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NONLINEAR VOLTERRA EQUATIONS FOR HEAT FLOW IN MATERIALS WITH ME--ETC(U)
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UNCLASSIFIED MRC-TSR-2081 NL

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MRC Technical Summary Report #2081

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197

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May 1980

(Received May 13, 1980)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

Consider the nonlinear Volterra equation

$$\text{(V)} \qquad u(t) + (b*Au)(t) \ni f(t) \qquad (0 \leq t < \infty)$$

in the general setting $b : [0,\infty) \to R$ a given kernel, A a nonlinear m-accretive operator on a real Banach space X, $f : [0,\infty) \to X$ a given function, and $*$ the convolution. This paper, based on lectures delivered at West Virginia University, discusses existing and recent results for the following problems concerning (V): 1. the global existence and uniqueness of solutions and their continuous dependence on the data, 2. the boundedness and asymptotic behaviour as $t \to \infty$ in the special cases when $X = H$ is a real Hilbert space and A is either a maximal monotone operator on H or A is a subdifferential of a proper, convex, lower semicontinuous function $\varphi : H \to (-\infty,+\infty]$, 3. the existence, boundedness, and asymptotic behaviour of positive solutions in the general setting. The theory is used to study one possible model problem for heat flow in a material with "memory" which can be transformed to the equivalent form (V) under physically reasonable assumptions; the latter provide a motivation for the natural setting of much of the theory developed here. This and various other models for heat flow in such materials are formulated from physical principals and discussed in an introductory chapter.

AMS (MOS) Subject Classifications: 45D05, 45J05, 45K05, 45G99, 45M05, 45M10, 47F05, 47H05, 47H15

Key Words: Nonlinear Volterra equation, m-accretive operators, Maximum monotone operators on a Hilbert space, Subdifferential of a proper, convex, l.s.c. function, Boundedness, Asymptotic behaviour, Limiting equation, Strong solutions, Generalized solutions, Energy methods, Frequency domain methods, Heat flow, Materials with memory, Positive solutions, Completely positive kernels

Work Unit Number 1 (Applied Analysis)

Research sponsored by the United States Army under Contract Nos. DAAG29-75-C-0024 and DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

These notes on lectures delivered at West Virginia University preceding the conference on Volterra and Functional Differential Equations in June 1979, concern the existence, uniqueness, positivity, boundedness, and asymptotic behaviour as $t \to \infty$ of solutions of the nonlinear Volterra equation

$$\text{(V)} \qquad u(t) + (b*Au)(t) \ni f(t) \qquad (0 < t < \infty) .$$

The general setting for V is as follows: $b : [0,\infty) \to R$ is a given kernel, A is a m-accretive, possibly multivalued, operator on a real Banach space X, $f : [0,\infty) \to X$ is a given function, and $*$ denotes the convolution; the integral in V is taken in the sense of Bochner. The special cases of A maximal monotone on a real Hilbert space H, and $A = \partial\varphi$, the subdifferential of a proper, convex, lower semicontinuous function $\varphi : H \to (-\infty,\infty]$ will also play a prominent role, primarily in the boundedness and asymptotic theory for (V).

It should be observed that if $b \equiv 1$ and $f \in W^{1,1}(0,\infty;X)$, where W denotes the usual Sobolev space, equation (V) is formally equivalent to the evolution problem

$$\text{(E)} \qquad \frac{du}{dt} + Au \ni f' \qquad (0 < t < \infty), \ u(0) = u_0 = f(0) .$$

Thus the theory for (V) is to a considerable extent a generalization of the theory of evolution equations, and uses most of the techniques for the latter combined with techniques for Volterra equations developed in recent years.

Chapter 1 is primarily intended for motivation. Beginning from simple physical principles equation (V) is derived as one possible mathematical model for nonlinear heat flow in a homogeneous body of material with memory following ideas of B. D. Coleman, M. E. Gurtin, R. C. Mac Camy, W. Noll, J. W. Nunziato, and A. C. Pipkin. While the derivation is restricted to one space dimension, the modification for heat flow in a homogeneous body Ω in R^2 or R^3 of isotropic material is also indicated. One purpose of Chapter 1 is to arrive at a physically reasonable set of conditions concerning the kernel b in (V), under which one can expect boundedness and the kind of asymptotic behaviour of solutions of (V) solutions studied later in Chapter 3. The kernel b in (V) does not arise directly from physical principles; rather, it is the case that b is expressed in terms of two physically measurable quantities (at least in principle) about which one can make appropriate assumptions based on physical considerations. The types of assumptions made concerning the operator A and the function f are also motivated. In particular, A is a nonlinear second-order elliptic partial differential operator in the space variables which incorporates the boundary conditions, and f depends on the external heat supply, the initial temperature distribution in the body and the history of temperature in the body. If $b \equiv 1$, then in the application (V) is equivalent to the classical nonlinear heat equation in an ordinary body.

Another purpose of Chapter 1 is to point out that heat flow in certain materials with memory can also be modelled by integrodifferential equations

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

other than (V). Limitations of space do not permit us to discuss the qualitative aspects of these alternative models in any detail; however, references to relevant existing literature are given.

Chapter 2 discusses the theory of existence, uniqueness, and continuous dependence of global solutions of (V), both in the general case of a real Banach space and in the special case of a real Hilbert space, and for $A = \partial\varphi$ where the results are stronger. The development is primarily based on recent joint work with M. G. Crandall [26], and partly on a recent paper by G. Gripenberg [34]. References to earlier and related literature are given.

Chapter 3 develops the theory of boundedness and asymptotic behaviour of solutions of (V) as $t \to \infty$, under assumptions partly motivated by the heat flow problem formulated in Chapter 1; application of the theory to this problem is given. The development is based on forthcoming joint work with P. Clément and R. C. Mac Camy [20]. References to other pertinent literature are given.

Chapter 4, based on recent and forthcoming joint work with P. Clément, [18], [19], as well as recent work by Clément [17], discusses the existence, boundedness and asymptotic behaviour as $t \to \infty$ of positive solutions of (V) under assumptions which are also motivated by the heat flow problem in Chapter 1. This problem is then used to illustrate the theory. The reader should recall that it is classical that solutions of the heat equation are positive, if the initial temperature distribution and the external heat supply are positive. Some of the results of this chapter were presented at the West Virginia Conference by Clément.

TABLE OF CONTENTS

NONLINEAR VOLTERRA EQUATIONS FOR HEAT FLOW IN MATERIALS WITH MEMORY

John A. Nohel

Chapter 1

Volterra Equations Occurring in Heat Flow in Materials with Memory

1.1. Introductory Remarks. The purpose of this chapter is to derive from physical considerations several mathematical models for nonlinear heat flow in materials with memory. For simplicity we shall limit most of our considerations to heat flow in one space dimension and remark about the situation in the multidimensional case. The primary objective is to arrive at equation (V) in Section 1.2 below under physically reasonable assumptions on the kernel b, the operator A and the forcing term f. This particular mathematical model motivates many of the considerations in Chapters 2, 3, and 4. In Sections 1.3 and 1.4 we shall also derive two other models for heat flow in materials with memory which have been and are being studied, but due to limitations of space we will only refer the reader to the relevant mathematical literature for their analysis.

The mathematical models are derived from the following general considerations of energy balance for heat transfer in a body B in R^n $(n = 1,2,3)$. If $\varepsilon(t,x)$ represents the internal energy, $\bar{q}(t,x)$ represents the heat flux, and $h(t,x)$ represents the heat supply, where t is the time and x is the position in the body, then the energy balance equation is

$$(1.1) \qquad \varepsilon_t = -\operatorname{div}\bar{q} + h \qquad (t > 0,\ x \in B)\ .$$

The classical linear heat equation which accurately describes heat transfer by conduction in many materials is derived from (1.1) by assuming that the heat flux obeys Fourier's law:

$$\bar{q} = -c_0 \operatorname{grad} u$$

where $c_0 > 0$ is the constant thermal conductivity and u represents the temperature in the body at time t and position x. It is also assumed that the internal energy depends linearly on the temperature

$$\varepsilon = \varepsilon_0 + b_0 u\ ,$$

where $b_0 > 0$ is the constant heat capacity and $\varepsilon_0 > 0$ is a constant. The energy balance (1.1) then yields the linear heat equation

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$$b_0 u_t = c_0 \nabla^2 u + h$$

which adequately describes the evolution of temperature in most homogeneous and isotropic rigid bodies.

However, some materials exhibit memory effects (materials of fading memory type, see Coleman and Mizel [23]) for which the classical theory is unable to account. Heat flow in such material is modelled by assuming that the internal energy ε and the heat flux q are respectively functionals (rather than functions) of the temperature and of the gradient of temperature. The considerations which follow are based on extensive research by Coleman, Gurtin, Noll, Pipkin, Mac Camy, Mizel, and Nunziato (see especially Coleman [21], Coleman and Gurtin [22], Coleman and Mizel [23], Gurtin and Pipkin [39], Mac Camy [57], [58], [59], Nunziato [68]).

2. A Model for Nonlinear Heat Flow in a Material with Memory. We consider nonlinear heat flow in a homogeneous bar of unit length of material with memory with the temperature $u = u(t,x)$ maintained at zero at $x = 0$ and $x = 1$. We shall assume that the history of u is prescribed for $t < 0$ and $0 < x < 1$. The equation satisfied by u in such a material is derived from the assumptions that the internal energy ε and the heat flux q are functionals (rather than functions) of u and of the gradient of u respectively. According to the theory developed by Coleman, Gurtin, Noll, Pipkin, Mac Camy and Nunziato (see e.g. Mac Camy [57], [59] and Nunziato [68]) for heat flow in materials of fading memory type the functionals ε and q are taken respectively as:

$$\varepsilon(t,x) = b_0 u(t,x) + \int_0^t \beta(t-s)u(s,x)ds \quad (t > 0,\ 0 < x < 1), \tag{1.2}$$

$$q(t,x) = -c_0\sigma(u_x(t,x)) + \int_0^t \gamma(t-s)\sigma(u_x(s,x))ds \quad (t > 0,\ 0 < x < 1). \tag{1.3}$$

In writing the functionals ε and q we have assumed for simplicity and without loss of generality that the history of the temperature u is prescribed as zero for $t < 0$ (if this is not the case and if the history of u is sufficiently smooth for $t < 0$ and $0 < x < 1$, this has the effect of altering the forcing term h in equation (1.4) below - and consequently also G in (1.6) below - by additional known forcing terms). In (1.2), (1.3) $b_0 > 0$, $c_0 > 0$ are given constants, $\beta, \gamma : [0,\infty) \to R$ are given sufficiently smooth functions which we call the internal energy and heat flux relaxation functions respectively.

The real function $\sigma : R \to R$ in (1.3) will be assumed to satisfy the assumptions (σ) of Lemma 1.3 below. It should be noted that the case $\sigma(r) \equiv r$ gives rise to the linear model derived in Nunziato [68], and that (1.3) is one reasonable generalization of the heat flux functional for nonlinear heat flow.

In the physical literature it is customary to define

$$a(t) = b_0 + \int_0^t \beta(s)ds \qquad (0 < t < \infty)$$

as the internal energy relaxation function; thus $\alpha(0) = b_0$ and $\alpha'(t) = \beta(t)$. Similarly,

$$\kappa(t) = c_0 - \int_0^t \gamma(s)ds \qquad (0 < t < \infty)$$

is defined as the heat flux relaxation function, so that $\kappa(0) = c_0$ and $\kappa'(t) = -\gamma(t)$. Then (1.2), (1.3) are replaced respectively by

$$\text{(1.2')} \qquad \varepsilon(t,x) = \varepsilon_0 + \alpha(0)u(t,x) + \int_0^t \alpha'(t-s)u(s,x)ds, \quad \varepsilon_0 > 0 ,$$

$$\text{(1.3')} \qquad q(t,x) = -\kappa(0)\sigma(u_x) - \int_0^t \kappa'(t-s)\sigma(u_x(s,x))ds ;$$

see [68, (5.13), (5.11)] where the linear case $\sigma(r) \equiv r$ is considered. The quantity $a(0)$ is called the instantaneous heat capacity while $\alpha(\infty)$ is the equilibrium heat capacity and similar definitions for the $\kappa(0)$ and $\kappa(\infty)$. It is shown in [68] that $\alpha(\infty) > \alpha(0) > 0$, and that $\kappa(0)$, $\kappa(\infty)$ are nonnegative.

In the physical literature β, γ are usually assumed to be decaying exponentials with positive coefficients. As we shall see the theory developed in Chapters 2-4 permits a much greater generality, and we shall merely have to require that $\beta(0) > 0$, $\gamma(0) > 0$, that β and $\gamma \in L^1(0,\infty)$, and that

$$b_0 + \int_0^t \beta(\tau)d\tau > 0, \quad c_0 - \int_0^t \gamma(\tau)d\tau > 0 \qquad (0 < t < \infty) ,$$

which corresponds to the physically reasonable assumptions $\alpha(t) > 0$, $\kappa(t) > 0$, $0 < t < \infty$. We shall also assume that the conditions

$$\text{(PW)} \qquad b_0 + \operatorname{Re} \hat{\beta}(i\eta) > 0 \qquad (\eta \in R) ,$$

$$(\gamma) \qquad c_0 - \int_0^\infty \gamma(\tau)d\tau > 0 ,$$

where $\hat{\beta}(i\eta) = \int_0^\infty \beta(t)\exp(-i\eta t)dt$, are satisfied; assumption (γ) states that $\kappa(\infty) > 0$. The above assumptions will be motivated presently. Remark 4.8 in Chapter 3 below shows that the physically reasonable assumptions $b_0 + \int_0^t \beta(\tau)d\tau > 0$ and $c_0 - \int_0^t \gamma(\tau)d\tau > 0$ $(0 < t < \infty)$ are actually not essential for the theory developed in Chapter 3 to apply.

If $h = h(t,x) \in L^1_{loc}(0,\infty;L^2(0,1))$ represents the external heat supply added to the rod for $t > 0$ and $0 < x < 1$, and if $u(0,x) = u_0(x)$, $0 < x < 1$, is the given initial temperature distribution, the law of balance of heat (1.1) shows that in one space dimension the temperature u satisfies the initial-boundary value problem

$$(1.4) \qquad \begin{cases} \dfrac{\partial}{\partial t} [b_0 u + \beta * u] = c_0 \sigma(u_x)_x - \gamma * \sigma(u_x)_x + h & (0 < t < \infty,\ 0 < x < 1) \\ u(0,x) = u_0(x) \quad (0 < x < 1), \quad u(t,0) = u(t,1) \equiv 0 & (t > 0) , \end{cases}$$

where subscripts denote differentiation with respect to x and where $*$ denotes the convolution on $[0,t]$. Note that in an ordinary material $\beta = \gamma \equiv 0$, and (1.4) becomes the nonlinear heat equation in one space dimension.

The next task is to transform (1.4) to the equivalent form (V) below which will be used for the analysis. Define

$$(1.5) \qquad C(t) = c_0 - \int_0^t \gamma(\tau)d\tau \qquad (0 < t < \infty) ,$$

$$(1.6) \qquad G(t,x) = b_0 u_0(x) + \int_0^t h(\tau,x)d\tau \qquad (0 < t < \infty,\ 0 < x < 1) .$$

Noting that

$$\frac{\partial}{\partial t}(C*\sigma(u_x)_x)(t,x) = c_0\sigma(u_x(t,x))_x - (\gamma*\sigma(u_x)_x)(t,x) ,$$

and integrating (1.4) using the initial condition, and (1.6) yields the equivalent Volterra equation (to (4.3)):

$$(1.7) \qquad b_0 u(t,x) + (\beta*u)(t,x) = (C*\sigma(u_x)_x)(t,x) + G(t,x) \qquad (0 < t < \infty,\ 0 < x < 1) ,$$

where u satisfies the boundary conditions $u(t,0) = u(t,1) \equiv 0\ (t > 0)$.

We next define the nonlinear operator A formally by the relation

$$(A) \qquad Au = -\sigma(u_x)_x \quad \text{where} \quad u(t,0) = u(t,1) \equiv 0 .$$

In order not to interrupt this development we postpone a precise definition of A to Lemma 1.3 below. Then the Volterra equation (1.7) has the abstract form

$$(V_1) \qquad b_0 u + \beta*u + C*Au = G \qquad (0 < t < \infty) .$$

To transform (V_1) to the equivalent form (V) below define $\rho : [0,\infty) \to R$ to be the unique solution of the linear Volterra equation (called the resolvent kernel of β):

$$(\rho) \qquad b_0\rho(t) + (\beta*\rho)(t) = -\frac{\beta(t)}{b_0} \qquad (0 < t < \infty) .$$

It is standard that if $b_0 > 0$ and $\beta \in L^1_{loc}[0,\infty)$, equation (ρ) has a unique solution $\rho \in L^1_{loc}(0,\infty)$. Applying the variation of constants formula for Volterra equations [63]

$$(b_0 y + \beta*y = g \iff y = \frac{g}{b_0} + \rho*g)$$

finally yields that (V_1) is equivalent to the abstract equation

$$(V) \qquad u + b*Au = f \qquad (0 < t < \infty)$$

with the definitions

(b) $$b(t) = \frac{C(t)}{b_0} + (\rho * C)(t) \qquad (0 < t < \infty)$$

(f) $$f(t) = \frac{G(t,\cdot)}{b_0} + (\rho * G)(t,\cdot) \qquad (0 < t < \infty) .$$

Similar considerations show that for heat flow in a bounded homogeneous body Ω of isotropic material with memory in R^2 or R^3 with a smooth boundary Γ, the temperature u will also satisfy the abstract Volterra equation (V) with the kernel b and forcing term f given exactly as above, but with the nonlinear operator A defined precisely in Remark 1.4.

We next comment on the significance of the assumptions concerning β, γ as well as (PW) and (γ). Since the relaxation functions β and γ are generally taken as decaying exponentials with positive coefficients in the physical literature, it is certainly reasonable to assume that $\beta, \gamma \in L^1(0,\infty)$ and that $\beta(0) > 0$, $\gamma(0) > 0$. We next motivate the assumption that $b_0 + \int_0^t \beta(\tau)d\tau > 0$ $(0 < t < \infty)$. A similar reasoning motivates $c_0 - \int_0^t \gamma(\tau)d\tau > 0$ $(0 < t < \infty)$. Consider the internal energy ε defined by (1.2) and suppose that the temperature u is maintained at zero up to time t_0 and at a state $\bar{u}_1(x) > 0$ $(0 < x \leq 1)$ for $t > t_0$. One would then expect the internal energy to be positive for $t > t_0$. If the function β is positive for $t > 0$ this is automatically the case. However if not, the assumption $b_0 + \int_0^t \beta(\tau)d\tau > 0$ $(0 < t < \infty)$ is natural in view of the fact that in this situation

$$\varepsilon(t,x) = \overline{u_1}(x)\left(b_0 + \int_{t_0}^t \beta(\tau)d\tau\right) \qquad (t_0 < t < \infty) .$$

Since $\beta \in L^1(0,\infty)$ equation (4.1) shows that ε is bounded whenever u is bounded. The assumption (PW) implies that $b_0 + \int_0^\infty \beta(t)dt > 0$ (take $\eta = 0$); thus if $u(x,t)$ tends to an equilibrium state $\bar{u}(x) > 0$ as $t \to \infty$, (1.2) implies that the corresponding limiting internal energy $\bar{\varepsilon}(x) > 0$ as is to be expected. For physical reasons it is also to be expected that if ε is bounded the temperature should be bounded. Applying the variations of constants formula to (1.2) yields

$$u(t,x) = \frac{\varepsilon(t,x)}{b_0} + (\rho*\varepsilon)(t,x) \qquad (0 < t < \infty,\ 0 < x < 1)\ ,$$

where ρ is the resolvent kernel of β defined by equation (ρ). Thus to have u bounded whenever ε is bounded it is sufficient to require that $\rho \in L^1(0,\infty)$. But by the Paley-Wiener theorem [69] applied to equation (ρ), $\beta \in L^1(0,\infty)$ implies that $\rho \in L^1(0,\infty)$ if and only if

$$b_0 + \hat{\beta}(z) \neq 0 \quad \text{for} \quad \mathrm{Re}\ z \geq 0\ .$$

The condition (PW) now results from taking the real part of this expression, noting that for physical reasons one wants $b_0 + \hat{\beta}(0) > 0$, and arguing as in the proof of Lemma 2.2, Chapter 3.

To motivate assumption (γ) suppose that $u(t,x) \to \bar{u}(x)$ as $t \to \infty$ and that $\frac{d}{dx}\bar{u}(x) > 0$, implying that $\sigma\left(\frac{d\bar{u}}{dx}\right) > 0$ (see assumptions (σ) below). One then expects that the limiting heat flux $\bar{q}(x)$ in equation (1.3) is strictly negative, if the process being modelled represents "forward" heat flow; condition (γ) insures that this is the case.

We shall next see that the physically reasonable assumptions $b_0 > 0$, $c_0 > 0$, (PW) and (γ), together with some mild additional technical assumptions, imply that the kernel b in the Volterra equation (V) defined (b) satisfies the assumptions

(H_b) $\qquad b(t) = b_\infty + B(t),\ b(0) > 0,\ b_\infty > 0,\ B, B' \in L^1(0,\infty)\ ;$

these will play an important role in the application of the boundedness and asymptotic theory developed in Chapter 3, Section 3, and in the application of that theory to heat flow described by the problem (1.4) above in Chapter 3, Section 4. One has the following result whose elementary proof is omitted:

Lemma 1.1. Let $b_0 > 0$ $c_0 > 0$, β, γ, $t\beta$, $t\gamma \in L^1(0,\infty)$, and let assumptions (PW) and (γ) be satisfied. Define

$$b_\infty = \frac{c_0 - \int_0^\infty \gamma(t)dt}{b_0 + \int_0^\infty \beta(t)dt}\ , \tag{1.8}$$

(1.9) $$B(t) = \frac{C(t)}{b_0} + (\rho * C)(t) - b_\infty ,$$

where $C(t) = c_0 - \int_0^t \gamma(\tau)d\tau$, *and* ρ *is the resolvent of* β *uniquely defined by equation* (ρ). *Then* $b_\infty > 0$ *and* $B, B' \in L^1(0,\infty)$, *and* $b(t) = b_\infty + B(t)$ *satisfies* (H_b) *with* $b(0) = \frac{c_0}{b_0} > 0$, $B(0) = \frac{c_0}{b_0} - b_\infty > 0$.

The next elementary result gives physically reasonable sufficient conditions on the relaxation function β, the initial temperature distribution u_0, and the external heat supply h in order that the forcing term f in (V) defined by equation (f) will satisfy the assumption

(H_f) $$f(t) = f_\infty + F(t),\ F \in W^{1,2}_{loc}([0,\infty);H),\ F' \in L^2(0,\infty;H) ,$$

where H is the real Hilbert space $L^2(0,1)$ and W is the usual Sobolev space. Assumption (H_f) will play an important role in the boundedness and asymptotic theory of Chapter 3.

Lemma 1.2. Let $H = L^2(0,1)$ *and* $u_0 \in H^1_0(0,1)$. *Let* $\beta \in L^1(0,\infty) \cap L^2(0,\infty)$ *and let assumption* (PW) *be satisfied. Finally, assume that*

(h) $$h \in L^1(0,\infty;H) \cap L^2(0,\infty;H) .$$

Then the function $f : [0,\infty) \times (0,1) \to H$, *defined by equations* (f), (G), *where* ρ *is the resolvent of* β, *satisfies* $f \in W^{1,2}_{loc}(0,\infty;H)$ *and* $f(0,x) = u_0(x) \in H^1_0(0,1)$. *Moreover,*

$$f(t,x) = f_\infty(x) + F(t,x) \qquad (0 \le t < \infty,\ 0 < x < 1) ,$$

where

(1.10) $$f_\infty(x) = (b_0u_0(x) + \int_0^\infty h(\tau,x)d\tau)(\frac{1}{b_0} + \int_0^\infty \rho(\tau)d\tau) ,$$

(1.11) $$F(t,x) = \frac{G(t,x)}{b_0} + (\rho * G)(t,x) - f_\infty(x) = -\frac{1}{b_0}\int_t^\infty h(\tau,x)d\tau$$
$$- \int_0^t \rho(t-s)\int_s^\infty h(\tau,x)d\tau ds - \int_t^\infty \rho(\tau)d\tau(b_0u_0(x) + \int_0^\infty h(\tau,x)d\tau) ,$$

and $\frac{\partial F}{\partial t} \in L^2(0,\infty;H)$. *If in addition* $t\beta \in L^1(0,\infty)$ *and* $th \in L^1(0,\infty;H)$, *then* $F \in L^2(0,\infty;H)$.

Sketch of Proof of Lemma 1.2. The assumptions $\beta \in L^1(0,\infty)$ and (PW), together with the Paley-Wiener theorem [69], applied to the resolvent equation (ρ) imply that $\rho \in L^1(0,\infty)$. But then the assumption $\beta \in L^2(0,\infty)$ and the fact that $\rho \in L^1(0,\infty)$ imply that also $\rho \in L^2(0,\infty)$ from the resolvent equation. These facts combined with the definition of f in (f) and assumption (h) yield the formulae (1.10) for f_∞ and (1.11) for F given in the statement, as well as $f \in W^{1,2}_{loc}(0,\infty;H)$. From formula (1.11) one easily proves that

$$(1.12) \qquad \frac{\partial F}{\partial t}(t,x) = \frac{1}{b_0} h(t,x) + b_0 u_0(x)\rho(t) + (\rho * h)(t,x) \quad (0 < t < \infty,\ 0 < x < 1)\ ;$$

then $\frac{\partial F}{\partial t} \in L^2(0,\infty;H)$ follows from $h \in L^2(0,\infty;H)$ and $\rho \in L^1(0,\infty) \cap L^2(0,\infty)$. Finally, (PW) and $t\beta \in L^1(0,\infty)$, together with $\rho \in L^1(0,\infty)$ imply that $t\rho \in L^1(0,\infty)$ from the resolvent equation. This, together with the assumption $th \in L^1(0,\infty)$ and routine estimates applied to the formula (1.11) yield $F \in L^2(0,\infty;H)$. This completes the proof.

The next task is to give a precise definition of the operator A in the abstract equation (V) for the heat flow problem under study. Let $H = L^2(0,1)$ be the real Hilbert space of square integrable *functions on* $(0,1)$. *Let* $\sigma : R \to R$ *satisfy the assumptions*

$$(\sigma) \qquad \sigma \in C^1(R),\ \sigma(0) = 0,\ \sigma'(\xi) \geq p_0 > 0 \qquad (\xi \in R)\ ,$$

for some $p_0 > 0$. Define $W : R \to R^+$ by

$$W(r) = \int_0^r \sigma(\xi)d\xi \quad \left(\geq \frac{p_0}{2} r^2\right) \qquad (r \in R)\ ,$$

and define $\varphi : H \to (-\infty,+\infty]$ by

$$(1.13) \qquad \varphi(y) = \begin{cases} \int_0^1 W\left(\frac{dy}{dx}(x)\right)dx & \text{if } y \in H^1_0(0,1) \\ +\infty & \text{otherwise}\ . \end{cases}$$

Lemma 1.3. Let the assumptions (σ) be satisfied and let $\varphi : H \to (-\infty,+\infty]$ be the function defined by (1.13). Then φ is convex, l.s.c. and proper on H, and

$$Ay = \partial\varphi(y) = -\frac{d}{dx}\,\sigma\Big(\frac{dy}{dx}\Big),\quad D(\partial\varphi) = \Big\{y \in H_0^1(0,1) : \frac{d}{dx}\,\sigma\Big(\frac{dy}{dx}\Big) \in L^2(0,1)\Big\}\ ,$$

where $\partial\varphi$ denotes the subdifferential. Moreover, $\varphi(y) \geq 0$, $\varphi(y) \to \infty$ as $\int_0^1 |y(x)|^2 dx \to \infty$, and $(y,\partial\varphi(y)) \geq p_0\pi^2 \int_0^1 |y(x)|^2 dx$, where $(\cdot,\cdot)$ denotes the scalar product in H.

Sketch of Proof of Lemma 1.3. The first result is standard see Brézis ([14],[15]). To prove the last two conclusions let $y \in H_0^1(0,1)$; then from the definition of φ and the Poincaré inequality one has

$$\varphi(y) \geq \frac{p_0}{2}\int_0^1 \Big|\frac{dy}{dx}(x)\Big|^2 dx \geq \frac{p_0}{2}\,\pi^2 \int_0^1 |y(x)|^2 dx \geq 0\ ,$$

and $\varphi(y) \to \infty$ as $\int_0^1 |y(x)|^2 dx \to \infty$. Moreover,

$$(y,\partial\varphi(y)) = -\int_0^1 y(x)\,\frac{d}{dx}\,\sigma\Big(\frac{dy}{dx}(x)\Big)dx\ ,$$

and an integration by parts, $y \in H_0^1(0,1)$ and the Poincaré inequality give

$$(y,\partial\varphi(y)) = \int_0^1 \frac{dy}{dx}(x)\sigma\Big(\frac{dy}{dx}(x)\Big)dx \geq p_0 \int_0^1 \Big|\frac{dy}{dx}(x)\Big|^2 dx \geq p_0\pi^2 \int_0^1 |y(x)|^2 dx\ .$$

While our considerations of the heat flow problem are primarily in one space dimension, we indicate how to define the nonlinear operator $A = \partial\varphi$ in (V) for the heat flow problem in two or three space dimensions such that the function φ satisfies the conclusions of Lemma 1.3 under the physically reasonable assumption (λ) below.

Remark 1.4. Let Ω be a bounded domain in R^n (for heat flow $n = 2$ or 3) with smooth boundary Γ. Let $\lambda : R^+ \to R$ be a given smooth function satisfying the assumption

$$(\lambda)\qquad \begin{cases} \lambda(0) > 0, \text{ there exists } p_0 > 0 \text{ such that } \lambda(\xi) \geq p_0 \text{ and} \\ \xi\lambda'(\xi) + \lambda(\xi) \geq p_0 \quad (\xi \in R)\ . \end{cases}$$

Define $\Lambda : R \to R^+$ by

$$A(r) = \int_0^r \xi\lambda(\xi)d\xi \quad \left(\geq \frac{p_0}{2} r^2\right) \quad (r \in R) .$$

Let $H = L^2(\Omega)$ and define

$$\varphi(u) = \begin{cases} \int_\Omega A(|\nabla u|)dx & \text{if } u \in H_0^1(\Omega) \\ +\infty & \text{otherwise} . \end{cases}$$

Then it is readily verified that $\varphi : H \to (-\infty,+\infty]$ is convex, l.s.c., and proper on H and

$$Au = \partial\varphi(u) = -\nabla\cdot(\lambda(|\nabla u|)\nabla u)$$

with

$$D(\partial\varphi) = \{u \in H_0^1(\Omega) : \nabla\cdot(\lambda(|\nabla u|)\nabla u) \in L^2(\Omega)\} .$$

Clearly $\varphi(u) \geq 0$ $(u \in H)$ and by the Poincaré inequality $\varphi(u) \to \infty$ as $|u| \to \infty$. Using integration by parts and the Poincaré inequality one also has the coercivity condition

$$(Au,u) \geq kp_0|u|_H^2 ,$$

where $k > 0$ is the constant in the Poincaré inequality.

The results of Lemmas 1.1-1.3 and of Remark 1.4 will be used in Chapter 3, Section 4, to discuss the global existence, uniqueness, boundedness and asymptotic behavior of the solution of the initial-boundary value problem (1.4) using the theory which will be developed for the equivalent abstract Volterra equation (V).

3. A General (Parabolic) Heat Flow Model. We consider the same heat flow problem as in Section 2. In this model we assume that the internal energy is given by the functional (1.2), but we assume the following more general form of the heat flux functional

$$q(t,x) = -\psi(u_x) - \int_0^t a(t-s)\sigma(u_x(s,x))ds \quad (t \geq 0) , \tag{1.14}$$

where we again assume that the history of temperature is prescribed as zero for $t < 0$ and $0 < x < 1$. The real function $\psi : R \to R$, $\psi(0) = 0$ satisfies the same assumptions as σ in Lemma 1.3, Section 2. If $\psi(\cdot) = c_0\sigma(\cdot)$, and $a = -\gamma$ (i.e. in the notation of

(1.3') $\kappa'(t) = a(t)$) (1.14) reduces to (1.3). We assume that β, a, and the external heat supply h satisfy the same type of smoothness assumptions respectively as β and γ in Section 2. We assume again that

$$b_0 + \int_0^t \beta(\tau)d\tau > 0 \qquad (0 < t < \infty) ,$$

and that $a(0) > 0$. For physical reasons it is also reasonable to assume that

$$\psi(\xi) + (\int_0^t a(\tau)d\tau)\sigma(\xi) \begin{cases} > 0 & \text{if } \xi > 0,\ t > 0 \\ < 0 & \text{if } \xi < 0,\ t > 0 \end{cases} ,$$

and that

$$\psi(\xi) + (\int_0^\infty a(\tau)d\tau)\sigma(\xi) \begin{cases} > 0 & \text{if } \xi > 0 \\ < 0 & \text{if } \xi < 0 \end{cases} .$$

Applying the energy balance equation (1.1), and using (1.2), (1.14) and h shows that the temperature u satisfies the initial-boundary value problem

$$(1.15) \quad \begin{cases} b_0 u_t + \beta(0)u + \beta' * u = \psi(u_x)_x + a*\sigma(u_x)_x + h & (0 < t < \infty,\ 0 < x < 1) \\ u(t,0) = u(t,1) \equiv 0 & (t > 0) \\ u(0,x) = u_0(x) & (0 < x < 1) . \end{cases}$$

Note that if $a = \beta \equiv 0$, (1.15) reduces to the nonlinear heat equation. Defining the opertors A, B, L by the relations:

$$Au = -\sigma(u_x)_x \quad \text{where} \quad u(t,0) = u(t,1) \equiv 0$$

$$Bu = -\psi(u_x)_x \quad \text{where} \quad u(t,0) = u(t,1) \equiv 0$$

$$Lu = \beta(0)u + \beta' * u ,$$

and taking (without loss of generality) $b_0 = 1$, one sees that the problem (1.15) has the abstract form

$$(V_L') \quad \begin{cases} \dfrac{du}{dt} + Bu + a*Au + Lu = h & (0 < t < \infty) \\ u(0) = u_0 . \end{cases}$$

If the functions σ and ψ satisfy the assumptions (σ), Lemma 1.3 shows that A and B are subdifferentials of proper convex, l.s.c. functions defined on the real Hilbert space $H = L^2(0,1)$. If the relaxation function β has $\beta' \in L^1(0,\infty)$, the linear operator L is well defined.

The abstract problem (V_L') has been investigated by a number of authors combining techniques of Volterra equations and the theory of monotone operators. If $\beta \equiv 0$ V. Barbu [5], [7], Barbu and Malik [11] studied the problem of global existence; a more complete existence theory for considerably more general kernels a, as well as a discussion of boundedness and asymptotic behaviour, was developed by Crandall, Londen, and Nohel [28]. The latter, see [28, p. 717], also permits in the existence theory Lipschitz type perturbations of the operator A; this essentially covers the case of the specific operator L in the present application; the details of this generalization of the existence theory have recently been worked out by M. J. Luo as a part of a forthcoming Ph.D. thesis at the University of Wisconsin-Madison.

The general assumptions for the existence theory in [28] concerning the operators A and B are roughly speaking as follows. The operators A and B are subdifferentials of proper, convex, l.s.c. functions defined on a real Hilbert space H; B dominates A in a certain precise sense (see [28], inequality (1.7); the case $B = kA$, $k > 0$ is not excluded), and B satisfies a compactness assumption (see (1.8) in [28]; this compactness condition excludes the possibility $B \equiv 0$. The kernel a satisfies an abstract condition (see [28; (1.10)]) which is shown to hold for two physically important classes of kernels:

$$(a_1)\qquad \begin{cases} a(0) > 0, \ a \text{ locally absolutely continuous on } [0,\infty)\,, \\ a' \text{ locally of bounded variation on } [0,\infty)\,; \end{cases}$$

$$(a_2)\qquad \begin{cases} a(0) > 0, \ a \in C[0,\infty) \cap C^2(0,\infty), \text{ and} \\ a \text{ nonnegative, nonincreasing, convex on } [0,\infty)\,. \end{cases}$$

It should be noted that no satisfactory uniqueness theorem has to date been discovered for the general problem (V_L') with $B \neq k A$, $k > 0$ a constant, even when $L \equiv 0$.

Other approaches to the study of (V_L') include Aizicovici [1], [2], [3] and Gripenberg [32], [36]. While these studies are in a more general setting than those referred to above, they do not seem to shed new light on the physical problem (1.15), and indeed the existence results of Aizicovici do not include those of [28]. An extension of the latter to nonconvolution kernels has been studied by Rennolet [71]. A semigroup approach to a special case of (V_L') has been investigated by Vrabie [83]. Another interesting "parabolic" heat flow problem involving an analysis of evolution inequalities has recently been studied by J. Naumann [64].

4. A Hyperbolic Nonlinear Volterra Equation for Heat Conduction with Finite Wave Speeds. The parabolic models for heat flow in materials with memory formulated in Sections 2 and 3 both predict that a thermal disturbance at any point of the body is instantly felt everywhere in the body (though not with equal strength). This implies that finite discontinuities propagate with infinite speed. This situation is unrealisitic for some materials, particularly at low temperatures.

Gurtin and Pipkin [39], see also Nunziato [68], have proposed a model for heat flow which exhibits a finite speed of propagation; they study of the linear model $\sigma(r) \equiv r$. We present briefly such a nonlinear model which has been investigated by Mac Camy [59] by the method of characteristics, and by Dafermos and Nohel [29] by an energy method. Another interesting variant of the energy method was recently developed by Staffans [80].

Consider the heat flow problem of Section 2. Define the internal energy by the functional (1.2), with β satisfying the conditions of Section 2. In place of (1.3) assume that the heat flux is given by the functional

$$(1.16) \qquad q(t,x) = -\int_0^t c(t-s)\sigma(u_x(s,x))ds \qquad (0 < t < \infty,\ 0 < x < 1)\ ,$$

where the relaxation function $c \in L^1(0,\infty)$, $c(0) > 0$, $\int_0^t c(s)ds > 0$ $(0 < t < \infty)$, $\int_0^\infty c(s)ds > 0$, and where the real function σ satisfies assumptions (σ) of Lemma 1.3 (actually σ, smooth, $\sigma(0) = 0$, $\sigma'(0) > 0$ is sufficient for the development in [29]). Note that from (1.16) the heat flux depends only on the history of the gradient of u, and is independent of the present value of the gradient of u. Evidently, this model of the heat flux results from (1.3) by taking $c_0 = 0$ and by replacing γ by $-c$, or from (1.3') by taking $\kappa(0) = 0$, $\kappa'(t) = c(t)$. The model (1.16) for the heat flux also results by taking $\psi \equiv 0$ in (1.14), a case which is excluded in the theory and the referenced accompanying mathematical literature in Section 1.3.

Applying the energy balance (1.1) and using (1.2), (1.16) and the external heat supply h leads to the equation

$$b_0 u_t + \frac{\partial}{\partial t}\, \beta * u = c * \sigma(u_x)_x + h \qquad (0 < t < \infty,\ 0 < x < 1)\ .$$

Noting that $\beta * u = u * \beta$, carrying out the differentiation, and imposing the boundary and initial conditions leads to the following initial-bounlary value problem for the heat flow problem under the assumptions of this section

$$(1.17)\quad \begin{cases} b_0 u_t + \beta * u_t = c * \sigma(u_x)_x + h(t,x) - \beta(t)u_0(x) & (0 < t < \infty,\ 0 < x < 1)\ , \\ u(t,0) = u(t,1) \equiv 0 & (t > 0) \\ u(0,x) = u_0(x)\ . & \end{cases}$$

The problem (1.17) is transformed to the more standard abstract form (V_H') below as follows. As in Section 2 define the resolvent kernel ρ of β by the linear Volterra equation (ρ); by assumption (PW) $\rho \in L^1(0,\infty)$. Define the function $a : [0,\infty) \to \mathbb{R}$ by

$$a(t) = \frac{1}{b_0}\, c(t) + (\rho * c)(t) \qquad (0 < t < \infty)\ ;$$

define the function $g : [0,\infty) \times (0,1) \to \mathbb{R}$ by

$$g(t,x) = \frac{1}{b_0}\,(h(t,x) - \beta(t)u_0(x)) + \rho * (h(t,x) - \beta(t)u_0(x)) \ (0 < t < \infty, 0 < x < 1)\ .$$

Finally define the operator A as in Section 2. Applying the variation of constants formula for Volterra equations, and these definitions to (1.17) shows that (1.17) is equivalent to the abstract nonlinear Volterra equation

$$(V_H') \qquad \begin{cases} \dfrac{du}{dt} + a*Au = g(t,\cdot) & (0 < t < \infty) \\ u(0) = u_0(\cdot) \ . \end{cases}$$

It is easy to see (compare Lemma 1.1) that under the present assumptions on β and c, $a \in L^1(0,\infty)$ and that

$$\int_0^\infty a(t)dt = \frac{\int_0^\infty c(t)dt}{b_0 + \int_0^\infty \beta(t)dt} > 0 \ ;$$

if also $\beta(0) > 0$, $c(0) > 0$, $c'(0) < 0$, then $a(0) > 0$ and $a'(0) < 0$. It is also evident that the forcing term $g \in L^2(0,\infty;L^2(0,1))$ if $h \in L^2(0,\infty;L^2(0,1))$ and also $\beta \in L^2(0,\infty)$.

Notice that if $\beta \equiv 0$ and $\gamma \equiv \gamma(0) > 0$ (or equivalently $a \equiv \frac{\gamma(0)}{b_0}$) the problem (1.17) reduces to the nonlinear wave equation problem

$$(W) \qquad \begin{cases} b_0 u_{tt} = \gamma(0)\ \sigma(u_x)_x + h_t & (0 < t < \infty,\ 0 < x < 1) \\ u(t,0) = u(t,1) \equiv 0 & (t > 0) \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) = g(0,x) \ . \end{cases}$$

If the real function σ is "genuinely nonlinear" ($\sigma''(\xi) \not\equiv 0$, $\xi \in R$), Lax [50] has shown that (W) fails to have global smooth (C^2) solutions in time, even if $h \equiv 0$, no matter how smooth and "small" one takes the initial functions u_0, u_1. If the function σ is convex the derivatives of the solution u of (W) develop singularities due to the crossing of characteristics in finite time ("shocks").

The objective of the analysis by Mac Camy [59] (which uses Riemann invariants and is therefore restricted to one space dimension), and a different analysis by Dafermos and Nohel [29, Theorem 4.1], Staffans [80], which are applicable to several space dimensions and both of which use energy methods, is to show that under the present assumptions on β and γ (together with some technical ones and some other physically reasonable ones which imply that the kernel a is strongly positive on $[0,\infty)$), the integral in (V_H') has the effect of a frictional damping mechanism which prevents the formation of shocks, provided the data u_0 and the forcing term g are sufficiently smooth and small in certain H norms. This analysis leads to global existence, uniqueness, and decay of smooth solutions of the problem (1.17) for sufficiently smooth and small data u_0 and h (see especially [29, Theorems 4.1, 6.1]; for a physically reasonable two-dimensional heat flow problem with the same kernel a see [29, Theorem 7.1]. It is also evident from the analysis in [29, Section 3] dealing with the local existence and uniqueness of the problem (1.17), resp. (V_H'), that solutions of (1.17), resp. (V_H'), possess the property of finite speed of propagation. For an analysis of the existence and uniqueness of classical solutions for "small" data in the simpler case of a nonlinear wave equation with frictional damping see Nishida [65], and Nohel [66].

We remark also that the abstract problem (V_H') has recently been studied by S. O. Londen [55], [56] for a class of kernels a which are positive, decreasing, and convex on $(0,\infty)$ and which satisfy the crucial (for his method) condition $a'(0+) = -\infty$. His method is a significant generalization of that of Crandall, Londen, and Nohel [28] for the parabolic problems discussed briefly in Section 3. However, the assumption $a'(0+) = -\infty$ is inappropriate in the present physical context; moreover, the type of solution obtained by Londen in [55] and [56] need not be regular in the sense of smooth solutions, and no decay results of solutions comparable to [29] are obtainable by his methods. It should also be remarked that for the linear problem (V_H') (e.g. $Au = -\nabla^2 u$), interesting and useful results using deep results of Shea and Wainger [73] have been obtained in a series of papers by K. B. Hannsgen [41-47] and by Carr and Hannsgen [16].

Finally, we note the model problem (V_H') is similar to a particular model problem for nonlinear viscoelastic motion in one space dimension in which, however, the kernel a has the form $a(t) = a_\infty + A(t)$, $a_\infty > 0$, A positive, decreasing convex on $[0,\infty)$, and for which the analysis is considerably more complicated. This problem has been extensively studied by Mac Camy [60], Dafermos and Nohel [29], and Staffans [80]; see also an interesting asymptotic result by Staffans [81] motivated by this problem.

Chapter 2

Existence and Uniqueness of Solutions of Abstract Volterra Equations

2.1. Introduction. In this chapter we study the abstract nonlinear Volterra equation

(V) $$u(t) + b*Au(t) \ni f(t) \qquad (0 < t < T) ,$$

where $T > 0$ is arbitrary, in the setting: A is an m-accretive (possibly multivalued) operator in a real Banach space X, the given kernel b is a real absolutely continuous function on $[0,T]$, $b*g(t) = \int_0^t b(t-s)g(s)ds$ with the integral in (V) interpreted as the usual Bochner integral, and the given function $f \in W^{1,1}(0,T;X)$ where $W^{1,1}$ is the usual Sobolev space.

We treat the problem of existence, uniqueness, dependence on data, and regularity of solutions of (V) on $[0,T]$ by means of a simple method developed jointly with M. G. Crandall [26] to which the reader is referred for more details; the results obtained for (V) generalize and simplify considerably earlier work on existence and uniqueness obtained by Barbu [6], [8], London [54], Gripenberg [30], for the case $X = H$ a real Hilbert space. A different approach to the study of (V) in the same general setting was developed independently by Gripenberg [31], [32]. The general theory will be used in Chapters 3 and 4. We will also comment on the special cases: (i) A maximal monotone on H, and (ii) $A = \partial\varphi$, where $\varphi : H \to (-\infty,+\infty]$ is a proper, convex, l.s.c. function and $\partial\varphi$ denotes the subdifferential of φ (see Brézis [14]); these special cases will be important in Chapter 3. We shall also consider briefly a recent generalization of [26] by Gripenberg [34], which will also be used in Chapter 4.

Our method involves reducing the study of (V) to that of an equivalent functional differential equation of the form

(FDE) $$\begin{cases} \frac{du}{dt} + Au \ni G(u) & (0 < t < T) \\ u(0) = x = f(0) , & \end{cases}$$

where $G : C([0,T]; \overline{D(A)}) \to L^1(0,T;X)$ is a particular mapping, and developing the theory for (FDE). Our results are also directly applicable to certain integrodifferential equations studied by Mac Camy [58] via a Galerkin argument which necessitates further restrictions.

We observe that if $b \equiv 1$, equation (V) is equivalent to the evolution problem

$$\text{(E)} \qquad \frac{du}{dt} + Au \ni f', \quad u(0) = x = f(0), \qquad (0 < t < T) .$$

Our method of studying (FDE) consists of generalizing known results for (E) due primarily to Benilan [13]; the latter are reviewed in Section 2. We recall also that the initial-boundary value problem for a linear or nonlinear diffusion problem is a special case of (E).

2.2. Preliminaries on Evolution Equations. For further background and details of this section we refer the reader to [7], [25], [26]. Let X be a real Banach space with norm $\|\cdot\|$. A mapping $A : X \to 2^X$ is called an operator in X; its domain $D(A) = \{x \in X : Ax \neq \phi\}$ and its range $R(A) = \cup \{Ax : x \in D(A)\}$; A is single-valued if Ax is a singleton. An operator A in X is accretive iff $J_\lambda = (I + \lambda A)^{-1}$ is a contraction in X for $\lambda > 0$. It follows immediately: A is accretive iff

$$\text{(2.1)} \qquad \|(x_1 + \lambda y_1) - (x_2 + \lambda y_2)\| \geq \|x_1 - x_2\| \quad \text{for} \quad y_i \in Ax_i \qquad (i = 1,2) .$$

An operator A in X is called m-accretive iff A is accretive and $R(I + \lambda A) = X$ for $\lambda > 0$.

We shall be concerned with applying some known facts about the abstract evolution equation

$$(E_g) \qquad \frac{dv}{dt} + Av \ni g, \quad v(0) = x$$

to the study of (FDE). We asume throughout that $g \in L^1(0,T;X)$, $T > 0$.

Definition 2.1. A function $v : [0,T] \to X$ is a strong solution of (E_g) on $[0,T]$ if $v(0) = x$, $v \in C([0,T];X) \cap W^{1,1}(0,T;X)$, $v(t) \in D(A)$ a.e. on $[0,T]$ and there exists $w \in Av$ such that $w(t) \in Av(t)$ and $v'(t) + w(t) = g(t)$ a.e. on $[0,T]$.

Definition 2.2. $v : [0,T] \to X$ is a weak solution of (E_g) on $[0,T]$ if there is a sequence $\{(v_n,g_n)\}_{n=1}^\infty \in C([0,T];X) \times L^1(0,T;X)$ such that v_n is a strong solution of (E_{g_n}) on $[0,T]$ and $(v_n,g_n) \to (v,g)$ in $C([0,T];X) \times L^1(0,T;X)$.

For our considerations we require a third concept of solution of (E_g), namely the notion of integral solution. First, let $[\ ,\ ]_\lambda : X \times X \to R$ be defined for $\lambda > 0$ by

$$[x,y]_\lambda = \frac{1}{\lambda}(\|x + \lambda y\| - \|x\|) ,$$

which is a nondecreasing function of λ. Define

$$[x,y]_+ = \lim_{\lambda \downarrow 0} [x,y]_\lambda = \inf_{\lambda > 0} [x,y]_\lambda$$

$$[x,y]_- = \lim_{\lambda \uparrow 0} [x,y]_\lambda = \sup_{\lambda < 0} [x,y]_\lambda .$$

Thus $\|x + \lambda y\| \geq \|x\|$ for $\lambda > 0$ iff $[x,y]_+ \geq 0$, so that A is accretive iff

(2.2) $$[x_1 - x_2, y_1 - y_2]_+ \geq 0 \quad \text{for} \quad y_i \in Ax_i .$$

<u>Definition 2.3.</u> $v : [0,T] \to X$ <u>is an integral solution of</u> (E_g) <u>on</u> $[0,T]$ <u>if</u> $v \in C([0,T];X$ <u>and</u>

(2.3) $$\|v(t) - x\| - \|v(s) - x\| \leq \int_s^t [v(\sigma) - x, g(\sigma) - y]_+ d\sigma$$

<u>for</u> $t \geq s$, $(t,s) \in [0,T]$, $x \in D(A)$ <u>and</u> $y \in Ax$. We note since $|[x,y]_+| \leq \|y\|$, and since $g \in L^1(0,T;X)$, the integral in (2.3) is well defined. A straightforward calculation, see [25], shows that the notion of integral solution only makes sense when A is accretive. We shall apply the following result on existence, uniqueness, dependence on data, and regularity about integral solutions of (E_g) due to Benilan [13].

<u>Theorem A.</u> <u>If</u> A <u>is m-accretive,</u> $x \in \overline{D(A)}$, <u>and</u> $g \in L^1(0,T;X)$ <u>then</u> (E_g) <u>has a unique integral solution</u> $v \in C([0,T]; \overline{D(A)})$ <u>on</u> $[0,T]$, <u>and if</u> $v,\hat{v}$ <u>are integral solutions of</u> (E_g), $(E_{\hat{g}})$ <u>on</u> $[0,T]$ <u>corresponding to initial values</u> $x,\hat{x}$ respectively then

(2.4) $$\|\hat{v}(t) - v(t)\| \leq \|x - \hat{x}\| + \int_0^t \|\hat{g}(\sigma) - g(\sigma)\| d\sigma, \qquad 0 \leq t \leq T .$$

<u>Moreover, if</u> $g \in BV([0,T];X)$ <u>and</u> $x \in D(A)$, <u>then</u>

(2.5) $$\|v(\xi) - v(\eta)\| \leq |\xi - \eta| (\|g(0^+) - y\| + var(g : [0,t]))$$

<u>for</u> $y \in Ax$, <u>and</u> $0 \leq \xi, \eta \leq t$, $t \in [0,T]$. <u>In particular, the integral solution</u> v <u>is</u>

Lipschitz continuous. If, in addition, X is reflexive, then v is a strong solution of (E_g) on $[0,T]$.

3. Discussion of Existence and Uniqueness Results. We shall reduce the study of existence and uniqueness of solutions of the nonlinear Volterra equation (V) on $[0,T]$ to studying the abstract functional differential equation

$$\text{(FDE)} \qquad \begin{cases} \dfrac{du}{dt} + Au \ni G(u) & (0 < t < T) \\ u(0) = x , \end{cases}$$

where A is a given m-accretive operator on X, and where G is a given mapping

$$G : C([0,T]; \overline{D(A)}) \to L^1(0,T;X) .$$

Let $v = H(g)$ denote the unique integral solution of (E_g). A solution of (FDE) is by definition a function $u \in C([0,T]; \overline{D(A)})$ such that $u = H(G(u))$. By analogy with Definition 2.1, we say that u is a strong solution of (FDE) on $[0,T]$ if $u(0) = x$, $u \in W^{1,1}(0,T;X) \cap C([0,T]; \overline{D(A)})$ and if $u'(t) + Au(t) \ni G(u)(t)$ a.e. on $[0,T]$.

Let $b \in L^1(0,T;R)$, $F \in L^1(0,T;X)$. We shall say that u is a strong solution of the Volterra equation (V) on $[0,T]$ if $u \in L^1(0,T;X)$ and if there exists $w \in L^1(0,T;X)$ such that $w(t) \in Au(t)$ and $u(t) + b*w(t) = F(t)$ a.e. on $[0,T]$. One establishes the following equivalence between strong solutions of (FDE) with a particular G and strong solutions of (V):

Proposition 3.1. Let $b \in AC([0,T];R)$, $b' \in BV([0,T];R)$, $F \in W^{1,1}(0,T;X)$ and $b(0) = 1$. Let u be a strong solution of (V) on $[0,T]$. Then u is a strong solution of (FDE) on $[0,T]$ with the identifications:

$$(3.1) \qquad \begin{cases} \text{(i)} & G(u)(t) = f'(t) - r*f'(t) - a(0)u(t) - r(t)x + \displaystyle\int_0^t u(t-s)dr(s) \\ \text{(ii)} & x = f(0) \\ \text{(iii)} & a = b' \\ \text{(iv)} & r \in L^1(0,T;R) \text{ is defined by } r + a*r = a . \end{cases}$$

Conversely, let $r \in BV([0,T];R)$, $f' \in L^1(0,T;X)$, $x \in \overline{D(A)}$ and G be given by (3.1) (i). Let u be a strong solution of (FDE) on $[0,T]$. Then u is a strong solution of (V) on $[0,T]$, where

$$(3.2) \qquad \begin{cases} \text{(i)} & f(t) = x + \int_0^t f'(s)ds \\ \text{(ii)} & b(t) = 1 + \int_0^t a(s)ds \\ \text{(iii)} & a - a*r = r \, . \end{cases}$$

We remark that if $b(t) \equiv 1$ and $F \in W^{1,1}(0,T;X)$, then the Volterra equation (V) is equivalent to the evolution equation (E_g) where $g = f'$ and where the initial value $x = f(0)$.

The proof of Proposition 1 is straightforward. The assumptions on b and F permit differentiation a.e. on $[0,T]$ of a strong solution u of (V). The differentiated equation is then "solved" for Au by means of the resolvent kernel r associated with $a = b'$, see (3.1) (iv), and the variation of constants formula for Volterra equations [63]. A known result [12] yields that $a \in BV([0,T];R)$ implies that $r \in BV([0,T];R)$, a fact which is used in arriving at the formula (3.1) (i) for $G(u)$. The converse is proved by reversing the steps. A part of Proposition 3.1 which motivates our approach is contained in Mac Camy [58] who, however, then studied (FDE) by an entirely different approach.

We remark that here we have chosen to define the resolvent kernel by (3.1) (iv), rather than by $r + a*r = -a$ as was done in [26]. This is more convenient for the theory in Chapter 4, and only causes a change of signs in the formula (3.1) (i) of some of the terms in $G(u)(t)$. Recall that if r is defined by (3.1) (iv), then the solution of the linear Volterra equation $w + a*w = v$ is given by $w = v - r*v$, while with the alternate definition of r, w would be given by $w = v + r*v$, as was used in [26].

We next use Benilan's theorem about solutions of (E_g) to obtain some general results concerning existence, uniqueness, dependence on data, and regularity of solutions of (FDE) of independent interest and use them to deduce corresponding results about solutions of (V).

Theorem 3.2. *Assume that* A *is m-accretive,* $x \in \overline{D(A)}$, *and let* $G : C([0,T]; \overline{D(A)}) \to L^1(0,T;X)$ *satisfy*

$$(3.3) \qquad \begin{cases} \|G(u) - G(v)\|_{L^1(0,t;X)} \le \int_0^t \gamma(s)\|u - v\|_{L^\infty(0,s;X)}\, ds \\ \text{for some } \gamma \in L^1(0,T;R^+), \ 0 \le t \le T, \text{ and } u,v \in C([0,T];\overline{D(A)}) \ . \end{cases}$$

Then (FDE) *has a unique solution* $u \in C([0,T]; \overline{D(A)})$ *on* $[0,T]$.

We remark that assumption (3.3) implies that the value of $G(u)$ at $t \in [0,T]$ depends only on the restriction of u to $[0,t]$. The idea of the proof is very simple. Let $v = H(g)$ denote the unique integral solution of (E_g) on $[0,T]$, $g \in L^1(0,T;X)$. We seek a fixed point of the map $K : C([0,T];\overline{D(A)}) \to C([0,T]; \overline{D(A)})$ defined by $K(u) = H(G(u))$. By property (2.4) of integral solutions

$$\|K(u)(t) - K(v)(t)\| \le \int_0^t \|G(u)(s) - G(v)(s)\|ds \qquad (0 \le t \le T) \ ,$$

for $u,v \in C([0,T]; \overline{D(A)})$, $u(0) = v(0) = x$. Applying assumption (3.3) it is now an easy matter to show that K^j is a strict contraction on $C([0,T]; \overline{D(A)})$ for j sufficiently large, so that the map K has a unique fixed point. For details see [26].

Under further assumptions one can apply the second part of Benilan's theorem to obtain greater regularity of solutions of (FDE).

Theorem 3.3. *In addition to the assumptions of Theorem 1 assume that there is a function* $k : [0,\infty) \to [0,\infty)$ *such that*

$$(3.4) \qquad \begin{cases} \mathrm{var}(G(u) : [0,t]) \le k(R)(1 + \mathrm{var}(u : [0,t])) \\ \text{and } \|G(u)(0^+)\| \le k(R) \qquad (0 \le t \le T) \ , \end{cases}$$

whenever $u \in C([0,T]; \overline{D(A)})$ is of bounded variation and $\|u\|_{L^\infty(0,T;X)} \le R$. If $x \in D(A)$, then the solution u of (FDE) is Lipschitz continuous on $[0,T]$. If X is also reflexive, then the solution u is a strong solution of (FDE) on $[0,T]$.

For the proof of Theorem 3.3 one defines $u_0 : [0,T] \to X$ by $u_0(t)x$ and $u_{n+1} = K(u_n) = H(G(u_n))$, $n = 0,1,\ldots$. These iterates converge uniformly and are uniformly bounded on $[0,T]$. By Benilan's theorem and assumption (3.4) one shows that there exists a constant $c > 0$ such that

$$\mathrm{var}(u_{n+1} : [0,t]) \le c(1 + \int_0^t \mathrm{var}(u_n : [0,s])ds)$$

for $0 \le t \le T$, so that $\mathrm{var}(u_{n+1} : [0,t]) \le c\exp(ct)$. Thus $\{\mathrm{var}(u_n : [0,T])\}$ and by (3.4) $\{\mathrm{var}(G(u_n)) : [0,T]\}$ are both bounded, and $\{u_n\}$, and hence also $u = \text{unif} \lim u_n$, is Lipschitz continuous on $[0,T]$. For more details see [26].

Finally, the solution u of (FDE) depends on the data A, G, x in the following sense:

Theorem 3.4. Let the assumptions of Theorem 1 be satisfied. Let m-accretive operators A_n in X, mappings $G_n : C([0,T];X) \to L^1(0,T;X)$, and $x_n \in \overline{D(A)}$ be given for $n = 1,2,\ldots$. Assume that the inequality (3.3) holds for G replaced by G_n, with the same γ, for $n = 1,2,\ldots$, and $u,v \in C([0,T]; \overline{D(A)})$. For $u \in C([0,T]; \overline{D(A)})$ assume that $\lim_{n\to\infty} G_n(u) = G(u)$ in $L^1(0,T;X)$, $\lim_{n\to\infty} x_n = x \in \overline{D(A)}$, and

$$\lim_{n\to\infty} (I + \lambda A_n)^{-1}z = (I + \lambda A)^{-1}z \qquad (z \in X,\ \lambda > 0) \ . \tag{3.5}$$

Let $u_n \in C([0,T]; \overline{D(A_n)})$ be solutions of (FDE) on $[0,T]$ with A replaced by A_n, G replaced by G_n, x replaced by x_n, and let $u \in C([0,T]; \overline{D(A)})$ be the solution of (FDE) on $[0,T]$. Then $\lim_{n\to\infty} u_n = u$ in $C([0,T];X)$.

The proof of Theorem 3.4 follows from the observation that under our assumptions the mapping $K(A,x,G)(u) = H(A,x,G(u))$ of Theorem 3.2 has the property that in the iterate K^j, which is a strict contraction for some j, both j and the contraction constant only on the function γ of (3.3), and the latter is assumed to be uniform in n; for details see [26].

We shall next apply Theorems 3.2, 3.3, 3.4 to study the nonlinear Volterra equation (V). If b and F in (V) satisfy the assumptions of Proposition 1, it follows from the definition of G in (3.1) (i) that

$$\|G(u)(t) - G(v)(t)\| \le (|r(0^+)| + \mathrm{var}(r : [0,t]))\|u - v\|_{L^\infty(0,t;X)} ,$$

where r is the resolvent kernel corresponding to $b' = a$ (recall that $a \in BV([0,T];R) \implies r \in BV([0,T];R)$). Thus assumption (3.3) of Theorem 3.2 is satisfied with

$$\gamma(s) = |r(0)| + \mathrm{var}(r : [0,s]) .$$

Moreover, if $f' \in BV([0,T];X)$, (3.1) (i), (ii), imply

$$\mathrm{var}(G(u) : [0,t]) \le C(1 + \mathrm{var}(u : [0,t])) \qquad (0 \le t \le T) ,$$

and $\|G(u)(0^+)\| \le C$, where C is a constant depending on $f(0)$, $f'(0^+)$, $\mathrm{var}(f' : [0,T])$, $r(0^+)$, and $\mathrm{var}(r : [0,T])$; thus assumption (3.4) of Theorem 3.3 is satisfied.

Let $\lambda > 0$ and define the Yosida approximation A_λ of the m-accretive operator A on X by

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda), \qquad J_\lambda = (I + \lambda A)^{-1} .$$

$A_\lambda : X \to X$ is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$, so a simple contraction argument shows that the approximating problem

$$(V_\lambda) \qquad u_\lambda + b*A_\lambda u_\lambda = f$$

has a unique strong solution u_λ on $[0,T]$, under the assumptions: $b \in L^1(0,T;R)$, and $f \in L^1(0,T;X)$. By Proposition 3.1 u_λ is a strong solution of

$$(\mathrm{FDE}_\lambda) \qquad \frac{du_\lambda}{dt} + A_\lambda u_\lambda = G(u_\lambda), \quad u_\lambda(0) = F(0) .$$

One also has $\lim\limits_{\lambda \downarrow 0} (I + \mu A_\lambda)^{-1} z = (I + \mu A)^{-1} z$, for $\mu > 0$, $z \in X$.

These considerations lead to the following result about solutions of (V).

Theorem 3.5 (see [26, Theorem 4]). Let $T > 0$ and let the following assumptions be satisfied:

b is absolutely continuous on $[0,T]$, $b(0) > 0$, and $b' \in BV[0,T]$.

A is m-accretive on X

$f \in W^{1,1}(0,T;X)$, $f(0) \in \overline{D(A)}$.

Then equation (V) has a unique (generalized) solution $u \in C([0,T]; \overline{D(A)})$ in the sense that (i) u is a unique solution of (FDE) on $[0,T]$ with the identifications (3.1) and (ii) $u = \lim_{\lambda \downarrow 0} u_\lambda$ in $C([0,T];X)$, where u_λ are strong solutions of the approximating equation (V_λ) on $[0,T]$.

If, moreover, $f' \in BV([0,T];X)$ and $f(0) \in D(A)$, then the generalized solution u is Lipschitz continuous on $[0,T]$. If also X is reflexive, then u is a strong solution of (V) on $[0,T]$.

Remarks 3.6. (i) We remark that if the Volterra equation (V) has a strong solution u on $[0,T]$ under the assumptions of Theorem 3.3, then from Theorem 3.3 and Proposition 3.1, $\lim_{\lambda \downarrow 0} u_\lambda = u$ in $C([0,T];X)$ exists, where u_λ are the strong solution of the approximating equation (V_λ). However, under our assumptions the solutions u_λ of (V_λ) converge to a limit u as $\lambda \downarrow 0$, whether or not (V) has a strong solution. For this reason we refer to the solution u of (V) of Theorem 3.5 as the generalized solution of V on $[0,T]$. Moreover, we note that if the assumption $b' \in BV[0,T]$ only holds on $[0,T_0]$ for some fixed $T_0 > 0$, then by a standard translation argument (see [55]) the solution can be extended to $[0,T]$.

(ii) A precise estimate giving the dependence of the generalized solution u of (V) on the data b and F is established in Theorem 5 of [26].

(iii) The assumption $b(0) = 1$ in Theorem 3.5 and in Proposition 3.1 is no loss of generality, provided $b(0) > 0$. For if $b(0) > 0$, defining $\tilde{b} = (b(0))^{-1}b$ and $\tilde{A} = b(0)A$ one has $b*Au = \tilde{b}*\tilde{A}u$.

(iv) Our method can be used to study the nonconvolution Volterra equation

$$u(t) + \int_0^t b(t,s)Au(s)ds \ni f(t) \qquad (0 < t < T) ,$$

where A and f are as in Theorem 3.5, provided the kernel b, which is defined on the region $\{(t,s) : 0 < s < t < T\}$, is sufficiently smooth and $b(t,t) > 0$. The technique for doing this is outlined in [26], and is carried out in detail by C. Rennolet [72]. For different nonconvolution equation results see Gripenberg [37], [38].

(v) In Theorem 3.5 above and 3.7 below it is important to note that the generalized (or strong) solution $u \in C([0,T];X)$, and therefore $u \in L^p(0,T;X)$ for $p > 1$.

(vi) If the assumptions of Theorems 3.5 above and 3.7 below are satisfied for every $T > 0$, then the conclusions hold on $[0,\infty)$.

(vii) The relation of Theorem 3.5 to other literature is explained in [26].

(viii) An interesting situation not covered by the theory discussed above arises if the kernel b in (V) has the property

$$\lim_{t \to 0^+} \frac{b(t) - b(0)}{t} = +\infty .$$

In the case $A = \partial\varphi$, where φ is convex, l.s.c. and proper, G. Gripenberg [36] extends the theory by replacing the assumption $b' \in BV[0,T]$ in Theorem 3.5 above by:

there exist $T_0 > 0$, $c_0 > 0$ such that if $0 < t < T_0$,
$\mathrm{Var}(b';[t,T_0]) \le c_0 \log t^{-1}$.

Recently G. Gripenberg [34, Theorem 2] obtained the following important generalization of Theorem 3.5 which will be used in Chapter 4. Such a result was established by Clément and Nohel [18] for the much simpler case $b_1 \equiv 0$ below and A a linear operator.

Theorem 3.7. Let the assumptions concerning f, A in Theorem 3.5 be satisfied. Let the kernel $b = b_1 + b_2$ in (V), where b_1 satisfies the assumptions of b in Theorem 3.5, and where $b_2 \in L^1(0,T)$ and b_2 is positive, nonincreasing, and $\log b_2$ is convex on $(0,T)$. Then the Volterra equation (V) has a generalized solution $u \in C([0,T];X)$.

Finally, we close this chapter with two important special cases. The first deals with the case of the operator A being maximal monotone on a real Hilbert space H and is a direct consequence of Theorem 3.5. Recall that if $X = H$, A is m-accretive iff A is maximal monotone.

Theorem 3.8. Let b satisfy the assumptions of Theorem 3.5 for every $T > 0$. Let A be maximal monotone on H. Let $f \in W^{1,2}_{loc}(0,\infty;H)$, $f(0) \in \overline{D(A)}$. Then (V) has a unique generalized solution $u \in C([0,\infty); \overline{D(A)})$. If, in addition $f' \in BV_{loc}([0,\infty);H)$, $f(0) \in D(A)$, then $u \in W^{1,2}_{loc}(0,\infty;H)$ and u is a strong solution of (V) on $[0,\infty)$.

Remark 3.9. By another theorem of Gripenberg [34, Theorem 1], this result also applies to (V) with kernels $b = b_1 + b_2$ which satisfy the assumptions of Theorem 3.7.

The second special case of Theorem 3.5 deals with $X = H$ a real Hilbert space and the operator $A = \partial\varphi$ (the subdifferential of φ), where $\varphi : H \to (-\infty,+\infty]$ is a proper, convex, l.s.c. function. The proof of the next result follows by combining Theorem 3.5 with known results for evolution equations (for details see [26, Section 4]).

Theorem 3.10. Let the kernel b satisfy the assumption of Theorem 3.5 for every $T > 0$. Let $A = \partial\varphi$, where $\varphi : H \to (-\infty,+\infty]$ is proper, l.s.c., and convex. Let $f \in W^{1,2}_{loc}(0,\infty;H)$. If $f(0) \in \overline{D(\varphi)}$, then (V) has a unique strong solution u on $[0,\infty)$ such that $\sqrt{t}\, u' \in L^2_{loc}(0,\infty;H)$; if $f(0) \in D(\varphi)$, then $u' \in L^2_{loc}(0,\infty;H)$.

Remark 3.11. In a different direction Kiffe and Stecher [48] study existence and uniqueness of $L^2(0,T)$ solutions of (V) in a Hilbert space setting. They assume that $f \in L^2(0,T;H)$ and they use techniques of Barbu [5] and Londen [54] to obtain their results without any differentiability assumptions on the forcing term f, but at the expense of drastically restricting the growth of the maximum monotone operator A in (V). In fact, this restriction rules the important possibility that A is a nonlinear differential operator in the spatial variables, and therefore, their results cannot be applied to the physical problem in Chapter 1.

An interesting and different variant of (V) was recently studied by Kiffe [49]. He obtains existence of global solutions of the equation

$$u + b*[Au + g(u)] \ni f \qquad (0 \le t \le T)$$

where A can be a nonlinear differential operator in the space variables and where the perturbation g is a discontinuous real function which is not necessarily monotone and satisfies certain growth conditions. The kernel b satisfies assumptions similar to those in this chapter; while $b' \in BV[0,T]$ is not assumed, certain monotonicity is required of b. The forcing term $f \in W^{1,2}$; the operator $A = \partial\varphi$, where the function φ is as in Theorem 3.10. The function φ satisfies a compactness assumption, and $f(0) \in D\varphi$.

Remark 3.12. Other interesting variants of the abstract equation (V), motivated by the heat flow problem formulated in Chapter 1, Section 2, have been studied by V. Barbu [8], [10], and by H. Attouch and A. Damlamian [4]. In [8,10] Barbu generalizes the dependence of the internal energy ε on the temperature; this leads him to study the equation

$$Bu + b*Au \ni f \qquad (0 \le t < \infty) ,$$

where B is a strictly monotone operator. In [4] the domain of the operator A in (V) is allowed to depend on time t; for the heat flow problem formulated in Chapter 1, Section 2, the temperature u is prescribed at each time t outside a body $\Omega(t)$ in x space. Assuming that $\Omega(t)$ depends smoothly on t, the temperature inside $\Omega(t)$ is determined.

Remark 3.13. While the results are still rather incomplete, an interesting study of numerical approximations of solutions of (V) has been initiated by Mac Camy and Weiss [61] where other references to numerical literature may be found.

Chapter 3

Boundedness and Asymptotic Behaviour by Energy Methods

3.1. Introduction. The purpose of this chapter is to discuss the boundedness and asymptotic behaviour as $t \to \infty$ of solutions of the nonlinear Volterra equation

$$\text{(V)} \qquad u(t) + (b*Au)(t) \ni f(t) \qquad (0 \le t < \infty) .$$

The setting for (V) is $b : [0,\infty) \to R$ is a given kernel, A is a (possibly multivalued) maximal monotone operator on a real Hilbert space H, and $f : [0,\infty) \to H$ is a given function. The exposition is largely based on a forthcoming paper by Clément, Mac Camy, and Nohel [20].

The following general assumptions will be assumed throughout:

$$(H_b) \qquad b(t) = b_\infty + B(t), \; b(0) > 0, \; b_\infty \ge 0, \; B,B' \in L^1(0,\infty) \; ;$$

$$(H_m) \qquad A \text{ maximal monotone on } H \; ;$$

$$(H_f) \qquad f(t) = f_\infty + F(t), \; F \in W^{1,2}_{loc}([0,\infty;H)), \; F' \in L^2(0,\infty;H), \; f_\infty \in H \; ;$$

here $' = d/dt$, H is a real Hilbert space with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$, and W denotes the usual Sobolev space. The special case of (H_m):

$$(H_\varphi) \qquad \begin{cases} A = \partial\varphi, \text{ where the function } \varphi : H \to (-\infty,\infty] \text{ is convex,} \\ \text{lower semicontinuous, and proper} \end{cases}$$

will also play an important role in the theory. For definitions and standard results concerning maximal monotone operators and the special case of a subdifferential the reader is referred to Brézis [14].

We remark that if one adds the assumption $B' \in BV_{loc}[0,\infty)$ to assumptions (H_b), (H_m), and (H_f), then by Theorem 3.8 of Chapter 2. (V) has a unique generalized solution $u \in C([0,\infty); \overline{D(A)})$, provided $f(0) \in \overline{D(A)}$; if also $F' \in BV_{loc}[0,\infty;H)$ and $f(0) \in D(A)$, then u is a strong solution of (V) on $[0,\infty)$. If in place of (H_m) assumption (H_φ) is satisfied, and if $f(0) \in \overline{D(\varphi)}$, then by Theorem 3.10 of Chapter 2, (V) has a unique strong solution u on $[0,\infty)$ such that $tu' \in L^2_{loc}(0,\infty;H)$; if $f(0) \in D(\varphi)$, then $u' \in L^2_{loc}(0,\infty;H)$.

The results on boundedness and asymptotic behaviour of solutions of (V) will be derived from a priori estimates obtained from the equivalent differentiated form of (V):

$$\text{(V')} \qquad \frac{du}{dt} + b(0)Au + B'*Au \ni F' \qquad (0 < t < \infty), \quad u(0) = f(0) .$$

We shall distinguish two cases: (i) A satisfies assumption (H_m) and (ii) $A = \partial\varphi$ with φ satisfying (H_φ). Case (i) is developed in Section 2, while Case (ii) is treated in Section 3; in each case a different energy method is used to deduce suitable a priori estimates under appropriate additional assumptions on the kernel b and the forcing term f. A number of examples illustrating each situation is presented. In particular, the theory developed in Section 3 is used in Section 4 to analyse the boundedness and asymptotic behaviour of solutions of the heat flow problem (1.4), Chapter 1, under the physically reasonable assumptions motivated in Chapter 1, Sec. 2.

The reader should note that if Assumption (H_φ) is satisfied, and if u is the strong solution of (V) of Theorem 3.10, Chapter 2, then (see Brézis [14, Lemma 3.3] $\varphi(u(t))$ is absolutely continuous and one has the "chain rule":

$$\frac{d}{dt}\varphi(u(t)) = (w, \frac{du}{dt}) \qquad (w \in \partial\varphi(u)(t)) .$$

Thus a plausible energy method for the case $A = \partial\varphi$ consists of taking the scalar product of (V') by $w \in \partial\varphi(u(t))$ for any solution u and integrating over an arbitrary interval $(0,T)$. Indeed, this method is used in Section 3. Unfortunately, there is no analogue for the chain rule when A is a maximal monotone operator, but $A \neq \partial\varphi$ for some proper, convex, l.s.c. function φ. For this reason the development of the theory in Section 2 is less direct in that the a priori estimates are derived from an equivalent equation to (V') resulting essentially from applying Proposition 3.1 and Remark 3.6 (iii), Chapter 2, to (V'), and then using a different energy method to obtain the a priori estimates.

It should be noted that Corollaries 2.4, 2.5, and Theorem 2.6 of Section 2 may be viewed as natural generalizations to Hilbert space of earlier results of Levin [51] and Londen [53] which describe the limiting behaviour as $t \to \infty$ of solutions of (V) in the scalar case in which the operator A is a real function.

2. <u>Boundedness and Asymptotic Properties when</u> A <u>is Maximal Monotone</u>. Throughout this section we assume that assumptions (H_b), (H_m), and (H_f) are satisfied and that u is a strong or generalized solution of the Volterra equation (V) on $[0,\infty)$. As explained in Chapter 2 (V) is equivalent to the Cauchy problem

$$\text{(V')} \qquad \frac{du}{dt} + b(0)Au + B'*Au \ni F' \quad (0 \le t < \infty), \quad u(0) = f(0) .$$

Let k be the resolvent kernel associated with B', defined to be the unique solution of the linear Volterra equation

$$\text{(k)} \qquad b(0)k(t) + (B'*k)(t) = -\frac{B'(t)}{b(0)} \quad \text{a.e. for } 0 < t < \infty ;$$

by standard results, Miller [63], assumption (H_b) implies that $k \in L^1_{loc}(0,\infty)$.

We now use the method of Proposition 3.1 and Remark 3.6 (iii) of Chapter 2 to transform (V'). Regarding (V') as a "linear" equation for Au, the variation of constants formula for Volterra equations [63] and an integration by parts show that (V') (and hence also (V)) is equivalent to the Cauchy problem

$$\text{(2.1)} \qquad \frac{1}{b(0)}\frac{du}{dt} + \frac{d}{dt}(k*u) + Au \ni f_1 \quad (0 < t < \infty), \quad u(0) = f(0) ,$$

where $f_1 : [0,\infty) \to H$ is the function given by either

$$\text{(2.2)} \qquad f_1(t) = \frac{1}{b(0)} F'(t) + f(0)k(t) + (k*F')(t) \quad (0 < t < \infty)$$

or

$$\text{(2.3)} \qquad f_1(t) = \frac{1}{b(0)} F'(t) + k(0)f(t) + (k'*f)(t) \quad (0 < t < \infty) .$$

We shall use an energy method based on taking the scalar product of (2.1) by u, and also by $\sqrt{t}\, u$, and we obtain a priori estimates by integrating over an arbitrary interval $[0,T]$. We will first state the general result for (2.1) and then interpret it for (V).

<u>Theorem 2.1.</u> <u>Let</u> u <u>be a strong or generalized solution of the Cauchy problem</u> (2.1) <u>on</u> $[0,\infty)$. <u>Let</u> $T > 0$ <u>be given and let there exist constants</u> $\varepsilon, \eta \in R$ <u>such that</u>

$$\text{(2.4)} \qquad \underline{\text{if}}\ v \in Au,\ \underline{\text{then}} \int_0^T (v,u)dt \ge \varepsilon \int_0^T |u|^2 dt \ (u \in D(A)) ,$$

$$\text{(2.5)} \qquad \underline{\text{for every}}\ w \in L^2(0,T;H) \int_0^T \left(w(t), \frac{d}{dt}(k*w)(t)\right)dt \ge \eta \int_0^T |w|^2 dt ,$$

(2.6) $$\varepsilon + \eta > 0 .$$

(a) If $f_1 \in L^2(0,\infty;H)$, then $u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$;

(b) If also $\sqrt{t}\, k' \in L^1(0,\infty)$ and $\sqrt{t}\, f_1 \in L^2(0,\infty;H)$, then $\sqrt{t}\, u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$. Consequently, $|u(t)| = O(\frac{1}{\sqrt{t}})$ as $t \to \infty$ and $u(t) \to 0$ strongly as $t \to \infty$.

We remark that no claim is made that the rate $|u(t)| = O(\frac{1}{\sqrt{t}})$ as $t \to \infty$ is optimal.

The coercivity assumption (2.4) concerning the maximal monotone operator A is natural for the problem in light of comparable assumptions in evolution equations. Assumption (2.5) and the hypotheses concerning k, k' will be justified in Lemmas 2.2, 2.3, below. Two different classes of kernels b in (V) are considered, each of which lead to the energy inequality (2.5), the first with $\eta = 0$, the second with $\eta > 0$, and for each of which $\sqrt{t}\, k' \in L^1(0,\infty)$. These technical lemmas, together with appropriate assumptions on the forcing function f in (V), permit an easy interpretation of Theorem 2.1 for solutions of (V). This will be done in Corollaries 2.4 and 2.5 below. The proof of Lemma 2.2 appears in Appendix 1. Lemma 2.3 is an extension of a result of Mac Camy [59] which in its present form was recently established by M. Tangredi [82].

Lemma 2.2. (a) Let b satisfy assumption (H_b) with $b_\infty > 0$, and let b satisfy the frequency domain condition

(F) there exists $\delta > 0$ such that $b_\infty + \operatorname{Inf}_{\eta \in \mathbb{R}} [-\eta \operatorname{Im} \hat{B}(i\eta)] > \delta$, where $\hat{B}(i\eta) = \int_0^\infty \exp(-i\eta t) B(t)\,dt$. Then the resolvent kernel k of B' satisfies $k \in L^1(0,\infty)$.

(b) If also $B' \in L^2(0,\infty)$, then $k \in L^2(0,\infty)$; if also $B'' \in L^1(0,\infty)$, then $k' \in L^1(0,\infty)$.

(c) If the assumptions of (a) are satisfied, $B'' \in L^1(0,\infty)$, and B is a kernel of positive type on $[0,\infty)$, then for every $T > 0$ and for every $w \in L^2(0,T)$

$$\int_0^T w(t) \frac{d}{dt} (k*w)(t)\,dt \geq 0 .$$

(d) If the assumptions of (a) and (b) are satisfied, and $\sqrt{t}\, B' \in L^1(0,\infty) \cap L^2(0,\infty)$, $\sqrt{t}\, B'' \in L^1(0,\infty)$, then $\sqrt{t}\, k \in L^1(0,\infty) \cap L^2(0,\infty)$, and $\sqrt{t}\, k' \in L^1(0,\infty)$.

Lemma 2.3. Let b satisfy assumption (H_b) with $b_\infty = 0$, and let

(i) $t^j B^{(m)} \in L^1(0,\infty)$ $\quad (j = 0,1,2;\ m = 0,1,2,3)$, $\ t^3 B \in L^1(0,\infty)$,

(ii) B be strongly positive on $[0,\infty)$.

Let k be the resolvent kernel of B'. Then:

(a) $k \in C^1[0,\infty)$;

(b) $k(t) = k_\infty + K(t)$, $\ k_\infty = (\int_0^\infty B(t)dt)^{-1} > 0$, $\ K^{(m)} \in L^1(0,\infty)$ $\quad (m = 0,1,2)$;

(c) if also $B, B', \sqrt{t}\, B, \sqrt{t}\, B' \in L^2(0,\infty)$ one has $K, \sqrt{t}\, K \in L^2(0,\infty)$;

(d) for every $T > 0$ and for every $w \in L^2(0,T)$ there exists $\eta > 0$ such that

$$\int_0^T w(t) \frac{d}{dt}(k*w)(t)dt \geq \eta \int_0^T |w(t)|^2 dt \ ;$$

(e) if assumptions (i) hold for $j,m = 0,1,2,3$, $t^4 B \in L^1(0,\infty)$, and assumptions (ii) hold, one has $\sqrt{t}\, k' \in L^1(0,\infty)$.

We shall mention some examples of kernels b which satisfy the assumptions of Lemmas 2.2 and 2.3.

Let

(2.7) $\qquad B : [0,\infty) \to R^+$ be positive, nonincreasing, and convex

and satisfy the smoothness and integrability assumptions in (H_b). Then B is a kernel of positive type on $[0,\infty)$ (see [67]), and

$$-\eta \operatorname{Im}\hat{B}(i\eta) = \eta \int_0^\infty \sin \eta t\ B(t)dt \geq 0 \quad (\eta \in R) .$$

Thus if $b_\infty > 0$ is any constant, $b(t) = b_\infty + B(t)$ satisfies the frequency domain condition (F) with $\delta = b_\infty$, and (see Lemma 2.2(a)) $k \in L^1(0,\infty)$. If, in addition, B satisfies the remaining smoothness and integrability assumptions of Lemma 2.2, all conclusions of Lemma 2.2, and assumptions (2.5) with $\eta = 0$, and $\sqrt{t}\, k' \in L^1(0,\infty)$ of Theorem 2.1 are satisfied.

Consider again B in (2.7). In addition, assume that

(2.8) the measure dB' has a nonzero absolutely continuous part;

then (see [67, Corollary 2.2]), B is strongly positive on $[0,\infty)$ (for example, $B \in C[0,\infty)$, $(-1)^k B^{(k)}(t) \geq 0$, $0 < t < \infty$, $k = 0,1,2$, $B'(t) \not\equiv 0$). Thus if B satisfies (2.7), (2.8), and the integrability and smoothness assumptions of Lemma 2.3, and if $b(t) \equiv B(t)$ ($b_\infty = 0$), then all conclusions of Lemma 2.3, assumptions (2.5) with $\eta > 0$, and $\sqrt{t}\, k' \in L^1(0,\infty)$ of Theorem 2.1 are satisfied.

Next, consider

$$B(t) = \sum_{j=1}^{m} B_j e^{-\lambda_j t} \cos w_j t \quad (B_j \geq 0,\ \lambda_j \geq 0,\ w_j \in R) \tag{2.9}$$

with strict inequalities holding for at least one j (if $w_j = 0$, $j = 1,\ldots,m$, B satisfies both (2.7), (2.8)). This function B is strongly positive on $[0,\infty)$ (see [67]), since by direct calculation

$$\operatorname{Re} \hat{B}(i\eta) = \frac{1}{2} \sum_{i=1}^{m} B_j \lambda_j \left(\frac{1}{\lambda_j^2 + (\eta - w_j)^2} + \frac{1}{\lambda_j^2 + (\eta + w_j)^2}\right) \quad (\eta \in R) .$$

Moreover, B satisfies all other assumptions of Lemma 2.3. Thus if $b(t) = B(t)$ ($b_\infty = 0$), all conclusions of Lemma 2.3, assumptions (2.5) with $\eta > 0$ and $\sqrt{t}\, k' \in L^1(0,\infty)$ of Theorem 2.1 are satisfied.

For the kernel B in (2.9) one has

$$-\eta \operatorname{Im} \hat{B}(i\eta) = \sum_{i=1}^{m} B_j \frac{\eta^2(\eta^2 + \lambda_j^2 - w_j^2)}{(\lambda_j^2 + w_j^2 - \eta^2)^2 + 4\lambda_j^2\eta^2} \quad (\eta \in R) .$$

Thus $b(t) = b_\infty + B(t)$, where $b_\infty > 0$ is any constant, satisfies the frequency domain condition (F) of Lemma 2.2 if $\lambda_j \geq w_j$ ($j = 1,\ldots,m$). Evidently, b is a kernel of positive type on $[0,\infty)$. Therefore, if $b(t) = b_\infty + B(t)$, $b_\infty > 0$, B defined by (2.9) with

$\lambda_j > w_j$ $(j = 1,\dots,m)$, all conclusions of Lemma 2.2 (but not of Lemma 2.3), assumptions (2.5) with $\eta = 0$ and $\sqrt{t}\, k' \in L^1(0,\infty)$ of Theorem 2.1 are satisfied.

Incidentally, if $b_\infty > 0$ is any constant, and if

$$B(t) = \sum_{j=1}^{m} B_j e^{-\lambda_j t} \sin w_j t \quad (\lambda_j > 0,\ w_j > 0) \tag{2.10}$$

with strict inequalities holding for at least one j, then the frequency domain condition (F) of Lemma 2.2 is satisfied with $\delta = b_\infty$. However, such a kernel b is not of positive type.

Lemmas 2.2 combined with appropriate assumptions on A and f yield the following easy interpretation of Theorem 2.1 for solutions of (V).

Corollary 2.4. Let assumptions (H_b) with $b_\infty > 0$, (H_m) and (H_f) with f_∞ arbitrary be satisfied. In addition, assume that b satisfies the hypotheses of Lemma 2.2, and that $\sqrt{t}\, F' \in L^2(0,\infty;H)$. Let u be a strong or generalized solution of (V) on $[0,\infty)$. If the coercivity assumption (2.4) holds with $\epsilon > 0$, then u and $\sqrt{t}\, u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$ and $u(t) \to 0$ strongly as $t \to \infty$.

Indeed, define f_1 by (2.2). By (H_f) and Lemma 2.2 $(k \in L^1(0,\infty) \cap L^2(0,\infty)$, $k' \in L^1(0,\infty))$ one trivially has $f_1 \in L^2(0,\infty;H)$. By Lemma 2.2 one also has $\sqrt{t}\, k \in L^1(0,\infty) \cap L^2(0,\infty)$ and $\sqrt{t}\, k' \in L^1(0,\infty)$. These together with the assumption $\sqrt{t}\, F' \in L^2(0,\infty;H)$ used in (2.2) show that $\sqrt{t}\, f_1 \in L^2(0,\infty;H)$; the fact that $\sqrt{t}\,(k*F') \in L^2(0,\infty;H)$ in (2.2) follows from the straightforward estimate

$$\int_0^T t\left|\int_0^t k(t-s)F'(s)\,ds\right|^2 dt \le 2\|k\|^2_{L^1(0,\infty)} \|\sqrt{t}\, F'\|^2_{L^2(0,\infty;H)} + 2\|\sqrt{t}\, k\|^2_{L^1(0,\infty)} \|F'\|^2_{L^2(0,\infty;H)} \quad (\forall T > 0)\ .$$

By Lemma 2.2 again, (2.5) holds with $\eta = 0$. Thus if $\epsilon > 0$ in (2.4), the result of Corollary 2.4 follows by applying Theorem 2.1.

Lemma 2.3 combined with appropriate assumptions on A and f yield a different interpretation of Theorem 2.1 for solutions of (V).

Corollary 2.5. Let assumptions (H_b) with $b_\infty = 0$, (H_m) and (H_f) with $f_\infty = 0$ be satisfied. In addition, assume that $b(t) = B(t)$ satisfies the assumptions of Lemma 2.3, and that $f \equiv F$ also satisfies $f, \sqrt{t}\, f, \sqrt{t}\, f' \in L^2(0,\infty;H)$. Let u be a strong or generalized solution of (V) on $[0,\infty)$. If the operator A satisfies the coercivity assumption (2.4) with $\varepsilon > 0$ (or even $\varepsilon > -\eta$, where $\eta > 0$ is the constant in Lemma 2.3d), then u and $\sqrt{t}\, u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$, and $u(t) \to 0$ strongly as $t \to \infty$.

The proof of Corollary 2.5 is similar to that of Corollary 2.4, except that f_1 must now be defined by (2.3), and Lemma 2.3 is used in place of Lemma 2.2. Note also that the additional assumptions concerning $f, \sqrt{t}\, f$ are essential.

The important case $b_\infty = 0$ in (H_b), $b \equiv B$ satisfying the assumptions of Lemma 2.3, and $f'_\infty \neq 0$ in (H_f) is not covered by Corollary 2.5. In this situation Theorem 2.1 must be modified in the following manner.

Theorem 2.6. Let the assumptions (H_b) $(b_\infty = 0)$, (H_m), (H_f) with f_∞ arbitrary, and the assumptions of Lemma 2.3 be satisfied. In addition, assume that $F, \sqrt{t}\, F, \sqrt{t}\, f' \in L^2(0,\infty;H)$. Let u be a strong or generalized solution of (V) on $[0,\infty)$, let u_∞ be the unique solution of the limit equation corresponding to (V):

$$(V_L) \qquad u_\infty + \left(\int_0^\infty B(t)dt\right)Au_\infty \ni f_\infty .$$

Let the operator A satisfy the coercivity condition:

$$(2.11) \qquad \begin{cases} \text{if } v \in Au \text{ and } v_\infty \in Au_\infty \text{ and } T > 0, \text{ then} \\ \int_0^T (v(t) - v_\infty, u(t) - u_\infty)dt \geq \varepsilon \int_0^T |u(t) - u_\infty|^2 dt \\ \text{for some } \varepsilon > 0 \ (\varepsilon > -\eta \text{ is sufficient; see Lemma 2.3d}) . \end{cases}$$

Then $u - u_\infty$ and $\sqrt{t}\,(u - u_\infty) \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$; consequently $u(t) \to u_\infty$ strongly as $t \to \infty$ and $|u(t) - u_\infty| = O(\frac{1}{\sqrt{t}})$ as $t \to \infty$.

Remark 2.7. Since $b \equiv B$ satisfies the hypothesis of Lemma 2.3, B is strongly positive on $[0,\infty)$, and therefore $\int_0^\infty B(t)dt > 0$. Since the operator A is maximum monotone on H the limit equation (V_L) has a unique solution for any $f_\infty \in H$; in particular, if $f_\infty = 0$, $u_\infty = 0$ and in this case Theorem 2.6 reduces to Corollary 2.5.

Corollaries 2.4, 2.5 and Theorem 2.6 together form the natural generalization to Hilbert space of corresponding scalar results for (V) due to Levin [51] and Londen [53].

Sketch of Proof of Theorem 2.1. (a) Take the scalar product of (2.1) with u and integrate from 0 to T. Using (2.4), (2.5) we obtain

$$\frac{1}{2b(0)}\,|u(T)|^2 + (\eta+\varepsilon)\int_0^T |u(t)|^2 dt \le \frac{1}{2b(0)}\,|f(0)|^2 + \int_0^T (f_1(t),u(t))dt\ .$$

Since $\eta + \varepsilon > 0$, and $f_1 \in L^2(0,\infty;H)$, the assertion (a) follows by standard estimates. (b) Next take the scalar product of (2.1) with tu and integrate from 0 to T. An integration by parts yields

$$\begin{aligned}(2.12)\qquad &\frac{T}{2b(0)}\,|u(t)|^2 + \int_0^T t\Big(u(t), \frac{d}{dt}(k*u)(t)\Big)dt + \varepsilon\int_0^T t|u(t)|^2 dt\\ &\le \frac{1}{2b(0)}\int_0^T |u(t)|^2 dt + \int_0^T (tf_1(t),u(t))dt\ .\end{aligned}$$

A straightforward calculation shows that (see [20])

$$\int_0^T t\Big(u(t), \frac{d}{dt}(k*u)(t)\Big)dt = I + J\ ,$$

where

$$I = \int_0^T \Big(\sqrt{t}\,u(t), \frac{d}{dt}(k*\sqrt{t}\,u)(t)\Big)dt \ge \eta\int_0^T t|u(t)|^2 dt\ ,$$

and where

$$J = \int_0^T \Big(\sqrt{t}\,u(t), \int_0^t k'(t-\tau)(\sqrt{t}-\sqrt{\tau})u(\tau)d\tau\Big)dt\ .$$

Using $\sqrt{t} - \sqrt{\tau} < \sqrt{t-\tau}$, $0 < \tau < t$, $u \in L^2(0,\infty; H)$, and $\sqrt{t}\, k' \in L^1(0,\infty)$ yields

$$|J| < \left(\int_0^T t|u(t)|^2 dt\right)^{1/2} \|u\|_{L^2(0,\infty;H)} \int_0^\infty \sqrt{t}|k'(t)|dt .$$

Substitution of the estimates for I, J in (2.12) gives the final inequality

$$\frac{T}{2b(0)} |u(t)|^2 + (\eta + \varepsilon) \int_0^T t|u(t)|^2 dt < \frac{1}{2b(0)} \|u\|^2_{L^2(0,\infty;H)}$$
$$+ \|u\|_{L^2(0,\infty;H)} \|\sqrt{t}\, k'\|_{L^1(0,\infty)} \left(\int_0^T t|u(t)|^2 dt\right)^{1/2} + \|\sqrt{t}\, f_1\|_{L^2(0,\infty;H)} \|u\|_{L^2(0,\infty;H)} .$$

The conclusion $\sqrt{t}\, u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$ follows by standard estimates using $\varepsilon + \eta > 0$, $u \in L^2(0,\infty;H)$ by (a), and the assumptions $\sqrt{t}\, k' \in L^1(0,\infty)$ and $\sqrt{t}\, f_1 \in L^2(0,\infty;H)$. This completes the sketch of the proof.

<u>Proof of Theorem 2.6</u>. The proof will be reduced to that of Theorem 2.1 by the following steps. First by Lemma 2.3 $\int_0^\infty B(t)dt = k_\infty^{-1} > 0$. Therefore, the limit equation (V_L) can be written in the form

$$k_\infty u_\infty + Au_\infty \ni k_\infty f_\infty ,$$

which is the same as

(2.13) $$\frac{1}{b(0)} \frac{d}{dt} u_\infty + \frac{d}{dt} (k*u_\infty) + Au_\infty \ni k_\infty f_\infty + (k(t) - k_\infty)u_\infty .$$

Next, subtracting (2.13) from (2.1) gives

(2.14) $$\frac{1}{b(0)} \frac{d}{dt} (u - u_\infty) + \frac{d}{dt} k*(u - u_\infty) + Au - Au_\infty \ni F_1(t) \quad (0 < t < \infty) ,$$

where by an elementary calculation

(2.15) $$F_1(t) = \frac{1}{b(0)} F'(t) + k(0)F(t) + K(t)f_\infty + (k'*F)(t) - u_\infty K(t) .$$

Lemma 2.3 and the assumptions concerning F clearly imply that F_1 satisfies the same assumptions as f_1 in Theorem 2.1. The method of proof of Theorem 2.1 applied to (2.14), (2.15), where the coercivity assumption (2.11) is used in place of (2.4), now yields the needed a priori estimates for $u - u_\infty$ and $\sqrt{t}\,(u - u_\infty)$, and completes the proof.

<u>Example 2.8</u>. We give an example of a maximum monotone operator A in (V) which is not a subdifferential, and for which the theory developed in this section is applicable. Let

$\Omega \subset R^n$ be a bounded open set with smooth boundary $\partial\Omega$. Let H be the Hilbert space $L^2(\Omega)$. Let β be a maximum monotone graph with $0 \in \beta(0)$ and with primitive j (i.e. $\beta = \partial j$). Let A_1 be the operator defined by

$$D(A_1) = \{u : u \in H_1^0(\Omega) \cap H^2(\Omega),\ \beta(u) \in L^2(\Omega)\} ,$$

$$A_1 u = -\Delta u + \beta(u) \qquad (u \in D(A_1)) .$$

It is clear [15] that A_1 is maximum monotone on H since $A_1 = \partial\varphi_1$, where $\varphi_1 : H \to (-\infty,\infty]$ is the proper, convex, l.s.c. function given by

$$\varphi_1(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega j(u)dx & \text{if } u \in H_0^1(\Omega) \text{ and } j(u) \in L^1(\Omega) \\ +\infty & \text{otherwise} . \end{cases}$$

Define $L(u) = \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}$ $(b_i \in R,\ u \in H_0^1(\Omega))$. By the divergence theorem $L(u)$ is monotone and $(u,L(u)) = \int_\Omega uL(u)dx = 0$. Finally, following Pazy [70, Ex. 3.5] define

$$A = A_1 + L .$$

By a perturbation theorem of Crandall and Pazy [27], A is maximum monotone on $H = L^2(\Omega)$, and by an easy calculation using Green's theorem and the Poincaré inequality *there exists a constant* $\varepsilon > 0$ *such that*

$$(Au,u) = (A_1u,u) = -\int_\Omega u\Delta u dx + \int_\Omega u\beta(u)dx \geq \int_\Omega |\nabla u|^2 dx \geq \varepsilon \|u\|^2_{L^2(\Omega)} .$$

Thus A satisfies the coercivity assumption (2.4) for every $T > 0$.

Remark 2.9. The concept of strong positivity of a kernel plays an important role in stability theory for Volterra equations (see Halanay [40], Nohel and Shea [67], Staffans [74], [75], [76], [77], [78], Crandall, Londen and Nohel [28]). As we have seen kernels B satisfying (2.7), (2.8), as well as oscillatory kernels B of the form (2.9) are strongly positive on $[0,\infty)$. If $b(t) = b_\infty + B(t)$ where $b_\infty > 0$ and B is strongly positive on $[0,\infty)$ one can extract some information about the behaviour of solutions of (V) as $t \to \infty$. In particular, one can establish $u \in L^2(0,\infty;H)$ (Theorem 2.1 part (a)) by another energy method directly from (V):

Proposition 2.10. Let the general assumptions (H_b), (H_m), (H_f) be satisfied. Let u be a strong or generalized solution of (V) on $[0,\infty)$. If $F \in L^2(0,\infty;H)$, if B is strongly positive on $[0,\infty)$, if the coercivity assumption (2.4) is satisfied with $\varepsilon > 0$, and if $f_\infty = 0$ whenever $b_\infty = 0$, then $u \in L^2(0,\infty;H)$.

Proof. Let $v \in Au$ and let $0 < T < \infty$ be given. Take the scalar product of (V) by v and integrate from 0 to T obtaining

$$(2.16) \qquad \int_0^T (v(t),u(t))dt + \frac{b_\infty}{2} \left|\int_0^T v(t)dt\right|^2 + Q_B[v;T] \le \int_0^T (v(t),f(t))dt ,$$

where

$$Q_B[v;T] = \int_0^T (v(t),(B*v)(t))dt .$$

Since $B \in L^1(0,\infty)$ is strongly positive and $F,F' \in L^1(0,\infty;H)$, a result of Staffans [70, Proposition 4.1] shows that there exists a constant $\gamma > 0$ such that

$$\left|\int_0^T (v(t),F(t))dt\right| \le \gamma\{Q_B[v;T]\}^{1/2} .$$

Using this, the coercivity assumption (2.4) $(\varepsilon > 0)$, and the obvious estimate

$$\left|\int_0^T (v(t),f_\infty)dt\right| \le \frac{b_\infty}{4} \left|\int_0^T v(t)dt\right|^2 + \frac{1}{b_\infty} |f_\infty|^2 \quad (b_\infty > 0)$$

in (2.16) yields the inequalities

$$(2.17) \quad \varepsilon \int_0^T |u(t)|^2 dt + \frac{b_\infty}{4} \left|\int_0^T v(t)dt\right|^2 + Q_B[v;t] \le \gamma\{Q_B[v;T]\}^{1/2} + \frac{|f_\infty|^2}{b_\infty} \quad (b_\infty > 0)$$

and

$$\varepsilon \int_0^T |u(t)|^2 dt + Q_B[v;T] \le \gamma\{Q_b[v;T]\}^{1/2} \qquad (b_\infty = 0,\ f_\infty = 0) \ . \tag{2.18}$$

The result of Proposition 2.9 now follows from (2.17), (2.18) by completing the square on the Q_B terms.

Incidentally, we have also shown that

$$\sup_{T>0} Q_B[v;T] < \infty \qquad (v \in Au) \ , \tag{2.19}$$

and if $b_\infty > 0$, also that

$$\sup_{T>0} \left|\int_0^T v(t)dt\right| < \infty \ .$$

It follows from (2.19) using another result of Staffans (see Crandall, Nohel, Londen [28, Lemma 3.1] in a Hilbert space setting) that

$$\sup_{T>0} |(B^*v)(T)| < \infty \qquad (v \in Au) \ .$$

Unfortunately, there appears to be no direct way to establish also that $u(t)$ is uniformly continuous on $[0,\infty)$ if assumption (H_m) is satisfied and $A \neq \partial\varphi$ (if $A = \partial\varphi$ this can be done as in Theorem 3.1 below). The uniform continuity together with $u \in L^2(0,\infty;H)$ would imply that $u \to 0$ as $t \to \infty$ strongly in H. This provides at least one motivation for the indirect method of Theorem 2.1.

3. Boundedness and Asymptotic Properties When $A = \partial\varphi$. Let the general assumptions (H_b), (H_φ), (H_f) be satisfied and let u be a strong or generalized solution of (V) or $(0,\infty)$. In this section we shall obtain different boundedness and asymptotic results for the case $A = \partial\varphi$, and when $b_\infty > 0$ in (H_b). These results which are motivated by the physical problem discussed in Chapter 1, Section 2 are deduced from a priori estimates which are

obtained directly from the equivalent Cauchy problem;

(V') $$\frac{du}{dt} + b(0)Au + B' * Au \ni F' \quad (0 < t < \infty), \quad u(0) = f(0) .$$

Theorem 3.1. *Let the general assumptions* (H_b) *with* $b_\infty > 0$, (H_φ), (H_f) *be satisfied and let* u *be a strong or generalized solution of* (V) *on* $[0,\infty)$. *If the kernel* b *satisfies the frequency domain condition* (F) *of Lemma 2.2 and if*

(3.1) $$\inf_{z\in H} \varphi(z) > -\infty ,$$

then

(3.2) $$\sup_{0<t<\infty} \varphi(u(t)) < \infty ;$$

if $v \in \partial\varphi(u)$, *then*

(3.3) $$v \in L^2(0,\infty;H) ,$$

(3.4) $$\frac{du}{dt} \in L^2(0,\infty;H) ,$$

and

(3.5) u *is strongly uniformly continuous on* $[0,\infty)$.

If also $\lim_{|u|\to\infty} \varphi(u) = +\infty$, *then*

(3.6) $$\sup_{0<t<\infty} |u(t)| < \infty ,$$

and

(3.7) $$\lim_{t\to\infty} \varphi(u(t)) = \varphi_\infty = \inf_{z\in H} \varphi(z) \text{ exists} .$$

Moreover, if the inclusion $\partial\varphi(w) \ni 0$ *implies* $w = 0$, *then*

(3.8) $$u(t) \longrightarrow 0 \text{ (weakly) as } t \to \infty .$$

The frequency domain condition (F) is satisfied by several classes of kernels b with $b_\infty > 0$ as was seen in Section 2 (see examples of $b = b_\infty + B$ with B given by (2.7), (2.9), (2.10)). Thus Theorem 3.1 generalizes a recent result of S. O. Londen [54, Corollary 2] and a result of V. Barbu [6, Theorem 2].

The assumptions concerning φ in Theorem 3.1 are not sufficient to obtain strong convergence of $u(t)$ to zero as $t \to \infty$. For this result one needs the coercivity condition (2.4) with $\varepsilon > 0$. If (2.4) is satisfied with $v \in \partial\varphi(u)$ it is a standard result (see Brézis [14]) that the inclusion $\partial\varphi(w) \ni 0$ has $w = 0$ as the only solution, and that $0 \in D(\partial\varphi)$. Then the definition of the subdifferential [14] implies that

$$(u) > \varphi(0) \qquad (u \in H) ,$$

and therefore assumption (3.1) of Theorem 3.1 holds. This motivates the following results which complement Corollary 2.4 for the case $A = \partial\varphi$. Note that in Theorem 3.2 below only the frequency domain condition (F), but not the assumption that B is a kernel of positive type (see Lemma 2.2), is needed. Also note that here the assumption on F is less restrictive.

Theorem 3.2. Let the general assumptions (H_b) with $b_\infty > 0$, (H_φ), (H_f) be satisfied, and let u be a strong or generalized solution of (V) on $[0,\infty)$. Let b satisfy the frequency domain condition (F), and for $v \in \partial\varphi(u)$ let the coercivity condition (2.4) with $\varepsilon > 0$ be satisfied. Then conclusions (3.2)-(3.5) of Theorem 3.1 hold, and $u \in L^2(0,\infty;H)$, which implies that $u(t) \to 0$ strongly as $t \to \infty$.

Remark 3.3. If $b(t) \equiv b_\infty > 0$ in (V), a case not excluded in Theorems 3.1 and 3.2, the above theorem and its proof yield a simple boundedness and asymptotic behaviour result for the evolution equation

$$\frac{du}{dt} + b_\infty \partial\varphi(u) \ni g, \quad u(0) = u_0 ,$$

where $g = F'$; compare Brézis [14, Theorem 3.11] where $g \in L^1(0,\infty;H)$.

Remark 3.4. If the coercivity condition (2.4) with $\varepsilon > 0$ and $A = \partial\varphi$ is replaced by the more general condition: for every $T > 0$ there exists $\varepsilon > 0$ such that

$$(2.4') \qquad \text{if } v \in \varphi(u), \text{ then } \int_0^T (v(t),u(t) - z)dt > \int_0^T |u(t) - z|^2 dt$$

for some $z \in H$, then it is easy to show that the inclusion $\partial\varphi(w) \ni 0$ has $w = z$ as the only solution, and that

$$\varphi(u) > \varphi(z) \qquad (u \in H) .$$

Then the method of proof of Theorem 3.2 easily yields that $u(t) \to z$ strongly as $t \to \infty$.

Remark 3.5. In Theorem 3.1 and 3.2 the assumption $b_\infty > 0$ in (H_b) is crucial; if $b_\infty = 0$ the frequency domain condition (F) cannot be satisfied (see examples (2.7), (2.9), (2.10)) for any $\delta > 0$. On the other hand, in these theorems f_∞ in (H_f) is arbitrary and the case $f_\infty = 0$ is not ruled out, provided $b_\infty > 0$. If $b_\infty = 0$ in (H_b), one can, of course, still apply Corollary 2.5 if $f_\infty = 0$, and Theorem 2.6 if $f_\infty \neq 0$, with $A = \partial\varphi$.

Proof of Theorem 3.1. (a) Let $0 < T < \infty$ be arbitrary; take the scalar product of (V') with $v \in \partial\varphi(u)$ and integrate over $[0,T]$. Using $\frac{d}{dt}\varphi(u(t)) = (v(t), \frac{du}{dt}(t))$ a.e. (see Brézis [14]) one obtains

$$(3.9) \qquad \varphi(u(T)) + b(0)\int_0^T |v(t)|^2 dt + Q_{B'}[v;T] = \int_0^T (F'(t),v(t))dt + \varphi(f(0)) \ ,$$

where

$$Q_{B'}[v;T] = \int_0^T (v(t),B'*v(t))dt \ .$$

We next apply a frequency domain method (see Nohel and Shea [67]) to $Q_{B'}$. Define $v_T(t) = v(t)\chi[0,T]$ and its Fourier transform

$$\tilde{v}_T(\eta) = \int_{-\infty}^{\infty} e^{-i\eta t} v_T(t)dt \ .$$

Extend B' evenly to $(-\infty,0)$ by $B'(-t) = B'(t)$ $(0 < t < \infty)$. In the following calculation use is made of the hypothesis $B' \in L^1(0,\infty)$, and the Parseval and convolution theorems:

$$Q_{B'}[v;T] = \int_0^T (v(t),B'*v(t))dt = \frac{1}{2}\int_0^T (v(t), \int_0^T B'(t-\tau)v(\tau)d\tau)dt$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} (v_T(t), \int_{-\infty}^{\infty} B'(t-\tau)v_T(\tau)d\tau)dt = \frac{1}{4\pi}\int_{-\infty}^{\infty} |\tilde{v}_T(\eta)|^2 \tilde{B}'(\eta)d\eta \ .$$

Since B' is even, $\tilde{B}'(\eta) = 2\mathrm{Re}\,\hat{B}'(i\eta)$ $(\eta \in \mathbb{R})$, where $\hat{\ }$ denotes the Laplace transform. The assumptions $B, B' \in L^1(0,\infty)$ and the familiar formula $\hat{B}'(i\eta) = i\eta\,\hat{B}(i\eta) - B(0)$ yield $\operatorname{Re}\hat{B}'(i\eta) = -\eta \operatorname{Im}\hat{B}(i\eta) - B(0)$. Therefore,

$$Q_{B'}[v;T] = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{v}_T(\eta)|^2[-\eta \operatorname{Im}\hat{B}(i\eta) - B(0)]\,d\eta .$$

Substituting this result into (3.9) and using $\int_0^T |v(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{v}_T(\eta)|^2 d\eta$, as well as $b(0) - B(0) = b_\infty$, the frequency domain condition (F), and Parseval's theorem again, yields

$$(3.10) \quad \varphi(u(T)) + \delta\int_0^T |v(t)|^2 dt \le |\varphi(f(0))| + \int_0^T |(F'(t),v(t))|\,dt \quad (0 \le T < \infty) .$$

The assumption $F' \in L^2(0,\infty)$, Cauchy-Schwarz and an elementary inequality give the estimate

$$(3.11) \quad \varphi(u(T)) + \frac{\delta}{2}\int_0^T |v(t)|^2 dt \le |\varphi(f(0))| + \frac{1}{2\delta}\int_0^\infty |F'(t)|^2 dt < \infty \quad (0 < T < \infty) .$$

Assumption (3.1) used in (3.11) yields conclusions (3.2), (3.3) and (3.6).

Returning to (V') and using $B' \in L^1(0,\infty)$, $v \in L^2(0,\infty;H)$, $F' \in L^2(0,\infty;H)$ gives conclusion (3.4). Combining (3.3), (3.4) with $\frac{d}{dt}\varphi(u(t)) = (v(t), \frac{du}{dt}(t))$ yields $\frac{d}{dt}\varphi(u(t)) \in L^1(0,\infty)$, and this together with assumption (3.1) implies that $\lim_{t\to\infty}\varphi(u(t))$ exists. To establish all of (3.7) we use the definition of subdifferential: for every $v \in \partial\varphi(u)$ and for every $w \in H$ $\varphi(u(t)) \le \varphi(w) + (v(t),u(t) - w)$, $0 \le t < \infty$. Since $u \in L^\infty(0,\infty;H)$ and $v \in L^2(0,\infty;H)$ there exists a sequence $\{t_n\} \to \infty$ as $n \to \infty$ such that $(v(t_n),u(t_n) - w) \to 0$ as $n \to \infty$; this proves (3.7), and from it easily (3.8). To prove (3.5) take $\tau < t$ and use (3.4) and Cauchy-Schwarz obtaining:

$$|u(t) - u(\tau)| \le \int_\tau^t \left|\frac{du}{dt}(s)\right| ds \le \sqrt{t-\tau}\ \left\{\int_0^\infty \left|\frac{du}{dt}(s)\right|^2 ds\right\}^{1/2}$$
$$\le K\sqrt{t-\tau} \quad (0 \le \tau \le t < \infty) .$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. As remarked in the paragraph preceding Theorem 3.2 the coercivity condition (2.4) implies that

$$\operatorname*{Inf}_{z \in H} \varphi(z) \geq \varphi(0) > -\infty ,$$

so that assumption (3.1) is satisfied. Thus conclusions (3.2)-(3.5) follow immediately from Theorem 3.1. In view of (3.5) the conclusion $u(t) \to 0$ strongly as $t \to \infty$ follows once it is shown that $u \in L^2(0,\infty;H)$. But using assumption (2.4) with $\varepsilon > 0$ and $v \in L^2(0,\infty;H)$ for $v \in \partial\varphi(u)$ (proved in (3.3)) one has

$$\varepsilon \int_0^T |u(t)|^2 dt \leq \int_0^T (v(t),u(t))dt \leq \frac{\varepsilon}{2} \int_0^T |u(t)|^2 dt + \frac{1}{2\varepsilon} \int_0^T |v(t)|^2 dt .$$

Thus

$$\frac{\varepsilon}{2} \int_0^T |u(t)|^2 dt \leq \frac{1}{2\varepsilon} \int_0^\infty |v(t)|^2 dt < \infty \qquad (0 < T < \infty) .$$

Since $\varepsilon > 0$, this completes the proof of Theorem 3.2.

4. Application to a Problem of Heat Flow in a Material with Memory. In this section we study the heat flow problem (1.4) in one space dimension formulated in Chapter 1, Section 2. We use the existence and uniqueness theory of Chapter 2 and the boundedness and asymptotic theory of Chapter 3, Section 3, to deduce the principal result, Theorem 4.5 below.

For clarity of exposition we restate the heat flow problem (1.4), Chapter 1, Section 2:

$$(4.1) \quad \begin{cases} \frac{\partial}{\partial t} [b_0 u + \beta * u] = c_0 \sigma(u_x)_x - \gamma * \sigma(u_x)_x + h & (0 \leq t < \infty,\ 0 < x < 1) \\ u(0,x) = u_0(x) \quad (0 < x < 1), & u(t,0) = u(t,1) \equiv 0 \quad (t \geq 0) , \end{cases}$$

where subscripts denote differentiation with respect to x. We assume that the conditions (which were motivated in Chapter 1):

(PW) $$b_0 + \mathrm{Re}\,\hat{\beta}(i\eta) > 0 \qquad (\eta \in \mathbb{R}),$$

(γ) $$c_0 - \int_0^\infty \gamma(\tau)d\tau > 0,$$

where $\hat{\beta}(i\eta) = \int_0^\infty \beta(t)\exp(-i\eta t)dt$, are satisfied. We also assume that the function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies assumptions (σ) of Lemma 1.3 (Chapter 1), $u_0 \in H_0^1(0,1)$, and that the external heat supply $h \in L^1(0,\infty;L^2(0,1)) \cap L^2(0,\infty;L^2(0,1))$. Under these assumptions we have seen in Chapter 1 that the initial-boundary value problem (4.1) is equivalent to the abstract Volterra equation

(V) $$u + b*Au \ni f \qquad (0 < t < \infty);$$

in the present application

(b) $$b(t) = \frac{C(t)}{b_0} + (\rho*C)(t) \qquad (0 < t < \infty),$$

(f) $$f(t) = \frac{G(t,\cdot)}{b_0} + (\rho*G)(t,\cdot) \qquad (0 < t < \infty),$$

where ρ is the unique solution of the resolvent equation

(ρ) $$b_0\rho(t) + (\beta*\rho)(t) = -\frac{\beta(t)}{b_0} \qquad (0 < t < \infty),$$

$$C(t) = c_0 - \int_0^t \gamma(\tau)d\tau \qquad (0 < t < \infty),$$

$$G(t,x) = b_0 x_0(x) + \int_0^t h(\tau,x)d\tau \qquad (0 < t < \infty,\ 0 < x < 1),$$

and the nonlinear operator $A = \partial\varphi$ satisfies assumption (H_φ) with φ given by the proper, convex, l.s.c. function defined in and satisfying the properties of Lemma 1.3 in Chapter 1. We recall that in two or three space dimensions the corresponding heat flow problem in a bounded domain Ω with smooth boundary Γ also satisfies the Volterra equation (V) with b and f as above, but with $L^2(0,1)$ replaced by $L^2(\Omega)$, and with $A = \partial\varphi$ where φ is the proper convex, l.s.c. function defined in and satisfying the properties of Remark 1.4 in Chapter 1. We recall also that the key properties of the kernel b and of the forcing term f for the heat flow problem are stated in Lemmas 1.1 and 1.2 of Chapter 1. With these properties in mind all that is needed in order to apply Theorems 3.1 and 3.2 of Chapter 3 to the problem under study is to show that the frequency domain assumption (F) of Lemma 2.2 can be satisfied for physically reasonable classes of relaxation functions β, γ. In this direction we have:

Lemma 4.1. *Let b_0, c_0, β, γ satisfy the assumptions of Lemma 1.1, Chapter 1. Define the kernel b in (V) by equation (b). Then the frequency domain assumption (F) of Lemma 2.2 is equivalent to the condition: there exists $\delta > 0$ such that*

$$\inf_{(\eta\in R)} \frac{(c_0 - \mathrm{Re}\hat{\gamma}(i\eta))(b_0 + \mathrm{Re}\hat{\beta}(i\eta)) - \mathrm{Im}\hat{\gamma}(i\eta)\mathrm{Im}\hat{\beta}(i\eta)}{|b_0 + \hat{\beta}(i\eta)|^2} > \delta \,. \tag{4.2}$$

Proof of Lemma 4.1. Define the constant $b_\infty > 0$ and the function B as in Lemma 1.1, Chapter 1. Taking the Laplace transform of B one computes

$$\hat{B}(i\eta) = -\frac{i}{\eta}\left(\frac{c_0 - \hat{\gamma}(i\eta)}{b_0 + \hat{\beta}(i\eta)} - b_\infty\right) \qquad (\eta \in R) \,.$$

Thus

$$b_\infty - \eta \, \mathrm{Im}\, \hat{B}(i\eta) = \mathrm{Re}\left(\frac{c_0 - \hat{\gamma}(i\eta)}{b_0 + \hat{\beta}(i\eta)}\right) \qquad (\eta \in R) \,,$$

from which the condition (4.2) is an immediate consequence.

Using Lemma 4.1 one can construct a large number of examples of functions β and γ such that assumption (F) is satisfied. In particular one has the following physically important special cases. Note that in Corollaries 4.2 and 4.3 below the physical conditions $b_0 + \int_0^t \beta(\tau)d\tau > 0$ $(0 < t < \infty)$, $c_0 - \int_0^t \gamma(\tau)d\tau > 0$ $(0 < t < \infty)$ are both satisfied (although they are not explicitly needed in the theory), because the functions β and γ are positive, and assumption (γ) is assumed to hold. For a different example in which (γ) is satisfied but the above physical conditions need not hold see Remark 4.8 below.

Corollary 4.2. *Let* $b_0 > 0$, $c_0 > 0$ *and* $\beta, \gamma, t\beta, t\gamma \in L^1(0,\infty)$. *Also assume that* β *and* γ *are positive, nonincreasing and convex on* $[0,\infty)$, *and that the assumption* (Υ) *is satisfied. Then assumption* (F) *is satisfied if either for a fixed* $b_0 > 0$ *the constant* $c_0 > 0$ *is chosen sufficiently large, or if for a fixed* $c_0 > 0$ *the constant* $b_0 > 0$ *is sufficiently large.*

Remark 4.3. (i) If $\beta = \gamma \equiv 0$ (the standard heat flow problem) (F) is satisfied for any choice of $b_0 > 0$, $c_0 > 0$ with $\delta = \frac{b_0}{c_0}$.

(ii) If $\beta \equiv 0$ and γ satisfies the assumptions of Corollary 4.2, (F) is satisfied for any choice of $b_0 > 0$, $c_0 > 0$ with $\delta = \dfrac{c_0 - \int_0^\infty \gamma(t)dt}{b_0}$.

(iii) If $\gamma \equiv 0$ and β satisfies the assumptions of Corollary 4.2, (F) is satisfied for any choice of $b_0 > 0$, $c_0 > 0$.

Sketch of Proof of Corollary 4.2. The proof will make use of Lemma 4.1; we establish (4.2). Since $\beta, \gamma \in L^1(0,\infty)$ and are positive, nonincreasing and convex, $\mathrm{Re}\,\hat\beta(i\eta)$ and $\mathrm{Re}\,\hat\gamma(i\eta)$ are nonnegative. The function

$$\mathrm{Im}\hat\gamma(i\eta)\mathrm{Im}\hat\beta(i\eta) = \int_0^\infty \gamma(t)\sin\eta t dt \int_0^\infty \beta(t)\sin\eta t dt \qquad (\eta \in R)$$

is even, continuous, zero when $\eta = 0$, nonnegative, and has limit zero as $\eta \to \infty$ (Riemann-Lebesgue lemma). The denominator in (4.2) satisfies

$$0 < b_0^2 < |b_0 + \hat\beta(i\eta)|^2 < 2b_0^2 + 3\left(\int_0^\infty \beta(t)dt\right)^2 \qquad (\eta \in R) .$$

Moreover,

$$b_0 + \mathrm{Re}\hat\beta(i\eta) > b_0 > 0 \qquad (\eta \in R) ,$$

(so that (PW) is satisfied), and

$$c_0 - \mathrm{Re}\hat\gamma(i\eta) > c_0 - \int_0^\infty \gamma(t)dt > 0 \qquad (\eta \in R) .$$

Therefore, the existence of $\delta > 0$ such that (4.2) holds is established for choices of b_0 and c_0 as asserted. This completes the proof.

Another physically important case for the heat flow problem is the following special case of Lemma 4.1 and Corollary 4.2.

Corollary 4.4. Let

$$(4.3)\qquad \begin{cases} \beta(t) = \sum_{k=1}^{n} b_k e^{-\beta_k t} & (0 < t < \infty)\,, \\ \gamma(t) = \sum_{k=1}^{m} c_k e^{-\gamma_k t} & (0 < t < \infty) \end{cases}$$

with $b_k > 0$, $\beta_k > 0$, $c_k > 0$, $\gamma_k > 0$ and strict inequalities hold for at least one pair b_k, β_k and one pair c_k, γ_k. Let $b_0 > 0$, $c_0 > 0$, and $c_0 - \sum_{k=1}^{m} \frac{c_k}{\gamma_k} > 0$. Then the frequency domain condition (F) is satisfied if

$$(4.4)\qquad b_0\Big(c_0 - \sum_{k=1}^{m} \frac{c_k}{\gamma_k}\Big) > \frac{5}{4}\Big(\sum_{k=1}^{n} \frac{b_k}{\beta_k}\Big)\Big(\sum_{k=1}^{m} \frac{c_k}{\gamma_k}\Big)\,.$$

The proof of Corollary 4.4 is a consequence of showing that there exists a $\delta > 0$ such that (4.2) holds. The inequality (4.4) follows by using elementary calculus to find the infimum over $\eta \in R$ of the expression in (4.2):

$$\frac{\Big(c_0 - \sum_{k=1}^{m} \frac{c_k\gamma_k}{\gamma_k^2 + \eta^2}\Big)\Big(b_0 + \sum_{k=1}^{n} \frac{b_k\beta_k}{\beta_k^2 + \eta^2}\Big) - \eta^2\Big(\sum_{k=1}^{m} \frac{c_k}{\gamma_k^2 + \eta^2}\Big)\Big(\sum_{k=1}^{n} \frac{b_k}{\beta_k^2 + \eta^2}\Big)}{\Big(b_0 + \sum_{k=1}^{n} \frac{b_k}{\beta_k^2 + \eta^2}\Big)^2 + \Big(\sum_{k=1}^{n} \frac{b_k\eta}{\beta_k^2 + \eta^2}\Big)^2}\,.$$

No claim is made that the constant $\frac{5}{4}$ in (4.14) is optimal.

We next combine the properties stated in Lemmas 1.1, 1.2 and 1.3 of Chapter 1 and Lemma 4.1, and Corollaries 4.2 and 4.3 above with the abstract theory to establish the following result for the physical heat flow problem (4.1) in a one-dimensional material with memory. To see that a more general result (not necessarily physical) with β and γ oscillatory can hold we refer to Remark 4.8 below.

Theorem 4.5. *Let* $b_0 > 0$, $c_0 > 0$, *let* $\beta, \gamma, t\beta, t\gamma \in L^1(0,\infty)$ *and let* $\beta \in L^2(0,\infty)$. *Assume that* β *and* γ *are positive, nondecreasing, convex, and that*

$$(\gamma) \quad c_0 - \int_0^\infty \gamma(t)dt > 0 \ .$$

Assume that $\sigma : R \to R$ *satisfies assumptions* (σ) *of, Chapter 1, that the initial temperature* $u_0 \in H_0^1(0,1)$, *and that the external heat supply* $h \in L^1(0,\infty;H) \cap L^2(0,\infty;H)$, *where* $H = L^2(0,1)$. *Then the heat flow problem* (4.1) *has a unique strong solution* u *on* $[0,\infty) \times (0,1)$ *such that* $\frac{\partial u}{\partial t} \in L^2_{loc}(0,\infty;H)$. *Moreover, if either for a fixed* $b_0 > 0$, *the constant* $c_0 > 0$ *is sufficiently large, or for a fixed* $c_0 > 0$, *the constant* $b_0 > 0$ *is sufficiently large, then the solution* u *has the properties:*

$$u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H), \ \frac{du}{dt} \in L^2(0,\infty;H) \ ,$$

and $\lim_{t\to\infty} u(t) = 0$ *strongly in* H.

Remark 4.6. For heat flow in more than one space dimension let Ω be a bounded body in R^2 or R^3 with smooth boundary Γ. Then the temperature u satisfies an equation of the form (4.1) with the operator $-\sigma(u_x)_x$ replaced by $-\nabla_0(\lambda|\nabla u|)\nabla u)$; the boundary condition is $u(t,x) = 0$ $(0 < t < \infty, x \in \Gamma)$, and the initial condition is $u(0,x) = u_0(x)$ $(x \in \Omega)$. If H is the Hilbert space $L^2(\Omega)$, if the function $\lambda : R \to R^+$ satisfies assumptions (λ) of, Chapter 1, if $u_0(x) \in H_0^1(\Omega)$, and $h \in L^1(0,\infty;H) \cap L^2(0,\infty;H)$, then the results of Theorem 4.5 holds, provided the constants $b_0 > 0$, $c_0 > 0$ and the relaxation functions β and γ satisfy the assumptions stated in Theorem 4.5.

Remark 4.7. Let

$$\beta(t) = \sum_{k=1}^{n} b_k e^{-\beta_k t} \qquad (0 < t < \infty)$$

$$\gamma(t) = \sum_{k=1}^{m} c_k e^{-\gamma_k t} \qquad (0 < t < \infty) \ ,$$

where $b_k > 0$, $\beta_k > 0$, $c_k > 0$, $\gamma_k > 0$ and strict inequalities hold for at least one pair b_k, β_k and one pair c_k, γ_k. Let $b_0 > 0$, $c_0 > 0$, and $c_0 - \sum_{k=1}^{m} \frac{c_k}{\gamma_k} > 0$. Let σ, u_0, h satisfy the assumptions of Theorem 4.5. Then by Corollary 4.4 all conclusions of Theorem 4.5 hold if the inequality (4.4) is satisfied.

<u>Proof of Theorem 4.5</u>. Under the assumptions of the theorem the heat flow problem (4.1) is equivalent to the abstract Volterra equation (V) with the kernel b given by equation (b), the forcing term given by equation (f), and the operator $A = \partial\varphi$ where $\varphi : H \to (-\infty,\infty]$ is the proper, convex, l.s.c. function defined in Lemma 1.3, Chapter 1 (or Remark 1.4, Chapter 1 in more than one space dimension). To establish the existence and uniqueness of a strong solution of (V) (equivalent to (4.3)), we apply Theorem 3.10, Chapter 2. Lemma 4.2, Chapter 1, shows that the assumptions of Theorem 3.10, Chapter 2 concerning f are satisfied with $f(0,x) = u_0(x) \in H_0^1(0,1) = D(\varphi)$. Moreover, (H_φ) is satisfied. Lemma 1.1, Chapter 1, shows that assumptions (H_b) are satisfied. Thus to apply Theorem 3.10, Chapter 2, we must still verify that $B' \in BV_{loc}[0,\infty)$. From the expression for B in Lemma 1.1, Chapter 1, we compute

$$B'(t) = -\frac{\gamma(t)}{b_0} + c_0\rho(t) - (\gamma*\rho)(t) \qquad (0 < t < \infty) . \tag{4.5}$$

Since β is monotone by hypothesis, the resolvent equation (ρ) and a standard argument (see e.g. Bellman and Cooke [12]) show that ρ is monotone. Finally, since γ is monotone, it follows that $B' \in BV[0,\infty)$. Thus Theorem 3.10, Chapter 2, yields the existence and uniqueness of a strong solution u of (V) on $[0,\infty)$ such that $u' \in L^2_{loc}[0,\infty;H)$.

We shall next apply Theorem 3.1 of this chapter. Concerning the kernel b Lemma 1.1, Chapter 1, shows that assumptions (H_b) are satisfied with $b_\infty > 0$. Moreover, Corollary 4.2 shows that b satisfies the frequency domain condition (F) if b_0 and c_0 are chosen as in the statement of Theorem 4.5.

Lemma 1.3, Chapter 1 (or Remark 1.4, Chapter 1 in the case of more than one dimension) shows that $\varphi(y) \geq 0$ $(y \in H)$, $\lim_{|y|\to\infty} \varphi(y) = +\infty$, and that the inclusion $\partial\varphi(w) \ni 0$ has $w = 0$ as the only solution. Lemma 1.2, Chapter 1, shows that assumptions (H_f) are satisfied. Therefore, by Theorem 3.1 the solution u has the properties:

$$\sup_{0<t<\infty} \varphi(u(t)) < \infty, \quad \sup_{0<t<\infty} |u(t)| < \infty, \quad \frac{du}{dt} \in L^2(0,\infty), \quad u(t) \text{ is uniformly}$$

continuous on $[0,\infty)$.

$$\lim_{t\to\infty} \varphi(u(t)) = 0, \quad \text{and} \quad u(t) \longrightarrow 0 \quad \text{as} \quad t \to \infty .$$

Lemma 1.3, Chapter 1 also shows that under assumption (σ) the coercivity assumption (2.4) is satisfied for every $T > 0$ with $\varepsilon = p_0\pi^2 > 0$ (or another positive constant in the case of more than one space dimension - see Remark 1.4, Chapter 1). Therefore, by Theorem 3.2 one also has $u \in L^2(0,\infty;H)$ and $u(t) \to 0$ as $t \to \infty$ strongly in H. This completes the proof.

Remark 4.8. Suppose

$$\beta(t) = b_1 e^{-\beta_1 t}\cos\lambda t \qquad (b_1, \beta_1 > 0,\ 0 < t < \infty)$$
$$\gamma(t) = c_1 e^{-\gamma_1 t}\cos\omega t \qquad (c_1, \gamma_1 > 0,\ 0 < t < \infty) ,$$

and assume that $b_0 > 0$, $c_0 > 0$. Also suppose that σ and h satisfy the assumptions of Theorem 4.5. Although the assumptions concerning β, γ in Theorem 4.5 are not satisfied, one still has by Lemma 1.1, Chapter 1, that assumptions (H_b) hold with $b_\infty > 0$ provided

$$(\gamma) \qquad c_0 - \frac{c_1\gamma_1}{\gamma_1^2 + \omega^2} > 0 .$$

Moreover, $B' \in BV_{loc}[0,\infty)$ from (4.5), and the existence and uniqueness of a strong solution of (V) (equivalent to (4.1)) such that $u' \in L^2_{loc}(0,\infty;H)$ follows from Theorem 3.10, Chapter 2. Thus to obtain all of the conclusions of Theorem 4.5 we need only verify that the frequency domain assumption ($\bar{F}$) holds in order to apply Theorems 3.1 and 3.2. By Lemma 4.1 it suffices to find sufficient conditions on the constants b_0, c_0, b_1, β_1, λ, c_1, γ_1, ω so that (4.2) holds. An elementary, but tedious calculation shows that $\delta > 0$

such that (4.2) holds exists in the case $\gamma_1^2 > \omega^2$, $\beta_1^2 > \lambda^2$, provided assumption (γ) above holds, and provided $b_0 > 0$ is chosen sufficiently large.

While no claim is made here that the above functions β and γ represent physically plausible relaxation functions, it is of some interest that the theory can still be applied. In this connection it may also be noted that here the function

$$C(t) = c_0 - \int_0^t \gamma(\tau)d\tau = c_0 - \frac{c_1\gamma_1}{\gamma_1^2 + \omega^2}(1 - e^{-\gamma_1 t}\cos\omega t) - \frac{c_1\omega}{\gamma_1^2 + \omega^2} e^{-\gamma_1 t}\sin\omega t \; .$$

In a genuinely physical problem as motivated above one would need to require $C(t) > 0$ $(0 < t < \infty)$, as well as assumption (γ). However, in the application of the theory the physical requirement $C(t) > 0$ $(0 < t < \infty)$ is not used and indeed, for example,

$$C\left(\frac{\pi}{2}\right) = c_0 - \frac{c_1\gamma_1}{\gamma_1^2 + \frac{\pi^2}{4}} - \frac{\frac{\pi}{2}c_1}{\gamma_1^2 + \frac{\pi^2}{4}} e^{-\gamma_1 \frac{\pi}{2}}$$

could be negative, even though $C(\infty) = c_0 - \dfrac{c_1\gamma_1}{\gamma_1^2 + \frac{\pi^2}{4}} > 0$ holds.

Chapter 4

Existence and Asymptotic Behaviour of Positive Solutions of Nonlinear Volterra Equation for Heat Flow

1. Introduction. In this chapter we discuss the positivity of solutions and their asymptotic behaviour as $t \to \infty$, of the nonlinear Volterra equation

$$(V) \qquad u(t) + (b*Au)(t) \ni f(t) \qquad (0 < t < \infty)$$

in the general setting: $b : [0,\infty) \to R$ is a given kernel, A is a nonlinear (possibly multivalued) m-accretive operator defined on a real Banach space X, $f : [0,\infty) \to X$ is a given function; the integral in (V) is understood in the sense of Bochner. The assumptions which are imposed on b, A, f are motivated by the problem of nonlinear heat flow in a material with memory formulated in Chapter 1 to which the general positivity and asymptotic theory developed in Section 3 will be applied in Section 4. A different application of general theory is given in Example 3.4, Section 3, to a nonlinear conservation law with memory.

This chapter which generalizes and complements earlier work of Clément and Nohel [18] on positivity and of Clément [17] on limiting behaviour of positive solutions of (V) is primarily based on a forthcoming paper by Clément and Nohel [19]. The generalization enables us to apply the theory to the physical problem in Section 4. General existence, uniqueness and continuous dependence results for solutions of (V) which need not be positive have been established by Crandall and Nohel [26] and by Gripenberg [34] (see also Theorems 3.5 and 3.7, Chapter 3); these will be referred to as needed.

We will motivate the assumptions on the kernel b which are needed for positivity of solutions of (V) and which will be needed throughout the analysis by means of a simple linear problem at the end of this section. These considerations suggest the concept of complete positivity of the kernel b (Definition 1.1 below) which plays an important role in the analysis. Some properties and a useful characterization of completely positive kernels are obtained in Section 2.

We shall consider equation (V) in the slightly less general form

$$(V_g) \qquad u(t) + (b*Au)(t) \ni u_0 + (b*g)(t) \qquad (0 \le t < \infty) .$$

We assume throughout the following minimal assumptions:

$$(H_1) \qquad \begin{cases} b \in L^1_{loc}(0,\infty), \\ A \text{ m-accretive in a real Banach space } X , \\ u_0 \in \overline{D(A)}, \text{ and } g \in L^1_{loc}(0,\infty;X) . \end{cases}$$

The motivation for taking $f = u_0 + b*g$ in (V) is given in Section 3 (see argument at the beginning of Section 3 following (V_g)). The main results of this chapter, described in Section 3, give a rather complete description of the asymptotic behaviour of the positive solutions of the abstract equation (V) as $t \to \infty$, including a priori estimates for their rates of decay. The results are then applied to the physical problem in Section 4.

The additional assumption we shall make on the kernel b in order to insure positivity of solutions was first introduced in [18]; it is motivated by the following remark. If $b \equiv 1$ then (V_g) reduces to the evolution equation

$$(DE) \qquad \begin{cases} \frac{du}{dt} + Au \ni g \\ u(0) = u_0 \end{cases}$$

It is well-known [13] that if the resolvent $J_\lambda = (I + \lambda A)^{-1}$ of A maps a closed convex cone P of X into itself for every $\lambda > 0$, then $u(t) \in P$ for all $t > 0$, provided that $u_0 \in P$ and $g(t) \in P$ a.e. on $[0,\infty)$. Let us take for instance

$$X = \{u \in C[a,b] \,|\, u(a) = u(b) = 0\}$$

equipped with the supremum norm; $D(A) = \{u \in X \mid u \in C^2[a,b] \text{ and } u_{xx} \in X\}$ and $Au(x) = -u_{xx}(x)$ for $u \in D(A)$. It is standard that A is m-accretive in X. Moreover, if $P = \{u \in X | u(x) \ge 0\ x \in [a,b]\}$, then $J_\lambda P \subset P$ for every $\lambda > 0$; thus, as is classical, the solution of the heat equation is nonnegative provided that the initial value u_0 and the forcing form g are nonnegative.

We want to consider a class of kernels b under which the solution (V), resp. (V_g), preserves this positivity property. This requirement is useful and natural in the application to the model of heat flow in a material with memory discussed in Section 4, and in Example 3.4 of Section 3.

Consider (V_g) with $Au = -u_{xx}$ with $D(A)$ as in the above example. It is easy to give necessary conditions to be imposed on b in order that positivity is preserved by (V_g) whenever u_0 and g are positive. Let $\bar{\lambda}$ denote the principal eigenvalue and $\bar{u}$ the corresponding principal eigenfunction of A, normalized by $\max_{x \in [a,b]} \bar{u}(x) = 1$. Clearly $\bar{\lambda} = \left(\frac{\pi}{b-a}\right)^2$ and $\bar{u}(x) = \sin\left(\frac{\pi}{b-a}\right)(x-a)$. If $u_0 = \alpha\bar{u}$, $g(t) = \bar{\lambda}\beta(t)\bar{u}$ with $\alpha > 0$, and $\beta(t) \geq 0$ where $\beta \in L^1_{loc}(0,\infty)$, then, as can be verified directly, the strong solution of (V_g) is

$$u(t) = [\alpha s(\bar{\lambda}b)(t) + (\beta * r(\bar{\lambda}b))(t)]\bar{u} \qquad (0 \leq t < \infty), \tag{1.1}$$

where the functions $s(b)$ and $r(b):[0,\infty) \to R$ are respectively solutions of the linear Volterra equations

$$s(b)(t) + (b*s(b))(t) = 1 \qquad (0 \leq t < \infty) \tag{s(b)}$$

$$r(b)(t) + (b*r(b))(t) = b(t) \qquad (0 \leq t < \infty). \tag{r(b)}$$

Recall the standard fact (see e.g. R. K. Miller [63]) that if $b \in L^1_{loc}(0,\infty)$, the functions $s(b)$, $r(b)$ are uniquely defined and $s(b), r(b) \in L^1_{loc}(0,\infty)$. Moreover, if $F \in L^1_{loc}(0,\infty)$ the unique solution of the linear Volterra equation

$$u(t) + (b*u)(t) = F(t) \qquad (0 \leq t < \infty) \tag{1.2}$$

is given by

$$u(t) = F(t) - (r(b)*F)(t) \qquad (0 \leq t < \infty). \tag{1.3}$$

In particular, taking $F \equiv 1$ in (1.2), one has

$$s(b)(t) = 1 - \int_0^t r(b)(\tau)d\tau \qquad (0 \leq t < \infty), \tag{1.4}$$

so that $s(b)$ is absolutely continuous on $[0,\infty)$ whenever $b \in L^1_{loc}(0,\infty)$. The function $s(b)$ is called the fundamental solution of (1.2), while the function $r(b)$ is called the resolvent kernel associated with b.

Returning to the solution (1.1) of (V_g) with $Au = -u_{xx}$, $\bar{\lambda} > 0$, $\bar{u}$, u_0, g defined above, we note that $\bar{u}(x) > 0$ for $x \in (a,b)$. Thus the solution $u(t)$ will be nonnegative for every $\alpha > 0$ and for every $\beta \in L^1_{loc}(0,\infty)$, $\beta > 0$, only if the functions $r(b)$, $s(b)$ are nonnegative on $[0,\infty)$. Moreover, if one imposes the requirement that the solution (1.1) of (V_g) should be nonnegative and independent of the length of the interval (a,b), it is clear that both of the functions $r(\lambda b)$ and $s(\lambda b)$ must be nonnegative for every $\lambda > 0$. We remark that these latter necessary conditions imposed on the kernel b have been shown to be sufficient to guarantee the preservation of positivity by the solution operator of the nonlinear equation (V) in the general case of A m-accretive on X (see [18, Theorem 4.5]).

The above considerations suggest the following concept of complete positivity of the kernel b:

Definition 1.1 We shall say that the kernel b is completely positive on $[0,T]$ if $b \in L^1(0,T)$ and if the functions $r(\lambda b)$ and $s(\lambda b) = 1 - 1*r(\lambda b)$ are nonnegative on $[0,T]$ for every $\lambda > 0$.

Some known sufficient conditions which insure the complete positivity of the kernel b on $[0,T]$ are:

(i) $b \in L^1(0,T)$ is nonnegative, nonincreasing, and $\log b$ is convex (see Miller [62], Levin [52], Clément and Nohel [18]).

(ii) (special case of (i)) $b \in L^1(0,T)$ and b is completely monotonic on $(0,T)$ (see Miller [62]).

2. Completely Positive Kernels.

In this section we give an alternate and useful characterization of completely positive kernels (Theorem 2.2) which will be needed for the development of the asymptotic properties of positive solutions of the abstract Volterra equation (V_g). For this purpose we consider the linear scalar Volterra equation (1.2) in the form

$$u + b*u = u_0 + b*g \tag{2.1}$$

where $b \in L^1(0,T)$, $u_0 \in R$, $g \in L^1(0,T)$, and $T > 0$. Its unique solution (see (1.3), (1.4)) is given by

$$u(t) = u_0 s(b)(t) + (r(b)*g)(t) \qquad (0 \le t \le T) . \tag{2.2}$$

In the following proposition we list some elementary properties of completely positive kernels which are needed in the sequel.

Proposition 2.1. *Assume that* b *is completely positive on* $[0,T]$ *for some* $T > 0$. *Then:*

1) b *is nonnegative on* $[0,T]$ *and for every* $\mu > 0$, $s(\mu b)$ *is nonnegative and nonincreasing on* $[0,T]$.

2) *For every* $\mu > 0$, $r(\mu b)$ *is itself completely positive on* $[0,T]$.

Next, assume b *is completely positive on* $[0,T]$ *for every* $T > 0$. *Then:*

3) *If* $b \in L^1(0,\infty)$, *then for every* $\mu > 0$

$$\lim_{t\to\infty} s(\mu b)(t) = (1 + \mu\int_0^\infty b(\tau)d\tau)^{-1} ,$$

$$\|r(\mu b)\|_{L^1(0,\infty)} = (\mu\int_0^\infty b(\tau)d\tau)(1 + \int_0^\infty b(\tau)d\tau)^{-1} .$$

4) *If* $b \notin L^1(0,\infty)$, *then for every* $\mu > 0$

$$\lim_{t\to\infty} s(\mu b)(t) = 0 \quad \text{and} \quad \|r(\mu b)\|_{L^1(0,\infty)} = 1.$$

5) *If* $b \notin L^1(0,\infty)$ *and* $b \in AC[0,\infty)$, *then for every* $\mu > 0$, $r(\mu b) \in C[0,\infty)$ *and* $\lim_{t\to\infty} r(\mu b)(t) = 0$.

The proof of Proposition 2.1 is elementary and is omitted; for details see [19].

In the next result we give an alternate and useful characterization of completely positive kernels b. Some arguments used are similar to those of [33].

Theorem 2.2. *Let* $T > 0$, $b \in L^1(0,T)$, $b \not\equiv 0$. *Then* b *is completely positive on* $[0,T]$ *if and only if there exists* $\alpha > 0$ *and* $k \in L^1(0,T)$ *nonnegative and nonincreasing satisfying:*

(2.3) $$\alpha b(t) + k*b(t) = 1 \quad t \in [0,T] .$$

Remarks: (i) It follows from (2.3) that $\alpha > 0$ if and only if $b \in L^{\infty}(0,T)$. If this is the case $b = \alpha^{-1}s(\alpha^{-1}k)$ and thus $b \in AC[0,T]$. Conversely if $b \in AC[0,T]$, then $\alpha = b(0)^{-1} > 0$. Moreover, observe that if $\alpha > 0$, then $k \in BV[0,T]$ (equivalently $k(0^+) < \infty$) if and only if $b' \in BV[0,T]$).

The importance of the remark $\alpha > 0$, ($k \in BV[0,T]$) is that for kernels b satisfying the assumption:

(H) $$b \in AC[0,T],\ b(0) > 0\ ,\ b' \in BV[0,T]$$

the existence and uniqueness of a generalized solution $u \in C([0,T]; \overline{D(A)})$ of the abstract Volterra equation (V) has been established by Crandall and Nohel (Chapter 2, Theorem 3.5), whenever the operator A is m-accretive, and $f(0) \in \overline{D(A)}$, $f \in W^{1,1}(0,T;X)$. For the special case $X = H$ a real Hilbert space and $A = \partial\varphi$ we refer to Chapter 2, Theorem 3.10. Recently Gripenberg [34, Theorem 2] has extended this result to the case of kernels $b = b_1 + b_2$, where b_1 satisfies the above regularity assumption and where $b_2 \in L^1(0,T)$, b_2 is positive, nonincreasing, and $\log b_2$ is convex on $(0,T)$ with A and f as above (see Chapter 2, Theorem 3.7). This result with $b_1 \equiv 0$ and A linear was already established by Clément and Nohel [18]. These more general completely positive kernels b correspond to the case $\alpha = 0$. The problem of existence of generalized solutions of (V) with only the assumption that b is completely positive is under study and will be treated elsewhere.

(ii) It follows from Theorem 2.2 and Remark (i) that if b is completely positive, then b need not be nonincreasing; it also need not be convex and a fortiori log convex. Choose $\alpha = 1$ and $k(t) = 1$ for $t \in [0,1]$ and $k(t) = 0$ for $t > 1$; then $b = s(k)$ is completely positive. But as shown in Levin [52], $b' = -r(k)$ is negative on some interval $[0,a)$ with $a \in (1,2)$ and positive in $(a,2]$. Thus b is not nonincreasing on $[0,2]$. Moreover, assume b to be convex on $[0,\infty)$. Then b is strictly increasing for $t > a$, and moreover, $\lim_{t\to\infty} b(t) = \infty$. But this is impossible, since $b(t) \le 1$ as seen from (2.3) and the fact that k,b are nonnegative and $\alpha = 1$. Thus b is not convex.

(iii) If b is completely positive and absolutely continuous on $[0,T]$, then it follows from (2.3) that $b(t) \leq b(0)$ for $t \in [0,T]$.

(iv) It follows from Theorem 2.2 that if $b \in L^1_{loc}(0,\infty)$, b is positive, decreasing, log b is convex and $b(0^+) = \infty$, then the linear Volterra equation of the first kind

$$k*b(t) = 1 \quad , \ t > 0 \tag{2.4}$$

possesses a unique solution $k \in L^1_{loc}(0,\infty)$ which is nonnegative and nonincreasing. However, given $k \in L^1[0,T]$, k nonnegative and nonincreasing, equation (2.4) may not have a solution in $L^1(0,T)$. (Take $k(t) \equiv 1$). Thus when $\alpha = 0$, equation (2.3) does not provide a way to generate completely positive kernels which are not absolutely continuous on $[0,T]$.

We omit the quite technical proof of Theorem 2.2. It makes repeated use of the following result due to Levin [52]. If u satisfies $u + b*u = f$ with $b \in L^1(0,T)$, b nonnegative and nonincreasing, $f \in L^1(0,T)$, nonnegative, nondecreasing, then u is nonnegative on $[0,T]$. The proof of Theorem 2.2 may be found in [19].

3. Qualitative properties of abstract Volterra equations with completely positive kernels.

In this section we study some properties of generalized solutions, including positivity and the asymptotic behaviour of positive solutions as $t \to \infty$, of the nonlinear abstract Volterra equation

$$u + b*Au \ni u_0 + b*g \qquad (t > 0) \ . \tag{V_g}$$

Although our results are stated for generalized solutions, it is obvious that the results hold for strong solutions, whenever strong solutions are shown to be generalized solutions (see Remark (i) following Theorem 3.1 below).

The justification for taking $f = u_0 + b*g$ in (V) is as follows. If b satisfies assumption (H) (Section 2, Remark (i) following Theorem 2.2), if $f \in W^{1,1}(0,T;X)$, and if $f(0) \in \overline{D(A)}$, then there exists a unique $u_0 \in \overline{D(A)}$ and a unique $g \in L^1(0,T;X)$ such that

$$f(t) = u_0 + (b*g)(t) \qquad (0 \leq t \leq T) \ . \tag{3.1}$$

Indeed, $u_0 = f(0) \in \overline{D(A)}$ and g is the unique solution of the linear equation

$$(3.2) \qquad b(0)g(t) + (b'*g)(t) = f'(t) \quad (0 < t < T) .$$

Conversely, if b satisfies assumption (H) and $u_0 \in \overline{D(A)}$, $g \in L^1(0,T;X)$, then f given by (3.1) satisfies $f(0) \in \overline{D(A)}$, $f \in W^{1,1}(0,T;X)$. We shall make the following general assumptions:

$$(\tilde{H}) \qquad \begin{cases} A \text{ is m-accretive in } X \\ u_0 \in \overline{D(A)} \\ g \in L^1_{loc}(0,\infty;X) \\ b \text{ is completely positive on } [0,\infty) . \end{cases}$$

The basic preliminary result assuming the global existence of solutions of (V_g) under assumption $(\tilde{H})$ is known:

<u>Theorem 3.1.</u> <u>If</u> A, u_0, g <u>and</u> b <u>satisfy assumption</u> $(\tilde{H})$ <u>then</u>:

(1) <u>if</u> u_1 <u>and</u> u_2 <u>are the generalized solutions of</u> (V_g) <u>corresponding to the data</u> $u_{0,i}$, g_i, $i = 1,2$, <u>then the following estimate holds:</u>

$$(3.3) \qquad \|u_1(t) - u_2(t)\| \le \|u_{0,1} - u_{0,2}\| + (b*\|g_1 - g_2\|)(t) \quad t > 0 \quad \text{a.e.}$$

(2) <u>if</u> P <u>is a closed convex cone in</u> X, <u>if</u> $J_\lambda(P) \subseteq P$ <u>for every</u> $\lambda > 0$, <u>and if</u> $u_0 \in P$ <u>and</u> $g(t) \in P$ a.e. <u>on</u> $[0,\infty)$, <u>then</u> $u(t) \in P$ a.e. <u>on</u> $[0,\infty)$; <u>moreover, if</u> $v - u \in P$ <u>implies</u> $J_\lambda v - J_\lambda u \in P$ <u>for every</u> $\lambda > 0$, $u,v \in X$, <u>and if</u> $u_{0,2} - u_{0,1} \in P$, $g_2(t) - g_1(t) \in P$ a.e. <u>on</u> $[0,\infty)$, <u>then</u> u_i, $i = 1,2$, <u>the corresponding generalized solutions of</u> (V_g) <u>satisfy</u> $u_2(t) - u_1(t) \in P$, a.e. <u>on</u> $[0,\infty)$.

<u>Remarks:</u> (i) The existence of a generalized solution in the linear case under the assumption b completely positive was proved in [18]. In the nonlinear case, when $b \in AC[0,T]$, $b(0) > 0$ and $\dot{b} \in BV[0,T]$, or when $b \in L^1(0,T)$, b is positive, nonincreasing and $\log b$ is convex on $(0,T)$, the existence of generalized solutions of (V_g) follows from results Crandall and Nohel [26] and Gripenberg [34], already discussed in Chapter 2 and in Remark (i) following Theorem 2.2. Moreover, if more regularity is assumed on b and f, then (see [26], [34]) the generalized solution is also a strong solution of (V_g).

(ii) Estimate (3.3) was proved in [17].

(iii) We sketch the proof of the positivity result asserted in (2); the details appear in Clement and Nohel [18]; the last assertion in (2) can be established in the same way.

Consider the approximating equation of (V_g) resulting from replacing the operator A by its Yosida approximation

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda), \quad J_\lambda = (I - \lambda A)^{-1};$$

$$(V_\lambda) \qquad u_\lambda + b*A_\lambda u_\lambda = u_0 + b*g \qquad (0 < t < \infty) .$$

By the definition of A_λ equation (V_λ) is the same as

$$u_\lambda + \frac{1}{\lambda} b*u_\lambda = u_0 + b*g + \frac{1}{\lambda} b*J_\lambda u_\lambda .$$

Using (2.2) with g replaced by $g + \frac{1}{\lambda} J_\lambda u_\lambda$, one easily checks that (V_λ) is equivalent to the integral equation

$$(\bar{V}_\lambda) \qquad u_\lambda = f_\lambda + W_\lambda(u_\lambda) ,$$

where

$$f_\lambda = s(\frac{1}{\lambda} b)u_0 + \lambda r(\frac{1}{\lambda} b)*g$$

and

$$W_\lambda(u_\lambda)(t) = r(\frac{1}{\lambda} b)*J_\lambda(u_\lambda) .$$

With

$$v_\lambda = u_\lambda - f_\lambda ,$$

the integral equation $(\bar{V}_\lambda)$ is equivalent to the nonlinear equation

$$(\bar{\bar{V}}_\lambda) \qquad v_\lambda = W_\lambda(v_\lambda + f_\lambda) .$$

Let $0 < T < \infty$ be arbitrary. By the complete positivity of b and by the hypothesis of Theorem 3.1, part (2), $f_\lambda(t) \in P$ a.e. on $[0,T]$. Noting that W_λ maps $L^1(0,T;X)$ into itself and recalling that the operator J_λ is a contraction (since A is m-accretive), one proves (for details see [18] and [26]) that some iterate W_λ^n of W_λ is a strict contraction on $L^1(0,T;X)$ for n sufficiently large. Thus the integral equation $(\bar{\bar{V}}_\lambda)$ has a unique solution $v_\lambda \in L^1(0,T;X)$ given in

$$v_\lambda = \lim_{n\to\infty} W_\lambda^n(v_0) \quad \text{for any } v_0 \in L^1(0,T;X) .$$

Therefore, the approximating equation (V_λ) has the unique solution $u_\lambda = f_\lambda + v_\lambda$. But $f_\lambda \in P$ a.e. on $[0,T]$, and if $v_0(t) \in P$ a.e. on $[0,T]$, the complete positivity of b and the assumption $J_\lambda(P) \subset P$ for every $\lambda > 0$ insure that $W_\lambda(v_0)(t) \in P$ a.e. on $[0,T]$ and the same holds for $W_\lambda^n(v_0)(t)$ for every n. Consequently $u_\lambda(t) \in P$ a.e. on $[0,T]$, and if u is a solution of (V_g) on $[0,T]$ such that $u = \text{weak}\lim_{\lambda \to 0} u_\lambda$ in $L^1(0,T;X)$, then $u(t) \in P$ a.e. on $[0,T]$. Since $T > 0$ is arbitrary this completes a sketch of the proof.

We next obtain some results concerning the asymptotic behaviour of solutions of (V_g) as $t \to \infty$. We first consider the case $b \in L^1(0,\infty)$.

Theorem 3.2. Let A, u_0, g, b, satisfy the general assumptions $(\tilde{H})$ with $b \not\equiv 0$ and $b \in L^1(0,\infty)$.

(1.) Let $g \in L^\infty(0,\infty;X)$ and assume there exists $g^\infty \in X$ such that $\lim_{t\to\infty} \|g(t) - g^\infty\| = 0$. Let u be the generalized solution of (V_g) and define $u^\infty = J_{\bar{b}}(u_0 + \bar{b}g^\infty)$, where $\bar{b} = \int_0^\infty b(t)dt > 0$. Then the following estimate, which implies strong convergence of $u(t)$ to u^∞ as $t \to \infty$, holds:

$$\|u(t)-u^\infty\| \le \frac{\int_t^\infty b(\tau)d\tau}{\bar{b}} \|u_0-u^\infty\| + (b*\|g - g^\infty\|)(t) \qquad (0 < t < \infty) \tag{3.4}$$

(2.) In addition, let $b \in L^\infty(0,\infty)$ and $\lim_{t\to\infty} b(t) = 0$. Let $g = g_1 + g_2$, where g_1 satisfies the assumptions of g in part 1, and where $g_2 \in L^1(0,\infty;X) + L^p(0,\infty;X)$, $p \in (1,\infty)$. Let u be the generalized solution of (V_g) and let $u^\infty = J_{\bar{b}}(u_0 + \bar{b}g_1^\infty)$. Then the following estimate, which implies strong convergence of $u(t)$ to u^∞ as $t \to \infty$, holds:

$$(3.5)\qquad \|u(t) - u^\infty\| \le \frac{\int_t^\infty b(s)ds}{\bar{b}} \|u_0 - u^\infty\| + (b^*\|g_1 - g_1^\infty\|)(t) + (b^*\|g_2\|)(t) \quad (0 \le t < \infty) ,$$

where $g_1^\infty = \lim_{t\to\infty} g_1(t)$.

Remark. Part 2 is proved below. Part 1 of Theorem 3.2 was proved in Clément [17] and the proof will be omitted; it uses ideas similar to the proof of part 2.

Next we consider the case where $b \notin L^1(0,\infty)$ which is needed for the application in Section 4. In order to establish the strong convergence of u to u^∞ as $t \to \infty$ we shall require that the nonlinear operator A in (V) satisfy a rather strong coercivity condition.

Theorem 3.3. Let A, u_0, g, and b satisfy the general assumptions $(\tilde{H})$ with $b \notin L^1(0,\infty)$ and suppose A is coercive in the sense that there exists $\omega > 0$ for which $A - \omega I$ is accretive in X.

1. Let g be in $L^\infty(0,\infty;X)$ and let $g^\infty \in X$ such that $\lim_{t\to\infty} \|g(t) - g^\infty\| = 0$. Let u be the generalized solution of (V_g) and let u^∞ be the unique element in X satisfying $Au^\infty \ni g^\infty$. Then the following estimate which implies strong convergence of $u(t)$ to u^∞ as $t \to \infty$, holds;

$$(3.6)\qquad \|u(t) - u^\infty\| \le \int_t^\infty r(\omega b)(\tau)d\tau \, \|u_0 - u^\infty\| + \omega^{-1}(r(\omega b)^*\|g - g^\infty\|)(t) \quad (0 \le t < \infty) .$$

2. In addition, let b be $AC[0,\infty]$ and $g = g_1 + g_2$ where g_1, g_2 satisfy the assumptions of Theorem 3.2, part (2), with $g_1^\infty = \lim_{t\to\infty} g_1(t)$. Let u be the generalized solution of (V_g), and let u^∞ be the unique element in X satisfying $Au^\infty \ni g_1^\infty$. Then the following estimate, which implies strong convergence of $u(t)$ to u^∞ as $t \to \infty$, holds;

$$\|u(t) - u^\infty\| \le \int_t^\infty r(\omega b)(\tau)d\tau \, \|u_0 - u^\infty\|$$

(3.7)
$$+ \omega^{-1}(r(\omega b)*\|g_1 - g_1^\infty\|)(t)$$
$$+ \omega^{-1}(r(\omega b)*\|g_2\|)(t) \ .$$

Remarks. (i) Since b is completely positive and $b \notin L^1(0,\infty)$, it follows (see Proposition 2.1) that $r(\omega b) \in L^1(0,\infty)$, and therefore, if the assumptions of part 1 hold, (3.6) implies $\lim_{t\to\infty} \|u(t) - u^\infty\| = 0$.

When b also satisfies $b \in AC[0,\infty]$ it follows [see Proposition 2.1] that $r(\omega b) \in L^1(0,\infty) \cap C[0,\infty)$, and $\lim_{t\to\infty} r(\omega b)(t) = 0$. Therefore (3.7) implies $\lim_{t\to\infty} \|u(t) - u^\infty\| = 0$, if the assumptions of part 2 hold.

(ii) As is clear from the proofs, the assumption $g \in L^\infty(0,\infty;X)$ and there exists g^∞ such that $\lim_{t\to\infty} \|g(t) - g^\infty\| = 0$ in part (1) of Theorem 3.2 can be weakened to $g \in L^1_{loc}(0,\infty;X)$ and there exists $g^\infty \in X$ such that $\lim_{t\to\infty} (b * \|g - g^\infty\|)(t) = 0$. Similar generalizations can be made in Theorem 3.2, part (2), and in Theorem 3.3.

Proof of Theorems 3.2, part 2. As in the proof of Theorem 3.2, part (1) in [17], we first prove the result with A replaced by A_λ, $\lambda > 0$, and then we pass to the limit as $\lambda \downarrow 0$. For $\lambda > 0$, let u_λ be the strong solution of the approximating equation

$$u_\lambda + b*A_\lambda u_\lambda = u_0 + b*g \qquad (t \in [0,\infty)) \ .$$

Using the definition of A_λ and applying (2.2) we see that u_λ satisfies the equation

(3.8)
$$u_\lambda = r(\lambda^{-1}b)*J_\lambda u_\lambda + s(\lambda^{-1}b)u_0 + \lambda r(\lambda^{-1}b)*g$$

for $t \in [0,\infty)$. Since A_λ is also m-accretive, there is a unique u_λ^∞ satisfying the limiting equation

(3.9)
$$u_\lambda^\infty + \bar{b}A_\lambda u_\lambda^\infty = u_0 + \bar{b}g_1^\infty \ .$$

Using the fact that $b \in L^1(0,\infty)$ and $g = g_1 + g_2$ we can rewrite (3.9) in the equivalent form

(3.10)
$$u_\lambda^\infty + b*A_\lambda u_\lambda^\infty = u_0 + b*g + b*(g_1^\infty - g_1) - b*g_2 - \xi w_\lambda^\infty \ ,$$

where

$$\xi(t) = \int_t^\infty b(s)ds \quad \text{and} \quad w_\lambda^\infty = \lambda u_\lambda^\infty - g_1^\infty .$$

Let $\eta : [0,\infty) \to R$ be the unique solution of the linear equation

$$\eta + \lambda^{-1} b * \eta = \xi \ ;$$

then obviously

$$\eta w_\lambda^\infty + \lambda^{-1} b * \eta w_\lambda^\infty = \xi w_\lambda^\infty \qquad (0 < t < \infty) . \tag{3.11}$$

Using (3.10), (3.11), (2.2) and the definition of A_λ we obtain

$$\begin{aligned} u_\lambda^\infty = {} & r(\lambda^{-1} b) * J_\lambda u_\lambda^\infty + s(\lambda^{-1} b) u_0 + \lambda r(\lambda^{-1} b) * g \\ & + \lambda r(\lambda^{-1} b) * (g_1^\infty - g_1) + \lambda r(\lambda^{-1} b) * g_2 - \eta w_\lambda^\infty . \end{aligned} \tag{3.12}$$

Subtracting (3.12) from (3.8) we obtain

$$\begin{aligned} \| u_\lambda^\infty(t) - u_\lambda^\infty \| \le {} & (r(\lambda^{-1} b) * \| u_\lambda^\infty - u_\lambda \|)(t) + \lambda (r(\lambda^{-1} b) * \| g_1^\infty - g_1 \|)(t) \\ & + \lambda (r(\lambda^{-1} b) * \| g_2 \|(t) + |\eta|(t) \| w_\lambda^\infty \| . \end{aligned} \tag{3.13}$$

It is shown in Clément [17] (see argument following (3.18) in [17]) that $\eta \ge 0$. Thus by using the same argument as in [17] one gets (take convolution of (3.13) with $\lambda^{-1} b$):

$$\begin{aligned} \| u_\lambda(t) - u_\lambda^\infty \| \le {} & \xi(t) \| w_\lambda^\infty \| + (b * \| g_1 - g_1^\infty \|)(t) \\ & + (b * \| g_2 \|)(t) \qquad (0 < t < \infty) . \end{aligned} \tag{3.14}$$

The conclusion (3.5) follows by using (3.9) and rewriting

$$\xi(t) \| w_\lambda^\infty \| = \frac{\int_t^\infty b(s)ds}{\int_0^\infty b(s)ds} \| u_0 - u_\lambda^\infty \| ,$$

and then letting $\lambda \downarrow 0$. Note that

$$u^\infty = (I + \bar{b}A)^{-1}(u_0 + \bar{b} g_1^\infty) = \lim_{\lambda \downarrow 0} (I + \bar{b}A_\lambda^{-1})(u_0 + \bar{b} g_1^\infty) = \lim_{\lambda \downarrow 0} u_\lambda^\infty .$$

Proof of Theorem 3.3. We first establish the results with A replaced by $A_{(\lambda)} = \omega I + B_\lambda$, $\lambda > 0$, where B_λ is the Yosida approximation of B, defined by $B = A - \omega I$. Note that B is m-accretive in X. Let u_λ be the strong solution of the approximating equation to (V_g) written in the form:

$$u_\lambda + \omega b * u_\lambda + \omega b * \omega^{-1} B_\lambda u_\lambda = u_0 + \omega b * \omega^{-1} g \;. \tag{3.15}$$

Since the kernel $b \notin L^1(0,\infty)$, we transform this equation into a form which has the property that its new kernel will be in $L^1(0,\infty)$ and completely positive. Indeed, if we take the convolution of (3.15) by $r(\omega b)$, subtract the result from (3.15), and use the definition of $r(\omega b)$ we get the approximating equation equivalent to (3.15):

$$u_\lambda + r(\omega b) * \omega^{-1} B_\lambda u_\lambda = u_0 + r(\omega b) * (\omega^{-1} g - u_0) \;. \tag{3.16}$$

From Proposition 2.1, $r(\omega b)$ is completely positive and $r(\omega b) \in L^1(0,\infty)$, with $\int_0^\infty r(\omega b)(\delta)d\delta = 1$.

To prove Theorem 3.3, part 1, we wish to apply Theorem 3.2, part 1, to (3.16). If g satisfies the assumptions of Theorem 3.3, part 1, so does $\omega^{-1}g - u_0$. Thus all assumptions of Theorem 3.2, part 1 are satisfied with b replaced by $r(\omega b)$, A replaced by $\omega^{-1}B_\lambda$, g replaced by $\omega^{-1}g - u_0$, and u by u_λ. We obtain (by (3.4)):

$$\|u_\lambda(t) - u_\lambda^\infty\| \le \int_t^\infty r(\omega b)(\tau)d\tau \, \|u_0 - u_\lambda^\infty\| + \omega^{-1}(r(\omega b) * \|g - g^\infty\|)(t) \quad (0 \le t < \infty) \;, \tag{3.17}$$

where u_λ^∞ is the unique solution of the limiting equation

$$u_\lambda^\infty + \omega^{-1} B_\lambda u_\lambda^\infty = \omega^{-1} g_\infty \;, \tag{3.18}$$

which exists because B_λ is m-accretive and $\omega > 0$. Note that (3.17) is the estimate (3.6) with u replaced by u_λ.

If g satisfies the assumptions of Theorem 3.3, part 2, and $b \in AC[0,\infty]$, if follows from Proposition 2.1. that $r(\omega b) \in L^1(0,\infty) \cap C[0,\infty]$. Thus with $g = g_1 + g_2$ we can apply Theorem 3.2, part 2, to (3.16) and we obtain (from (3.5)) the estimate:

$$\begin{aligned} \|u_\lambda(t) - u_\lambda^\infty\| &\le \int_t^\infty r(\omega b)(\tau)d\tau \, \|u_0 - u_\lambda^\infty\| \\ &+ (\omega^{-1} r(\omega b) * \|g_1 - g_1^\infty\|)(t) + (\omega^{-1} r(\omega b) * \|g_2\|)(t) \qquad (0 \le t < \infty) \;, \end{aligned} \tag{3.19}$$

where u_λ^∞ is the unique solution of the limiting equation

$$u_\lambda^\infty + \omega^{-1}B_\lambda u_\lambda^\infty = \omega^{-1}g_1^\infty \; . \tag{3.20}$$

Note that (3.19) is the estimate (3.7) with u replaced by u_λ.

Since B is m-accretive $\lim\limits_{\lambda\downarrow 0} u_\lambda^\infty = u^\infty$, where in the case of (3.18) u^∞ satisfies the limiting equation

$$u^\infty + \omega^{-1}Bu^\infty \ni \omega^{-1}g^\infty \; , \tag{3.21}$$

or equivalently u^∞ satisfies the limiting equation

$$Au^\infty \ni g^\infty \; . \tag{3.22}$$

Similary, in the case of (3.20) we find that u^∞ satisfies the limiting equation

$$Au^\infty \ni g_1^\infty \; . \tag{3.23}$$

It remains to prove that $\lim\limits_{\lambda\downarrow 0} u_\lambda = u$ in $L^1(0,T;X)$ for every $T > 0$, where u is the generalized solution of (V_g). Having done so it is immediate that the estimates (3.17), (3.19) hold with u_λ replaced by u thus obtaining (3.6) and (3.7). We know that $\lim\limits_{\lambda\downarrow 0} \int_0^T \|\tilde{u}_\lambda(t) - u(t)\|dt = 0$, where $\tilde{u}_\lambda \in L^1_{loc}[0,\infty;X]$ satisfies

$$\tilde{u}_\lambda + b*A_\lambda\tilde{u}_\lambda = u_0 + b*g \; . \tag{3.24}$$

Introduce the notation $\delta = \lambda(1 + \lambda\omega)^{-1}$. Since $A = \omega I + B$, one easily checks that

$$A_\lambda = \omega(1 + \omega\lambda)^{-1}I + B_\delta \; .$$

Thus the solution $\tilde{u}_\lambda$ of (3.24) satisfies the equation

$$\tilde{u}_\lambda + b*B_\delta \tilde{u}_\lambda = u_0 + b*(g - \omega(1 + \omega\lambda)^{-1}\tilde{u}_\lambda) \; , \tag{3.25}$$

or by (3.15):

$$u_\delta + b*B_\delta u_\delta = u_0 + b*(g - \omega u_\delta) \; . \tag{3.26}$$

To compare solutions of (3.25) and (3.26) we apply the inequality (3.3) of Theorem 3.1, and we obtain (note that $u_{01} = u_{02} = u_0$)

$$\|\tilde{u}_\lambda - u_\delta\| \le \omega b*(1 + \lambda\omega)^{-1}\|\tilde{u}_\lambda - u_\delta\| + \lambda\omega^2(1 + \lambda\omega)^{-1}b*\|\tilde{u}_\lambda\| \; ,$$

and hence also the estimate

$$\|\tilde{u}_\lambda - u_\delta\| \le \omega b*\|\tilde{u}_\lambda - u_\delta\| + \delta\omega^2 b*\|\tilde{u}_\lambda\| \; . \tag{3.27}$$

It follows from (3.27) that for every $T > 0$

$$(3.28) \qquad \int_0^T \|\tilde{u}_\lambda - u_\delta\|(t)dt \le \delta\omega^2 \|b\|_{L^1(0,T)} \cdot \left(1 + \|r(-\omega b)\|_{L^1(0,T)}\right) \int_0^T \|\tilde{u}_\lambda(t)\|dt \ .$$

Since $\tilde{u}_\lambda$ converges to u in $L^1(0,T;X)$, we finally obtain $\lim_{\lambda\downarrow 0} \int_0^T \|u_\delta - u\|(t)dt = 0$, which proves that the u_λ converges in $L^1[0,T;X]$ to the generalized solution u of (V_g) for every $T > 0$. This completes the proof of Theorem 3.3.

Example 3.4. A Conservation Law with Memory. As an illustration of Theorems 3.1 and 3.2 we consider the existence and qualitative properties of positive solution of the problem

$$(c) \qquad u(t,x) + \int_0^t b(t-s)\phi(u(s,x))_x ds = u_0(x) \qquad (t > 0,\ x \in R)\ .$$

We assume that $\phi \in C^1(R)$ is a given function. If $b \equiv 1$ problem (c) is equivalent to the nonlinear conservation law in one space dimension:

$$u_t + \phi(u)_x = 0\ ,\quad u(0,x) = u_0(x) \qquad (x \in R)\ .$$

Although no particular physical significance is claimed for (c), it evidently contains the usual conservation law as a special case. The latter has been studied extensively from special points of view. Crandall [24] has shown that if $\phi: R \to R$ is a given smooth strictly increasing function (actually ϕ continuous is sufficient) such that $\phi(0) = 0$, then the operator A defined by $Au = \phi(u)_x$ on the Banach space $X = L^1(R)$, with $D(A) = \{u \in L^1(R): \phi(u_x) \in L^1(R)\}$ (see [24, Definition 1.1 and Theorem 1.1]), is m-accretive on X, and $\overline{D(A)} = X$. Moreover, one has: $J_\lambda(0) = 0$ and $J_\lambda u \le J_\lambda v$ $(\lambda > 0)$, whenever $u \le v$, $u, v \in L^1(R)$.

In (c) assume that $b \in L^1_{loc}(0,\infty)$, $b \not\equiv 0$ completely positive on $[0,\infty)$, and $u_0 \in \overline{D(A)}$; to be specific take b nonnegative, nonincreasing and $\log b$ convex on $(0,\infty)$, $b \not\equiv 0$. Then by Gripenberg's result [34], see Theorem 3.7, Chapter 2, and Remark (i) following Theorem 2.2, and by Theorem 3.1, problem (c) has a unique generalized solution u; u is nonnegative whenever u_0 is nonnegative, and $u_1 \ge u_2$ whenever $u_{01} \ge u_{02}$. If, in addition, $b \in L^1(0,\infty)$, then this generalized solution u converges strongly in $L^1(R)$ as $t \to \infty$ to the element $u_\infty \in D(A)$ which is the unique solution of the limit equation

$$u_\infty(x) + (\int_0^\infty b(t)dt)\ \phi(u_\infty(x))_x = u_0(x) \qquad (x \in R)\ ;$$

u_∞ exists and is uniquely defined since $\int_0^\infty b(t)dt > 0$ and A is m-accretive.

4. Application to Nonlinear Heat Flow in a Material with Memory. In this section we apply the theory developed in Sections 2 and 3 of this chapter to discuss the global existence, uniqueness, positivity, and decay of positive solutions of the nonlinear heat flow problem formulated in Chapter 1, Section 2. This problem was discussed from a different point of view in Chapter 3, Section 4. In Chapter 3, Section 4, boundedness and decay results were obtained without consideration of the positivity of the data. Here the principal concern is the existence and decay of positive solutions when the data are positive.

Referring to Chapter 1, Section 2, we again study the heat flow initial-boundary value problem (see (1.4) Chapter 1, also (4.1) Chapter 3):

$$(4.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}[b_0\, u(t,x) + (\beta * u)(t,x)] = c_0\ \sigma(u_x(t,x))_x - (\gamma * \sigma(u_x)_x)(t,x) \\ \qquad\qquad + h(t,x) \qquad (0 < t < \infty,\ 0 < x < 1) \\ u(t,0) = u(t,1) \equiv 0 \qquad (t > 0) \\ u(0,x) = u_0(x) \qquad (0 < x < 1) \end{cases}$$

As before we assume that the constants b_0, c_0 are positive, that $\beta, \gamma \in L^1(0,\infty)$, and that

$$(\beta)\qquad b_0 + \int_0^\infty \beta(t)dt > 0$$

and

$$(\gamma) \qquad c_0 - \int_0^\infty \gamma(t)dt > 0 \ .$$

We also assume that the function $\sigma : R \to R$ satisfies assumptions (σ) of Lemma 1.3, Chapter 1, and that the external heat supply satisfies (at least) the assumption $h \in L^1_{loc}(0,\infty;L^2(0,1))$. Defining the functions

$$(C) \qquad C(t) = c_0 - \int_0^t \gamma(\tau)d\tau \qquad (0 < t < \infty) \ ,$$

$$(G) \qquad G(t,x) = b_0 \ u_0(x) + \int_0^t h(\tau,x)d\tau \qquad (0 < t < \infty, \ 0 < x < 1) \ ,$$

the heat flow problem (4.1) is equivalent to (see Chapter 1, Section 2) the abstract Volterra equation

$$(V_1) \qquad u + \beta * u + C * Au = G \qquad (0 < t < \infty, \ 0 < x < 1) \ ;$$

here we have taken the constant b_0 as 1 without loss of generality. The nonlinear operator $A = \partial\varphi$, defined in Lemma 1.3, Chapter 1, satisfies the properties established in Lemma 1.3, Chapter 1, provided σ satisfies assumptions (σ), and if $u_0 \in H^1_0(0,1)$, then $G(0,x) = u_0(x) \in H^1_0(0,1) = D(\varphi)$. We note also that in the case of more than one space dimension, the relevant heat flow problem formulated in Chapter 1, Section 2 is equivalent to the Volterra equation (V_1) with the nonlinear operator A defined in Remark 1.4, Chapter 1; the theory developed below applies equally well to this case.

The Volterra equation (V_1) may be written in the standard form (V_g) of Section 3 above by defining the resolvent kernel $r(\beta)$ of β to be the unique solution of the linear equation:

$$(r(\beta)) \qquad r(\beta) + \beta * r(\beta) = \beta \qquad (0 < t < \infty) \ ;$$

clearly, if $\beta \in L^1(0,\infty)$, then $r(\beta) \in L_{loc}(0,\infty)$ (at least). Next, define $b : [0,\infty) \to R$ by

$$(b) \qquad b = C - r(\beta) * C \ ,$$

where C is the function defined in (C). Then the variation of constants formula shows that (V_1) is equivalent to the Volterra equation

$$u + b*Au = G - r(\beta)*G \qquad (0 < t < \infty) ;$$

taking $b_0 = 1$ in (G) one sees that (V_1), as well as the heat flow problem (4.1), is equivalent to the Volterra equation

$$u + b*Au = u_0 + 1*(h - r(\beta)*h - u_0 r(\beta)) \qquad (0 < t < \infty) . \tag{4.2}$$

The result of applying the theory of Section 3 on positivity and decay of solutions of (V_g) to the heat flow problem (4.1) is:

Theorem 4.1. *Let* β *be bounded, nonnegative, nonincreasing and convex on* $[0,\infty)$. *Let* γ *be nonnegative, nonincreasing, log convex and bounded on* $[0,\infty)$. *Let assumption* (γ) *hold, and let*

$$\beta'(t) + \frac{\gamma(0)}{c_0}\beta(t) < 0 \quad \text{a.e. for } t \in [0,\infty) . \tag{4.3}$$

Let the assumption (σ) *be satisfied, and let* $A = \partial\varphi$ *where* φ *is defined in* Lemma 1.3, Chapter 1.

1. *If* $u_0 \in L^2(0,1)$ *and if the forcing function* $h \in L^2_{loc}([0,\infty) \times [0,1))$, *then the nonlinear Volterra equation* (4.2) (equivalent to the heat flow problem (4.1)) *possesses a unique strong solution* u on $[0,\infty)$, *such that* $t\,u' \in L^2_{loc}(0,\infty;L^2(0,1))$; *if* $u_0 \in H^1_0(0,1)$, *then* $u' \in L^2_{loc}(0,\infty;L^2(0,1))$.

2. *If the data* u_0 *and* h *satisfy* $u_{0,1}(x) < u_{0,2}(x)$ a.e. on $[0,1]$ *and if* $h^1(t,x) < h^2(t,x)$ a.e. *on* $[0,\infty) \times [0,1]$; *then the corresponding strong solutions* u_i $(i = 1,2)$ *satisfy* $u_1(t,x) < u_2(t,x)$ a.e. $[0,\infty) \times [0,1]$; *in particular, if* $u_0(x) > 0$ *and* $h(t,x) > 0$ a.e. *on* $[0,1]$ *and* $[0,\infty) \times [0,1)$ *respectively, then* $u(t,x) > 0$ a.e. *on* $[0,\infty) \times [0,1]$.

3. *If, in addition,* $\beta \in L^1(0,\infty)$, *and if* $h = h_1 + h_2$, *where* $h_1 \in L^\infty(0,\infty;L^2(0,1))$ *and there exists* $h_1^\infty \in L^2(0,1)$ *such that* $\lim_{t\to\infty} \|h_1(t) - h_1^\infty\|_{L^2(0,1)} = 0$, *and where* $h_2 \in L^p(0,\infty;L^2(0,1))$ *for some* $p > 1$, *then the strong solution* u *of* [4.2] *converges strongly in* $L^2(0,1)$ *as* $t \to \infty$ *to the element* $u^\infty \in L^2(0,1)$; u^∞ *is the unique solution of the limit equation* $Au^\infty = g_1^\infty$, *where*

$$g_1^\infty = \frac{h_1^\infty}{c_0}\left(1 + \frac{\bar\gamma}{C(\infty)}\right),\ \bar\gamma = \int_0^\infty \gamma(t)dt,\quad C(\infty) = c_0 - \int_0^\infty \gamma(t)dt > 0\ .$$

<u>In particular, if</u> $h_1^\infty = 0$, <u>then</u> $u^\infty = 0$.

Before giving the proof of Theorem 4.1 we state a lemma which establishes some properties of the kernel b defined by (b).

<u>Lemma 4.2.</u> <u>Let</u> β, γ, C <u>satisfy the assumptions of Theorem 4.1. Then</u> b <u>defined by</u> (b) <u>is completely positive on</u> $[0,\infty)$, b <u>satisfies the assumption</u> (H) (Remark (i) following Theorem 2.2), <u>and</u> α, k <u>associated with</u> b <u>in Theorem 2.2 satisfy</u> $\alpha = c_0^{-1} > 0$ <u>and</u> $k \in L^1(0,\infty)$ <u>with</u>

$$\int_0^\infty k(\tau)d\tau = \frac{1}{c_0}\left[\frac{1}{C(\infty)}\bar\gamma(1 + \bar\beta) + \bar\beta\right], \tag{4.4}$$

<u>where</u> $\bar\beta = \int_0^\infty \beta(t)dt$, $\bar\gamma = \int_0^\infty \gamma(t)dt$. <u>Moreover,</u> $b \notin L^1(0,\infty)$ <u>and</u> $b' \in L^1(0,\infty)$.

<u>Proof.</u> Since the functions β and $C \in AC_{loc}[0,\infty)$ it follows that the functions $r(\beta)$ and $b \in AC_{loc}[0,\infty)$ (see definitions $(r(\beta))$ and (b) respectively). Note that $b(0) = c_0 > 0$. Define $\alpha = b(0)^{-1}$ and let k be the solution of the linear Volterra equation

$$b(0)y + b'*y = \frac{b'}{b(0)} \qquad (0 < t < \infty)\ . \tag{k}$$

Since $b' \in L^1_{loc}[0,\infty)$, $k \in L^1_{loc}(0,\infty)$, and since

$$\frac{d}{dt}(b*k)(t) = b(0)k(t) + (b'*k)(t) = -\frac{b'(t)}{b(0)}\ ,$$

one has by integration that k satisfies the linear Volterra equation

$$\alpha b(t) + (k*b)(t) = 1 \qquad (0 < t < \infty)\ . \tag{4.5}$$

In order to show that b is completely positive it suffices, by Theorem 2.2, to show that k is nonnegative, nonincreasing and bounded on $[0,\infty)$. We first observe that the assumptions made on γ imply that C, $-C'$ are convex and $\log(-C')$ is convex on $(0,\infty)$. This in turn implies that $\log C$ is convex on $(0,\infty)$, see G. Gripenberg [33].

Since C is nonnegative, nonincreasing and belongs to $L^1_{loc}(0,\infty)$, C is completely positive on $[0,\infty)$. Moreover C also satisfies assumption (H) (see Remark (i) following Theorem 2.2). It follows from Theorem 2.2 that there exists $\alpha_c > 0$, and $k_c \in L^1_{loc}(0,\infty)$, nonnegative, nonincreasing and bounded satisfying

(4.6) $$\alpha_c C(t) + (k_c * C)(t) = 1 \qquad (0 < t < \infty) .$$

Note that $\alpha_c = c_0^{-1} = b^{-1}(0) = \alpha$. From the definitions of b in (b), of $r(\beta)$, and from (1.2) and (1.3) (see Section 1, Chapter 4) it follows that

(4.7) $$C(t) = b(t) + (\beta * b)(t) \qquad (0 < t < \infty) .$$

Substituting (4.7) into (4.6) yields

$$\alpha b + (k_c + \alpha\beta + k_c * \beta) * b = 1 ,$$

and thus (4.5) implies that

(4.8) $$k(t) = k_c(t) + \alpha\beta(t) + (k_c * \beta)(t) \qquad (0 < t < \infty) .$$

Since $k_c \in BV(0,\infty)$ we have

$$\frac{d}{dt} [\alpha\beta + k_c * \beta](t) = \alpha\beta'(t) + k_c(0)\beta(t) + \int_0^t \beta(t - \tau) dk_c(\tau) \quad \text{a.e.}$$

on $[0,\infty)$. Hypothesis (4.3) and the identity (4.6) imply that $\alpha\beta'(t) + k_c(0)\beta(t) = \frac{1}{c_0} [\beta'(t) + \frac{\gamma(0)}{c_0} \beta(t)] \le 0$. Moreover, since k_c is nonincreasing and β is nonnegative

$$\int_0^t \beta(t - \tau) dk_c(\tau) \le 0 \qquad (0 < t < \infty) .$$

Thus k is nonnegative and nonincreasing on $[0,\infty)$. Therefore, one also has $k \in BV[0,\infty)$ if $k \in L^\infty[0,1]$. But k_c and β are bounded and $\beta \in L^1(0,1)$ imply $k \in L^\infty[0,1]$ (note that here the assumption $\beta \in L^1(0,\infty)$ is not needed). Using Theorem 2.2 again it follows that b is completely positive and satisfies assumption (H).

We next establish that $b \notin L^1(0,\infty)$. Since $\dot{C} = -\gamma \in L^1(0,\infty)$, and $\lim_{t\to\infty} C(t) = C(\infty) = c_0 - \int_0^\infty \gamma(s)ds > 0$, it follows that $C \notin L^1(0,\infty)$. If $b \in L^1(0,\infty)$, it would follow from (4.7) and the assumption $\beta \in L^1(0,\infty)$ that $C \in L^1(0,\infty)$, a contradiction. Thus $b \notin L^1(0,\infty)$.

We next prove that $b' \in L^1(0,\infty]$. Indeed, from (b) it follows that

$$b'(t) = C'(t) - C(0)r(\beta)(t) - (r(\beta)*C')(t) .$$

But $C' = -\gamma \in L^1(0,\infty)$; moreover, $r(\beta) \in L^1(0,\infty)$, since β is nonnegative, nonincreasing, convex and $\beta \in L^1(0,\infty)$ (use the Paley-Wiener theorem and the fact that β is positive definite).

Finally, we show that $k \in L^1(0,\infty)$. From (4.8) and the fact that $\beta \in L^1(0,\infty)$, it is sufficient to prove $k_c \in L^1(0,\infty)$. From (4.6) and the fact that C is positive, nonincreasing, $C(\infty) > 0$ and k is nonnegative, we have

$$C(\infty) \int_0^t k_c(\tau)d\tau \leq C*k_c \leq 1 \qquad (0 \leq t < \infty) ,$$

which proves that $k_c \in L^1(0,\infty)$. Formula (4.4) follows easily from (4.8) and the differentiated form of (4.6). This completes the proof of Lemma 4.2.

<u>Remark.</u> In Lemma 4.2, if $\beta(t) = \sum_{k=1}^{n} b_k e^{-\beta_k t}$ with $b_k > 0$ and $0 < \beta_1 < \beta_2 \dots < \beta_n$, then condition (4.3) is satisfied if $\beta_1 > \frac{\gamma(0)}{c_0}$ holds. Indeed, since $\log \beta$ is convex and nonincreasing, it suffices to require

$$\lim_{t\to\infty} \frac{\beta'(t)}{\beta(t)} < -\frac{\gamma(0)}{c_0} .$$

<u>Proof of Theorem 4.1.</u> We begin with the proof of the existence and uniqueness of strong solutions of the Volterra equation (4.2). Defining $f\colon [0,\infty) \times L^2(0,1) \to L^2(0,1)$ by

$$f = u_0 + 1*(h - r(\beta)*h - u_0 r(\beta)) \tag{4.9}$$

we have

$$f' = h - r(\beta)*h - u_0 r(\beta) \qquad (0 \leq t < \infty,\ 0 < x < 1) . \tag{4.10}$$

It follows from Lemma 4.2 that the kernel b satisfies the assumption

(H) $b(0) > 0,\ b \in AC_{loc}[0,\infty),\ b' \in BV_{loc}[0,\infty)$,

and that $f \in W^{1,2}_{loc}(0,\infty;L^2(0,1))$ whenever $h \in L^2_{loc}((0,\infty) \times [0,1])$. Since under assumptions (σ), $A = \partial\varphi$, where φ is defined in Lemma 1.3, Chapter 1, the existence and uniqueness of strong solutions u with the properties asserted in conclusion 1 of Theorem 4.1 follows from the result of Crandall and Nohel [26, Theorem 4 and Theorems 3.5 and 3.10, Chapter 2].

We next establish the asymptotic results asserted in Theorem 4.1. Since $r(\beta) \in L^1(0,\infty)$ (proved in the demonstration of Lemma 4.2), and since $h = h_1 + h_2$ one has from (4.10) that

$$f' = (h_1 - r(\beta)*h_1) + (h_2 - r(\beta)*h_2) - r(\beta)u_0 \qquad (0 < t < 0,\ 0 < x < 1) ,$$

where $h_1 - r(\beta)*h_1 \in L^\infty(0,\infty;L^2(0,1))$, and

$$\lim_{t\to\infty} (h_1 - r(\beta)*h_1)(t) = (1 - \int_0^\infty r(\beta)(\tau)d\tau)h_1^\infty = s(\beta)(\infty)h_1^\infty = (1 + \bar{\beta})^{-1}h_1^\infty ;$$

moreover,

$$(h_2 - r(\beta)*h_2 - r(\beta)u_0) \in L^1(0,\infty;L^2(0,1)) + L^p(0,\infty;L^2(0,1)) ,$$

with $1 < p < \infty$.

By using (4.10), the fact that the kernel b defined by (b) is by Lemma 4.2 completely positive, and Theorem 2.2, we can write the Volterra equation (4.2) in the equivalent form

$$u + b*Au = u_0 + b*(\alpha f' + k*f') \qquad (0 < t < \infty) . \tag{4.11}$$

To arrive at (4.11) we use the relation $\alpha b + k * b = 1$ in the right hand side of (4.9) and recombine terms making use of (4.10). Thus (4.11) is in the basic form (V_g) of Section 3 with

$$g = \alpha f' + k*f' \qquad (0 < t < \infty) . \tag{4.12}$$

From Lemma 4.2 $k \in L^1(0,\infty)$, and $g = g_1 + g_2$, where (using $h = h_1 + h_2$ in (4.10))

$$g_1 = \alpha(h_1 - r(\beta)*h_1) + k*(h_1 - r(\beta)*h_1) \qquad (0 < t < \infty] , \tag{4.13}$$

$$g_2 = g_{2,1} + g_{2,2} \qquad (0 < t < \infty) \tag{4.14}$$

with

$$g_{2,1} = -\alpha u_0 r(\beta) - \alpha r(\beta)*h_2 + k*(h_2 - r(\beta)*h_2 - u_0 r(\beta),\ g_{22} = \alpha h_2 .$$

Clearly $g_{21} \in L^1(0,\infty;L^2(0,1))$ and $g_{22} \in L^p(0,\infty;L^2(0,1))$. From Lemma 4.2 one has that b is completely positive on $[0,\infty)$, $b \notin L^1(0,\infty)$, and $b' \in L^1(0,\infty)$. Thus all assumptions of Theorem 3.3, part 2, are satisfied. We conclude that estimate (3.7) of Theorem 3.3, Section 3, holds, and therefore $\lim_{t\to\infty} \|u(t) - u^\infty\| = 0$, where $u^\infty = A^{-1}g_1^\infty$ with g_1^∞ given in the statement of Theorem 4.1; note that to evaluate g_1^∞ use is made of (4.4) and of Proposition 2.1.

Finally, we establish the "comparison" result asserted in Theorem 4.1, part 2. Let $P = \{u \in L^2(0,1) : u \geq 0\}$; P is a closed convex cone in $L^2(0,1)$ and $v - u \in P$ if and only if $u \leq v$. Moreover, it is standard that if $u \leq v$ then $J_\lambda u \leq J_\lambda v$ for every $\lambda > 0$, where $J_\lambda = (I + \lambda A)^{-1}$. We shall prove the result for solutions of the Volterra equation (V_1) which is equivalent to (4.2). As usual we shall prove the result for solutions of the approximating equation $(V_{1\lambda})$ of (V_1) in which A is replaced by the Yosida approximation A_λ, $\lambda > 0$, and then obtain the result by letting $\lambda \downarrow 0$.

Let $u_{0,i} \in L^2(0,1)$, $h_i \in L^1(0,1)$, $i = 1,2$, satisfy $u_{01} \leq u_{02}$ and $h_1(t) \leq h_2(t)$ a.e. on $[0,T]$; let $u_{\lambda,i}$ be the strong solutions of the approximating equation

$$(V_{1\lambda}) \qquad u_{\lambda,i} + \beta*u_{\lambda,i} + \lambda^{-1}C*u_{\lambda,i} = \lambda^{-1}C*J_\lambda u_{\lambda,i} + u_{0,i} + 1*h_i$$
$$(i = 1,2;\ \lambda > 0;\ 0 \leq t \leq T) .$$

It follows by an elementary calculation (which uses the definitions of b, $r(\lambda^{-1}b)$, and the relation $C \equiv b + \beta*b$) that

$$(4.15) \qquad u_{\lambda,i} = r(\lambda^{-1}b)*J_\lambda u_{\lambda,i} + f_{\lambda,i} \qquad (i = 1,2;\ \lambda > 0) ,$$

where $f_{\lambda,i}$ are solutions of the linear Volterra equation

$$(4.16) \qquad f_{\lambda,i} + \beta*f_{\lambda,i} + \lambda^{-1}C*f_{\lambda,i} = u_{0i} + 1*h_i \quad (i = 1,2) .$$

Hence by a familiar calculation one has

$$f_{\lambda,i} = u_{0,i}s(\beta + \lambda^{-1}C) + h_i*S(\beta + \lambda^{-1}C) \quad (i = 1,2) ,$$

and from (4.15), (4.16) the difference $u_{\lambda,2} - u_{\lambda,1}$ satisfies the Volterra equation

(4.17) $$u_{\lambda,2} - u_{\lambda,1} = r(\lambda^{-1}b)*(J_\lambda u_{\lambda,2} - J_\lambda u_{\lambda,1}) + (u_{0,2} - u_{0,1})s(\beta + \lambda^{-1}C) + (h_2 - h_1)*s(\beta + \lambda^{-1}C) .$$

Since $\beta + \lambda^{-1}C$ is positive, nonincreasing, it follows from Levin's result (see Section 2) that $s(\beta + \lambda^{-1}C)(t) > 0$. Thus

$$z_\lambda = (u_{0,2} - u_{0,1})\, s(\beta + \lambda^{-1}C) + (h_2 - h_1)*s(\beta + \lambda^{-1}C)$$

satisfies $z_\lambda(t) > 0$ a.e. on $[0,\infty)$.

Next, define $v_\lambda = u_{\lambda,2} - u_{\lambda,1} - z_\lambda$; using (4.17) we have that v_λ satisfies

(4.18) $$v_\lambda = r(\lambda^{-1}b)*(J_\lambda(v_\lambda + u_{\lambda,1} + z_\lambda) - J_\lambda(u_{\lambda,1})) .$$

As in [18] (see also Remark (iii) following the statement of Theorem 3.1, Section 3, Chapter 4) one shows that $v_\lambda = \lim_{n\to\infty} v_{\lambda,n}$, where

(4.19) $$v_{\lambda,n+1} = r(\lambda^{-1}b)*(J_\lambda(v_{\lambda,n} + u_{\lambda,1} + z_\lambda) - J_\lambda(u_{\lambda,1}))$$

where $v_{\lambda,1} \in L^1(0,T;L^2(0,1))$ is arbitrary. Choosing $v_{\lambda,1}(t) \in P$ a.e. on $[0,T]$, $T > 0$ *arbitrary, one shows easily* that $v_{\lambda,n}(t) \in P$ a.e. on $[0,T]$ for all positive integers n. This also uses the fact that $r(\lambda^{-1}b) > 0$, that J_λ is an increasing map with respect to the ordering $<$, and that $z_\lambda(t) \in P$ a.e. on $[0,T]$. Thus $v_\lambda(t) \in P$ a.e. on $[0,T]$, and for $\lambda > 0$:

(4.20) $$u_{\lambda,2}(t) - u_{\lambda,1}(t) = z_\lambda(t) + v_\lambda(t) > 0 \quad \text{a.e. on } [0,T] .$$

Since $T > 0$ is arbitrary, (4.20) holds on $[0,\infty)$, and the conclusion follows by letting $\lambda \downarrow 0$. This completes the proof of Theorem 4.1.

Appendix 1

Proof of Lemma 2.2: (a) Consider the resolvent equation (k) of B'. Since $B' \in L^1(0,\infty)$, the Paley-Wiener theorem [69] yields that $k \in L^1(0,\infty)$ if and only if

$$P(z) = b(0) + \hat{B}'(z) = b_\infty + z\hat{B}(z) \neq 0 \qquad (\mathrm{Re}\ z \geq 0) .$$

With $z = x + iy$

$$\mathrm{Re}\ P(z) = b_\infty + x\ \mathrm{Re}\ \hat{B}(z) - y\ \mathrm{Im}\ \hat{B}(z) \qquad (x \geq 0)$$

$$\mathrm{Im}\ P(z) = x\ \mathrm{Im}\ \hat{B}(z) + y\ \mathrm{Re}\ \hat{B}(z) \qquad (x \geq 0) .$$

Since $P(z)$ is analytic in $\mathrm{Re}\ z > 0$ and continuous in $\mathrm{Re}\ z \geq 0$, $\mathrm{Re}\ P(z)$ and $\mathrm{Im}\ P(z)$ are harmonic for $x > 0$. Hence by the maximum principle for harmonic functions, $P(z) \neq 0$ for $x \geq 0$ if either $\mathrm{Re}\ P(iy) = b_\infty - y\ \mathrm{Im}\ \hat{B}(iy)$, or $\mathrm{Im}\ P(iy) = y\ \mathrm{Re}\ \hat{B}(iy)$ are different from zero for $-\infty < y < \infty$. But by the frequency domain condition (F) $\mathrm{Re}\ P(iy) > 0$ for $-\infty < y < \infty$, and thus $k \in L^1(0,\infty)$.

(b) Since $B' \in L^1(0,\infty) \cap L^2(0,\infty)$ and $k \in L^1(0,\infty)$, one has $B' * k = k * B' \in L^2(0,\infty)$, and the result $k \in L^2(0,\infty)$ follows by inspection of equation (k). If also $B'' \in L^1(0,\infty)$, then $B' \in C[0,\infty)$ (so that $|B'(0)| < \infty$) and we may differentiate (k) to obtain

$$b(0)k'(t) + B'(0)k(t) + (B''*k)(t) = -\frac{B''(t)}{b(0)} \qquad (0 \leq t < \infty) ,$$

and clearly $k' \in L^1(0,\infty)$.

(c) If, as is the case here $k' \in L^1(0,\infty)$, the energy inequality in (c) is derived by the following simple argument (see the method of [67, Theorem 1]). Extend k' evenly for $t < 0$, and let

$$w_T(t) = \begin{cases} w(t) & \text{if } t \in [0,T] \\ 0 & \text{otherwise} . \end{cases}$$

Then

$$\int_0^T w(t)\frac{d}{dt}(k*w)(t)dt = k(0)\int_0^T w^2(t)dt + \int_0^T w(t)(k'*w)(t)dt$$

$$= k(0)\int_0^T w^2(t)dt + \frac{1}{2}\int_0^T w(t)\int_0^T k'(t-\tau)w(\tau)d\tau dt$$

$$= k(0)\int_{-\infty}^{\infty} w_T^2(t)dt + \frac{1}{2}\int_{-\infty}^{\infty} w_T(t)\int_{-\infty}^{\infty} k'(t-\tau)w_T(\tau)d\tau dt \ .$$

Letting $\tilde{w}_T(\eta) = \int_{-\infty}^{\infty} e^{i\eta t}w_T(t)dt$, $(\eta \in R)$, the Parseval and convolution theorems give

$$\int_0^T w(t)\frac{d}{dt}(k*w)(t)dt = \frac{k(0)}{2\pi}\int_{-\infty}^{\infty}|\tilde{w}_T(\eta)|^2 d\eta + \frac{1}{4\pi}\int_{-\infty}^{\infty}|\tilde{w}_T(\eta)|^2\tilde{k}'(\eta)d\eta \ .$$

But $\tilde{k}'(\eta) = 2\text{Re}\ \hat{k}'(i\eta)$, where $\hat{\ }$ is the Laplace transform, and $\text{Re}\ \hat{k}'(i\eta) = \text{Re}[i\eta\hat{k}(i\eta) - k(0)]$. Therefore,

$$\int_0^T w(t)\frac{d}{dt}(k*w)(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\tilde{w}_T(\eta)|^2\text{Re}[i\eta\hat{k}(i\eta)]d\eta \ .$$

Now an easy calculation from equation (k) yields

$$\text{Re}[i\eta\hat{k}(i\eta)] = \text{Re}\frac{1}{\hat{b}(i\eta)} = \frac{\text{Re}\ \hat{B}(i\eta)}{(\text{Re}\ \hat{B}(i\eta))^2 + \left(\text{Im}\ \hat{B}(i\eta) - \frac{b_\infty}{\eta}\right)^2}$$

$$= \frac{\eta^2\text{Re}\ \hat{B}(i\eta)}{\eta^2(\text{Re}\ \hat{B}(i\eta))^2 + (\text{Im}\ \hat{B}(i\eta)\eta - b_\infty)^2} \geq 0 \qquad (-\infty < \eta < \infty) \ ,$$

where the last inequality follows from the assumption that B is a kernel of positive type on $[0,\infty)$ (which is equivalent to $\text{Re}\ \hat{B}(i\eta) \geq 0$ [67, Thm. 2]; note that it is impossible to bound $\text{Re}[i\eta\hat{k}(i\eta)]$ away from zero, even if B is strongly positive on $[0,\infty)$).

(d) Multiply equation (k) by $\sqrt{t}$:

$$b(0)\sqrt{t}\,k(t) + \sqrt{t}\,(B'*k)(t) = -\frac{\sqrt{t}\,B'(t)}{b(0)} \qquad (0 < t < \infty) .$$

An elementary calculation involving $\sqrt{t}(B'*k)$ shows that $\sqrt{t}\,k$ satisfies

$$b(0)\sqrt{t}\,k(t) + \int_0^t B'(t-\tau)\sqrt{\tau}\,k(\tau)d\tau = -\frac{\sqrt{t}\,B'(t)}{b(0)} - \int_0^t (\sqrt{t}-\sqrt{\tau})B'(t-\tau)k(\tau)d\tau$$

$$(0 < t < \infty) .$$

Since $\sqrt{t} - \sqrt{\tau} < \sqrt{t-\tau}$ for $0 < \tau < t$, and since also $\sqrt{t}\,B' \in L^1(0,\infty)$ by assumption and $k \in L^1(0,\infty)$ by (a), the integral on the right side of the last equation defines a function in $L^1(0,\infty)$. Then $\sqrt{t}\,k \in L^1(0,\infty)$ by the argument of part (a). The additional assumptions and elementary estimates also yield $\sqrt{t}\,k \in L^2(0,\infty)$.

Finally, differentiating the equation (k) as in part (b), multiplying the resulting equation by $\sqrt{t}$, and using elementary estimates yields $\sqrt{t}\,k' \in L^1(0,\infty)$. This corpletes the proof of Lemma 2.2.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2081	2. GOVT ACCESSION NO. AD-A089634	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) NONLINEAR VOLTERRA EQUATIONS FOR HEAT FLOW IN MATERIALS WITH MEMORY		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) John A. Nohel		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE May 1980
		13. NUMBER OF PAGES 90
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Nonlinear Volterra equation, m-accretive operators, Maximum monotone operators on a Hilbert space, Subdifferential of a proper, convex, l.s.c. function, Boundedness, Asymptotic behaviour, Limiting equation, Strong solutions, Generalized solutions, Energy methods, Frequency domain methods, Heat flow, Materials with memory, Positive solutions, Completely positive kernels

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Consider the nonlinear Volterra equation

$$\text{(V)} \qquad u(t) + (b * Au)(t) \ni f(t), \qquad (0 \le t < \infty)$$

in the general setting $b : [0,\infty) \to R$ a given kernel, A a nonlinear m-accretive operator on a real Banach space X, $f : [0,\infty) \to X$ a given function, and $*$ the convolution. This paper, based on lectures delivered at West Virginia

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AD-A089 634 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 20/13
NONLINEAR VOLTERRA EQUATIONS FOR HEAT FLOW IN MATERIALS WITH ME--ETC(U)
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20. ABSTRACT - Cont'd.

University, discusses existing and recent results for the following problems
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concerning ~~(V)~~: 1. the global existence and uniqueness of solutions and their continuous dependence on the data; 2. the boundedness and asymptotic behaviour as $t \to \infty$ (approaches infinity) in the special cases when $X = H$ is a real Hilbert space and A is either a maximal monotone operator on H or A is a subdifferential of a proper, convex, lower semicontinuous function, and ~~$\varphi : H \to (-\infty, +\infty]$, 3.~~ the existence, boundedness, and asymptotic behaviour of positive solutions in the general setting. The theory is used to study one possible model problem for heat flow in a material with "memory" which can
of the equation
be transformed to the equivalent form ~~(V)~~ under physically reasonable assumptions; the latter provide a motivation for the natural setting of much of the theory developed here. This and various other models for heat flow in such materials are formulated from physical principals and discussed in an introductory chapter.