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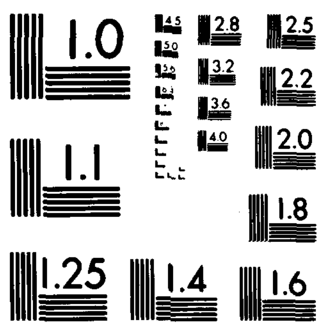

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ODD ZONAL HARMONICS IN THE GEOPOTENTIAL, FROM ANALYSIS OF 28 SATELLITE ORBITS

by

D.G. King-Hele
C.J. Brookes
G.E. Cook

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Procurement Executive, Ministry of Defence
Farnborough, Hants

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SUMMARY

The geopotential is usually expressed as an infinite series of spherical harmonics, and the odd zonal harmonics are the terms independent of longitude and antisymmetric about the equator: they define the 'pear-shape' effect. The coefficients J_3, J_5, J_7, ... of these harmonics have been evaluated by analysing the variations in eccentricity of 28 satellite orbits from near-equatorial to polar. Most of the orbits from our previous determination in 1973 are used again, but three new orbits are added, including two at inclinations between 62° and 63°, which have been specially observed for more than five years by the Hewitt cameras. With the help of the new orbits and revised theory, we have obtained sets of J-coefficients with standard deviations about 40% lower than before. A 9-coefficient set is chosen as representative, and is as follows:

$10^9 J_3 = -2530 \pm 4$	$10^9 J_9 = -90 \pm 7$	$10^9 J_{15} = -20 \pm 15$
$J_5 = -245 \pm 5$	$J_{11} = 159 \pm 9$	$J_{17} = -236 \pm 14$
$J_7 = -336 \pm 6$	$J_{13} = -158 \pm 15$	$J_{19} = -27 \pm 19$

With this set of values, the pear-shape asymmetry of the geoid (north polar minus south polar radius) amounts to 45.1 m instead of the previous 44.7 m. The accuracy of the longitude-averaged geoid profile is estimated as 50 cm, except at latitudes above 86°. The geoid profile and predicted amplitude of the oscillation in eccentricity are compared with those from other sources.

Departmental Reference: Space 576

* Earth Satellite Research Unit, University of Aston in Birmingham

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LIST OF CONTENTS

1 INTRODUCTION

The accepted form of expression for the Earth's gravitational potential is in terms of a double infinite series of spherical harmonics dependent on latitude and longitude. The terms of order zero in this expansion, which are independent of longitude, are called zonal harmonics. The even zonal harmonics, those of degree 2, 4, 6, ..., are symmetric about the equator; the odd zonal harmonics, those of degree 3, 5, 7, ..., which are the subject of this paper, are anti-symmetric about the equator and define the pear-shape tendency of the Earth.

The odd zonal harmonics give rise to various perturbations in satellite orbits, usually sinusoidal oscillations with the same period as the argument of perigee ω. The most accurately measurable of these perturbations is the oscillation in eccentricity, e, as a result of which the perigee distance $a(1-e)$, where a is the semi major axis, undergoes an oscillation which has an amplitude of order 10 km. The amplitude of this oscillation varies greatly with the orbital inclination i, and if accurate orbits could be determined for satellites at orbital inclinations of, say, 1°, 2°, 3°, ..., 90°, their observed amplitudes could be analysed to produce reliable values of the odd zonal coefficients J_3, J_5, J_7, ... up to high degree. In practice, satellite launches tend to be confined to particular bands of inclination, and there are still wide gaps in the coverage.

In the past 12 years, one of the main aims of the British programme for optical tracking of satellites, using the Hewitt cameras (particularly that at Malvern), has been to obtain accurate orbits of the satellites which would be most useful in improving the values of odd zonal harmonics. The first results of this work[1] emerged in 1974, when the previous determination of odd zonal harmonics[2] made in 1968 was improved by the addition of eight new orbits, including two determined at RAE from Hewitt camera and other observations, those of Cosmos 44* [3] at inclination 65° and Cosmos 248 [4] at inclination 62°. These orbits were both at inclinations close to the critical inclination of 63.4° where the most serious gap existed. Results from three satellites at low inclinations (3°, 5° and 15°) were also included, to help fill the gap in this region.

Even after the addition of these satellites, however, the lack of orbits at inclinations very close to 63.4° remained a major limitation on the accuracy of the values of the odd zonal harmonics, because the amplitude of the oscillation in eccentricity becomes extremely large near this inclination. Also the motion

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* The international designations of all the satellites used are given in Table 1 on page 11.

023

of perigee is very slow, and the observations have to be made over an interval of several years before the orbit can be determined over a half-cycle of the argument of perigee. So the observation of satellites at inclinations near 63.4° with the Hewitt cameras has continued, and orbits have been determined at the University of Aston from Hewitt camera and other observations on two satellites, Cosmos 373 at 62.9° inclination[5] and Cosmos 248 over a longer time interval than before[6].

The present paper takes advantage of the results from these two satellites and also utilizes NASA orbits on one further satellite at a previously unrepresented inclination, Explorer 46 at inclination 38°. The basic methods used are unchanged; but there are some significant improvements in the theory, and the method of solution has been modified by including an optional number of constraint equations. The new sets of values of odd zonal harmonics obtained should be considerably better than the previous set, and should give better values for high-degree coefficients than solutions which do not utilize satellites so near the critical inclination.

2 THEORY

2.1 The geopotential

The Earth's gravitational potential may be expressed as the sum of (i) an infinite series of zonal harmonics, independent of longitude, and (ii) a double infinite series of tesseral harmonics dependent on both latitude and longitude. In the absence of resonance effects[7], the tesseral harmonics do not contribute to the long-period variations in eccentricity which we are analysing, so we are only concerned with the longitude-averaged potential, $\bar{U}$. At an exterior point distant r from the Earth's centre and having geocentric latitude ϕ , the standard form for $\bar{U}$ is[8]

$$\bar{U} = \frac{\mu}{r}\left\{1 - \sum_{\ell=2}^{\infty} J_{\ell}\left(\frac{R}{r}\right)^{\ell} P_{\ell}(\sin\phi)\right\} \qquad (1),$$

where μ is the gravitational constant for the Earth[9], 398600 km^3/s^2, R is the equatorial radius[10], 6378.14 km, and $P_{\ell}(\sin\phi)$ is the Legendre polynomial of degree ℓ and argument $\sin\phi$. The J_{ℓ} are constant coefficients and we seek to determine J_3, J_5, J_7, ...

2.2 Perturbations of low-eccentricity orbits

Since many of the orbits utilized are of low eccentricity, the perturbations caused by odd zonal harmonics may most usefully be expressed in terms of $\xi = e \cos \omega$ and $\eta = e \sin \omega$, rather than in terms of e and ω. The variations in ξ and η are given by[11]:

$$\dot{\xi} = -k\eta + C \tag{2}$$

$$\dot{\eta} = k\xi \tag{3}$$

where C is a function of the odd zonal harmonics $J_3, J_5, J_7, \ldots$, k is a function of the even zonal harmonics $J_2, J_4, J_6, \ldots$ and both C and k are assumed constant*.

We may write C explicitly in the form[11]

$$C = \frac{3}{2}\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}}\left(\frac{R}{a}\right)^3 \sin i\left(1 - \frac{5}{4}f\right)\left[-J_3 + \frac{2}{4-5f}\sum_{n=2}^{\infty} E_{2n+1}J_{2n+1}\left(\frac{R}{p}\right)^{2n-2}\right] \tag{4}$$

where $f = \sin^2 i$, and the E_{2n+1} are functions of e and f specified by recurrence relations given as equations (6) to (9) of Ref 12. The first two E-functions are[13]

$$E_5 = 5\left(1 + \frac{3}{4}e^2\right)\left(1 - \frac{7}{2}f + \frac{21}{8}f^2\right) \tag{5}$$

$$E_7 = -\frac{35}{4}\left(1 + \frac{5}{2}e^2 + \frac{5}{8}e^4\right)\left(1 - \frac{27}{4}f + \frac{99}{8}f^2 - \frac{429}{64}f^3\right) \tag{6}$$

and subsequent E-functions are of a similar form but lengthier. The e^2 and e^4 terms in (5) and (6) would be dropped for a near-circular orbit but are included here to show the proper forms of E_5 and E_7 for use with orbits of higher eccentricity. In equation (4), $p = a(1-e^2)$; and p would be taken equal to a for a near-circular orbit.

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* Both are functions of a, e and i, but these elements do not usually change enough to cause significant changes in C or k during the time when the orbit is being analysed.

The parameter k depends on $J_2, J_4, J_6, \dots$; and, since $J_2 = 1.1 \times 10^{-3}$ while $J_4, J_6, \dots$ are all of order 10^{-6} or less, the J_2-term is usually dominant. In previous papers[1,2,11], only the J_2 contribution to k has been taken into account. But at inclinations near 63.4° the terms in $J_4, J_6, \dots$ contribute significantly: their contribution amounts to 2% of the main term for one of the new satellites, 1970-87A at inclination 62.9°. It is therefore necessary to evaluate further terms in k .

If ℓ is even, equation (7) of Ref 11 gives the J_ℓ term in $\bar{U}$ as

$$U_\ell = -J_\ell\left(\frac{\mu}{R}\right)\left(\frac{R}{a}\right)^{\ell+1}\left[P_\ell(\cos i)P_\ell(0)X_0^{-(\ell+1),0} - \frac{2(\ell-2)!}{(\ell+2)!}P_\ell^2(\cos i)P_\ell^2(0)X_0^{-(\ell+1),2}\cos 2\omega + O(e^4)\right] \quad (7),$$

where the Hansen coefficient $X_0^{-(\ell+1),s}$ is of order e^s , so that only terms with $s < 4$ need be considered: of these terms, those with $s = 1$ and $s = 3$ do not arise for ℓ even because the associated Legendre function $P_\ell^s(0) = 0$ if $\ell + s$ is odd; thus only the $s = 2$ term appears in (7). When e is small

$$\left.\begin{aligned} X_0^{-(\ell+1),0} &= 1 + \frac{\ell(\ell+1)}{4}e^2 + O(e^4) \\ \text{and} \qquad X_0^{-(\ell+1),2} &= \frac{(\ell-1)(\ell-2)}{8}e^2 + O(e^4) \end{aligned}\right\} \quad (8).$$

From equation (3) of Ref 11,

$$\dot{\xi} = -(\mu a)^{-\frac{1}{2}}\left[\left\{\frac{1}{e}\frac{\partial U_\ell}{\partial e} - \cot i\,\frac{\partial U_\ell}{\partial i}\right\}\eta + \frac{1}{e}\frac{\partial U_\ell}{\partial \omega}\cos\omega\right]\left[1 + O(e^2)\right] \quad (9).$$

On inserting the partial differentials of U_ℓ using (7) and (8), and writing $\cos i = u$, equation (9) becomes

$$\dot{\xi} = J_\ell\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}}\left(\frac{R}{a}\right)^\ell\left[P_\ell(0)\left[\frac{\ell(\ell+1)}{2}P_\ell(u) + u\frac{\partial}{\partial u}\left\{P_\ell(u)\right\}\right]\eta + \frac{(\ell-2)\xi}{2\ell(\ell+1)(\ell+2)}P_\ell^2(u)P_\ell^2(0)\sin 2\omega\right]\left[1 + O(e^2)\right] \quad (10).$$

For $\ell = 2$, the ξ term in (10) vanishes. For $\ell = 4$, the ξ term has a numerical factor $\frac{1}{16}$, while the main η term has a numerical factor $\frac{15}{4}$, and is therefore 60 times larger. Thus, for $\ell = 4$, the ξ term is much smaller than the η term, which is itself much smaller than the main J_2 term. So the ξ term in equation (10) may be ignored for $\ell = 4$, and, for similar reasons, for $\ell = 6, 8, 10, \ldots$. On inserting the explicit forms for P_2, P_4, P_6, P_8 and P_{10} (given, for example, in Ref 14), we find the contribution of even harmonics to $\dot{\xi}$, the term $-k\eta$ in equation (2), is defined by

$$k = \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} \left(\frac{R}{a}\right)^2 3J_2' \left(1 - \frac{5}{4}f\right) \qquad (11),$$

where J_2' , which may be regarded as a value of J_2 adjusted for the effects of higher even harmonics, is given by

$$\frac{J_2'}{J_2} = 1 - \frac{5(R/a)^2}{8(5\lambda - 1)J_2} \left[J_4(49\lambda^2 - 36\lambda + 3) - \frac{7}{8}\left(\frac{R}{a}\right)^2 J_6(297\lambda^3 - 375\lambda^2 + 115\lambda - 5)\right.$$

$$+ \frac{21}{128}\left(\frac{R}{a}\right)^4 J_8(7865\lambda^4 - 14014\lambda^3 + 7700\lambda^2 - 1330\lambda + 35)$$

$$\left. - \frac{231}{2048}\left(\frac{R}{a}\right)^6 J_{10}(54587\lambda^5 - 125307\lambda^4 + 99918\lambda^3 - 32214\lambda^2 + 3591\lambda - 63)\right] \quad \ldots\ldots(12),$$

where $\lambda = 1 - f = \cos^2 i$, and terms beyond J_{10} have not been evaluated because they are likely to be negligible.

The general solution of equations (2) and (3), with C given by (4) and k by (11) and both being assumed constant, is

$$\xi = A\cos(kt + \alpha) \qquad (13)$$

$$\eta = A\sin(kt + \alpha) + \beta \qquad (14),$$

where A and α are constants of integration, dependent on the initial conditions, and

$$\beta = \frac{C}{k} = \frac{R\sin i}{2aJ_2'}\left\{-J_3 + \frac{2}{4 - 5f}\sum_{n=2}^{\infty} E_{2n+1}J_{2n+1}\left(\frac{R}{p}\right)^{2n-2}\right\} \qquad (15)$$

from equations (4) and (11).

Equations (13) and (14) show that the variation of e and ω takes the form of a circle in the (ξ, η) plane. The circle has the centre $(0, \beta)$ and radius A, and is swept out at a constant angular rate k. If the observational values of ξ and η, after removal of other perturbations, are plotted in the (ξ, η) plane and fitted with a circle, the centre of the circle will give a value of β and hence, via equation (15), one linear equation between the coefficients $J_3, J_5, J_7, \ldots$ which it is convenient to rewrite in the form

$$F_3J_3 + F_5J_5 + F_7J_7 + \ldots = (Y \pm \sigma) \times 10^{-6} \qquad (16),$$

where $F_3 = -1$ and $F_{2n+1} = \dfrac{2}{4 - 5f}\left(\dfrac{R}{p}\right)^{2n-2} E_{2n+1}$ for $n \geqslant 2$ (17),

$$Y = \frac{2aJ_2'\beta}{R \sin i} \times 10^6 \qquad (18)$$

and σ is the standard deviation in Y.

In our previous determinations of odd zonal harmonics[1,2], the values of Y were calculated using J_2 rather than J_2' in equation (18). The use of J_2' changes the value of Y by an amount which is significant for inclinations near 63°, but negligible at other inclinations (see section 3).

2.3 Orbits of higher eccentricity ($e \gg \beta$)

Equations (13) and (14) may be written

$$\left.\begin{aligned} e\cos\omega &= A\cos(kt + \alpha) \\ e\sin\omega - \beta &= A\sin(kt + \alpha) \end{aligned}\right\} \qquad (19).$$

On squaring and adding equations (19), we have

$$(e - \beta\sin\omega)^2 = A^2 - \beta^2\cos^2\omega \qquad (20).$$

Thus if $\beta \ll A$,

$$e = \beta\sin\omega + A\left(1 - \frac{\beta^2}{4A^2}\right) - \frac{\beta^2}{4A}\cos 2\omega + O\left(\frac{\beta^4}{A^3}\right) \qquad (21).$$

023

If the $\cos 2\omega$ term is small enough, it is most convenient to evaluate β by fitting a sinusoidal curve to the observed variation of e and determining its amplitude. Such a fitting would be acceptable if $\beta/4A$ was less than about a quarter of the fractional error in β : a typical value of β is $(500 \pm 5) \times 10^{-6}$, and this suggests that the sinusoidal fitting should not be used unless $A > 0.05$. In practice, however, comparison with the circle-fitting method shows that the sinusoidal fitting may be used successfully for values of A down to 0.03, presumably because the error term, $(\beta^2/4A) \cos 2\omega$, has a different frequency.

Another possible method of evaluating β would be to utilize the equation

$$e^2 = A^2 + \beta^2 + 2\beta A \sin(kt + \alpha) \qquad (22)$$

obtained by squaring and adding (13) and (14). This equation has the advantage of being exact, in the absence of perturbations. With a real orbit, however, perturbed by drag, the parameter k is not constant and the fitting of a sinusoidal variation to the observed values of e^2 , assuming constant k , would be subject to an error which is very difficult to estimate. It is better to fit a circle in the (e, ω) plane, or to fit equation (21), so that k only arises subsequently, in expressing β in terms of the J coefficients through equation (15): a mean value of k can then be used, with an error which is easily assessed and usually negligible.

2.4 Treatment of the effect of even harmonics

When $e > 0.03$ and the sinusoidal fitting is used, the perturbation due to even harmonics, given as equation (9) of Ref 1, may be appreciable. This perturbation varies as $\cos 2\omega$, and a term of this form has been included in the fitting when necessary, as explained in Ref 1.

In the fitting of the low-eccentricity orbits, the effects of even zonal harmonics are *not* separated out, and the appropriate expression for β is equation (15), which includes a contribution from even zonal harmonics within J_2' . For orbits of higher eccentricity, when the even harmonic contribution is fitted separately by including a $\cos 2\omega$ term, the proper expression for β is equation (15) with J_2' replaced by J_2 , that is

$$\beta = \frac{R \sin i}{2aJ_2}\left\{- J_3 + \frac{2}{4 - 5f} \sum_{n=2}^{\infty} E_{2n+1} J_{2n+1}\left(\frac{R}{p}\right)^{2n-2}\right\} \qquad (23).$$

023

The difference between (15) and (23) is negligible except at inclinations near 63.4°; but, strictly, equation (23) should be regarded as the 'correct' expression for the amplitude of the oscillation in eccentricity caused by *odd* zonal harmonics only; it would be identical with (15) if J_4, J_6, J_8, ... were zero.

3 THE ORBITS USED

3.1 General

If a reliable set of odd zonal harmonic coefficients J_3, J_5, J_7, ... is to be derived from equations of the form (16), the complete range of inclinations from 0° to 90° should be covered, and, since the E and F functions depend primarily on $f = \sin^2 i$, the ideal would be a large number of orbits with values of f uniformly distributed between 0 and 1, except for the region near $f = 0.8$, where a greater density of data is needed, because Y becomes very large. Although the inclinations of the orbits actually available have gaps, the distribution of the 27 orbits previously used was fairly uniform: there were, however, too many orbits with values of f near 1.0, and none with values between 0.1 and 0.2, or between 0.3 and 0.4 (see Fig 10 of Ref 1). Three new orbits have now been added, one of which is Cosmos 248, and supersedes the orbit of that satellite previously used. Also, one of the previous orbits, that of Explorer 20 at inclination 79.9°, has been dropped because there are three other orbits with similar inclinations which are more accurate and fitted better. So we are left with 28 orbits, which are listed in Table 1 in order of increasing f. The Table gives the relevant orbital parameters, and the values of Y with their nominal standard deviations, σ. In the course of the computations it proved advantageous to increase some of the values of σ (see section 4.3): these values of σ are marked with an asterisk in Table 1 and the amended values σ' are given in the final column.

For most of the satellites in Table 1, the values of Y are the same as before; for some, however, the values of β were re-determined by fitting a circle to the values of ξ and η instead of the previous sinusoidal fitting of e (see section 3.3).

For the orbits with inclinations outside the range 59° to 66°, the values of Y were calculated using equation (18) with J_2 rather than J_2' : a change from J_2 to J_2' would not have altered any of the values of Y by more than 0.1σ. For the five orbits with inclinations between 59° and 66°, the values of Y were calculated by using equation (18) with J_2' rather than J_2. The

Table 1

Mean values of a, e, and i, and values of f, Y and σ for the 28 orbits

Satellite		a km	e	i deg	f ($= \sin^2 i$)	Y	σ	σ'
Explorer 42	1970-107A	6900.0	0.0025	3.03	0.0027	0.49	0.62	
Dial	1970-17A	7339.0	0.0880	5.41	0.0089	1.19	0.22	
Peole	1970-109A	7007.7	0.0166	15.00	0.0670	2.51	0.05	
Explorer 11	1961ν1	7512.6	0.0862	28.80	0.2321	2.424	0.02	
LCS 1	1965-34C	9165.5	0.0013	32.14	0.2830	2.351	0.1*	0.2
OSO 3	1967-20A	6931.9	0.0022	32.86	0.2945	2.292	0.05	
Vanguard 2	1959α1	8300.0	0.1641	32.88	0.2947	2.304	0.05*	0.2
Explorer 46	1972-61A	7028.0	0.0225	37.69	0.3738	2.166	0.05	
Explorer 27	1965-32A	7507.2	0.0251	41.19	0.4336	2.272	0.03	
Telstar 1	1962αε1	9672.1	0.2423	44.80	0.4965	2.460	0.02	
Echo 1 rocket	1960ι2	7971.5	0.0114	47.23	0.5389	2.407	0.05*	0.1
Anna 1B	1962βμ1	7507.8	0.0070	50.13	0.5890	2.561	0.03	
Ariel 2	1964-15A	7192.4	0.0730	51.65	0.6150	2.721	0.05	
Tiros 5	1962αα1	7158.7	0.0263	58.10	0.7208	4.177	0.05	
Explorer 29	1965-89A	8073.1	0.0717	59.38	0.7406	4.038	0.1	
Cosmos 248	1968-90A	6841.9	0.0060	62.238	0.7830	11.25	0.03*	0.3
Cosmos 373	1970-87A	6875.9	0.0170	62.910	0.7926	21.83	0.08*	0.5
Explorer 32	1966-44A	7709.0	0.1372	64.66	0.8168	-2.917	0.03	
Cosmos 44	1964-53A	7109.5	0.0174	65.07	0.8223	-2.092	0.03*	0.1
Transit 4A	1961ο1	7318.2	0.0080	66.80	0.8448	0.811	0.08*	0.1
Secor 5	1965-63A	8159.0	0.0791	69.23	0.8743	1.857	0.05	
Geos 2	1968-02A	7705.2	0.0320	105.79	0.9260	2.359	0.02	
FR-1	1965-101A	7128.6	0.0010	75.88	0.9405	2.371	0.05	
Alouette 2	1965-98A	8120.0	0.1528	79.82	0.9688	2.736	0.05*	0.2
Prospero	1971-93A	7441.9	0.0692	82.06	0.9809	2.910	0.03*	0.2
Essa 1	1966-08A	7147.3	0.0097	97.91	0.9810	2.903	0.1*	0.2
Midas 4	1961αδ1	10005.0	0.0121	95.86	0.9896	2.697	0.02	
Transit	1963-49B	7473.1	0.0036	89.96	1.0000	3.070	0.02	

* These values of σ are increased in the course of the computations, to the value σ' shown in the last column.

values of J_2'/J_2 on each of these five orbits were: 0.997 for Explorer 29; 0.990 for Cosmos 248; 0.976 for Cosmos 373; 1.009 for Explorer 32; and 1.007 for Cosmos 44. In retrospect it appears that it would have been more logical to have used J_2 rather than J_2' for Explorer 32, because the even zonal harmonic effect had already been allowed for: the change is small, but worth making in a future analysis.

3.2 The three new orbits

3.2.1 Cosmos 373, 1970-87A

The orbit of Cosmos 373, at inclination 62.9°, has been determined by Brookes[5] at 25 epochs between February 1971 and July 1975, from more than 1500 optical and radar observations, including Hewitt camera observations for all 25 orbits. The orbit determination covered nearly half a cycle of the argument of perigee, and the values of eccentricity had standard deviations corresponding to accuracies between 30 and 120 m in perigee height.

The values of eccentricity were cleared of lunisolar perturbations and the effects of air drag; a circle was then fitted to the resulting values of ξ and η, giving $\beta = (8.53 \pm 0.03) \times 10^{-3}$. This value is of excellent accuracy, and it is also the largest observational value of β obtained from any orbit of high accuracy: it corresponds to an amplitude of 58.7 km for the oscillation in perigee height. On using equation (18), Cosmos 373 gives $Y = 21.83 \pm 0.08$, although the sd has to be increased when computing the solutions for the J coefficients, because of the errors incurred by neglecting odd harmonics of high degree (see section 4.3).

3.2.2 Cosmos 248, 1968-90A

The orbit of Cosmos 248, at inclination 62.2°, has been determined by Brookes and Holland[6] at 57 epochs over nearly 1¼ cycles of the argument of perigee, from January 1972 to December 1975. The orbit was computed from about 3000 optical and US Navy observations, using the RAE orbit refinement program PROP, and the accuracies achieved in eccentricity correspond to between 30 and 120 m in perigee height.

The values of eccentricity were cleared of lunisolar perturbations and the effects of air drag, and a circle was fitted to the resulting values of ξ and η. The fitting was excellent and the value of β obtained was $(4.33 \pm 0.01) \times 10^{-3}$. This was a satisfactory improvement on, and confirmation of, the value 4.30 ± 0.02 obtained previously[1] from the orbit determination in the years 1969 to 1971. On using equation (18), we find $Y = 11.25 \pm 0.03$, but, as with Cosmos 373, the sd has to be increased to allow for neglected odd harmonics of high degree (see section 4.3).

3.2.3 Explorer 46, 1972-61A

In our previous evaluation of odd zonal harmonics, there was no orbit with an inclination between 33° and 41°. This serious gap has now been filled by using Explorer 46 at inclination 37.7°, for which NASA orbits were kindly supplied by S.M. Klosko.

Explorer 46, 1972-61A, also known as Meteoroid Technology Satellite, was launched on 13 August 1972 into an orbit with perigee height 500 km and apogee 820 km. The satellite was a cylinder of mass 175 kg, 3.2 m long and about 50 cm in diameter[15]. The mass/area ratio exceeds 100 kg/m^2 and the orbit is nearly circular, so the effects of solar radiation pressure should not be significant. The effect of air drag is however quite considerable, and consequently it is better to work with the perigee height than with the eccentricity.

There were 27 orbits of Explorer 46 available, running from MJD 41543 (1972 August 14) to MJD 41817 (1973 May 15), and covering 6 cycles of the argument of perigee. In Fig 1 selected values of semi major axis a are shown by triangles joined by a broken line. For this satellite it was expected that the oscillation in perigee height due to odd zonal harmonics would be about $4.1 \sin \omega$ km, and the circles in Fig 1 show the values of $\{a(1 - e) + 4.1 \sin \omega\}$, which should be almost free of the odd harmonic oscillation. This is confirmed by Fig 1, which also shows that the variation of $\{a(1 - e) + 4.1 \cdot \sin \omega\}$ is of the same form as the variation in semi major axis, as it should be. The amount $(\Delta r_p)_{AD}$ to be added to $a(1 - e)$ to compensate for the decrease due to air drag is taken to be as given by the unbroken curve in Fig 1: the curve has a slope proportional to da/dt . If $a(1 - e) + (\Delta r_p)_{AD}$ is written as $a_0(1 - e')$, the values of e' provide a set of values of e cleared of air drag perturbations, and this set can be fitted either by a circle or a sine curve, to obtain a value of β . Lunisolar perturbations to the perigee height were calculated, but were ignored because they were always less than 80 m.

The 27 values of e' are plotted against ω in Fig 2: it can be seen that the drag effects have been successfully removed, and that the points on all 6 cycles of ω lie close to a sine curve with an amplitude β of 0.555×10^{-3} . The points have been numbered chronologically, so that the successive cycles can be followed.

The values of $e' \cos \omega$ and $e' \sin \omega$ have been fitted with a circle by least-squares, as shown in Fig 3, with a representative selection of the points. The fitting gives the centre of the circle at the point $(-0.007, 0.555) \times 10^{-3}$. The mean value of eccentricity, given by the radius of the circle, is 0.022501. Although the fitting of a circle is necessary on theoretical grounds, the form of presentation in Fig 2 is visually preferable, because its wider scale gives an impression of the residual errors, which are scarcely perceptible in Fig 3. The value of β obtained from the fitting of the circle is $(0.555 \pm 0.003) \times 10^{-3}$, from which $Y = 2.166 \pm 0.012$. The sd of 0.012 is increased to 0.05 in conformity with the treatment of the other satellites in Ref 1.

3.3 Orbits with β re-determined by fitting a circle

Three orbits with eccentricity less than 0.02, for which β was previously determined by fitting a sine curve, have been fitted with a circle in the (ξ, η) plane. As explained in section 2, this procedure is theoretically preferable for eccentricities less than about 0.03, although in practice the changes proved to be quite small. For these three orbits the effect of air drag is negligible.

The first of the three satellites is Echo 1 rocket at 46° inclination, for which the fitted circle is shown in Fig 4, with a small selection of the 61 points fitted. The centre C of the circle is at the point $(0.003, 0.652) \times 10^{-3}$, and the value of $10^{3}\beta$ obtained is 0.652 ± 0.004 as compared with 0.650 ± 0.002 from the original fitting of a sine curve (Fig 4 of Ref 13), and 0.653 ± 0.002 in the revised fitting of Ref 2. The new value of β gives $Y = 2.404 \pm 0.015$, and the sd is increased to 0.05 in accordance with the procedures of Ref 1. (The old value of Y was 2.707 and the change is thus only 0.06σ: the old value was retained in the calculations.) The radius of the circle in Fig 4, giving the mean value of eccentricity, is 0.01147.

The second of the three satellites is Cosmos 44 at inclination 65.1°, for which the fitted circle is shown in Fig 5, with a selection of the 28 points fitted. The mean value of eccentricity is 0.01730, the centre C of the circle is at the point $(-0.025, -0.781) \times 10^{-3}$, and the value of β obtained from the fitting is $(-0.781 \pm 0.003) \times 10^{-3}$, giving $Y = -2.092 \pm 0.008$, as compared with -2.106 ± 0.024 previously. The sd is increased to 0.03 in accordance with the procedures of Ref 1.

The third satellite is Transit 4A at inclination 66.8°, for which the fitted circle is shown in Fig 6, with a small selection of the 137 points fitted. The mean value of eccentricity is 0.00805, the centre C of the circle is at

(-0.02, 0.30) × 10^{-3} and the value of $10^3\beta$ derived from the fitting is 0.30 ± 0.03 , as compared with 0.297 ± 0.003 from the sine curve. It is rather surprising that the fitting of the circle gives an sd so much larger, but it is of little consequence since the value of β is the same, and the sd is increased in the course of the computations. The new value of β gives γ = 0.811 ± 0.08 .

4 RESULTS

4.1 The equations to be fitted

We have 28 equations of the form (16), one for each of the 28 satellites. The 28 values of γ are given in Table 1; the values of $F_3, F_5, \ldots, F_{33}$ for all the satellites previously used are given in Table 3 of Ref 2 and Table 3 of Ref 1, and will not be repeated here. The values of the F coefficients for the three new orbits are given in Table 2 below. The values for Cosmos 248 are very slightly larger than before, because of the decrease in semi major axis.

Table 2

Values of $F_3, F_5, \ldots, F_{33}$ for the three new orbits

Satellite	F_3	F_5	F_7	F_9	F_{11}	F_{13}	F_{15}	F_{17}
Explorer 46	-1	0.2264	0.8056	-0.5524	-0.3465	0.5486	0.0082	-0.3890
Cosmos 248	-1	-13.4313	-13.0732	-1.7304	8.9327	10.3108	3.2719	-4.7544
Cosmos 373	-1	-29.1780	-30.4216	-7.0630	17.1225	23.0910	10.3971	-7.0499

Satellite	F_{19}	F_{21}	F_{23}	F_{25}	F_{27}	F_{29}	F_{31}	F_{33}
Explorer 46	0.1603	0.2029	-0.1976	-0.0573	0.1614	-0.0284	-0.0999	0.0619
Cosmos 248	-7.2384	-3.5129	2.0714	4.7330	3.0944	-0.5524	-2.9014	-2.4427
Cosmos 373	-15.0760	-9.9842	1.2485	8.7969	7.9864	1.4818	-4.5044	-5.6835

In recent determinations of geopotential coefficients from analysis of orbital resonances (for example, Ref 16), it has been found useful to apply constraints when solving for the values of the coefficients, by adding constraint equations based on Kaula's rule of thumb[17], that the normalized coefficients of degree ℓ are likely to have values of order $10^{-5}/\ell^2$. The first evaluation[18] of harmonics up to order and degree 180 has provided a test of this rule, and the general indications are that the actual values of the coefficients are smaller than $10^{-5}/\ell^2$ for $10 < \ell < 40$. The value $10^{-5}/\ell^2$ therefore seems to provide a reasonable upper limit, though it should be emphasized that the value represents an average over all orders, whereas the zonal harmonics are of a specific order (zero) and may not always conform. Since the J coefficients are not normalized, the normalization factor $\sqrt{2\ell + 1}$ must be inserted and, writing $\ell = 2n + 1$, the appropriate constraint equations are

$$J_{2n+1} = 0 \pm \frac{\sqrt{4n + 3} \times 10^{-5}}{(2n + 1)^2} \qquad (24).$$

These equations are included among those to be fitted for all values of n greater than or equal to a chosen minimum value n_0 . The value of n_0 is chosen after testing the effects of various choices (see section 4.2).

So the equations to be fitted are 28 equations of the form (16), with values of Y given in Table 1, and $(n_m - n_0 + 1)$ equations of the form (24), where n_m is the value of n for the highest-degree coefficient evaluated and is also a matter of choice, like n_0 . Thus the number of equations is $(29 + n_m - n_0)$ and they are solved for up to n_m coefficients.

4.2 The number of constraint equations

The effect of the constraints was tested by re-calculating the solutions of the equations of Ref 1 for 4, 5, 6, ..., 12 coefficients, first with constraints on the coefficients of degree 21, 23 and 25; then with constraints on degrees 15, 17, 19, ..., 25; and then on degrees 9, 11, 13, ..., 25 . The presence of the constraints slightly improved the fitting (as was inevitable, because the previous values of the high-degree coefficients were on average less than half the values specified by the constraints); but the values of the coefficients were not significantly altered. Since neither extreme offered advantage, the middle choice was adopted initially, with $n_0 = 7$, *ie* with constraints for degree 15 and above.

Most of the solutions were calculated with $n_0 = 7$; but since J_{17} remained large whether or not it was constrained, there was no point in keeping this constraint, and the final solutions all have $n_0 = 9$, so that there are constraints on the coefficients $J_{19}, J_{21}, \ldots, J_{2n_m+1}$. With $n_0 = 9$, the number of equations being solved is 28 for $n_m \leqslant 8$, and $20 + n_m$ for $n_m \geqslant 8$.

4.3 Progress of the solutions

The solutions were obtained by least-squares fitting, the measure of fit being assessed by the value of ε, where ε^2 is the sum of the squares of the weighted residuals, divided by the number of degrees of freedom - the weighted residual being the observational value of Y minus the value given by the left-hand side of equation (16), divided by σ (or σ'). In all, 572 solutions were computed.

First, as already mentioned, Explorer 20 was omitted because its fitted sine curve (Fig 13 of Ref 2) is unsatisfactory, and because there is a better satellite (Alouette 2) at the same inclination. The omission of Explorer 20 reduced ε, and the standard deviations of the values obtained in the solutions, by 1½% on average.

In the previous solutions, Cosmos 44 and Transit 4A had been in conflict, so σ was increased to 0.1 for both: this led to an improvement of nearly 15% in the standard deviations. Then the new values of Y for these two satellites (section 3.3) were inserted: the resulting change was not significant.

Of the three new satellites, Explorer 46 fits well and offers no problems, but the values for Cosmos 248 and 373 cannot be used as they stand because the F coefficients remain large up to beyond degree 33 (see Table 2), and our final solutions do not go beyond J_{29}. So σ must be increased to take account of the neglect of high-degree coefficients. The numerical values of the coefficients $F_{31}, F_{33}, \ldots, F_{49}$ for Cosmos 373 are: -4.5, -5.7, -2.4, 1.9, 3.7, 2.3, -0.4, -2.2, -1.9, -0.3; while the corresponding values of $\sqrt{2\ell+1}\ 10^{-5}/\ell^2$ are $(83, 75, 69, \ldots, 41) \times 10^{-9}$. The arithmetic mean of the ten values of $|F_\ell\sqrt{2\ell+1}\ 10^{-5}/\ell^2|$ is 165×10^{-9}, so the expected order of magnitude of the sum of these neglected terms is $\sqrt{10} \times 165 \times 10^{-9} \approx 0.5 \times 10^{-6}$. If the J coefficients are smaller than given by Kaula's rule, this value could be reduced; on the other hand, it needs to be increased to allow for the neglected harmonics of degree greater than 50. Thus $\sigma = 0.5$ seems a reasonable figure and in fact turns out to be a practical value. For Cosmos 248, the values of the F coefficients are about half as large as for Cosmos 373, giving $\sigma \approx 0.3$ for Cosmos 248; again this value proved satisfactory in practice and was adopted.

The other increases in σ, indicated by the asterisks in Table 1, were made because the values appeared to be ill-fitting, and increases in σ led to lower values of ε. For LCS 1, σ was increased from 0.1 to 0.2; this improved ε and the standard deviations of the coefficients by about 2%. For Vanguard 2,

σ was first relaxed to 0.1, which reduced ε by 2%; and then to 0.2, which reduced ε by another 2%. For Echo 1 rocket, σ was increased from 0.05 to 0.1; this reduced ε by 5%. In the previous solution, the three satellites with i, or $(180^\circ - i)$, near 80° were persistently ill-fitting, so the values of σ for these satellites were further increased: for Prospero, σ was increased from 0.1 to 0.2, which reduced ε by 3%; for Alouette 2 and Essa 1, σ was also increased to 0.2, and this reduced ε by another 3%. The effect of omitting individual satellites was also tested, but no advantage was gained: for example, the omission of Prospero increased ε by 1%.

All the percentages given in this section are averages over the solutions for 4, 5, 6, ..., 16 coefficients, but the effect is usually similar whatever the number of coefficients evaluated: a change tends to improve (or worsen) all solutions.

4.4 The final solutions

After the alterations specified in section 4.3, the solutions for n_m = 4, 5, 6, ..., 16 coefficients gave the following values of ε :

n_m	4	5	6	7	8	9	10	11	12	13	14	15	16
ε	3.17	2.79	2.14	2.06	0.571	0.543	0.537	0.526	0.518	0.518	0.495	0.494	0.494

For $n_m \leqslant 7$, the value of ε is too large to be acceptable: the inclusion of J_{17} is essential and greatly reduces ε. As n increases from 8 to 16, ε steadily decreases, with a larger decrease whenever the new J-value is relatively large.

Three of the solutions are selected for presentation here (Table 3).

The 8-coefficient solution is chosen because (a) it represents the field well with a minimal number of coefficients, (b) it serves as a basis for comparison with solutions having more coefficients, and (c) it provides a comparison with the recommended 8-coefficient solution of Ref 1.

The 9-coefficient solution is chosen because (a) it provides a significantly better fit than the 8-coefficient solution, with ε reduced by 5%, and (b) the standard deviations of the values are the lowest of any solution.

The 14-coefficient solution is chosen as the representative solution having a large number of coefficients, because there is a decrease in ε of 4.4% between the 13- and 14-coefficient solutions, but no further significant decrease on going to 15 or 16 coefficients.

Table 3

Selected solutions for J coefficients

	Previous 8-coefficient	New 8-coefficient	New 9-coefficient	New 14-coefficient
$10^9\ J_3$	-2531 ± 7	-2529 ± 5	-2530 ± 4	-2528 ± 6
J_5	-246 ± 9	-247 ± 5	-245 ± 5	-250 ± 7
J_7	-326 ± 11	-334 ± 7	-336 ± 6	-329 ± 9
J_9	-94 ± 12	-92 ± 7	-90 ± 7	-93 ± 7
J_{11}	159 ± 16	161 ± 10	159 ± 9	158 ± 12
J_{13}	-131 ± 22	-147 ± 14	-158 ± 15	-157 ± 20
J_{15}	-26 ± 24	-25 ± 15	-20 ± 15	-24 ± 27
J_{17}	-258 ± 19	-238 ± 15	-236 ± 14	-232 ± 24
J_{19}			-27 ± 19	-12 ± 23
J_{21}				-12 ± 33
J_{23}				31 ± 39
J_{25}				29 ± 39
J_{27}				-6 ± 35
J_{29}				54 ± 38

It will be seen from Table 3 that, for a particular ℓ, the values of any pair of J_ℓ differ by less than the sum of their standard deviations. So the solutions are self-consistent, and immune from instability as higher harmonics are evaluated. In the new 9-coefficient solution the standard deviations are about 40% lower than in the previous (8-coefficient) solution.

We select the 9-coefficient solution as our recommended set of harmonics, because it has the lowest sd; but there is little to choose between the three solutions, and the 8- or 14-coefficient solutions may be preferred for some purposes.

The weighted residuals for each orbit, as defined in section 4.3, are listed in Table 4.

Table 4

Weighted residuals in the three solutions of Table 3

Satellite	Weighted residual		
	8-coefficient	9-coefficient	14-coefficient
Explorer 42	0.682	0.515	0.395
Dial	-0.263	-0.442	-0.294
Peole	-0.004	0.178	0.078
Explorer 11	0.147	0.053	0.011
LCS 1	-0.288	-0.288	-0.276
OSO 3	0.062	0.439	0.167
Vanguard 2	-0.281	-0.260	-0.265
Explorer 46	-0.904	-0.652	-0.522
Explorer 27	0.529	0.511	0.418
Telstar 1	0.778	0.734	0.605
Echo 1 rocket	-0.438	-0.461	-0.510
Anna 1B	-0.243	-0.305	-0.519
Ariel 2	0.381	0.473	0.314
Tiros 5	-0.210	0.013	0.127
Explorer 29	0.582	0.646	0.630
Cosmos 248	-0.309	-0.556	-0.650
Cosmos 373	1.010	0.700	0.513
Explorer 32	0.207	0.135	0.162
Cosmos 44	-0.571	-0.567	-0.657
Transit 4A	0.828	0.719	0.594
Secor 5	0.025	-0.057	-0.072
Geos 2	-0.406	-0.403	-0.217
FR-1	-0.635	-0.561	-0.235
Alouette 2	0.275	0.298	0.293
Prospero	0.471	0.500	0.460
Essa 1	0.377	0.418	0.352
Midas 4	-0.342	-0.318	-0.313
Transit	-0.145	-0.327	-0.100

Table 4 shows that there are no startling changes in the fit as the number of coefficients increases, and when one residual in a conflicting pair fits better, it is usually at the expense of the other, *eg* Cosmos 248/Cosmos 373 and Cosmos 44/Transit 4A.

The weighted residuals for Cosmos 373 are rather large with the 8- and 9-coefficient solutions: this is to be expected, because the value of σ' for Cosmos 373 was taken as 0.5, which was the expected error due to neglecting J_{31}, J_{33}, ..., J_{49} , whereas the errors due to neglecting J_{21}, J_{23}, ..., J_{29} ought also to be included, giving a value of σ' about twice as large. A re-run of the 8- and 9-coefficient solutions with $\sigma' = 1.0$ for Cosmos 373 and 0.5 for Cosmos 248 gave nearly identical solutions, with standard deviations about 5% less for the 8-coefficient solution, and almost the same for the 9-coefficient solution. Although these solutions will not be used, the 9-coefficient solution is recorded here for reference (all $\times 10^9$): $J_3 = -2529$; $J_5 = -246$; $J_7 = -336$; $J_9 = -90$; $J_{11} = 159$; $J_{13} = -157$; $J_{15} = -22$; $J_{17} = -234$; $J_{19} = -22$.

4.5 The variation of β with inclination

Fig 7 shows the values of β given by the 9-coefficient solution, as obtained from equation (23) with $R/a = R/p = 0.9$ (which corresponds to a near-circular orbit at a mean height of just over 700 km). The right-hand scales on Fig 7 show the values of $a\beta$, the amplitude of the oscillation in perigee distance, which is less sensitive to variations in a than is β itself. Usually β is positive, and perigee is then closest to the Earth's centre at northern apex ($\omega = 90^\circ$); when β is negative, perigee is closest at southern apex ($\omega = 270^\circ$).

Fig 7 shows that the amplitude of the oscillation in perigee height increases from zero for an equatorial orbit to 5.7 km at an inclination of 50°, and then increases rapidly: for inclinations between 55° and 70°, see the diagram on the right, which has a semi-logarithmic scale for β . At inclinations between 63.4° and 66.1°, β is negative, *ie* the perigee is closest to the Earth at southern apex, and the amplitude reduces to zero at 66.1°. For inclinations between 66.1° and 90° the situation reverts to normal, with perigee closest in the northern hemisphere, and the amplitude of the oscillation increases from zero at 66.1° to 9.5 km at 90° inclination.

Fig 8 shows how β and $a\beta$ vary when $R/a = R/p = 0.8$, which corresponds to a near-circular orbit at a mean height of 1600 km. The amplitude of the oscillation in perigee height, $a\beta$, is generally slightly smaller: for example, at inclinations of 30°, 60° and 90°, the values are 3.5, 10.9 and 8.7 km respectively for $R/a = 0.8$, as compared with 3.5, 13.6 and 9.5 km respectively for $R/a = 0.9$. Another significant difference is that, when $R/a = 0.8$, β is negative over a smaller range of inclination, from 63.4° to 65.7° (instead of from 63.4° to 66.1° for $R/a = 0.9$). This is because the higher harmonics have less effect on a more distant orbit, and cannot cancel out the positive effect of J_3 over such a wide range of values of inclination.

Fig 9 gives an alternative version of Figs 7 and 8, avoiding the infinities. The quantity plotted is $\beta(4 - 5f)$, which remains within reasonable limits. The value of β can be obtained on dividing by $(5 \cos^2 i - 1)$.

4.6 Geoid height

The zonal harmonics serve to define the profile of the equipotential surface of the Earth's meridional section, averaged over all longitudes. If $V(r,\phi)$ is the external potential at latitude ϕ , including the rotational acceleration, we have

$$V(r,\phi) = \frac{\mu}{r}\left\{1 - \sum_{n=2}^{N} J_n\left(\frac{R}{r}\right)^n P_n(\sin\phi)\right\} + \tfrac{1}{2}r^2 w^2 \cos^2\phi \qquad (25),$$

where w is the Earth's angular velocity (72.92115×10^{-6} rad/s) and N is the degree of the highest-degree harmonic in the set. (The atmosphere is assumed to be internal.) The equipotential surface is found by setting $V(r,\phi) = V(R,0)$ and solving for r iteratively at each value of ϕ , giving a value r_G say. The value of r on a spheroid of flattening F is

$$r_s = \frac{R(1 - F)}{\{(1 - F\cos^2\phi)^2 + F^2\cos^2\phi\sin^2\phi\}^{\frac{1}{2}}} \qquad (26),$$

and we define the geoid height above the spheroid, h , as

$$h = r_G - r_s \qquad (27).$$

The values of geoid height h are, of course, affected by the even harmonics as well as the odd harmonics at most latitudes. Here our aim is to display and compare odd zonal harmonic values, so we choose a standard set of even zonal harmonics, that of GEM 10B[10] (given in Table 5), and indicate later the effect of changes in the even zonal harmonics.

Fig 10 shows the variation of geoid height with latitude given by our 9-coefficient set of odd harmonics and the GEM 10B even harmonics, with $F = 1/298.25$. This value of the flattening has been retained to allow comparison with Ref 1, but the change produced by altering F to $1/298.257$, in conformity with GEM 10B, is very small: the maximum change is at the poles, and is only 50 cm, as indicated by the arrows.

In Fig 10 the south polar depression of the geoid is 27.23 m and the north polar hump is 17.84 m, so that the north polar radius R_N of the geoid exceeds the south polar radius R_S by 45.1 m: in other words, sea level at the north pole is 45.1 m further from the equator than is sea level at the south pole. This 'pear-shape asymmetry', $(R_N - R_S)$, depends solely on the odd zonal harmonics, and is not affected by the particular choice of flattening or of even zonal harmonics.

5 COMPARISONS

5.1 Comparisons between our three solutions

5.1.1 Values of β

Fig 11 shows the variation of $\Delta\beta$, the value of β given by our 14-coefficient solution minus the value given by the standard 9-coefficient solution. The broken curve in Fig 11 gives the variation of $\Delta\beta$ for the 8-coefficient solution. Both solutions have $R/a = 0.9$. The maximum difference $a\Delta\beta$ is only 60 m, except at inclinations near 63.4° (where $\Delta\beta$ inevitably tends to infinity). Since few of the orbits are determined to an accuracy better than 100 m, agreement to within 60 m implies that we have exploited the available data as fully as possible, and that all three solutions give essentially the same results.

5.1.2 Geoid height

Geoid heights have been computed for the 8- and 14-coefficient solutions and are found to differ from those of Fig 10 by less than 30 cm for latitudes up to 86°: so the three profiles are virtually the same, and it seems reasonable to assess the errors in the longitude-averaged profile of Fig 10 due to errors in odd zonal harmonics as about 40 cm, at latitudes up to 86° - *ie* over more than 99% of the Earth.

Very close to the poles, however, the 14-coefficient solution diverges significantly, and Fig 12 shows the difference Δh between the geoid height given by the 14-coefficient solution and that given by the 9-coefficient solution, for latitudes greater than 80°. The (much smaller) differences for the 8-coefficient solution are shown by a broken line. The changes at the pole do significantly alter the pear-shape asymmetry: it is 43.5 m for the 14-coefficient solution and 44.6 m for the 8-coefficient solution, as compared with 45.1 m for the 9-coefficient solution. Thus the difference between the north polar and south polar radius of the geoid proves to be a sensitive parameter for assessing sets of odd zonal harmonics; conversely, it is a difficult parameter to estimate accurately, so that it is quite likely that the largest errors in many geoid maps will occur at the poles.

5.2 Effect of even zonal harmonics on geoid height

As already mentioned, the geoid height profile depends on the values of the even zonal harmonics as well as the odd, and Fig 13 shows the change when, instead of the GEM 10B even harmonics[10], we use a set of even harmonics determined at the Smithsonian Astrophical Observatory in 1974 (E.M. Gaposchkin and Y. Kozai, private communication), which will be called 'SAO 74', and is a revised version of the set used in the Smithsonian Standard Earth IV.3[19]. In Fig 13 the average numerical difference between the geoid heights is only 20 cm, so a change to the SAO 74 even harmonics would have little effect on Fig 10. From the evidence of Fig 13, it seems reasonable to assign an error of about 30 cm due to even zonal harmonics, giving an overall rms error of 50 cm in the geoid profile of Fig 10, for latitudes up to 86°.

The GEM 10B and SAO 74 even zonal harmonics are listed in Table 5.

Table 5

Values of even zonal harmonic coefficients from GEM 10B and SAO 74

	GEM 10B	SAO 74		GEM 10B	SAO 74
$10^9 J_2$	1082627	1082639	$10^9 J_{20}$	-157	-135
J_4	-1623	-1609	J_{22}	26	92
J_6	543	535	J_{24}	9	173
J_8	-208	-183	J_{26}	-14	
J_{10}	-242	-258	J_{28}	109	
J_{12}	-196	-179	J_{30}	12	
J_{14}	125	110	J_{32}	58	
J_{16}	36	16	J_{34}	62	
J_{18}	-63	-77	J_{36}	3	

5.3 Comparison with other sets of odd harmonics (GEM 10B, SAO 74, GRIM 2, WGS 72)

5.3.1 Values of β

The value of the amplitude β of the oscillation in eccentricity due to odd zonal harmonics has also been calculated for the sets of odd zonal harmonics of GEM 10B and SAO 74, which are listed in Table 6.

Table 6

Values of odd zonal harmonic coefficients from GEM 10B and SAO 74 (with our 9-coefficient set for comparison)

	GEM 10B	SAO 74	9-coefficient
$10^9 J_3$	-2536	-2553	-2530
J_5	-226	-207	-245
J_7	-363	-395	-336
J_9	-117	-97	-90
J_{11}	229	246	159
J_{13}	-223	-267	-158
J_{15}	-14	32	-20
J_{17}	-93	-141	-236
J_{19}	-11	55	-27

	GEM 10B	SAO 74
$10^9 J_{21}$	6	-56
J_{23}	140	153
J_{25}	-3	-219
J_{27}	-48	
J_{29}	51	
J_{31}	6	
J_{33}	31	
J_{35}	-78	

Fig 14 shows the variation of $\Delta\beta$, the value of β given by GEM 10B minus the value given by our 9-coefficient solution, with the corresponding value of $\Delta\beta$ from SAO 74 as a broken line. The differences are an order of magnitude greater than the differences between our three solutions, so it is quite acceptable to make the comparisons only with the 9-coefficient solution. Fig 14 shows that, away from 63.4°, the values of β predicted by GEM 10B differ from ours by up to about 350 m, while the SAO 74 values differ from ours by up to about 600 m. At inclinations up to 23° our values agree well with GEM 10B, while SAO 74 diverges from both (partly because it gives negative values of β for inclinations less than 5°). For inclinations between 25° and 32°, and again from 35° to 43°, GEM 10B and SAO 74 agree with each other better than with our values. Between 44° and 58°, our values and GEM 10B agree extremely well. As the inclination approaches 63.4°, the solutions diverge: at 63.0°, for example, the values of $a\beta$ from GEM 10B and SAO 74 fall short of ours by 2.0 and 4.4 km respectively. At inclinations greater than 64° the three solutions scarcely ever agree, and the divergences near 69° are the worst of all (apart from the infinities at 63.4°).

Fig 15 shows $\Delta\beta$ for the European geopotential model GRIM 2 (full line)[20] and for the World Geodetic System 1972 (WGS 72, broken line)[21]. For GRIM 2, the values are fairly close to those of SAO 74 for inclinations up to 60°, and fairly close to those of GEM 10B for 65° to 90°. The maximum divergence from our values is 600 m at an inclination of 7°. The WGS 72 model is rather an old one, and has only three coefficients, J_3, J_5 and J_7, but it is still used quite widely. The agreement with WGS 72 with our values is surprisingly good at inclinations between 16° and 53°, the difference being less than 250 m; but at other inclinations there are large discrepancies, and WGS 72 could not be recommended for inclinations between 56° and 71°, where $a\Delta\beta$ exceeds 700 m (apart from a chancy zero near 63.7° en route to infinity at 63.4°).

5.3.2 Discussion

It may seem surprising that GEM 10B, which utilizes thousands of laser observations accurate to better than 1 m, should predict values of β which reveal discrepancies of order 200 m when compared with our solution. However, it appears likely that, although GEM 10B incorporates a million observations on 30 satellites, only a few of these satellites were tracked by laser, and not all were observed over a complete cycle of the argument of perigee, as is necessary for a strong determination of odd zonal harmonics. Furthermore, the absence of satellites with inclinations near 63.4° in GEM 10B (the nearest is 115.0°, equivalent to 65.0°) removes the most powerful determinant of high-degree zonal harmonics. The SAO 74 solution, like ours, relied mainly on orbits determined from photographic observations; but again the absence of orbits with inclinations near 63.4° (59° and 67° are the nearest) weakens the solution for high-degree coefficients.

The inquiring reader may ask, "which of the three solutions is the best?". To hope for an unbiassed answer from us is, perhaps, expecting too much, but we shall try to be impartial. The following points seem relevant:

(1) Our solutions include two satellites at inclinations considerably closer to 63.4° than the other two solutions. Our solutions fit the results from these two satellites; the other solutions do not, and give amplitudes which fall short of the observational value of 57.1 km for Cosmos 373 by 2 km (for GEM 10B) and 4 km (for SAO 74).

(2) Our inclusion of these 'difficult' satellites near 63.4° may have affected the results at other inclinations. Our set of harmonics does not fit the orbits at inclinations near 80° as well as was hoped, and for these orbits (see Table 2) we had to increase σ' to 0.2.

Whatever the answer to the question posed, it is obvious that further determinations of odd zonal harmonics, utilizing more accurate orbits more closely spaced in inclination, are essential if the centimetric accuracy hoped for in the late 1980s is ever to be achieved. Odd zonal harmonics form an important part of the geopotential, and errors of order 200 m or more in their predicted effects will not be acceptable in future orbital models.

In the course of our solutions, the two most potent satellites, Cosmos 248 and Cosmos 373, had to be degraded in accuracy because of the influence of neglected high-degree harmonics. Thus the primary need is *not* for orbits of better accuracy, but for orbits of 100-m accuracy which cover the range of inclination more thoroughly. The higher harmonics (degree 31 and above) can then justifiably be included, and the full accuracy of the existing orbits can be exploited.

5.3.3 Geoid height

Fig 16 shows the difference Δh between the geoid heights given by the odd harmonics of GEM 10B, and those given by our 9-coefficient solution, with the corresponding values from SAO 74 shown by the broken line. The even harmonics are kept the same for all three. At latitudes up to 85° the GEM 10B geoid differs from ours by less than 80 cm, the average difference being about 25 cm, but there is greater divergence at the poles (1.6 m), and GEM 10B gives a pear-shape' asymmetry of 41.9 m as compared with values between 43.5 and 45.1 m from our three solutions. The SAO 74 geoid agrees with ours better than does GEM 10B, the maximum difference (including the poles) being 70 cm, and the average about 25 cm. The pear-shape asymmetry given by SAO 74 is 44.6 m.

These comparisons substantiate the estimate made in section 5.1.2, that the geoid shape from our 9-coefficient solution should be accurate to 40 cm, except possibly in a small region near the poles. The difference of 1.6 m between GEM 10B and our solution at the poles confirms that further improvement of the odd harmonic values is needed.

6 CONCLUSIONS

We have analysed 28 satellite orbits, including several close to the critical inclination of 63.4°, to determine improved sets of values of odd zonal

023

harmonic coefficients. The three sets obtained, with 8, 9 and 14 coefficients, do not differ significantly, and the 9-coefficient solution is chosen as representative. It is as follows:

$10^9 J_3$	$= -2530 \pm 4$	$10^9 J_{13}$	$= -158 \pm 15$
J_5	$= -245 \pm 5$	J_{15}	$= -20 \pm 15$
J_7	$= -336 \pm 6$	J_{17}	$= -236 \pm 14$
J_9	$= -90 \pm 7$	J_{19}	$= -27 \pm 19$
J_{11}	$= 159 \pm 9$		

The inclusion of higher-degree coefficients does not appreciably alter these values (see Table 3). This set of harmonics is considerably better than our previous set derived in 1973, the standard deviations being about 40% smaller.

The predicted amplitude of the oscillation in eccentricity (β) and in perigee distance ($a\beta$) experienced by satellites at any inclination between 0° and 90° is shown in Figs 7 to 9. The geoid height profile from pole to pole, averaged over all longitudes, is shown in linear form in Fig 10, and in radial form in Fig 17. The pear-shape asymmetry, the north polar radius minus the south polar radius of the geoid, is a sensitive measure of the effect of odd zonal harmonics, and has a value of 45.1 m with our 9-coefficient solution.

Comparisons with the sets of odd zonal harmonics of GEM 10B and SAO 74 show significant differences in the predicted values of the amplitude of the oscillation in perigee distance: see Fig 14. At inclinations near 63.4°, the differences inevitably tend to infinity; away from 63.4°, the maximum differences are about 350 m for GEM 10B (at inclination 70°) and about 600 m for SAO 74 (at inclination 69°). The implications are discussed in section 5.3.2: the main point worth emphasizing here is that much better sets of odd zonal harmonic values, utilizing satellites in chosen orbits specially observed, will be needed if orbital models are ever to be developed which are capable of giving orbits accurate to a few centimetres.

Comparison of our geoid profile with those given by GEM 10B and SAO 74 shows an average difference of 25 cm, with a maximum of 80 cm at latitudes up to 85°. Changes in even harmonics change the profile by an average of 20 cm. On this basis the rms error of our profile is assessed as 50 cm, except very near the poles, where errors up to 1.5 m are possible.

Acknowledgment

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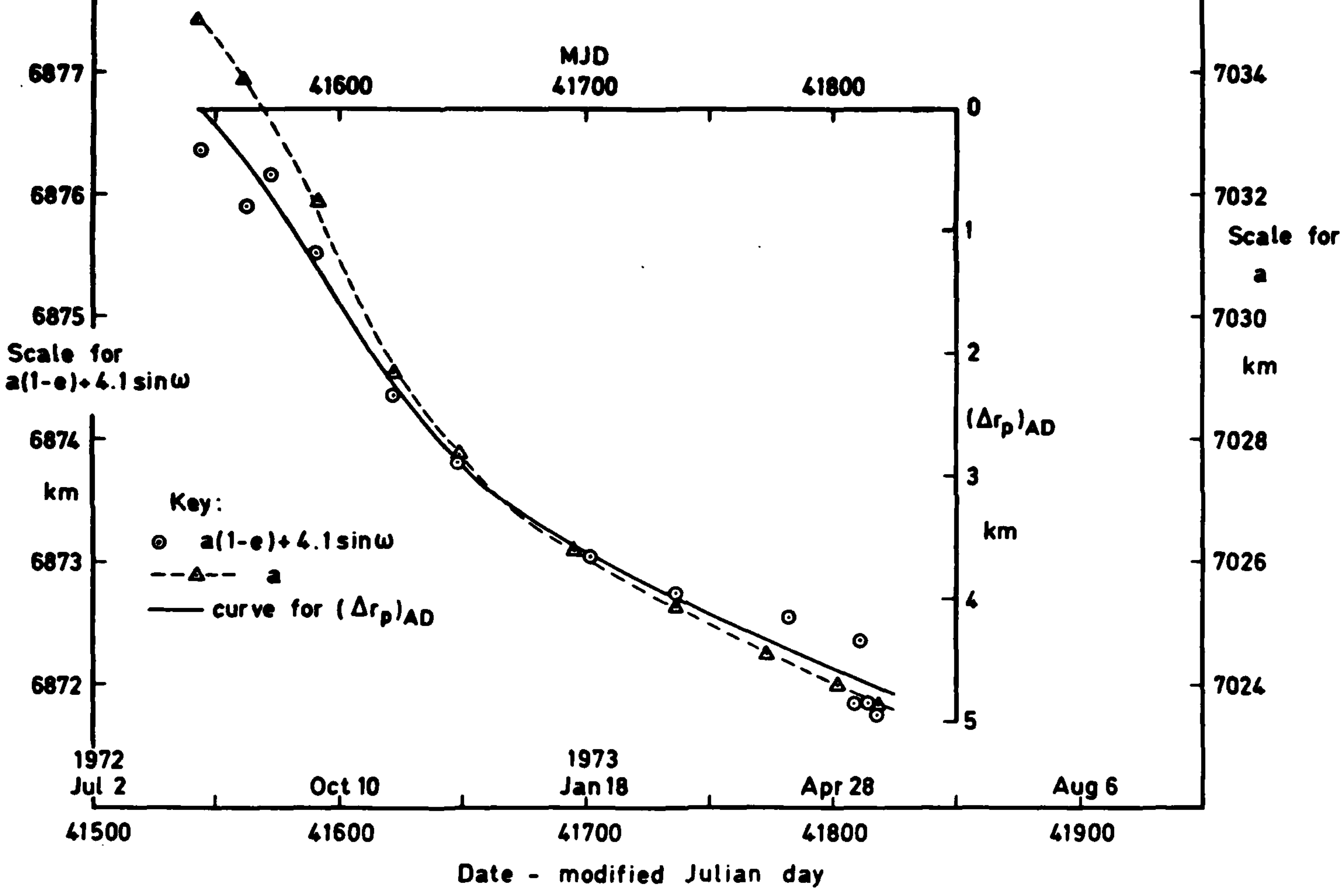

Fig 1 Explorer 46, 1972-61A: values of $a(1 - e) + 4.1 \sin \omega$ and a, with curve for $(\Delta\gamma_p)_{AD}$

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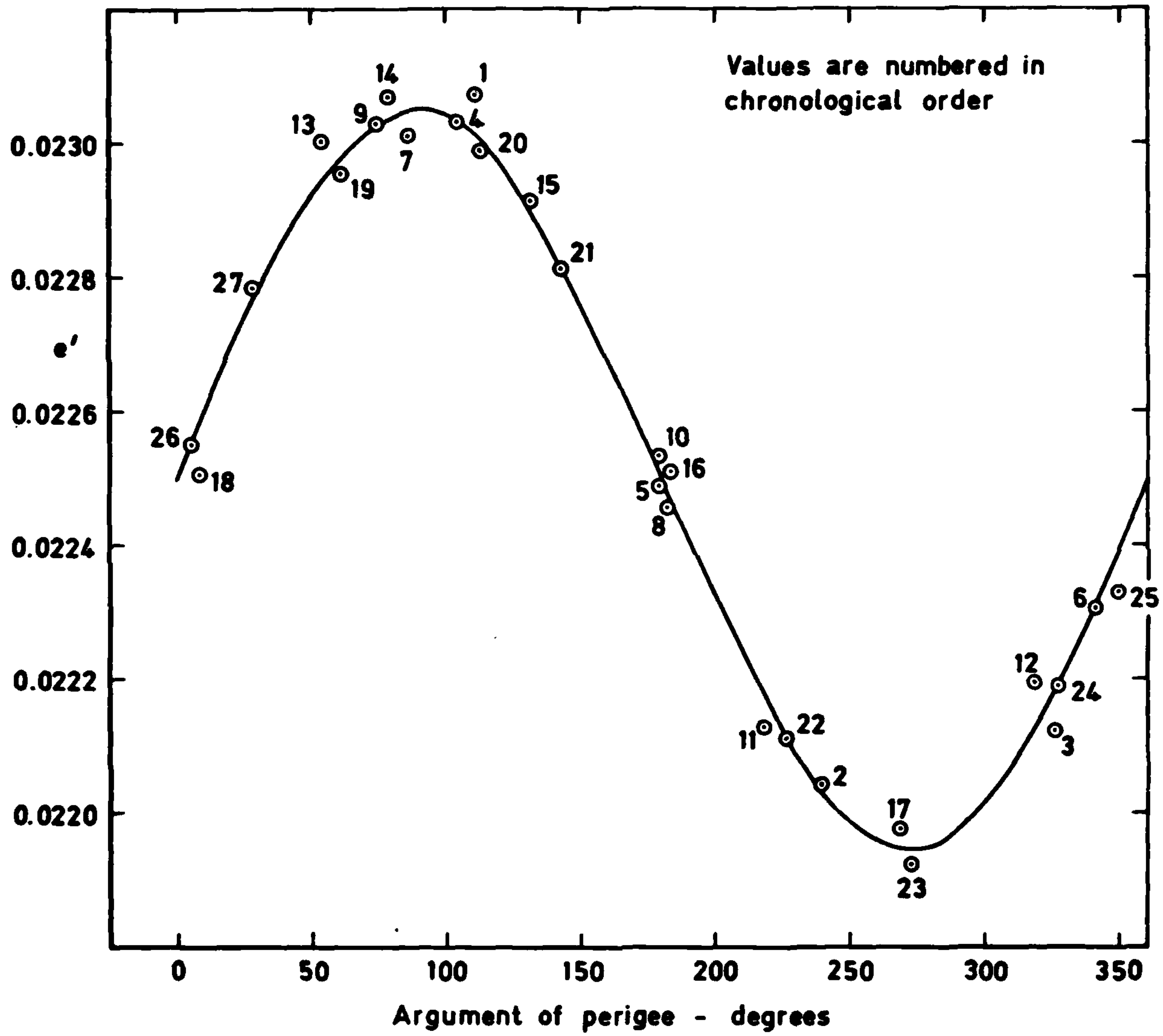

Fig 2 Explorer 46, 1972-61A: values of eccentricity after correction for air drag, e', plotted against ω

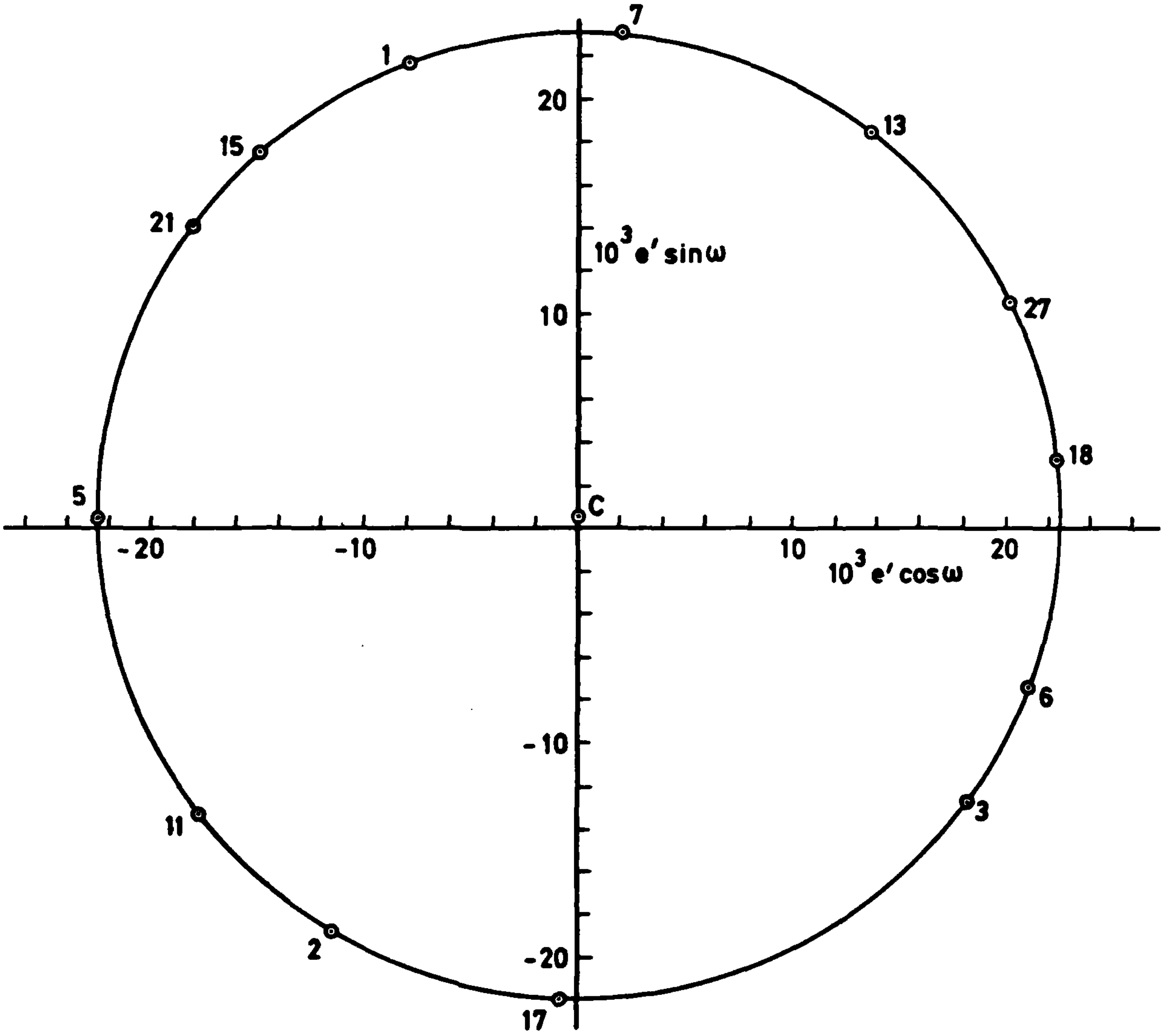

Fig 3 Explorer 46, 1972-61A: circle fitted to the 27 values of $e' \cos\omega$ and $e' \sin\omega$, with selected values

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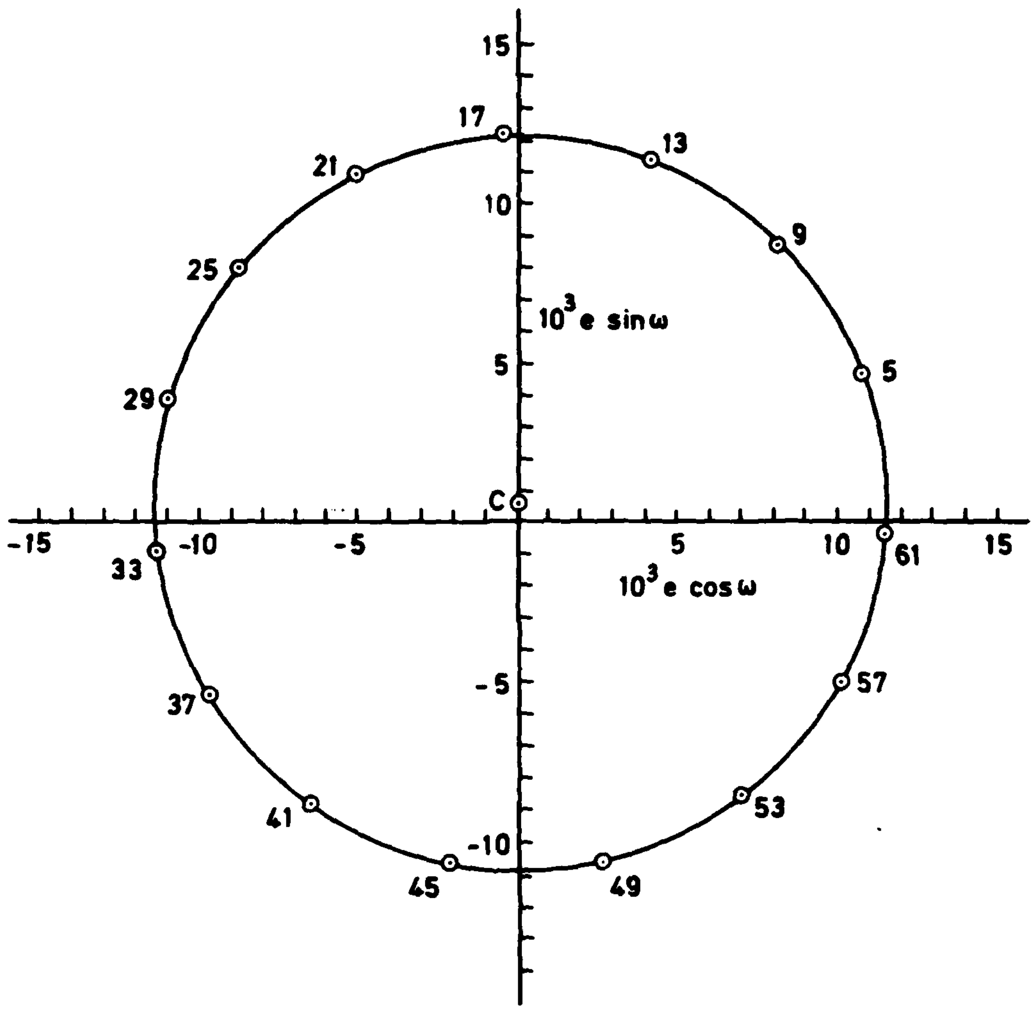

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Fig 4 *Echo 1 rocket, 1960ι2*: circle fitted to 61 values of $e \cos\omega$ and $e \sin\omega$, with selected values

Fig 5

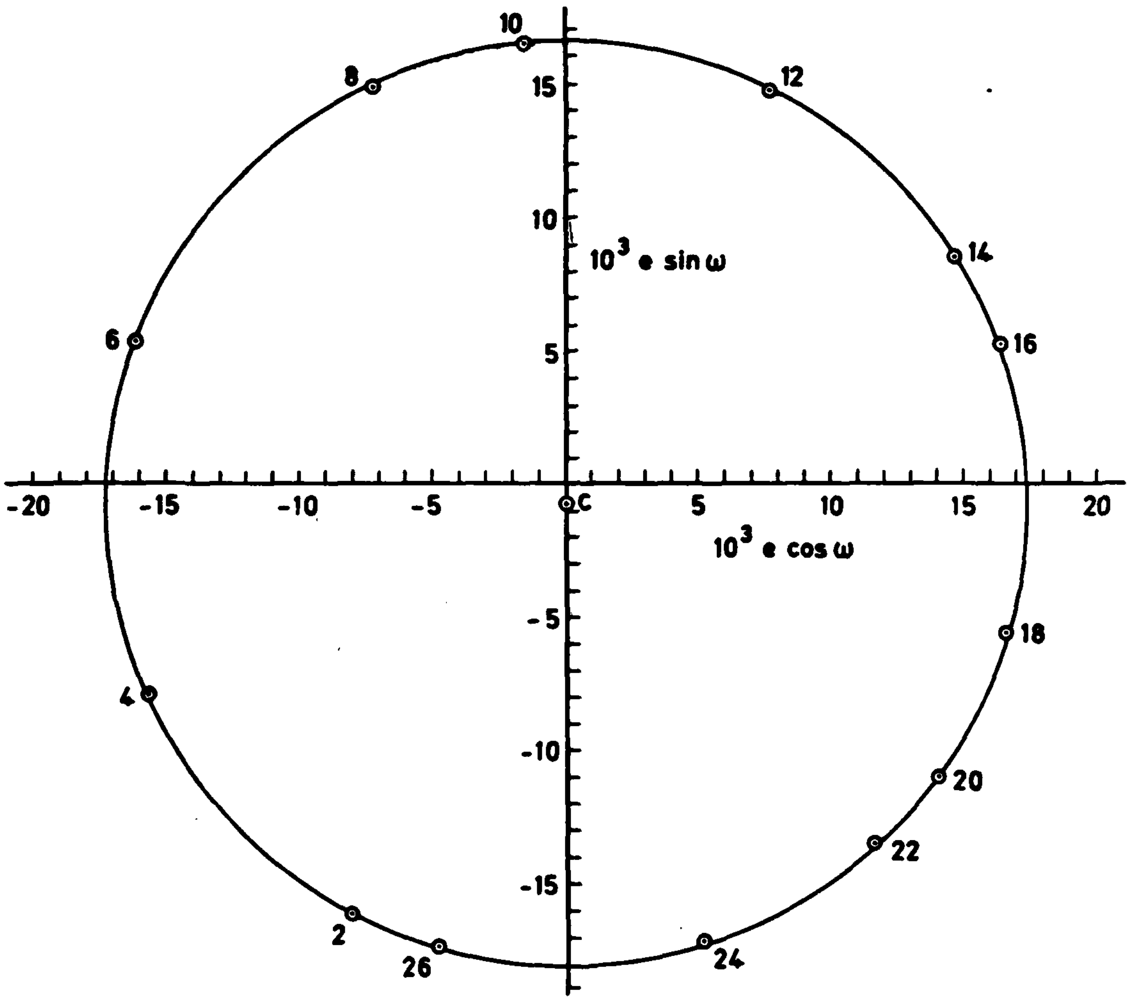

Fig 5 Cosmos 44, 1964-53A: circle fitted to 28 values of e cos ω and e sin ω, with selected values

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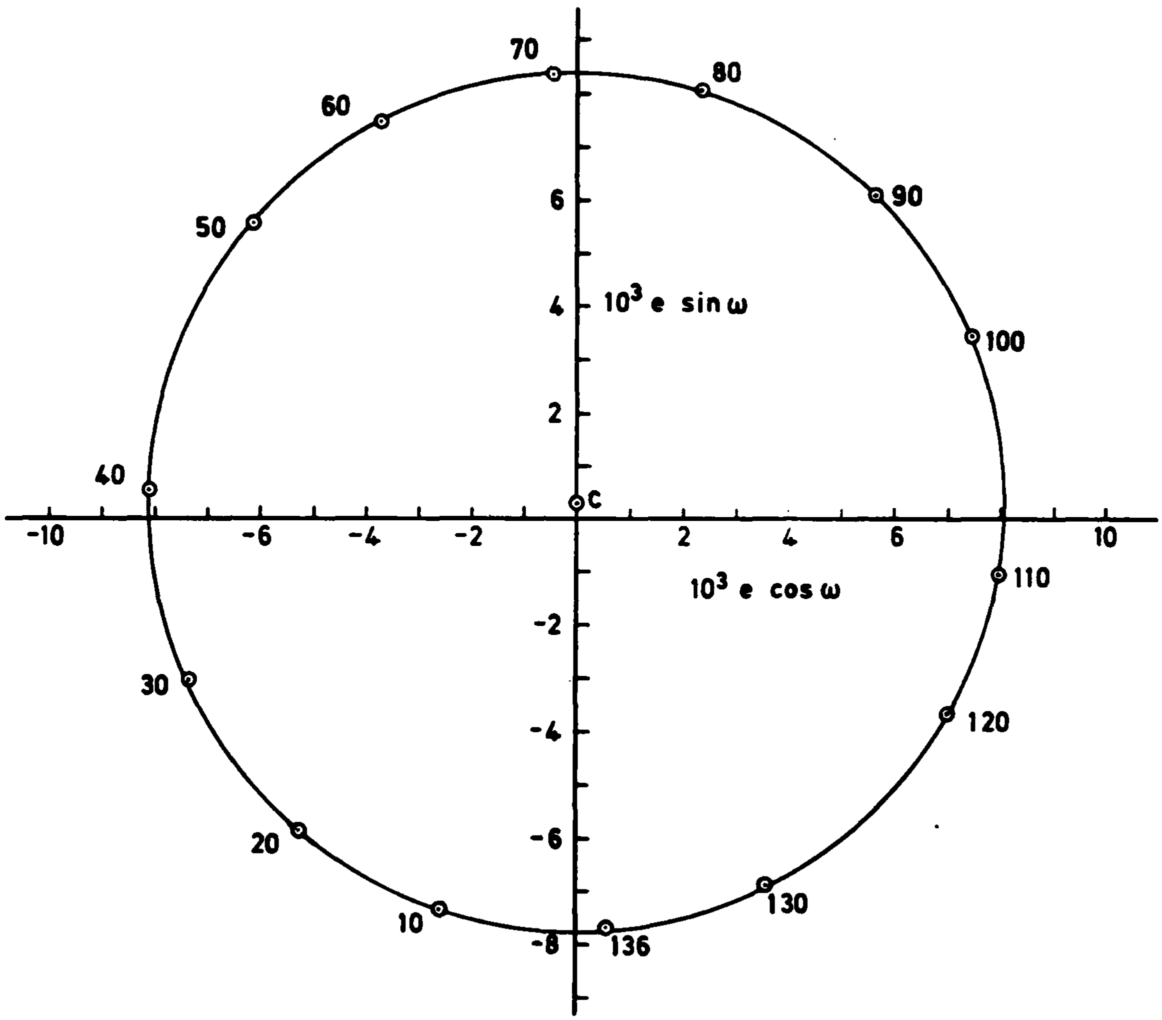

Fig 6 Transit 4A, 1961o1: circle fitted to 137 values of e cos ω and e sin ω, with selected values

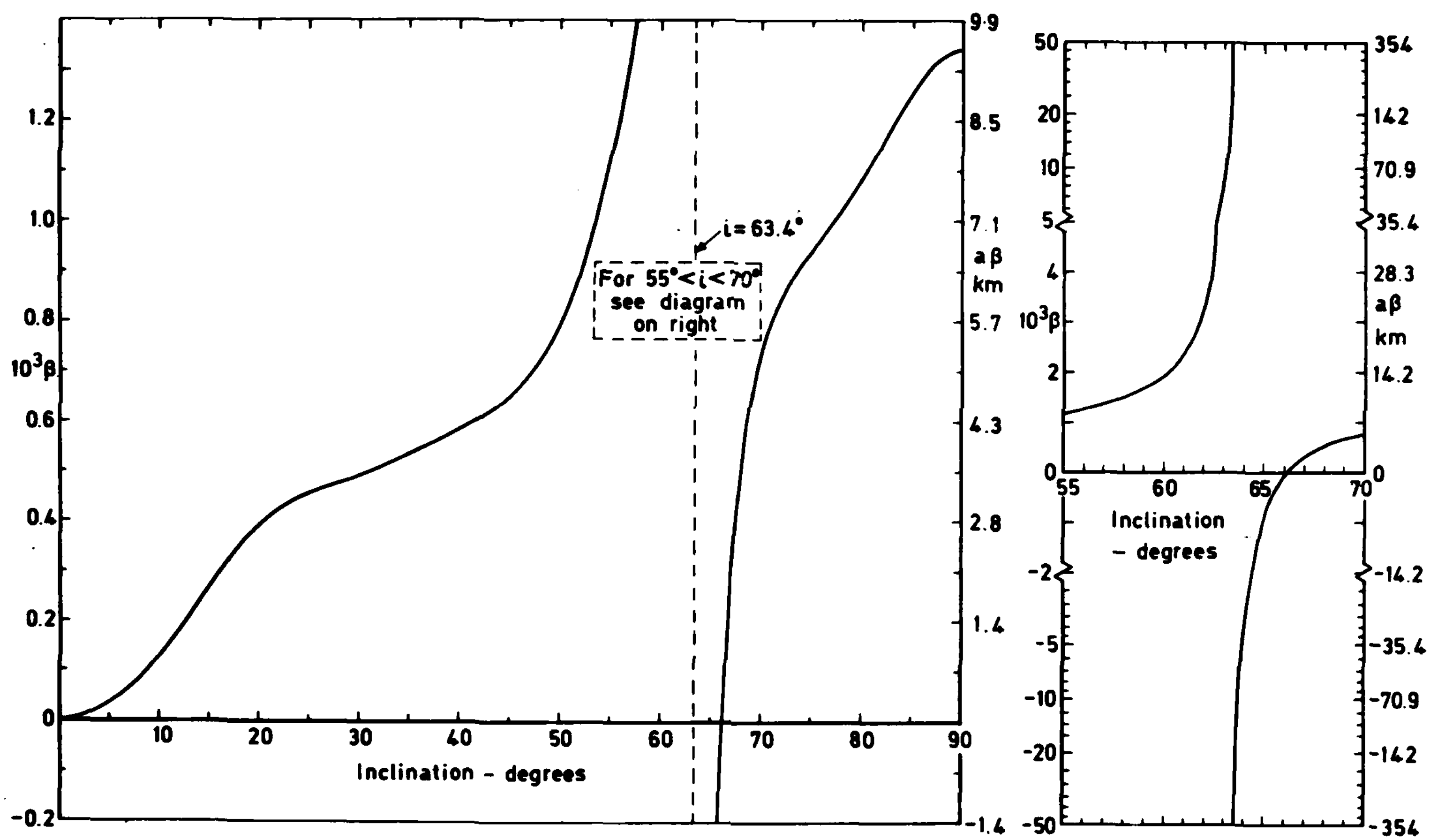

Fig 7 Values of β (and $a\beta$) given by the 9-coefficient solution, for $R/a = 0.9$

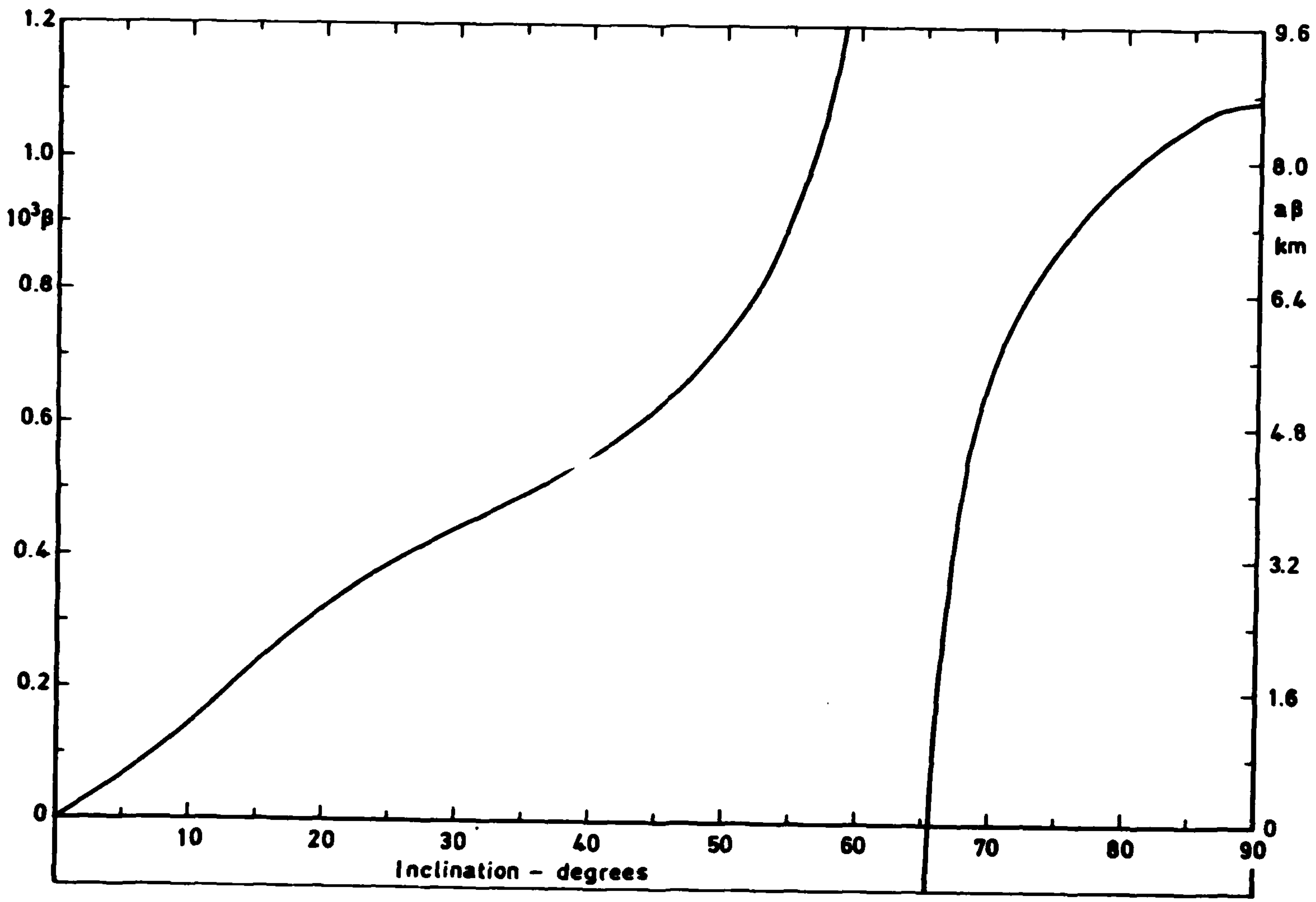

Fig 8 Values of β (and $a\beta$) given by the 9-coefficient solution, for $R/a = 0.8$

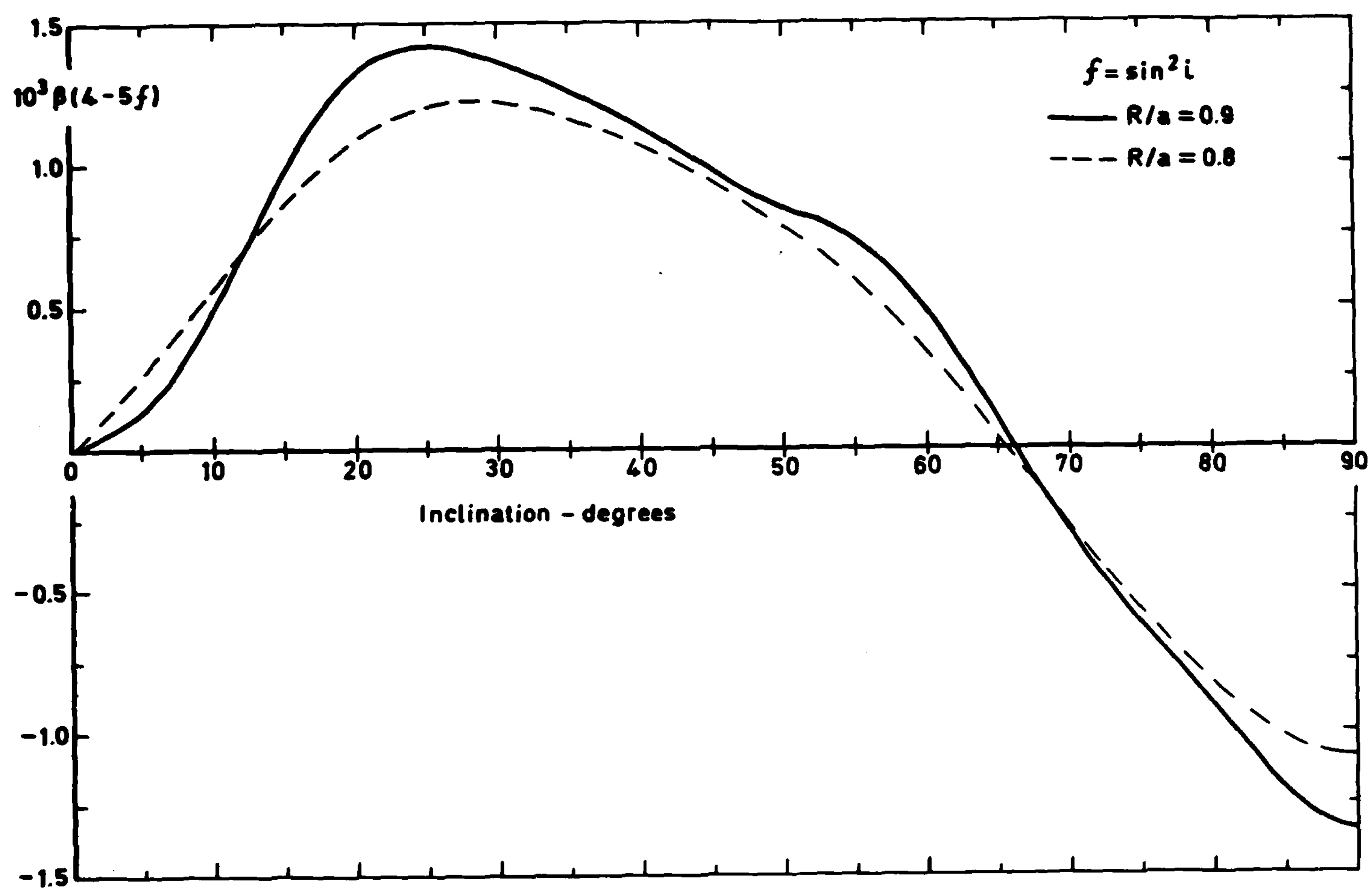

Fig 9 Values of $\beta(4-5f)$ given by the 9-coefficient solution with R/a = R/p = 0.9 or 0.8

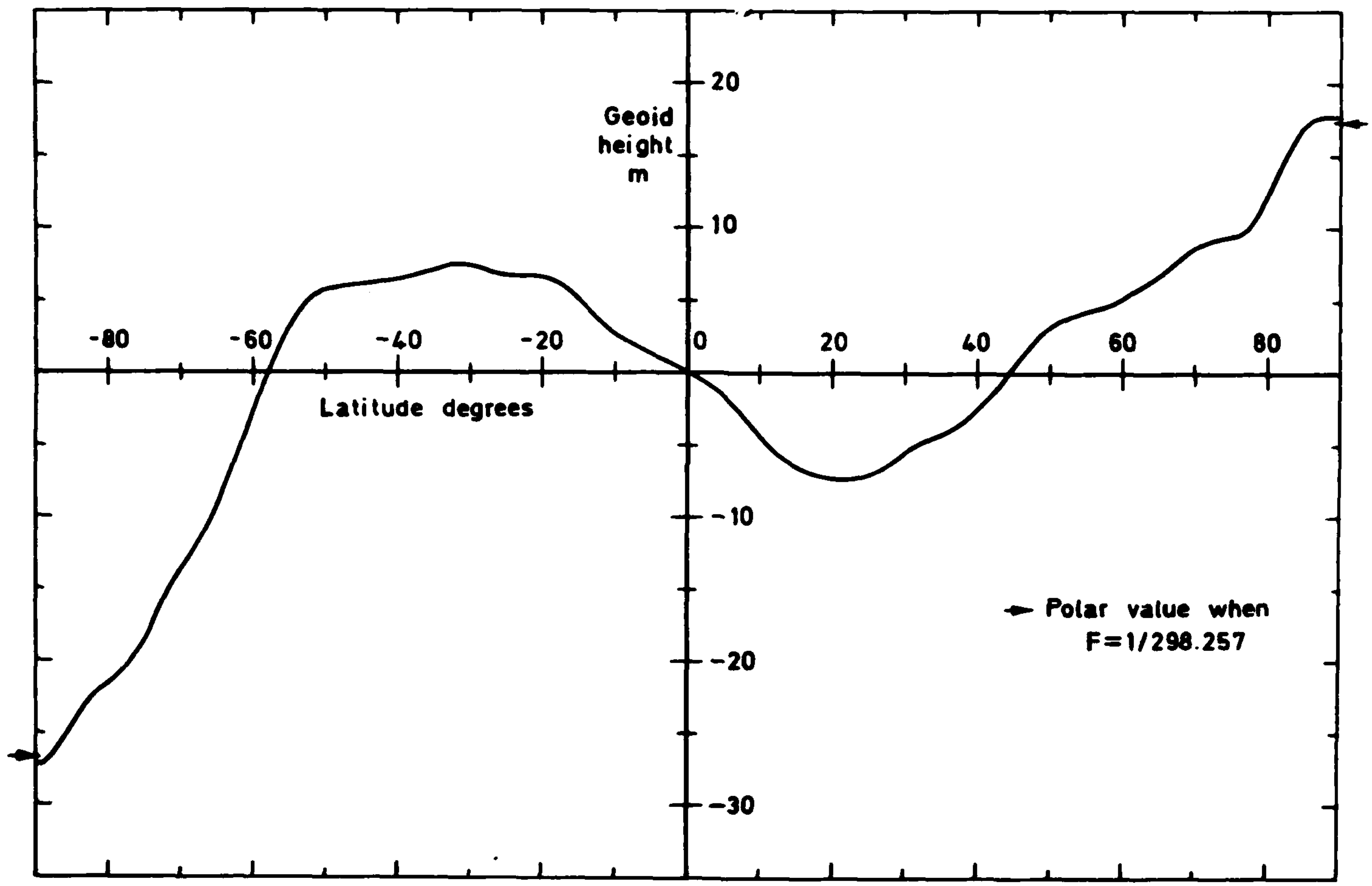

Fig 10 Geoid height above a spheroid of flattening 1/298.25, as given by our 9-coefficient set of odd harmonics and GEM 10B even harmonics

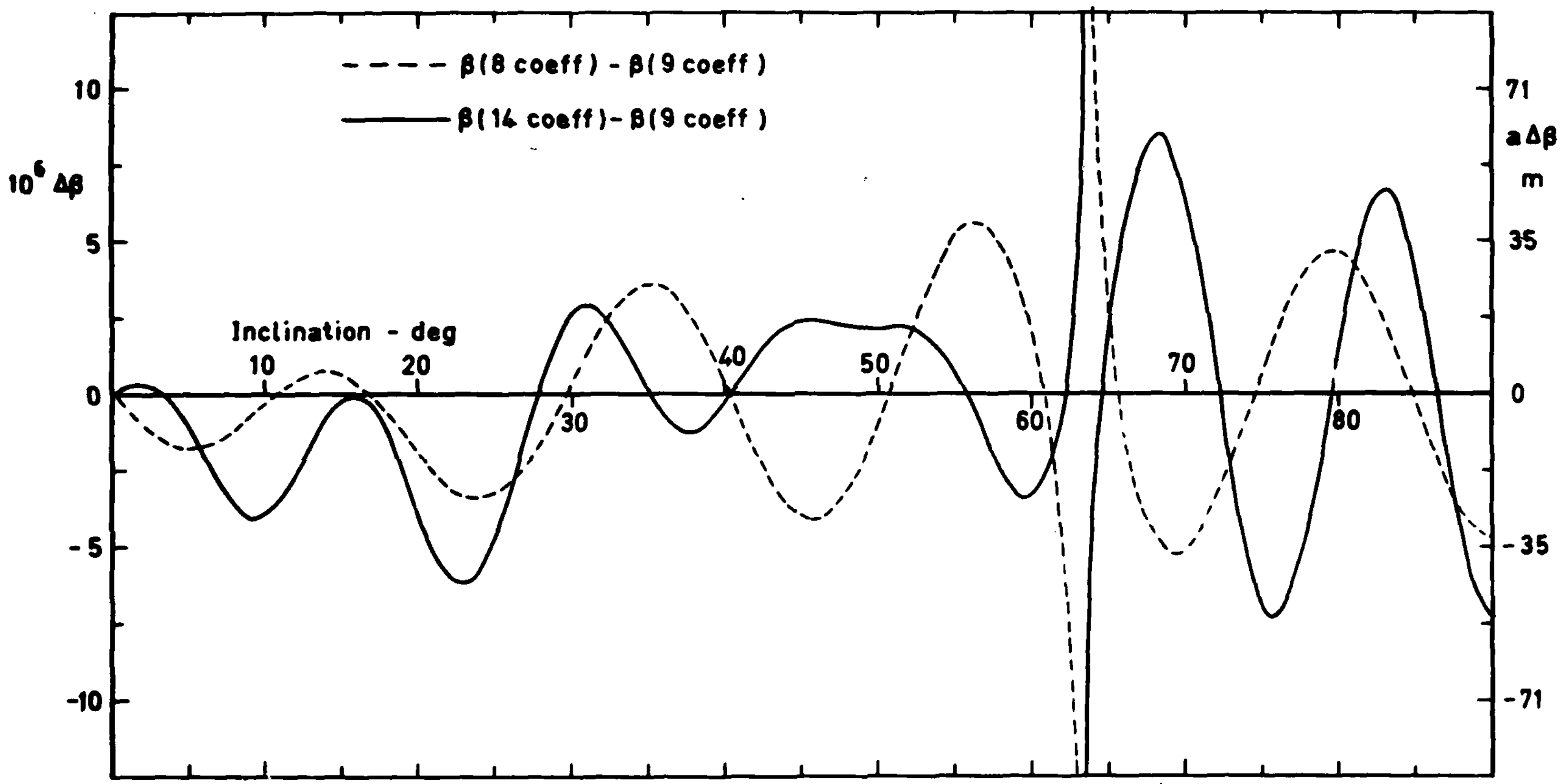

Fig 11 The values of β given by the 8- and 14-coefficient solutions, expressed as their difference $\Delta\beta$ from the 9-coefficient solution (R/a = 0.9)

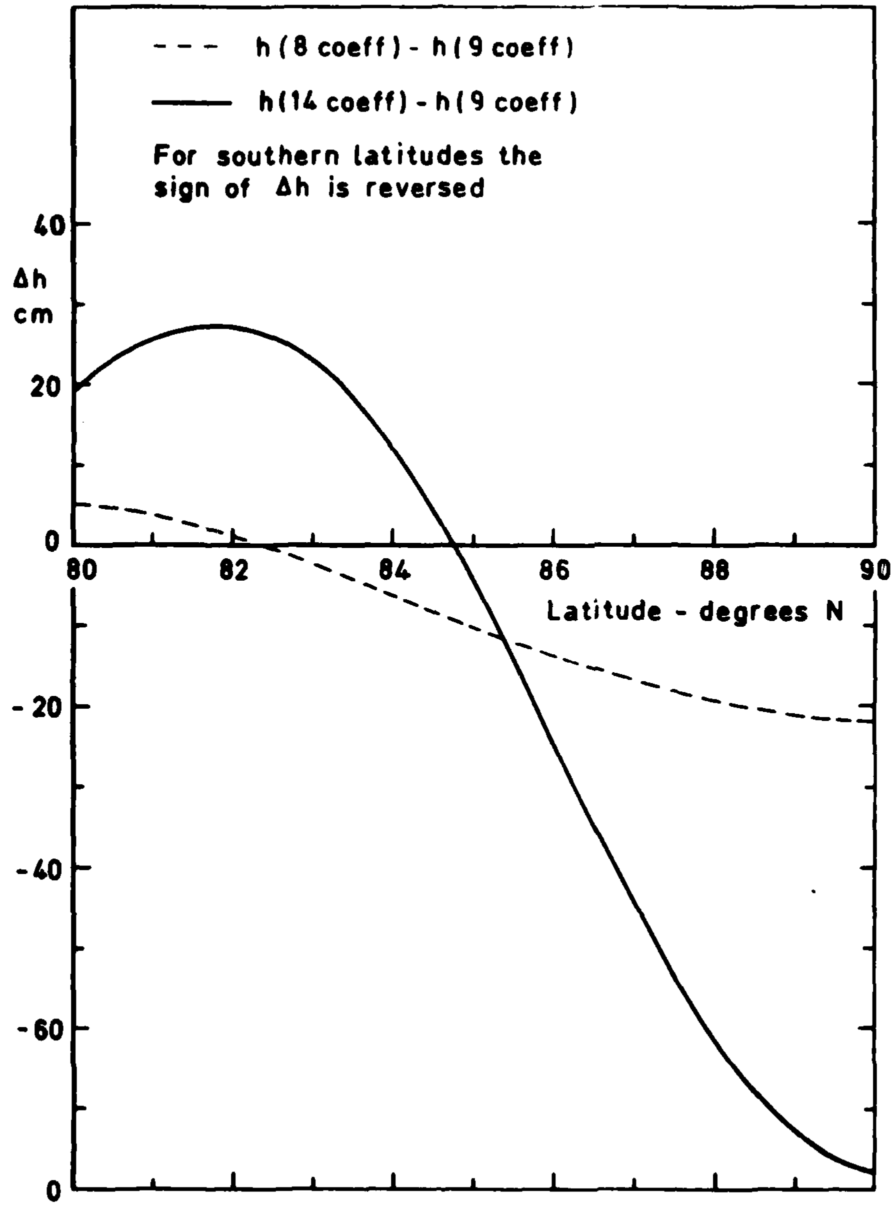

Fig 12 The difference Δh between the geoid heights given by the 14-coefficient and 9-coefficient solutions (and between 8- and 9-coefficient solutions) at latitude 80-90°

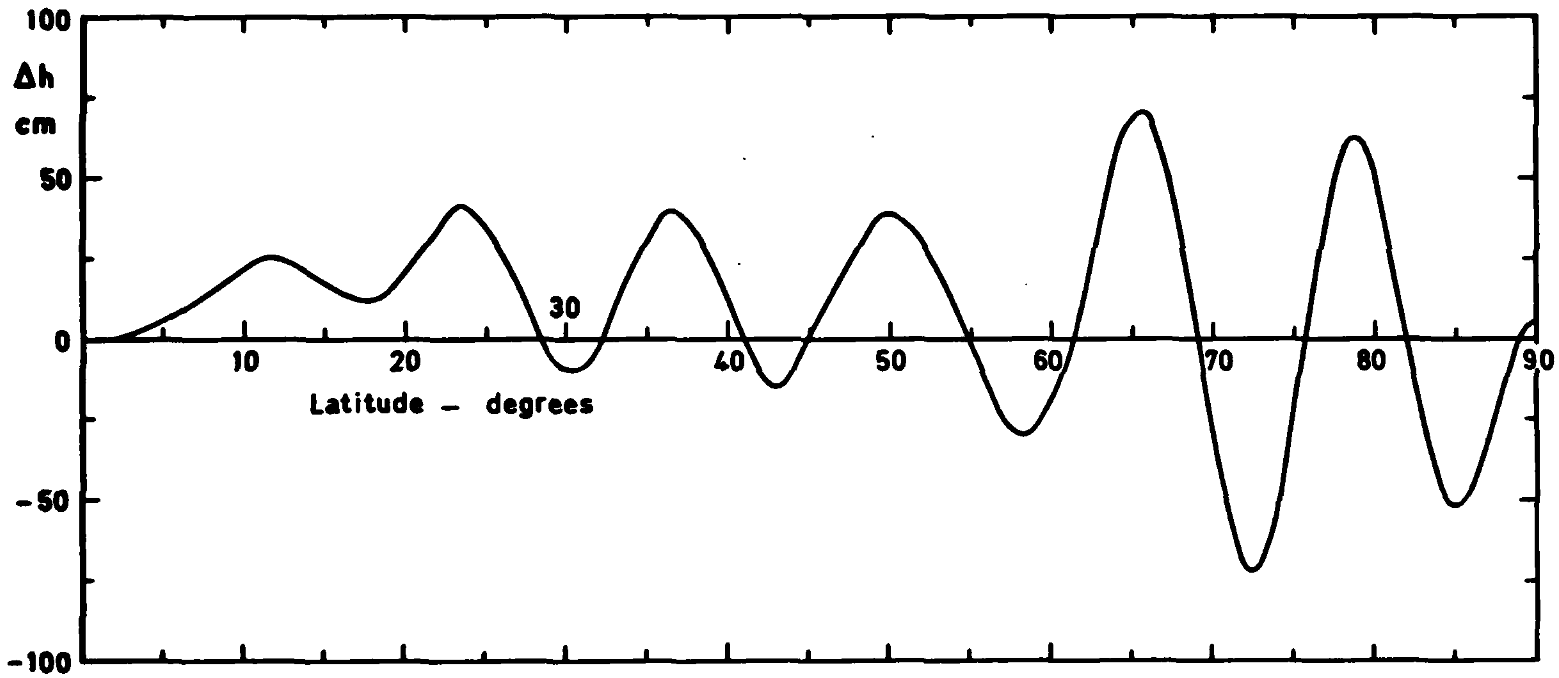

Fig 13 The geoid height Δh given by SAO 74 even harmonics, after subtraction of the geoid height given by GEM 10B even harmonics

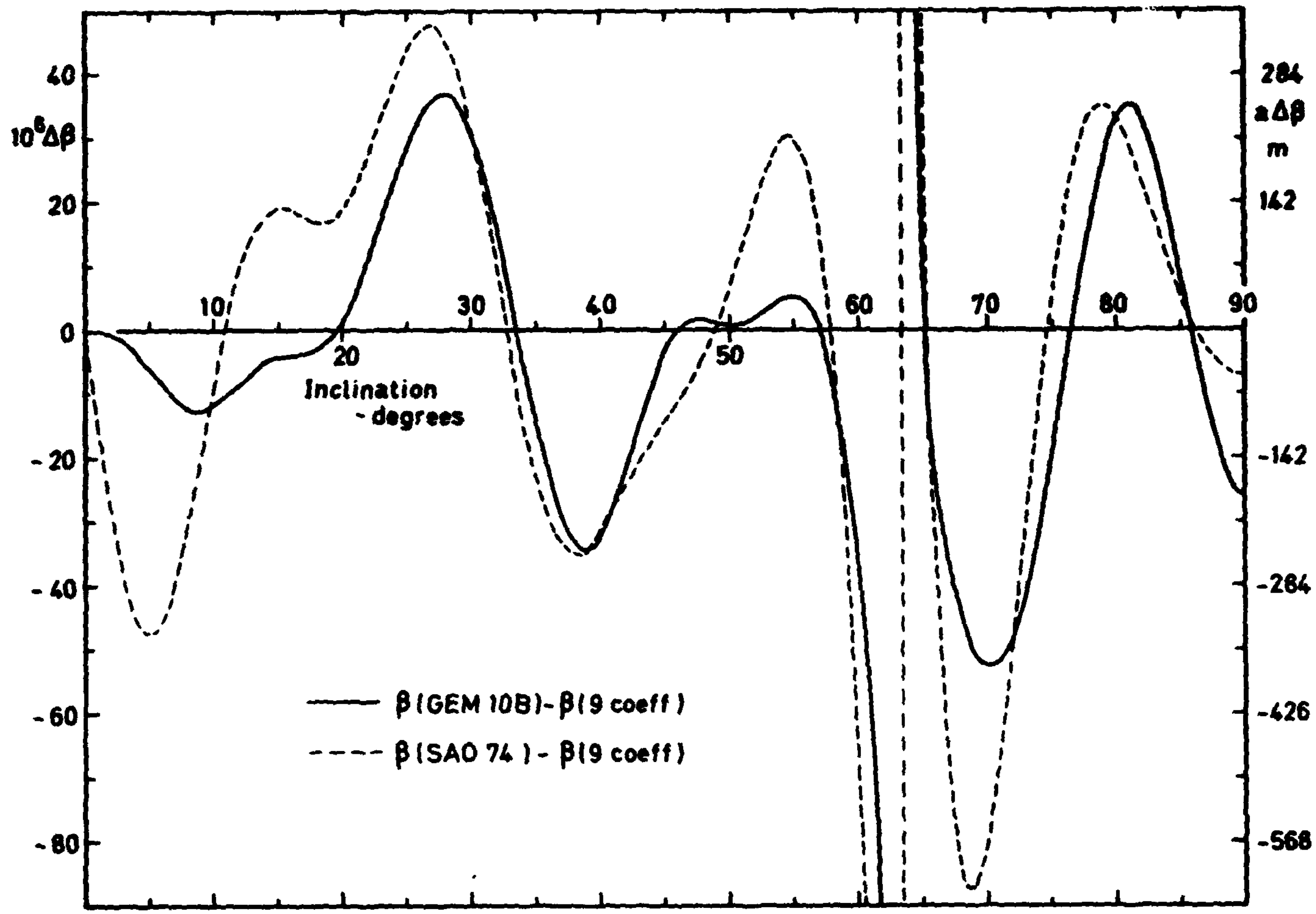

Fig 14 The values of β given by the GEM 10B and SAO 74 sets of odd zonal harmonics, expressed as their difference $\Delta\beta$ from the 9-coefficient solution (R/a = 0.9)

Fig 14

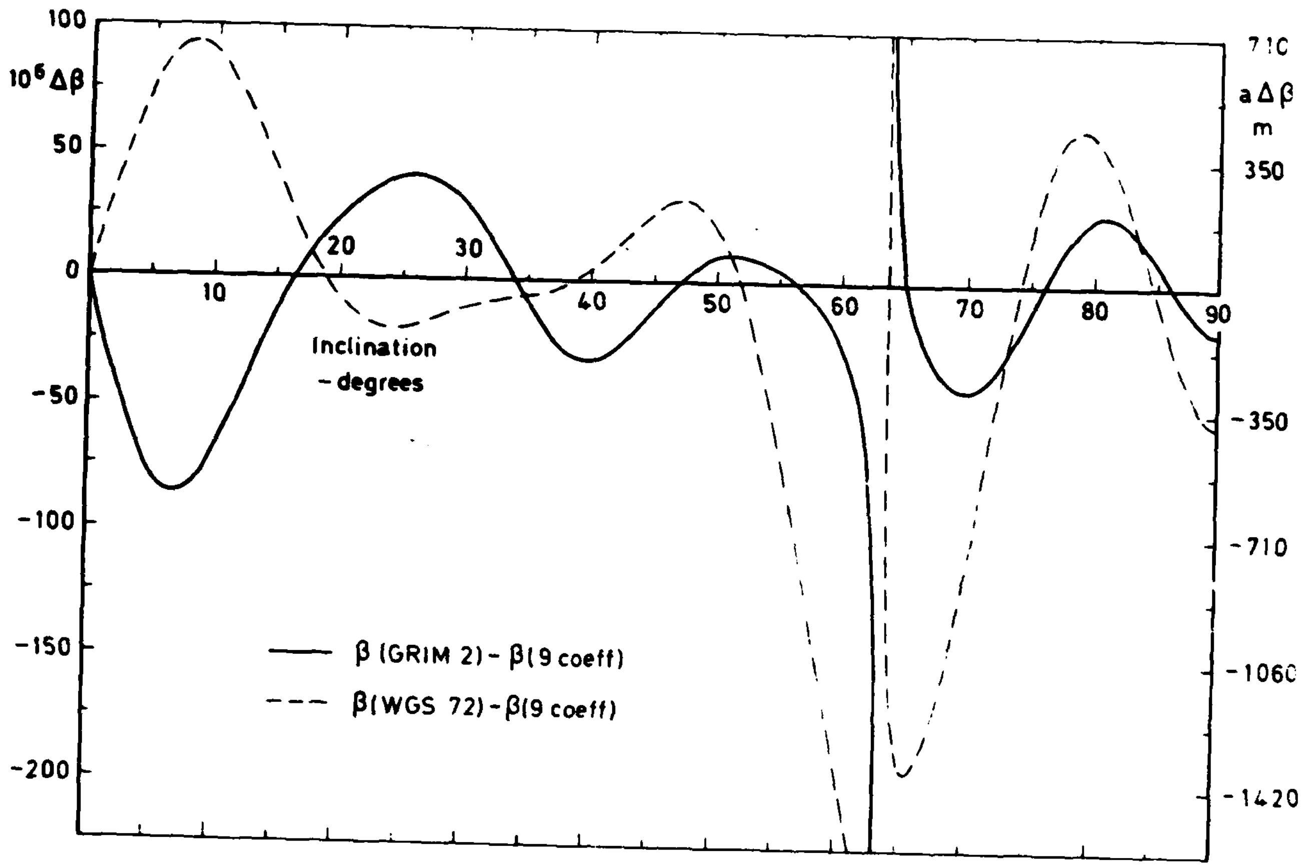

Fig 15 The values of β given by the GRIM 2 and WGS 72 sets of odd zonal harmonics expressed as their difference $\Delta\beta$ from the 9-coefficient solution (R/a = 0.9)

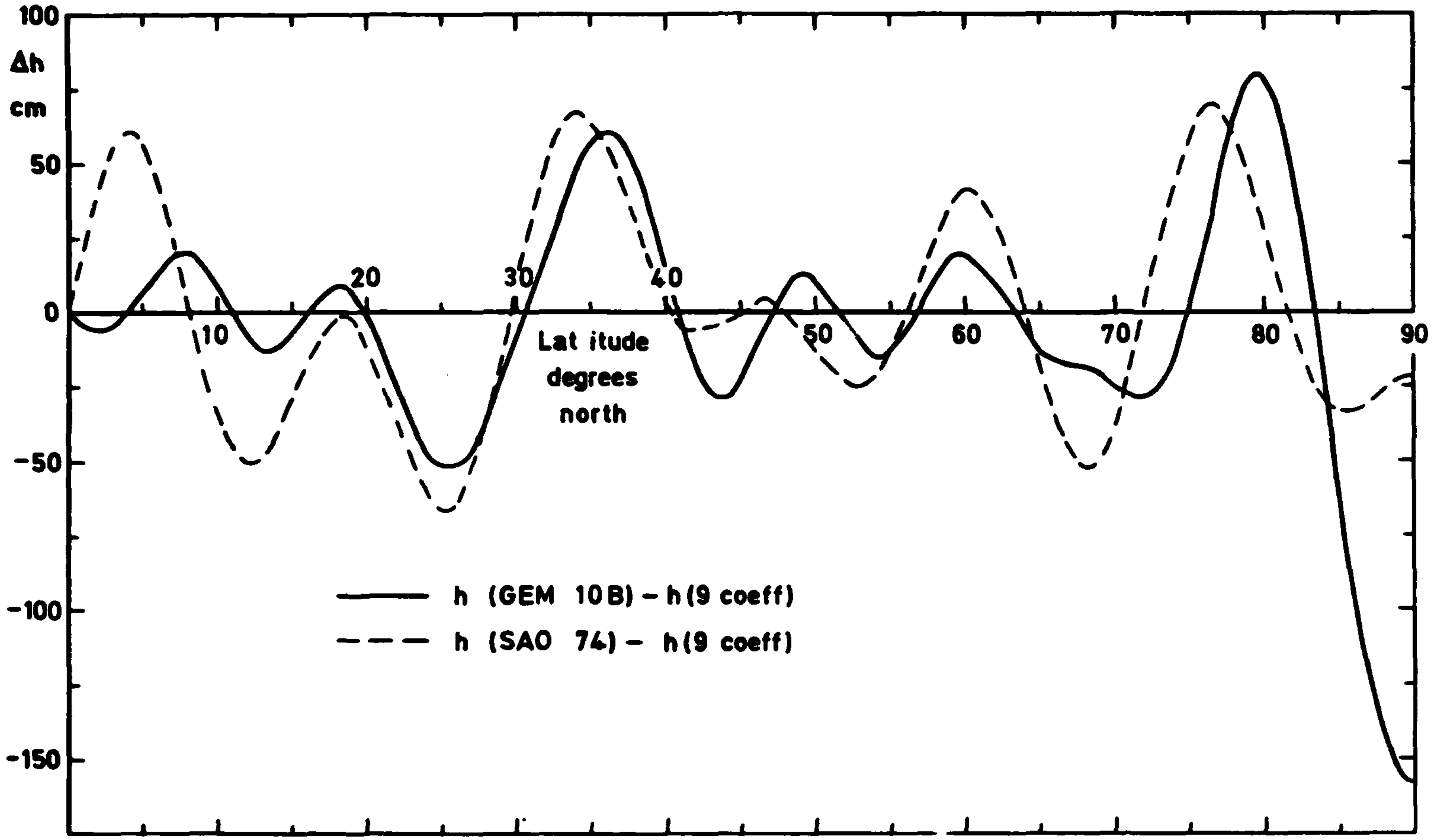

Fig 16 The difference Δh between the geoid height given by GEM 10B (or SAO 74) and the 9-coefficient solution

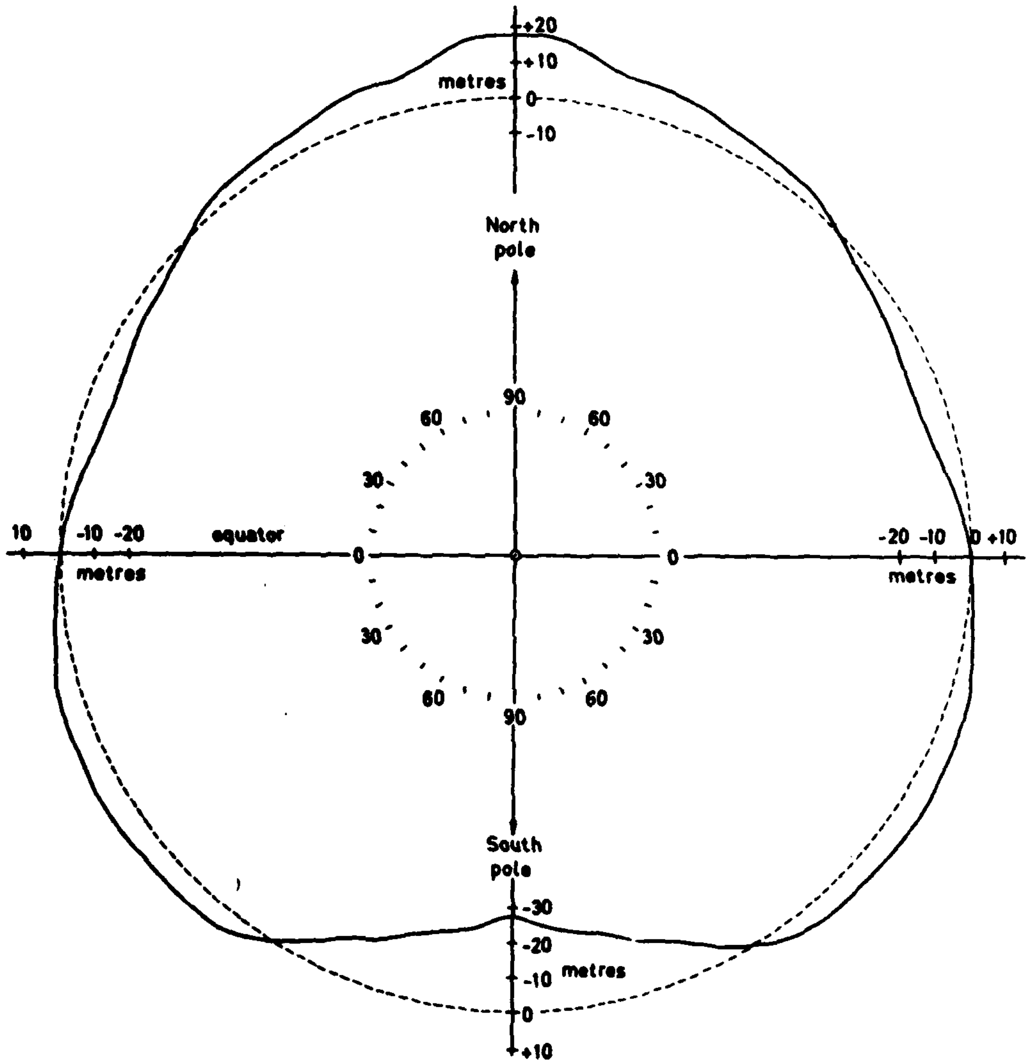

Fig 17 Height of the meridional geoid section (solid line) relative to a spheroid of flattening 1/298.25, as given by our 9-coefficient set of odd harmonics and GEM 10B even harmonics

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17. Abstract The geopotential is usually expressed as an infinite series of spherical harmonics, and the odd zonal harmonics are the terms independent of longitude and antisymmetric about the equator: they define the 'pear-shape' effect. The coefficients J_3, J_5, J_7, ... of these harmonics have been evaluated by analysing the variations in eccentricity of 28 satellite orbits from near-equatorial to polar. Most of the orbits from our previous determination in 1973 are used again, but three new orbits are added, including two at inclinations between 62° and 63°, which have been specially observed for more than five years by the Hewitt cameras. With the help of the new orbits and revised theory, we have obtained sets of J-coefficients with standard deviations about 40% lower than before. A 9-coefficient set is chosen as representative, and is as follows:

$$10^9 J_3 = -2530 \pm 4 \qquad 10^9 J_9 = -90 \pm 7 \qquad 10^9 J_{15} = -20 \pm 15$$
$$J_5 = -245 \pm 5 \qquad J_{11} = 159 \pm 9 \qquad J_{17} = -236 \pm 14$$
$$J_7 = -336 \pm 6 \qquad J_{13} = -158 \pm 15 \qquad J_{19} = -27 \pm 19$$

With this set of values, the pear-shape asymmetry of the geoid (north polar minus [illegible] polar radius) amounts to 45.1 m instead of the previous 44.7 m. The accuracy of the longitude-averaged geoid profile is estimated as 50 cm, except at latitudes [illegible]. The geoid profile and predicted amplitude of the oscillation in eccentricity are compared with those from other sources.