int_{\Omega} |u(x)|^2 dx\}, \quad u \in W.$$

The subgradient  $A = \partial\varphi$  is given by Lemma 7.1 and we have  $A = A_0 = A_0$ ,  $R_g(A) = L^2(\Omega)$ . Finally, let  $v_0 \in L^2(\Omega)$ ,  $f_0 \in L^2(0, T; L^2(\Omega))$ ,  $g_0 \in L^2(0, T; H^{-1/2}(\Gamma))$  be given and define

$$f(t)(v) = \int_{\Omega} f_0(x, t)v(x)dx + g_0(t)(\gamma v), \quad v \in V.$$

Then we obtain a weak solution of

$$(7.5) \quad \begin{cases} \frac{\partial v}{\partial t} + B_0(u) \ni f_0 & \text{in } L^2(0, T; H^{-1}(\Omega)) \\ v(x, 0) = v_0(x) & \text{in } L^2(\Omega) \\ \partial g(u) \ni g_0 & \text{in } L^2(0, T; H^{-1/2}(\Gamma)) \end{cases}$$

where  $v$  is determined by

$$v \in \begin{cases} \{0\}, & \text{if } \int_{\Omega} |u|^2 dx < 1, \\ \{\lambda u : 0 < \lambda < 1\} & \text{if } \int_{\Omega} |u|^2 dx = 1, \\ \{u\}, & \text{if } \int_{\Omega} |u|^2 dx > 1. \end{cases}$$

Thus, the type of the equation is either elliptic (with parameter  $t$ ) or parabolic and depends on the total energy  $\int_{\Omega} |u|^2 dx$ .

(d) A Flux-Dependent Equation. Take  $V = H_0^1(\Omega)$ ,  $W = L^2(\Omega)$  and  $T = \{0\}$ . Let the convex function  $\varphi_0$  and its bounded subgradient  $\alpha_0 = \partial\varphi_0$  be given as above in (a), and define  $A = \alpha_0$  in  $L^2(\Omega)$ ; cf. (7.2). Denoting the gradient of  $u$  by  $\hat{\nabla}u$ , we define the continuous convex

$$\psi(u) = \frac{1}{2} \max\{N, \int_{\Omega} |\hat{\nabla}u(x)|^2 dx\}, \quad u \in V.$$

Let  $\hat{b} \in \mathbb{R}^n$  and define  $B : V \rightarrow 2^{V^*}$  by

$$B(u) = \hat{b} \cdot \hat{\nabla}u + \partial\psi(u).$$

Note that  $B$  is maximal monotone, bounded and coercive. Let  $v_0 \in R_g(A)$  and

$f_0 \in L^2(0, T; H^{-1}(\Omega))$ . From Theorem 2 we obtain existence of a solution of the problem

$$(7.6) \quad \begin{cases} u \in L^2(0, T; H_0^1(\Omega)), v \in H^1(0, T; H^{-1}(\Omega)), w \in L^2(0, T; H^{-1}(\Omega)) \\ \frac{\partial v}{\partial t} + \hat{b} \cdot \hat{\nabla}u - K(\int_{\Omega} |\hat{\nabla}u|^2 dx) \Delta_n u = f_0 \\ v(x, t) \in \alpha_0(u(x, t)), v(x, 0) = v_0(x) \end{cases}$$

where the maximal monotone  $K : \mathbb{R} \rightarrow \mathbb{R}$  is given by

$$K(s) = \begin{cases} \{0\}, & s < N, \\ [0, 1], & s = N, \\ \{1\}, & s > N. \end{cases}$$

Remarks. In the region where  $\int_{\Omega} |\hat{\nabla}u|^2 dx < N$  the equation in (7.6) is a conservation law of the form

$$(7.7) \quad \frac{\partial v}{\partial t} + \hat{b} \hat{\nabla} g(v) \ni f_0$$

where the maximal monotone  $g : \mathbb{R} \rightarrow \mathbb{R}$  is the inverse to  $\alpha_0$ . Thus (7.6) suggests a penalty method [18] to approximate solutions of (7.7). We shall develop these observations elsewhere.

In order to consider (7.6) in the form (6.1) it is essential that  $\hat{B}$  is not required to be a subgradient.

(e) Elliptic-Parabolic Systems. Our final example consists of a pair of equations of the type given above in (a) that are (nonlinearly) coupled. For  $i = 0, 1$  and  $k = 0, -1$ , let  $\varphi_k^i : \mathbb{R} \rightarrow \mathbb{R}$  be convex and continuous with subgradient,  $\alpha_k^i \equiv \partial \varphi_k^i$ , satisfying

$$(7.8) \quad |w| \leq C(|s| + 1) \text{ for } w \in \alpha_k^i(s), \quad s \in \mathbb{R}.$$

On the product space  $W \equiv H^r(\Omega) \times H^r(\Omega)$ ,  $1/2 < r < 1$ , we have the continuous trace operator  $\gamma([u^1, u^2]) = [\gamma(u^1), \gamma(u^2)]$  which maps  $W$  into  $L^2(\Gamma) \times L^2(\Gamma)$ . Thus we define by

$$\varphi(v) = \sum_{i=1}^2 \int_{\Omega} \varphi_0^i(v^i(x)) dx + \sum_{i=1}^2 \int_{\Gamma} \varphi_{-1}^i(\gamma v^i(s)) ds, \quad v = [v^1, v^2] \in W,$$

a continuous and convex function whose subgradient is given by

$$A(u) \equiv \partial \varphi(u) = [\alpha_0^1(u^1) + \gamma^*(\alpha_{-1}^1(\gamma(u^1))), \alpha_0^2(u^2) + \gamma^*(\alpha_{-1}^2(\gamma(u^2)))] , \\ u = [u_1, u_2] \in W.$$

The operator  $A : W \rightarrow 2^{W^*}$  is bounded; its formal and boundary parts are given, respectively, by (see (7.2))

$$(7.9) \quad A_0(u) = [\alpha_0^1(u^1), \alpha_0^2(u^2)], \quad \partial_A(u) = [\alpha_{-1}^1(\gamma(u^1)), \alpha_{-1}^2(\gamma(u^2))].$$

Hereafter we restrict  $\gamma$  to the product space  $V \equiv H^1(\Omega) \times H^1(\Omega)$ . Assume we are given a set of continuous and convex functions  $\psi_k^i : \mathbb{R} \rightarrow \mathbb{R}$  for  $i = 1, 2$ ,  $-1 \leq k \leq n$ , whose

subgradients  $\beta_k^i \equiv \partial \psi_k^i$  all satisfy the estimate (6.7). For  $i = 1, 2$  we define

$\psi^i : H^1(\Omega) \rightarrow \mathbb{R}$  as in Section 6; its subgradient is then given by (see (6.8))

$$S^i(u^i) = \partial \psi^i(u^i) = \sum_{k=0}^n D_k^* \beta_k^i(D_k u^i) + \gamma^* \beta_{-1}^i(\gamma u^i), \quad u^i \in H^1(\Omega).$$

The formal and boundary parts of  $B^i$  are given by (6.9) and (6.10) for each of  $i = 1, 2$ . Thus we have two pairs of operators similar to the pair in Example (a). The coupling of the corresponding equations will be attained by a maximal monotone graph  $\mu : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  which is bounded, i.e., (7.8) holds for  $w \in \mu(s)$ . Then we define a maximal monotone operator  $M$  on  $\mathbb{R} \times \mathbb{R}$  by

$$M([s_1, s_2]) = \{[w, -w] : w \in \mu(s_1 - s_2)\}, \quad [s_1, s_2] \in \mathbb{R} \times \mathbb{R}.$$

This operator  $M$  induces a corresponding operator on  $L^2(\Omega) \times L^2(\Omega)$ , hence, from  $V$  into  $V^*$ , which we also denote by  $M$ . Finally we define

$$B([u_1, u_2]) = [B^1(u_1), B^2(u_2)] + M(u_1, u_2), \quad [u_1, u_2] \in V.$$

This  $B$  is the sum of maximal monotone operators, each of which is defined on all of  $V$ , so  $B$  is maximal monotone. Similarly  $B$  is bounded, and we note that  $B$  is coercive if both of  $B^1$  and  $B^2$  are coercive.

Assume that we are given the following data:

$$\begin{aligned} f_0^i &\in L^2(0, T; H^{-1}(\Omega)), \quad g_0^i \in L^2(0, T; H^{-1/2}(\Gamma)), \quad i = 1, 2, \\ [v_0^1, v_0^2] &\in R_g(A_0), \quad [v_{-1}^1, v_{-1}^2] \in R_g(\partial_A). \end{aligned}$$

If the functions  $\{\beta_k^i : -1 \leq k \leq n\}$  satisfy (6.11) for both  $i = 1$  and  $i = 2$ , then from Theorem 2 it follows there exists a weak solution of the system

$$(7.10) \quad \left\{ \begin{aligned} \frac{\partial}{\partial t} \alpha_0^1(u^1(x, t)) + B_0^1(u^1(x, t)) + \mu(u^1(x, t) - u^2(x, t)) &\ni f_0^1(x, t), \\ \frac{\partial}{\partial t} \alpha_0^2(u^2(x, t)) + B_0^2(u^2(x, t)) - \mu(u^1(x, t) - u^2(x, t)) &\ni f_0^2(x, t), \\ &\text{in } L^2(0, T; H^{-1}(\Omega)), \\ \frac{\partial}{\partial t} \alpha_{-1}^i(\gamma u^i(s, t)) + \beta_{-1}^i(u^i(s, t)) &\ni g_0^i, \quad i = 1, 2, \text{ in } L^2(0, T; H^{-1/2}(\Gamma)), \\ \alpha_0^i(u^i(x, 0)) &\ni v_0^i(x), \quad i = 1, 2, \text{ in } L^2(\Omega), \\ \alpha_{-1}^i(\gamma u^i(s, 0)) &\ni v_{-1}^i(s), \quad i = 1, 2, \text{ in } L^2(\Gamma). \end{aligned} \right.$$

Remarks. All of the operators in this system are (possibly) multi-valued so each of the "equations" should be made precise as was done in our preceding examples. See (6.9), (6.10), (7.2) and (7.9) for related computations.

The only requirement on the  $\alpha_k^1$  is that they be maximal monotone graphs in  $\mathbb{R}$  which satisfy the bound (7.8). Thus much degeneracy is possible in the leading operator given by (7.9). Related Stefan-type free boundary problems can be so considered.

Interesting examples of the coupling term arise in applications to diffusion problems. These include problems with a semi-permeable membrane,  $\mu(s) = s^+$  (where  $s^+$  denotes  $s$  if  $s > 0$  and  $0$  otherwise), or those with a threshold phenomena,  $\mu(s) = (s - \epsilon)^+ - (-s - \epsilon)^+$ . The operator  $M$  as given above is a subgradient; this is easily verified by showing it is cyclic monotone [1]. However we may add to  $M$  non-symmetric monotone terms, for example,  $[-s_2, s_1]$ , and thereby obtain systems of the form (6.1) in which  $\mathcal{B}$  is not a subgradient.

Systems of equations of pseudoparabolic type can be resolved similarly by Theorem 1. For example we can choose  $V = H_0^1(\Omega) \times H_0^1(\Omega)$  with scalar-product on each factor as given in Example (b) and obtain existence of a solution of the problem

$$(7.11) \left\{ \begin{array}{l} \frac{\partial}{\partial t} (\alpha_0^1(u^1(x,t)) - \Delta_n u^1(x,t)) + B_0^1(u^1(x,t)) + \mu(u^1(x,t) - u^2(x,t)) \ni f_0^1(x,t), \\ \frac{\partial}{\partial t} (\alpha_0^2(u^2(x,t)) - \Delta_n u^2(x,t)) + B_0^2(u^2(x,t)) - \mu(u^1(x,t) - u^2(x,t)) \ni f_0^2(x,t), \\ \text{in } L^2(0,T;H^{-1}(\Omega)), \\ u^j \in H^1(0,T;H_0^1(\Omega)), u^j(x,0) = u_j(x), \alpha_0^j(u^j(x,0)) \ni v_j(x), \quad j = 1,2, \\ \text{in } L^2(\Omega) \end{array} \right.$$

where the data is given as above with  $v_j \in A_0(u_j)$  for  $j = 1,2$ .

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20. ABSTRACT - Cont'd.

proved when  $A$  is a subgradient and either  $A$  is strongly-monotone or  $B$  is coercive; existence is established also in the case where  $A$  is strongly-monotone and  $B$  is subgradient. Uniqueness is proved when one of  $A$  or  $B$  is continuous self-adjoint and the sum is strictly-monotone; examples of non-uniqueness are given. Applications are indicated for various classes of degenerate nonlinear partial differential equations or systems of mixed elliptic-parabolic-pseudoparabolic types and problems with non-local nonlinearity.