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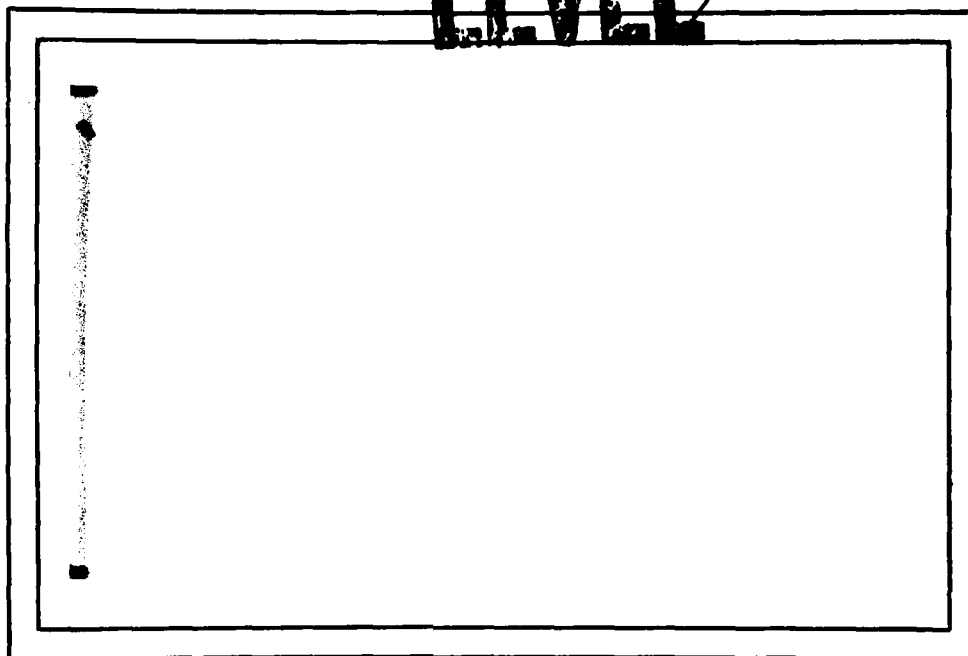
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6 ON CELLULAR STRAIGHT LINE SEGMENTS.

10 Chul E. Kim
Department of Computer Science
University of Maryland
College Park, MD 20742

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ABSTRACT

We discuss a scheme for digitizing curves that is consistent with a scheme for digitizing regions. It is shown that the cellular image of a region is determined by the cellular image of its boundary by the scheme. It is proved that the chord property is a necessary and sufficient condition for a cellular arc to be a cellular straight line segment. By showing that the chord property and the cellular convexity condition are equivalent, we prove that a cellular arc is a cellular straight line segment if and only if it is cellularly convex. This leads to an algorithm to determine whether or not a cellular complex is a cellular straight line segment in time linear in the number of rows of cells. Finally it is proven that a cellular complex is cellularly convex if and only if any pair of its cells is connected by a cellular straight line segment in the cellular complex.

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if and only if it is cellularly convex. This leads to an algorithm to determine whether or not a cellular complex is a cellular straight line in time linear in the number of rows of cells. Finally it is proven that a cellular complex is cellularly convex if and only if any pair of its cells is connected by a cellular straight line segment in the cellular complex.

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1. Introduction

A standard scheme for digitizing curves is the grid-intersection scheme for chain coding [1]. It was used in [7] to study digital curves and digital straight line segments and in [4] to characterize convex digital regions in terms of digital straight line segments. More specifically, in [7] the chord property was shown to be a necessary and sufficient condition for a digital arc to be a digital straight line segment. It was proved in [4] that a digital region is convex if and only if any pair of points in the region can be connected by a digital straight line segment within it. In deriving the latter result, however, two different digitization schemes were used, one for digitizing curves and another for digitizing regions. The reason was that the scheme for digitizing curves could not be used to digitize regions.

There are two slightly different schemes for digitizing regions. One has been used widely [2,6,9,10] and the other was introduced recently and used in [3,4,5]. We use the latter because the digitization of a region is unique under this scheme while not unique under the former scheme.

In this paper we present a new scheme for digitizing curves. It will be shown that the new scheme can be used to digitize regions as well and the digitization by this method results in the same digitization as under the usual scheme for digitizing

regions. In fact, the digitization of a curve under the new scheme is the same as the digitization of a region obtained by thickening the curve infinitesimally on one side. In the sequel, we use "cellular" instead of "digital" for the following two reasons: (1) it distinguishes the new scheme from the old one, and (2) it is more convenient to discuss the scheme in terms of cells and lattice points than in terms of lattice points only.

The chord property was introduced in [7] and used to characterize digital straight line segments. We prove that a cellular arc is a cellular straight line segment if and only if it has the chord property. This is an interesting characterization of a cellular straight line segment but does not easily lead to an efficient algorithm to determine the straightness of a cellular arc.

In Euclidean plane geometry, an arc is a straight line segment if and only if it is convex. Here it will be shown that the same holds in the case of cellular arcs; that is, a cellular arc is a cellular straight line segment if and only if it is cellularly convex. This in turn leads to an efficient algorithm to determine the straightness of a cellular arc because there exists an efficient algorithm to check cellular convexity.

A region is said to be convex if for every pair of points in the region, the line segment connecting them lies entirely within it. In the sequel, we show that a cellular complex is cellularly convex if and only if every pair of cells in the

complex is connected by a cellular straight line segment lying within the complex.

In the next section, the new scheme for digitizing curves is introduced and its relation to a scheme for digitizing regions is discussed. Section 3 is concerned with characterizing cellular straight line segments in terms of the chord property. An efficient algorithm is presented in Section 4 to determine whether or not a cellular complex is a cellular straight line segment. In the next section, the relation between cellular convexity and a cellular straight line segment connecting a pair of cells is discussed.

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2. Two schemes for digitizing curves

Consider a coordinate grid on the plane and the set of all lattice points. With each lattice point $d=(h,k)$ is associated a unit square whose center is the lattice point. The square associated with d is denoted by c and is called a cell.

Let D be a set of lattice points. Then $\bar{D}$ denotes its complement. We denote by C the set of cells that are associated with the points of D . The set of (real) points in C is denoted by $s(C)$, and its boundary by $\partial s(C)$. An interior point of D is a point all of whose eight 8-neighbors [8] are points of D . A boundary point of D is a point of D which is not an interior point. An interior cell and a boundary cell are defined accordingly. A finite set D of lattice points is called a digital region and a finite set C of cells is called a cellular complex.

Digital image of a curve [1,7]

Consider a curve on the plane with a coordinate grid. Whenever the curve crosses a grid line, the lattice point nearest to the crossing becomes a point of the digital image of the curve. When the crossing is exactly midway between two lattice points, the one having smaller coordinate becomes a point of the digital image.

Digital arc [7]

A digital arc R is an 8-connected digital region in which every point except two has exactly two 8-neighbors in R and the exceptional two, called endpoints, each have exactly one 8-neighbor in R .

It is easy to see that the digital image of an arc is not necessarily a digital arc. The larger the radius of curvature of an arc at every point, the more likely is its digital image to be a digital arc. As was shown in [7], if the radius is infinite, that is, the arc is a straight line segment, then its digital image is always a digital arc.

Cellular image of a curve

A set C of cells is the cellular image of a curve f if $f \subseteq s(C)$ and for every element c of C ,

- (i) $c^o \cap f \neq \emptyset$, where c^o is the interior of c , or
- (ii) $c \cap f \neq \emptyset$, assuming $c^o \cap f = \emptyset$, and c lies to the right of f , where c is the boundary of f .

(Note that a curve has a direction and its right side is with respect to its direction.)

Cellular arc

A cellular complex S is a cellular arc if

- (i) it is 4-connected, and
- (ii) every cell except two has exactly two 4-neighbors in S , and the exceptional two, called end cells, each have one 4-neighbor in S .

Again, the cellular image of an arc is not necessarily a cellular arc. It is shown in the next section that if an arc is a straight line segment, then its cellular image is a cellular arc.

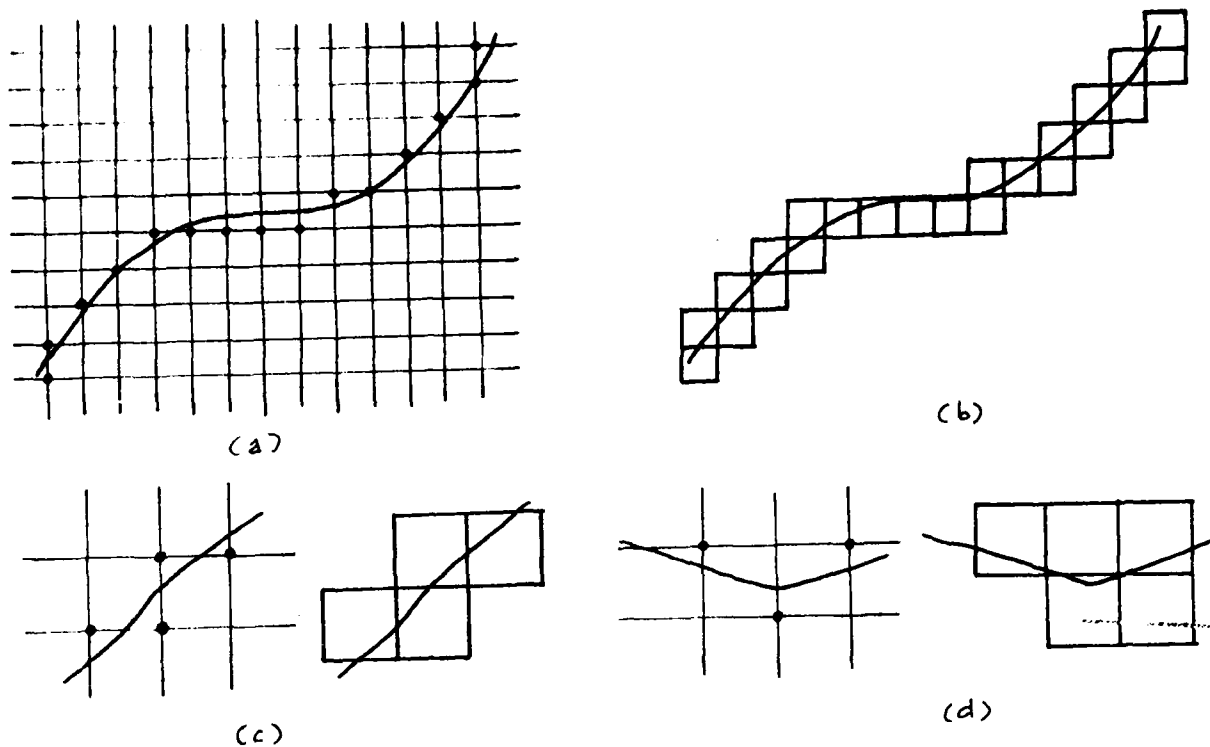


Figure 1. Digital and cellular images of arcs.

Figures 1-(a) and 1-(b) show the digital and cellular images of an arc. These images are a digital arc and cellular arc, respectively. Figure 1-(c) illustrates the cellular image of an arc that is a cellular arc, while its digital image is not a digital arc. The opposite case is shown in Figure 1-(d). We note from Figure 1-(c) that an arc may have an arbitrarily large radius of curvature at every point but still have a digital image that is not a digital arc. A similar remark for the case of the cellular image may be made from Figure 1-(d).

The digital image of a curve is a digitization of the curve. This method of digitization can be used for digitizing curves but not for digitizing regions. Since the boundary determines a region, it is reasonable to expect that the digitization of the boundary determines the digitization of the region. In the sequel, we show that the cellular image of a boundary indeed determines the digitization of a region.

Cellular image of a region [3]

A cellular complex C is said to be the cellular image of a region q , and q a preimage of C , if

- (i) $q \subseteq s(C)$ and
- (ii) for each element c of C , $c^0 \cap q \neq \emptyset$.

We denote the unique cellular image of a region q by $I(q)$. A slightly different definition has been used widely [2,6,9,10]. However, we use the definition given above because the cellular image of a region is not unique under the other definition.

Let q be a simply 4-connected region and ∂q its boundary. Then ∂q is a closed curve that does not meet itself. We assume that its direction is such that the interior of q lies to the right of the closed curve. Let $J(\partial q)$ be the cellular image of ∂q . $J(\partial q)$ is a closed sequence of cells. Let $J^0(\partial q)$ be the set of all cells that are bounded by the closed sequence of cells $J(\partial q)$. Then the points of $J^0(\partial q)$ are in q , that is, $s(J^0(\partial q)) \subseteq q$. Let the cellular complex $J(q)$ be the union of $J(\partial q)$ and $J^0(\partial q)$. Then, obviously, $q \subseteq s(J(q))$.

Suppose that c is an element of $I(q)$. Then $c^0 \cap q \neq \emptyset$. Thus, c is an element of $J(q)$, since otherwise there exist points of q that are not in $s(J(q))$. Conversely, suppose that c is an element of $J(q)$. Either c is an element of $J(\partial q)$ or c is an element of $J^0(\partial q)$. In both cases, it is obvious that $c^0 \cap q \neq \emptyset$. Hence, c is also an element of $I(q)$. Therefore, $I(q) = J(q)$. Summarizing the argument above we obtain:

Theorem 1. Let q be a simply 4-connected region. Then the cellular image of its boundary determines the cellular image of the region.

Due to the above theorem the digitization scheme for a curve by the use of cellular images is consistent with the digitization scheme for a region. Thus for a given curve f , we denote its cellular image by $I(f)$.

Let f be a curve. Build a parallel curve f' to the right side of f such that the distance between them is as small as desired. Add line segments between corresponding end points of the two curves to obtain an elongated region g . It is not difficult to see that $I(f) = I(g)$. Thus, the scheme for digitizing curves by their cellular images may be considered equivalent to the scheme for digitizing regions.

3. Cellular straight line segments

A cellular (digital) arc is called a cellular (digital) straight line segment if there is a straight line segment whose cellular (digital) image is the cellular (digital) arc. In [7], it was shown that the chord property is a necessary and sufficient condition for a digital arc to be a digital straight line segment. In this section, we derive an identical result for the case of cellular straight line segments.

Let C be a cellular complex. If a cell c is an element of C , then its center d is called a lattice point of C . If d_1 and d_2 are two lattice points, the line segment between them is denoted by $\overline{d_1 d_2}$.

Chord property [7]

Let C be a cellular complex, and d_1, d_2 be lattice points of C . We say that $\overline{d_1 d_2}$ lies near C if, for any real point (x, y) of $\overline{d_1 d_2}$, there exists a lattice point $d = (h, k)$ of C such that $\max\{|h-x|, |k-y|\} < 1$. We say that C has the chord property if, for every d_1, d_2 of C , the chord $\overline{d_1 d_2}$ lies near C .

Lemma 2. The cellular image of a straight line segment is a cellular arc and has the chord property.

Proof: Let ℓ be a straight line segment and $I(\ell)$ its cellular image. If ℓ is either horizontal or vertical, then $I(\ell)$ is a row or column of cells. It is obvious that $I(\ell)$ is a cellular arc and has the chord property.

Now suppose that ℓ is neither horizontal nor vertical. Without loss of generality, assume that ℓ has a slope α , $0 < \alpha \leq 1$.

It is easy to see that $I(\ell)$ is a cellular arc. It remains to show that $I(\ell)$ has the chord property. Let d_1 and d_2 be lattice points of $I(\ell)$, u_1 and u_2 the upper left corner points and v_1 and v_2 the lower right corner points of c_1 and c_2 , respectively. Then ℓ lies between the line segments $\overline{u_1 u_2}$ and $\overline{v_1 v_2}$. (See Figure 2.)

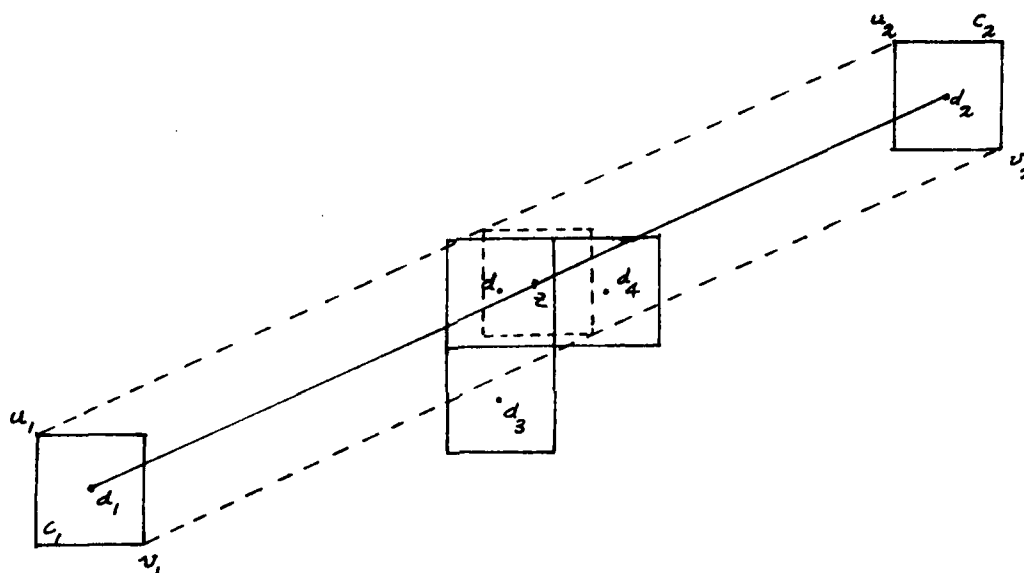


Figure 2.

Let $z = (x, y)$ be any point of $\overline{d_1 d_2}$ and c the cell containing z . If c is an element of $I(\ell)$, $\max\{|x-h|, |y-k|\} < 1$, where $d = (h, k)$ is the center of c . Suppose c is not an element of $I(\ell)$. Note that d is not on $\overline{d_1 d_2}$, since if it were, ℓ would pass through c and c would be an element of $I(\ell)$. Assume without loss of generality that d lies above $\overline{d_1 d_2}$. Then the cell c_3 just below c is an element of $I(\ell)$. If z lies strictly

below d , that is, $y < k$, then $\max\{|x-h_3|, |y-k_3|\} < 1$, where $d_3 = (h_3, k_3)$. Suppose $k \leq y$, and consider the unit square c' whose center is z . Since ℓ must lie below c and pass through c' , it passes through the cell c_4 immediately to the right of c . Then $\max\{|x-h_4|, |y-k_4|\} < 1$, where $d_4 = (h_4, k_4)$. Thus, $\overline{d_1 d_2}$ lies near $I(\ell)$ and therefore, $I(\ell)$ has the chord property. $\square$

A run of cells in a cellular complex C is a row or column of cells of C . In a cellular arc S , horizontal and vertical runs of cells alternate and the first cell of a run is a 4-neighbor of the last cell of the previous run. An example is shown in Figure 3.

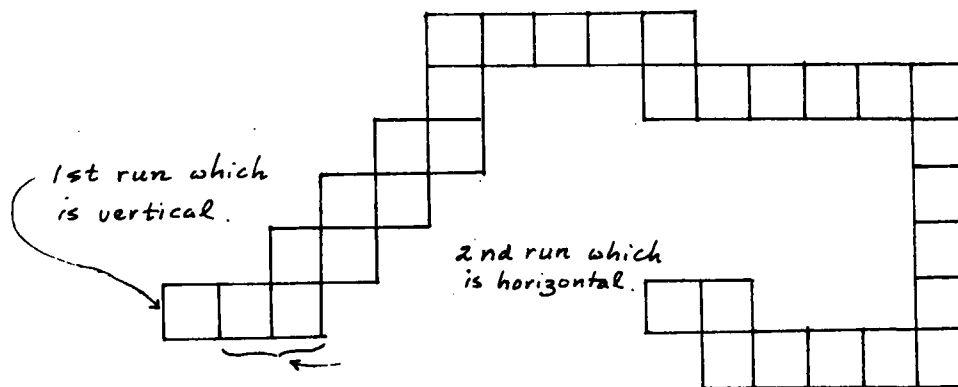


Figure 3. A cellular arc.

The first run is always assumed to consist of a single cell. If it has a horizontal 4-neighbor, it is a vertical run; otherwise, it is a horizontal run. The length of a run is the number of the cells in the run.

Lemma 3. If a cellular arc has the chord property, then it is a cellular straight line segment.

Proof: Let S be a cellular arc that has the chord property.

No two successive runs may both have a run length larger than 1. Otherwise S does not have the chord property as can be seen easily in the examples of Figure 4. Either all horizontal

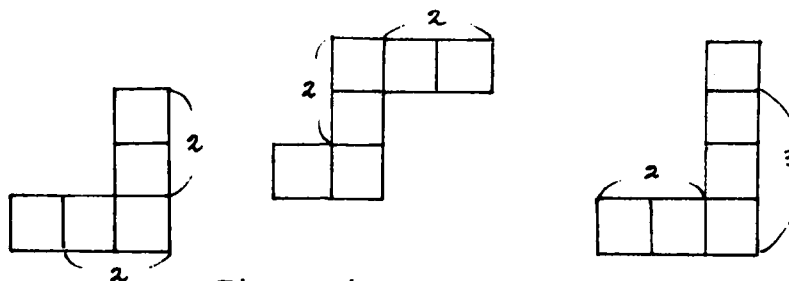


Figure 4

runs or all vertical runs are of length 1. Thus, S must have the shape of a step as shown in Figure 5, since it is a cellular arc.

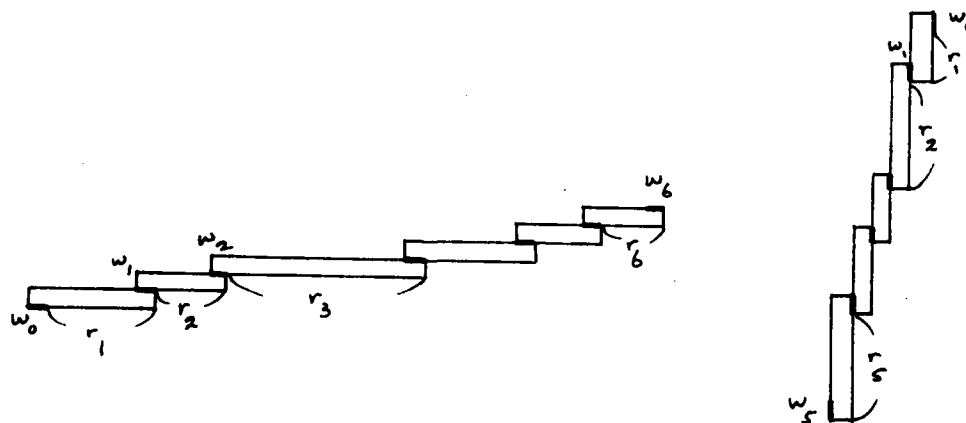


Figure 5.

Without loss of generality, assume that the vertical runs of S are of length 1 as in Figure 5-(a). If S has only one run, it is a cellular straight line segment. Assume that S has h , $h > 1$, horizontal runs. Let r_i , $1 \leq i \leq h$, denote the length of the i -th horizontal run and w_i , $1 \leq i \leq h$, the top edge of the last cell of the i -th row and w_0 the bottom edge of the cell of the first vertical run. The line segment w_i is closed to the left and open to the right, that is, the right corner point of the edge is excluded from w_i . All of these notions may be easily modified for the case when all horizontal runs have length 1 or the other end cell of the arc is used as the first run.

(a) S has two horizontal runs.

Then it is obvious that S is a cellular straight line segment.

(b) S has more than two horizontal runs.

We claim that $|r_i - r_j| \leq 1$ for all $2 \leq i, j \leq h-1$. Suppose not and assume without loss of generality that $r_j - r_i \geq 2$ for some i and j , $2 \leq i < j \leq h-1$, and $r_k = r_i + 1$ for all k , $i < k < j$. Let d_{i-1}, d_i be the centers of the last cells of the $(i-1)$ st and i -th runs, respectively. The center of the $(r_i + 2)$ nd cell of the j -th run is denoted by d_j . Then $\overline{d_{i-1}d_j}$ passes through the lattice point d immediately to the right of d_i , and d is not a lattice point of S . Thus, $\overline{d_{i-1}d_j}$ does not lie near S , and S does not have the chord property. Let m be an integer such that for all

$i, 2 \leq i \leq h-1, r_i = m \text{ or } m+1$. It may be that $r_i = m$ for all i . By an argument similar to the above, it can be shown that $r_1, r_h \leq m+1$.

Let W be the set that consists of $w_1, w_2, \dots, w_{h-1}$, as well as w_0 if $r_1 = m+1$, and w_h if $r_h = m+1$. First we show that if there exists a line ℓ passing through every w_i in W , then S is the cellular image of a segment of ℓ and is a cellular straight line segment. If W contains every $w_i, 0 \leq i \leq h$, then obviously S is a cellular image of a segment of ℓ . If w_0 is not in W , then $r_1 \leq m$. Since ℓ passes through w_i and $r_i \geq m$ for all $i, 1 \leq i \leq h-1$, its slope is less than or equal to $1/m$. Also it does not touch the top-right corner point of the last cell of the first run. Hence, ℓ passes through every cell of the first run and S is the cellular image of a segment of ℓ . A similar argument applies when w_h is not in W .

It remains to be shown that such a line exists. Suppose that no such line exists and ℓ is a line that passes through the most w_i 's in W . Let W' be the subset of W that consists of all w_i 's through which ℓ passes and $|W'| = n$, where $|W'|$ is the number of elements in W' . Then there is a w_j through which ℓ does not pass.

We consider the case when w_j lies above ℓ as shown in Figure 6. Displace ℓ by upward parallel translation until it reaches the left end point of some w_i in W' for the first time.

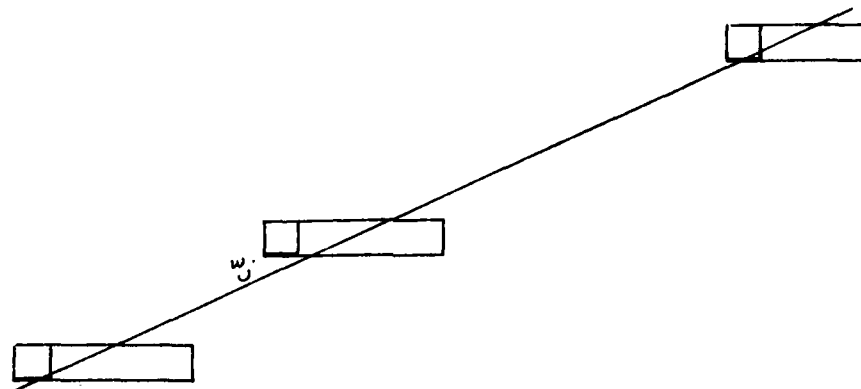


Figure 6. w_j lies above ℓ .

The line still does not pass through w_j , since no line passes through more than n w 's.

Case 1: ℓ passes through the left endpoint of more than one w_i .

(i) Suppose there are two integers i and k such that $i < j < k$ and ℓ passes through the left end points of w_i and w_k and does not pass through w_j as shown in Figure 7. (The line ℓ may pass through the right corner point of the edge of w_j .) Note that i cannot be 0, since the slope of ℓ is larger than $1/(m+1)$, and for it to pass through both w_1 and the left end point of w_0 , its slope must be at most $1/(m+1)$. Let d_i , d_j and d_k be the lattice points just below w_i , w_j and w_k , respectively. Consider $\overline{d_i d_k}$ and the point $z=(x,y)$ at which the horizontal grid line passing through d_j intersects $\overline{d_i d_k}$. Then there is no lattice point $d=(u,v)$ such that $\max\{|u-x|, |v-y|\} < 1$. Therefore $\overline{d_i d_k}$ does not lie near S and S does not have the chord property.

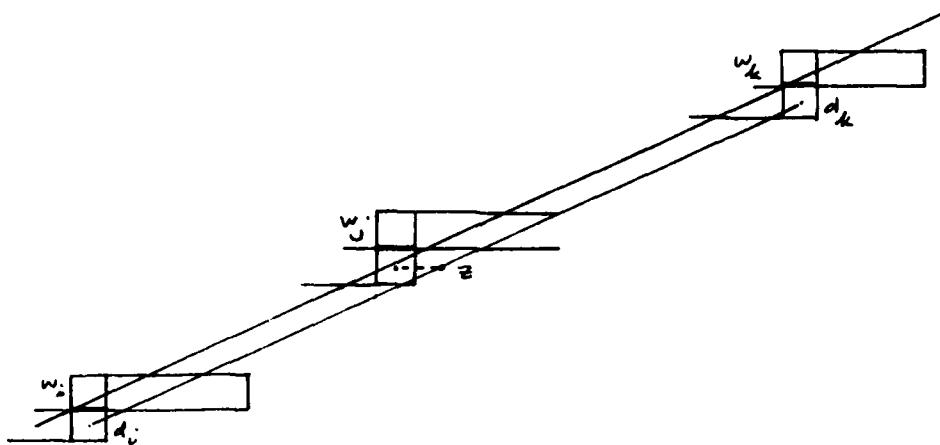


Figure 7.

(ii) Suppose that if ℓ passes through the left end point of w_i , then $j < i$. (The case when $i < j$ is dealt with identically.) Let i be the smallest such index. Using the left end point of w_i as the fixed point, rotate ℓ clockwise until ℓ touches either the right end point of the edge of w_k or the left end point of w_k for some k .

If ℓ touches the right end point of the edge of w_k , then w_k is in W' and $i < k$ as shown in Figure 8. Let d_j , d_i and d_k be

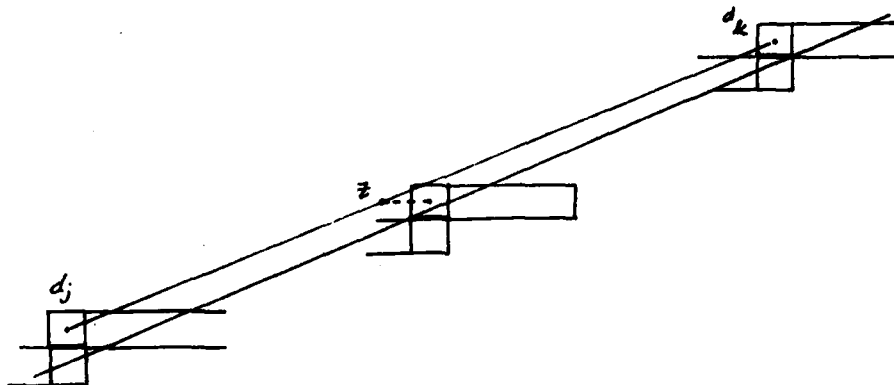


Figure 8.

the lattice points of S just above w_j , w_i and w_k , respectively. Consider $\overline{d_j d_k}$ and the point $z=(x,y)$, the intersection of $\overline{d_j d_k}$ and the horizontal grid line through d_i . There is no lattice point $d=(u,v)$ of S such that $\max\{|u-x|, |v-y|\} < 1$. Thus $\overline{d_j d_k}$ does not lie near S and so S does not have the chord property.

If ℓ touches the left end point of w_k , then w_k is in W' and $k < i$. If $k < j < i$, then we have the case 1-(i) and S does not have the chord property. If $j < k$, we come back to the case 1-(ii), with smaller i . Thus, eventually either the case in Figure 7 or the case in Figure 8 is achieved.

Case 2: There is one w_i through whose left end point ℓ passes.

Suppose $j < i$. The opposite case is handled similarly. Using the left end point of w_i as the fixed point, rotate ℓ clockwise until ℓ touches either the right end point of the edge of w_k or the left end point of w_k for some k . This happens before ℓ reaches w_j . The first is the case shown in Figure 8, and the second is Case 1. This completes the proof for the case when w_j lies above ℓ . The poof for the case when w_j lies below ℓ is almost identical. Care must be taken because of the fact that the w_i 's are open to the right. But these considerations can be handled without difficulty. $\square$

Combining Lemmas 2 and 3, we obtain a necessary and sufficient condition for a cellular arc to be a cellular straight line segment as stated in the following theorem:

Theorem 4. A cellular arc is a cellular straight line segment if and only if it has the chord property.

Next we briefly describe a relationship between digital and cellular straight line segments. Let ℓ be a straight line segment and let $J(\ell)$ and $I(\ell)$ denote the digital and cellular images of ℓ , respectively. For any cellular complex C , let $D(C)$ denote the set of the centers of all cells of C . By Lemma 2, $I(\ell)$ is a cellular arc and has the chord property. In the proof of Lemma 3, it was shown that in $I(\ell)$, either all the vertical runs or all the horizontal runs have length 1. Let $L(I(\ell))$ be the set of last cells of all the runs. $L(I(\ell))$ contains all the cells of either vertical runs or horizontal runs. We state a relationship between $J(\ell)$ and $D(I(\ell))$ as a theorem without proof.

Theorem 5. $D(I(\ell)) - D(I(\ell) - L(I(\ell))) \subseteq J(\ell) \subseteq D(I(\ell))$. Thus, the center of any cell of $I(\ell)$ which is not the last cell of a run is a point of $J(\ell)$. Moreover, except for the last cells of the first and the last runs, the center of exactly one of two adjacent cells in $L(I(\ell))$ is a point of $J(\ell)$.

Given a digital straight line segment R , there may be line segments ℓ and ℓ' such that $J(\ell) = J(\ell') = R$ but $I(\ell) \neq I(\ell')$. Therefore, the digital image of the boundary of a region does not determine the digital image of the region.

4. Algorithm for recognition of cellular straight line segments

In this section we present an algorithm that determines whether or not a cellular complex is a cellular straight line segment. Suppose that a cellular complex C resides in n rows of m cells and is represented by a run length code [8]. The algorithm has a time complexity of $O(n)$, and is optimal up to a constant factor, since it takes $O(n)$ time for any sequential algorithm to scan a cellular complex.

Informally, the algorithm checks if a given cellular complex is a cellular arc with a special configuration and then determines whether or not it is cellularly convex. The part of the algorithm that determines the cellular convexity of a cellular complex was presented in [3]. If a cellular complex passes both tests, it is a cellular straight line segment; otherwise, it is not. To see why the algorithm works, we need a preliminary result, which is interesting in its own right.

Cellular convexity [3]

Let d_1, d_2 be lattice points of a cellular complex C . $P(C; d_1, d_2)$ denotes the set of polygons each of whose boundaries consists of a nonempty subarc of $\overline{d_1 d_2}$ and $\partial s(C)$, and whose interior is a subset of $s(\overline{C})$. A cellular complex is said to be cellularly convex if there are no lattice points d_1, d_2 of C such that $P(C; d_1, d_2)$ contains a lattice point of $\overline{C}$.

A result given in [3] is stated as a lemma, which will be used to prove our next theorem.

Lemma 6 (Theorem 5 in [3]): A cellular complex C is cellularly convex if and only if there is no triplet of collinear lattice points (d_1, d_2, d_3) , such that d_1 and d_3 are in C and d_2 is in $\bar{C}$.

Theorem 7. A cellular complex has the chord property if and only if it is cellularly convex.

Proof: Let C be a cellular complex, and suppose C does not have the chord property. Then there exist lattice points d_1, d_2 of C such that $\overline{d_1 d_2}$ does not lie near C . That is, there is a point $z=(x,y)$ on $\overline{d_1 d_2}$ such that for any lattice point (h,k) of C , $\max\{|x-h|, |y-k|\} \geq 1$. Consider the center of the cell c that contains z . Then d is not in C . If d is in $P(C; d_1, d_2)$, then C is not cellularly convex. Assume that d is not in $P(C; d_1, d_2)$ as shown in Figure 9. If z is on $\overline{vw}$, then $\max\{|x-h'|, |y-k'|\} < 1$ where $d'=(h',k')$. Then d' is a

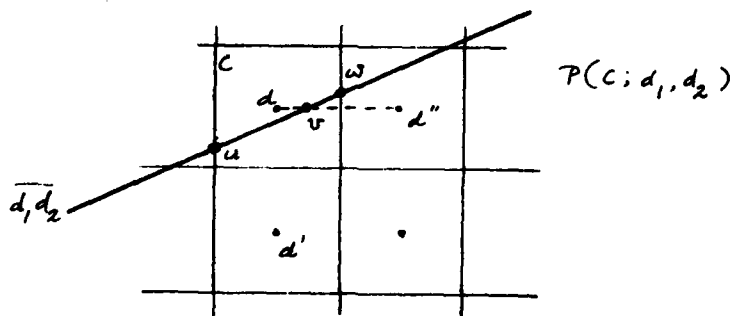


Figure 9.

lattice point of both $\bar{C}$ and $P(C; d_1, d_2)$. If z is on $\overline{uv}$ excluding v , then $\max\{|x-h'|, |y-k'|\} < 1$, where $d'=(h',k')$. Thus d' is a lattice point of both $\bar{C}$ and $P(C; d_1, d_2)$. Therefore C is not cellularly convex.

Now suppose C is not cellularly convex. By Lemma 6, there are lattice points d_1, d_3 of C and d_2 of $\bar{C}$ such that $d_2 = (x, y)$ is a point on $\overline{d_1 d_3}$. Then for any lattice point $d = (h, k)$ of C , $\max\{|x-h|, |y-k|\} \geq 1$. Thus $\overline{d_1 d_3}$ does not lie near C and C does not have the chord property. $\square$

Our algorithm is based on the following result, which is obtained by combining Theorems 4 and 7.

Theorem 8. A cellular arc is a cellular straight line segment if and only if it is cellularly convex.

Algorithm LINE(C)

1. Check if C is a cellular arc in which horizontal and vertical runs alternate and if so, see if either all horizontal runs or all vertical runs have a length of 1. If not, output (False); stop.
2. Construct $H(C)$, the convex hull of the set of lattice points of C .
3. If $H(C)$ contains a lattice point of $\bar{C}$, then output (False); stop.
4. Output (True); stop.

Theorem 9. Algorithm LINE determines whether or not a cellular complex is a cellular straight line segment and has a time complexity of $O(n)$.

Proof: The correctness of the algorithm is due to Theorems 4 and 8. It is obvious that step 1 takes $O(n)$ time and it was shown in [3] that step 2 also takes $O(n)$ time. $\square$

5. Cellular convexity and cellular straight line segments

In Euclidean plane geometry, a region q is convex if and only if, for any pair of points z, z' of q , $\overline{zz'}$ is a subset of q . Here we shown that an equivalent statement holds for cellular complexes.

A cellular arc is said to connect two cells c_1 and c_2 if they are its end cells.

Theorem 10. A simply 4-connected cellular complex C is cellularly convex if and only if, for any pair of cells c_1 and c_2 of C , there is a cellular straight line segment that connects them and is a subset of C .

Proof: Suppose C is not cellularly convex. Then by Lemma 6, there exist cells c_1, c_2 of C and c of $\bar{C}$ such that d is a point on $\overline{d_1 d_2}$. Let $d=(h,k)$. For any lattice point $d'=(h',k')$ of C , $\max\{|h-h'|, |k-k'|\} \geq 1$. Thus, $\overline{d_1 d_2}$ does not lie near C . Let S be a cellular arc in C that connects c_1 and c_2 . Then $\overline{d_1 d_2}$ does not lie near S , and S does not have the chord property. By Theorem 4, S is not a cellular straight line segment.

Now suppose that C is cellularly convex. Let c_1, c_2 be cells of C , and assume without loss of generality that the slope α of $\overline{d_1 d_2}$ is between 0 and 1, that is, $0 \leq \alpha \leq 1$. If $\alpha=0$, then the set of cells S between c_1 and c_2 inclusive is a subset of C , since otherwise C is not cellularly convex. S is a cellular straight line segment that connects c_1 and c_2 . If $\alpha=1$,

it is easy to see that the cellular image S of $\overline{d_1 d_2}$ in one of two directions, either from d_1 to d_2 or from d_2 to d_1 , is a subset of C . So assume that $0 < \alpha < 1$. In the following, the diagonal of c_i , $i=1,2$, means the one with the top-left corner point as an end point and will include the left but not the right end point. We claim that there is a line segment ℓ such that (i) its end points e_1 and e_2 are on the diagonals of c_1 and c_2 , and (ii) it does not go through the interior of any cell of $\bar{C}$ to its left and does not touch any cell of $\bar{C}$ to its right. If there is such a line segment ℓ , then $I(\ell)$ is a cellular straight line segment that is a subset of C and connects c_1 and c_2 . It remains to prove our claim. Suppose there is no such line segment. Let ℓ be a line segment with end points e_1 and e_2 on the diagonals of c_1 and c_2 that does not go through the interior of any cell in $\bar{C}$ to its right and passes through the fewest number of cells in $\bar{C}$ to its right. Translate ℓ upward in parallel until it reaches either the end point of the diagonal of c_i , $i=1$ or 2 , or a corner point of a cell in $\bar{C}$ to its left. (The end points of ℓ are extended or retracted so that they lie on the diagonals of c_1 and c_2 .) The line ℓ still passes through all the cells in $\bar{C}$ to its right that it did before the parallel translation because otherwise the translated ℓ passes through fewer cells in $\bar{C}$ to its right than the original ℓ .

(a) ℓ reaches the end point of the diagonal of c_1 .

Case 1: ℓ also reaches the end point of the diagonal of c_2 .

Let c be a cell in $\bar{C}$ to the right of ℓ through which ℓ passes. The $P(C; d_1, d_2)$ contains d and C is not cellularly convex.

Case 2: ℓ touches a corner point e' of a cell in $\bar{C}$ to its left and e' is farther from e_1 than from d as shown in Figure 10. Let d' be the lattice point just below and to the right of e' . Since C is 4-connected, d' is a lattice point of C . Then $P(C; d_1, d')$ contains d and C is not cellularly convex.

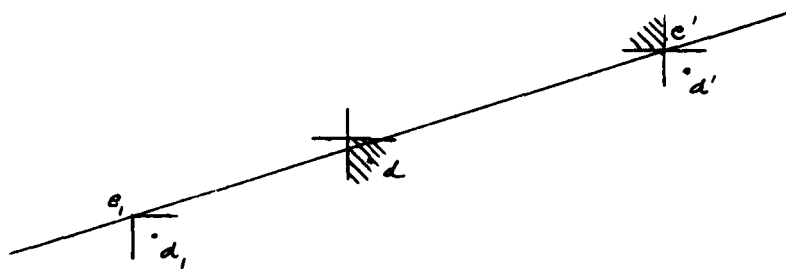


Figure 10.

Case 3: ℓ touches a corner point e' of a cell in $\bar{C}$ to its left and e' is always nearer to e_1 than d as shown in Figure 11.

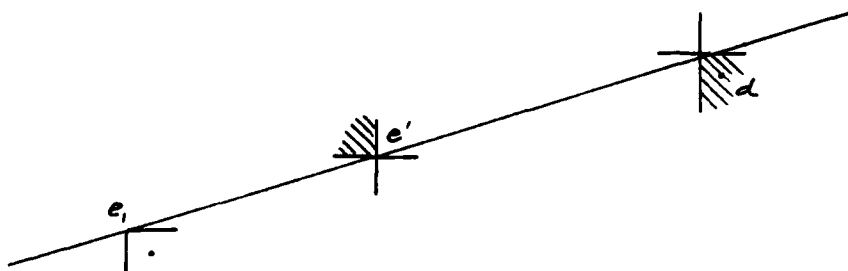


Figure 11.

Let e' be such a point farthest from e_1 . Rotate ℓ around e' counterclockwise until (i) it reaches e_2 , (ii) it reaches the opposite end of e_1 on the diagonal of c_1 , (iii) it touches the corner point of a new cell in $\bar{C}$ to its right, or it touches a corner point e'' of a cell in $\bar{C}$ to its left. In case (iii), e' lies between C and the new cell. The corner point e'' is such that either (iv) c lies between e' and e'' or (v) e'' lies between e' and c . For cases 3-(i), (ii), (iii) and (iv), it is easy to show that C is not cellularly convex. In case 3-(v), rotate ℓ counterclockwise around e'' until one of the above four cases arises again.

Case 4: ℓ does not touch a corner of any cell in $\bar{C}$ to its left.

Rotate ℓ counterclockwise around e_1 until it touches e_2 or a corner point of a cell in $\bar{C}$ to its left. Both result in cases discussed already.

The case where ℓ reaches the end point of the diagonal of c_2 is handled identically.

(b) ℓ reaches a corner point of a cell in $\bar{C}$ to its left.

Case 1: There are two such points and between them there is a cell in $\bar{C}$ to the right of ℓ through which ℓ passes. Then it is easy to show that C is not cellularly convex.

Case 2: There is such a point and on both sides of it, there are two cells in $\bar{C}$ to the right of ℓ through which ℓ passes. Again it is easy to see that C is not cellularly convex.

Case 3: None of the cases above. Let e be such a point nearest to a cell in $\bar{C}$ through which ℓ passes. Assume without loss of generality that e lies to the left of any cell in $\bar{C}$ through which ℓ passes. This case can be treated as was case 3 of (a). This completes the proof. $\square$

6. Conclusions

The results in this paper are the "cellular" analogies of the "digital" results given in [7] and [4]. In [7] it was suggested that such results be proved for the scheme to digitize regions. We accomplished this in Sections 2 and 3 by introducing a new scheme for digitizing curves and proving that the chord property is a necessary and sufficient condition for a cellular arc to be a cellular straight line segment. The results in Sections 4 and 5 correspond to those in [4]. Even though the results are analogous, the proofs are different because of the differences between the digitization schemes.

As is shown in [3] and [5], three independent definitions of digital convexity are equivalent. Hence, the concept of digital convexity seems well defined and universal. Many equivalent properties of convex regions in Euclidean geometry have been shown to hold for convex digital regions under two different schemes for digitizing curves. These further confirm the soundness of the definitions of digital straightness and convexity in digital pictures.

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