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A DERIVATION OF THE NONLINEAR PLATFORM ORIENTATION ERROR EQUATI--ETC(U)
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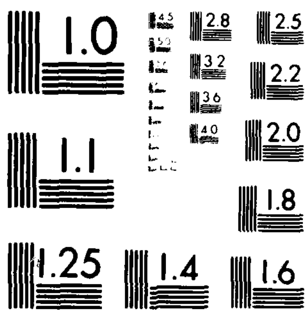
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This report documents a derivation of the nonlinear platform orientation error equations for a local level inertial navigation system (INS). These differential equations govern the Euler angles that relate the actual orientation of the INS platform axes to the ideal orientation of these axes. The derivation deals with specific coordinate representations from the outset, by explicitly including the time-dependent transformations that relate the particular coordinate frames involved in the problem. Thus, the elements (Cont'd)

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of the required angular velocity vectors are obtained in a very natural fashion, and the resulting equations incorporate the transformations needed to ensure computational consistency. A tutorial description of this approach to problems involving rotating coordinate frames is included in this report.

FOREWORD

The work reported herein was conducted in the Space and Surface Systems Division of the Naval Surface Weapons Center. Appreciation is expressed to O. T. Schultz and S. L. Lipscomb for reviewing a draft copy of this report and providing many helpful suggestions.

The final version of this report has been reviewed and approved by D. R. Brown, Jr., Head, Space and Surface Systems Division.

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STRATEGIC SYSTEMS DEPARTMENT

A

I. INTRODUCTION

This report documents a derivation of the *nonlinear* platform orientation error equations for a local level inertial navigation system (INS). These differential equations govern the Euler angles that relate the actual orientation of the INS platform axes to the ideal orientation of these axes. Pinson (Reference 1) discusses the causes of the orientation error and derives the *linear* differential equations for the error angles.

The need to document this derivation, and the resulting equations, arises because the platform orientation error equations form an important part of the formulation of NAVSHIP, a nonlinear, deterministic computer simulation of a local level INS. This simulation, which is currently being used at the Naval Surface Weapons Center (NSWC), will be fully documented in another report.

A second reason for writing this report is tutorial in nature. Most introductory courses in classical mechanics cover the kinematics of rotating coordinate frames by deriving the vector differential operator,*

$$\left(\frac{d}{dt}\right)_{fixed} = \left(\frac{d}{dt}\right)_{rotating} + \underline{\omega} \times ,$$

as quickly as possible, and then use this formalism to solve various fairly simple problems. The required angular velocity vector, $\underline{\omega}$, is often easy to determine by inspection in typical exercises. The beauty and utility of vector formalism, in general, derive from the ability to formulate a physical theory in terms of vector equations that are independent of any particular coordinate frame. We are forced to relinquish this generality and return to coordinate representations, however, whenever we need to do calculations involving these equations.

An alternate approach to the treatment of rotating coordinate frames is often useful. This approach deals with specific coordinate representations from the outset, by explicitly including the time-dependent transformations that relate the particular coordinate frames involved in the problem. Since these transformations are considered when any differentiation is performed, the resulting equations contain the transformations needed to ensure computational consistency. Moreover, the elements of the angular velocity vectors are available in a very natural way. These properties of the alternate approach are very useful when formulating a complicated problem for computer solution. In texts on classical mechanics, this approach is usually treated only in the sections dealing with rigid body motion. A particularly concise and well-motivated presentation may be found in Reference 2.

Thus, the author intends to outline carefully the alternate approach and to provide a nontrivial example of its application. It should be stressed that neither the approach nor the error equations to be derived are original. I have not, however, seen them in combination before.

Throughout this report, the following assumptions are made, often without comment:

(a) All vectors used are elements of a three-dimensional Euclidean vector space.

(b) All coordinate frames are right-hand Cartesian systems.

(c) All transformations represent rigid rotations and, hence, are linear and orthogonal.

*Vector quantities will be denoted either by bold face type or by an underbar.

II. ROTATING COORDINATE FRAMES

Suppose Σ_a, Σ_b, and Σ_c are three coordinate frames and let **v** denote an arbitrary vector. In this section, we will treat Σ_a as fixed and assume that Σ_b and Σ_c are rotating with respect to (wrt) Σ_a. Now, **v** may be represented in terms of Σ_a, Σ_b, or Σ_c by appropriate 3-tuples composed of the components of **v** along the axes of Σ_a, Σ_b, or Σ_c. These 3-tuples will be distinguished by the following notation: "$\mathbf{v}^a$" will denote the 3-tuple whose elements are the components of **v** along the axes of Σ_a. That is,

$$\mathbf{v}^a \equiv \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix}.$$

In general, superscripts on quantities (e.g., vectors or operators) will denote an element representation wrt a particular coordinate frame (e.g., Σ_a, Σ_b).

To handle vector differentiation, we must carry this notation a little further. The symbol "$\dot{\mathbf{v}}^a$" is defined by

$$\dot{\mathbf{v}}^a \equiv d\mathbf{v}^a/dt = \begin{pmatrix} dv_1^a/dt \\ dv_2^a/dt \\ dv_3^a/dt \end{pmatrix} \equiv \begin{pmatrix} \dot{v}_1^a \\ \dot{v}_2^a \\ \dot{v}_3^a \end{pmatrix}.$$

Suppose that W is the matrix representation of the transformation from Σ_a to Σ_b. Then for an arbitrary vector **v**,

$$\mathbf{v}^b = W\mathbf{v}^a .$$

It is important to keep in mind that in terms of the notation defined above,

$$(\dot{\mathbf{v}}^a)^b \equiv W\dot{\mathbf{v}}^a ,$$

and that in general,

$$(\dot{\mathbf{v}}^a)^b \neq \dot{\mathbf{v}}^b .$$

Specifically, $(\dot{\mathbf{v}}^a)^b = \dot{\mathbf{v}}^b$ iff. the transformation from Σ_a to Σ_b is time independent (i.e., iff. $\dot{W} = 0$).

Now that our notation has been defined, let's derive the angular velocity operator associated with the rotation of Σ_b wrt Σ_a. Suppose that **u** is fixed in Σ_b (i.e., that the components of **u** along the axes of Σ_b are constant). Then (from our definition of W given above),

$$\mathbf{u}^b = W\mathbf{u}^a . \tag{1}$$

Differentiating Equation (1) wrt time (remembering that $\mathbf{u}$ is assumed fixed in Σ_b), we find

$$\dot{\mathbf{u}}^b = 0 = \dot{W}\mathbf{u}^a + W\dot{\mathbf{u}}^a , \tag{2}$$

where, of course, $\dot{W} \equiv dW/dt$. Solving Equation (2) for $\dot{\mathbf{u}}^a$ yields

$$\dot{\mathbf{u}}^a = -W^{-1}\dot{W}\mathbf{u}^a = -W^T\dot{W}\mathbf{u}^a , \tag{3}$$

where the superscript T denotes the transpose of a matrix. The last equality in Equation (3) follows from the fact that W is an orthogonal matrix.

The matrix operator, $W^T\dot{W}$, that appears on the right-hand side of Equation (3) is skew-symmetric. To see this, consider the equation,

$$W^TW = I , \tag{4}$$

where I denotes the identity matrix. Differentiating Equation (4) wrt time yields

$$\dot{W}^TW + W^T\dot{W} = 0 ,$$

so

$$W^T\dot{W} = -\dot{W}^TW = -(W^T\dot{W})^T ,$$

as claimed. Since $W^T\dot{W}$ is skew-symmetric, it may be written in the form,

$$W^T\dot{W} = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix} . \tag{5}$$

At this point, we may make the connection between the operator, $W^T\dot{W}$, and the angular velocity vector, $\underline{\alpha}$, that is associated with the rotation of Σ_b wrt Σ_a in the vector operator formalism. Since $\mathbf{u}$ is, by assumption, fixed in Σ_b, the usual vector operator formalism tells us that

$$\dot{\mathbf{u}}^a = \underline{\alpha}^a \times \mathbf{u}^a = \begin{pmatrix} \alpha_2^a u_3^a - \alpha_3^a u_2^a \\ \alpha_3^a u_1^a - \alpha_1^a u_3^a \\ \alpha_1^a u_2^a - \alpha_2^a u_1^a \end{pmatrix} , \tag{6}$$

where the superscript "a" on $\underline{\alpha}$ indicates that it is given in terms of components along the axes of Σ_a. But from Equations (3) and (5) we see that

$$\dot{\mathbf{u}}^a = -W^T\dot{W}\mathbf{u}^a = \begin{pmatrix} \beta_2 u_3^a - \beta_3 u_2^a \\ \beta_3 u_1^a - \beta_1 u_3^a \\ \beta_1 u_2^a - \beta_2 u_1^a \end{pmatrix} \tag{7}$$

Equating the right-hand sides of Equations (6) and (7), we have $\beta_1 = \alpha_1^a$, $\beta_2 = \alpha_2^a$, and $\beta_3 = \alpha_3^a$. It follows that

$$A^a \equiv W^T\dot{W} = \begin{pmatrix} 0 & \alpha_3^a & -\alpha_2^a \\ -\alpha_3^a & 0 & \alpha_1^a \\ \alpha_2^a & -\alpha_1^a & 0 \end{pmatrix} \tag{8}$$

is the angular velocity operator associated with the rotation of Σ_b wrt Σ_a, in terms of elements given wrt the Σ_a axes, or, $A^a = \mathcal{O}[a, b; a]$. (The new notation, $\mathcal{O}[x, y; z]$, denotes the angular velocity operator associated with the rotation of Σ_y wrt Σ_x, in terms of elements given wrt the Σ_z axes.) It is clear from Equation (8) that when W is known, $\underset{\sim}{\alpha}^a$ may be determined from

$$\left.\begin{aligned} \alpha_1^a &= W_{12}\dot{W}_{13} + W_{22}\dot{W}_{23} + W_{32}\dot{W}_{33} , \\ \alpha_2^a &= W_{13}\dot{W}_{11} + W_{23}\dot{W}_{21} + W_{33}\dot{W}_{31} , \\ \alpha_3^a &= W_{11}\dot{W}_{12} + W_{21}\dot{W}_{22} + W_{31}\dot{W}_{32} . \end{aligned}\right\} \tag{9}$$

We may transform A^a from Σ_a to any other coordinate frame, say Σ_n, so long as a nonsingular matrix is associated with the transformation from Σ_a to the new frame. Letting U denote this matrix, then for an arbitrary vector **v**,

$$\mathbf{v}^n = U\mathbf{v}^a ,$$

so

$$(\dot{\mathbf{u}}^a)^n = -UA^a\mathbf{u}^a = -UA^aU^{-1}\mathbf{u}^n . \tag{10}$$

If U is an orthogonal matrix, Equation (10) becomes

$$(\dot{\mathbf{u}}^a)^n = -UA^aU^T\mathbf{u}^n ,$$

or

$$(\dot{\mathbf{u}}^a)^n = -A^n\mathbf{u}^n ,$$

where

$$A^n = UA^aU^T . \tag{11}$$

We claim that $A^n = \mathcal{O}[a, b; n]$. The first step in showing this is to demonstrate that A^n is skew-symmetric. Remembering that $A^a = -A^{aT}$, we have

$$A^n = UA^aU^T = -UA^{aT}U^T = -(UA^aU^T)^T = -A^{nT}, \tag{12}$$

so A^n is skew-symmetric. Proceeding just as we did before, it is easy to show that if

$$\underline{\alpha}^n = \begin{pmatrix} \alpha_1^n \\ \alpha_2^n \\ \alpha_3^n \end{pmatrix} \tag{15}$$

is the angular velocity vector associated with the rotation of Σ_b wrt Σ_a in terms of elements given wrt the Σ_n axes, then

$$(\dot{\mathbf{u}}^a)^n = -A^n\mathbf{u}^n = \underline{\alpha}^n \times \mathbf{u}^n, \tag{16}$$

and

$$A^n = \begin{pmatrix} 0 & \alpha_3^n & -\alpha_2^n \\ -\alpha_3^n & 0 & \alpha_1^n \\ \alpha_2^n & -\alpha_1^n & 0 \end{pmatrix} \tag{17}$$

A little more manipulation shows that

$$\underline{\alpha}^n = U\underline{\alpha}^a. \tag{18}$$

Before we consider the specific problem of deriving the platform orientation error equations, let's carry the general development a little further. As mentioned earlier, we assume that Σ_a is fixed, and Σ_b and Σ_c are rotating wrt Σ_a. These three frames are related by transformations, whose representative matrices are shown in Figure 1. In other words, for an arbitrary vector, $\mathbf{v}$, the 3-tuples representing it in these three frames are related by the following equations:

$$\left.\begin{aligned} \mathbf{v}^b &= W\mathbf{v}^a, \\ \mathbf{v}^c &= G\mathbf{v}^a, \\ \mathbf{v}^c &= F\mathbf{v}^b. \end{aligned}\right\} \tag{19}$$

In the preceding paragraphs, we determined that the operator $\mathcal{O}[a, b; a]$ is A^a of Equation (8). Using the technique illustrated by that development, it follows from the second of Equation (19) that $\mathcal{O}[a, c; a]$ is just

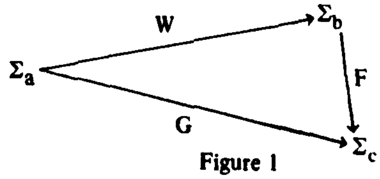

Figure 1

$$B^a = G^T\dot{G}\,, \tag{20}$$

and the associated angular velocity vector may be written as

$$\mathbf{b}^a = \begin{pmatrix} b_1^a \\ b_2^a \\ b_3^a \end{pmatrix} = \begin{pmatrix} G_{12}\dot{G}_{13} + G_{22}\dot{G}_{23} + G_{32}\dot{G}_{33} \\ G_{13}\dot{G}_{11} + G_{23}\dot{G}_{21} + G_{33}\dot{G}_{31} \\ G_{11}\dot{G}_{12} + G_{21}\dot{G}_{22} + G_{31}\dot{G}_{32} \end{pmatrix}. \tag{21}$$

Similarly, $\mathcal{O}[b, c; b]$ is given by

$$C^b = F^T\dot{F}\,. \tag{22}$$

The objective at this point is to show that these angular velocity operators may be treated additively. Specifically, we will show that

$$\mathcal{O}[a, c; k] = \mathcal{O}[a, b; k] + \mathcal{O}[b, c; k]\,, \tag{23}$$

with Σ_k denoting an arbitrary frame related to Σ_a, Σ_b, and Σ_c by rigid rotations.

Using the matrices associated with the angular velocity operators, Equation (23) may be derived as follows. First,

$$\left.\begin{aligned} B^a &= G^T\dot{G} && \text{(by Equation (20))} \\ &= W^TF^T(\dot{F}W + F\dot{W}) && \text{(since } G = FW\text{)} \\ &= W^TF^T\dot{F}W + W^T\dot{W} && \text{(since } F^TF = I\text{)} \\ &= W^TC^bW + A^a && \text{(by Equations (8) and (22))} \\ &= C^a + A^a\,. && \text{(using Equation (11))} \end{aligned}\right\} \tag{24}$$

Thus,

$$\mathcal{O}[a, c; a] = \mathcal{O}[a, b; a] + \mathcal{O}[b, c; a]\,. \tag{25}$$

To complete the deviation, we simply transform both sides of Equation (24) from Σ_a to Σ_k.

The outline given in the preceding pages of this section covers the material needed in the derivation of the nonlinear differential equations that govern the INS's platform orientation error. It should be pointed out, however, that the outlined approach may be carried further to determine the matrix operators needed to write the equations of motion for a particle in terms of rotating coordinates. This extension of the present section will be the subject of another applications report.

III. PLATFORM ORIENTATION ERROR EQUATIONS

The technique outlined in Section II will be used in this section to derive the nonlinear differential equations governing the Euler angles that relate the actual platform orientation to the ideal platform orientation of a local level INS. The particular local level configuration assumed is an "up," "east," "north" system. Thus, the ideal platform axes are defined by unit vectors $\mathbf{u}_h$, $\mathbf{u}_\lambda$, and $\mathbf{u}_\phi$ in the direction of increasing height, increasing longitude, and increasing latitude, respectively, at the system's true position.

Five coordinate frames will be used in the derivation. They are defined as follows:

Σ_e ≡ Earth Fixed Frame – Z-axis through the North Pole, x-axis in the equatorial plane and passing through the Greenwich meridian, y-axis such that Σ_e is a right-hand system

Σ_I ≡ Inertial Frame – Coincident with Σ_e at $t = 0$ and fixed wrt the fixed stars

Σ_P ≡ Ideal Platform Frame – X-axis along the normal to the reference ellipsoid at the true position of the INS, y-axis level, and east pointing at the true position, z-axis level, and north at the true position

Σ_C ≡ Computer Frame – Analog of Σ_P, except that the (erroneous) position indicated by the INS, rather than the true position, defines its orientation

Σ_M ≡ Actual Platform Frame – Frame actually defined by the platform hardware

The five coordinate frames are related by transformations whose representative matrices are shown in Figure 2. In defining these transformations, I will use the following notation: for any axis, ξ, and any angle, ζ, $R_{+\xi}(\zeta)$ will denote a positive rotation about the ξ-axis, through the angle ζ. If a rotation is negative, it will be denoted by $R_{-\xi}(\zeta)$. Finally, $\mathcal{T}_{J2K}$ will denote the transformation from Σ_J to Σ_K.

$$
\begin{tikzcd}
\Sigma_I \arrow[r, "E"] & \Sigma_e \arrow[r, "F"] \arrow[dr, "G"'] & \Sigma_P \arrow[dr, "H"] & \\
 & & \Sigma_C \arrow[r, "Q"'] & \Sigma_M
\end{tikzcd}
$$

Figure 2

From the definitions of the frames, it follows that the transformations shown in Figure 2 are completely defined by

$$\mathcal{T}_{I2e} \equiv R_{+Z}(\Omega_e t), \text{ with associated matrix} \equiv E,$$

$$\mathcal{T}_{e2P} \equiv R_{-y'}(\phi) R_{+Z}(\lambda), \text{ with associated matrix} \equiv F,$$

$$\mathcal{T}_{e2C} \equiv R_{-y'}(\phi^s) R_{+Z}(\lambda^s), \text{ with associated matrix} \equiv G,$$

$$\mathcal{T}_{P2M} \equiv R_{+Z''}(\theta_3) R_{+y'}(\theta_2) R_{+x}(\theta_1), \text{ with associated matrix} \equiv H,$$

where

$$
\begin{aligned}
t &\equiv \text{time.} \\
\Omega_e &\equiv \text{magnitude of the Earth's angular velocity,} \\
\phi &\equiv \text{geodetic latitude of the true position,} \\
\lambda &\equiv \text{geodetic longitude of the true position,} \\
\phi^s &\equiv \text{geodetic latitude of the indicated* position,} \\
\lambda^s &\equiv \text{geodetic longitude of the indicated position,} \\
\theta_1, \theta_2, \theta_3 &\equiv \text{Euler angles relating } \Sigma_P \text{ to } \Sigma_M \,.
\end{aligned}
$$

The claim that these four transformations completely define the matrices shown in Figure 2 follows from the fact that Q may be derived from F, G, and H.

Given the transformations in terms of the rotation operators, the associated matrix representations may be written explicitly. Using the notation, $C\zeta \equiv \cos\zeta$ and $S\zeta \equiv \sin\zeta$, the matrices are

$$
E = \begin{pmatrix} C(\Omega_e t) & S(\Omega_e t) & 0 \\ -S(\Omega_e t) & C(\Omega_e t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}
$$

$$
F = \begin{pmatrix} C\phi C\lambda & C\phi S\lambda & S\phi \\ -S\lambda & C\lambda & 0 \\ -S\phi C\lambda & -S\phi S\lambda & C\phi \end{pmatrix} \tag{26}
$$

$$
G = \begin{pmatrix} C\phi^s C\lambda^s & C\phi^s S\lambda^s & S\phi^s \\ -S\lambda^s & C\lambda^s & 0 \\ -S\phi^s C\lambda^s & -S\phi^s S\lambda^s & C\phi^s \end{pmatrix} \tag{27}
$$

$$
H = \begin{pmatrix} C\theta_2 C\theta_3 & S\theta_1 S\theta_2 C\theta_3 + C\theta_1 S\theta_3 & -C\theta_1 S\theta_2 C\theta_3 + S\theta_1 S\theta_3 \\ -C\theta_2 S\theta_3 & -S\theta_1 S\theta_2 S\theta_3 + C\theta_1 C\theta_3 & C\theta_1 S\theta_2 S\theta_3 + S\theta_1 C\theta_3 \\ S\theta_2 & -S\theta_1 C\theta_2 & C\theta_1 C\theta_2 \end{pmatrix} . \tag{28}
$$

It should be noted that, while $\mathcal{T}_{I2e}$, $\mathcal{T}_{e2P}$, and $\mathcal{T}_{e2C}$ are completely determined by the definitions of Σ_I, Σ_P, and Σ_C, the particular form chosen for $\mathcal{T}_{P2M}$ (and its associated matrix, H) is somewhat arbitrary. That is, a different set of rotations and associated Euler angles could have been chosen.

*The "indicated" value of a quantity is the value available from the INS. It will usually differ from the true value.

At this point, a precise statement of the problem addressed in this section can be given and its solution outlined. The problem is: Given the form chosen for $\mathcal{T}_{P2M}$, determine the differential equations that, together with suitable initial conditions, specify the values of θ_1, θ_2, and θ_3 at any time $t \geq 0$. The solution may be broken down into five steps. They are:

Step (1) – Determine the angular velocity operator $\Gamma^P \equiv \mathcal{O}[P, M; P]$. The three independent elements of Γ^P (i.e., γ_1^P, γ_2^P, and γ_3^P) will be functions of θ_1, θ_2, θ_3, $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$.

Step (2) – Given the system of equations,

$$\begin{aligned}
\gamma_1^P &= f_1(\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)\,, \\
\gamma_2^P &= f_2(\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)\,, \\
\gamma_3^P &= f_3(\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)\,,
\end{aligned}$$

determined in Step (1), solve this system for $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$. This yields the expressions,

$$\begin{aligned}
\dot{\theta}_1 &= g_1(\theta_1, \theta_2, \theta_3, \gamma_1^P, \gamma_2^P, \gamma_3^P)\,, \\
\dot{\theta}_2 &= g_2(\theta_1, \theta_2, \theta_3, \gamma_1^P, \gamma_2^P, \gamma_3^P)\,, \\
\dot{\theta}_3 &= g_3(\theta_1, \theta_2, \theta_3, \gamma_1^P, \gamma_2^P, \gamma_3^P)\,.
\end{aligned}$$

Step (3) – Determine the angular velocity operators, $\Delta^I \equiv \mathcal{O}[I, M; I]$ and $\Psi^I \equiv \mathcal{O}[I, P; I]$.

Step (4) – Transform Δ^I and Ψ^I to Σ_P to obtain $\Delta^P \equiv \mathcal{O}[I, M; P]$ and $\Psi^P \equiv \mathcal{O}[I, P; P]$, and, on the basis of Equation (24), determine γ_1^P, γ_2^P, and γ_3^P from the equation

$$\Gamma^P = \Delta^P - \Psi^P\,.$$

Step (5) – Substitute the expressions found for γ_1^P, γ_2^P, and γ_3^P in Step (4) into the equations found for $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in Step (2). This yields the desired nonlinear differential equations governing θ_1, θ_2, and θ_3.

* * *

Since the transformation, $\mathcal{T}_{P2M}$, is represented by the matrix, H, it follows that

$$\Gamma^P = H^T \dot{H}\,, \tag{29}$$

where H is given by Equation (28).

Using Equation (28) to determine H^T and $\dot{H}$, we find upon substitution of these matrices in Equation (29) that

$$
\left.\begin{aligned}
\gamma_1^P &= \dot\theta_1 + \dot\theta_3 \sin\theta_2 \,, \\
\gamma_2^P &= \dot\theta_2 \cos\theta_1 - \dot\theta_3 \sin\theta_1 \cos\theta_2 \,, \\
\gamma_3^P &= \dot\theta_2 \sin\theta_1 + \dot\theta_3 \cos\theta_1 \cos\theta_2 \,.
\end{aligned}\right\} \tag{30}
$$

Solving Equation (30) for $\dot\theta_1$, $\dot\theta_2$, and $\dot\theta_3$ yields

$$
\left.\begin{aligned}
\dot\theta_1 &= \gamma_1^P + \gamma_2^P \tan\theta_2 \sin\theta_1 - \gamma_3^P \tan\theta_2 \cos\theta_1 \\
\dot\theta_2 &= \gamma_2^P \cos\theta_1 + \gamma_3^P \sin\theta_1 \\
\dot\theta_3 &= \gamma_3^P \cos\theta_1 \sec\theta_2 - \gamma_2^P \sin\theta_1 \sec\theta_2
\end{aligned}\right\} \tag{31}
$$

This completes Steps (1) and (2) of the solution outline.

To accomplish Step (3), the matrices associated with the transformations $\mathcal{T}_{12P}$ and $\mathcal{T}_{12M}$ must first be determined. By inspection of Figure 2, we see that for an arbitrary vector, **v**,

$$
\mathbf{v}^P = FE\mathbf{v}^I \,, \tag{32}
$$

and

$$
\mathbf{v}^M = QGE\mathbf{v}^I \,. \tag{33}
$$

Thus, the matrices associated with $\mathcal{T}_{12P}$ and $\mathcal{T}_{12M}$ are FE and QGE, respectively, which implies that

$$
\begin{aligned}
\Psi^I &= E^T F^T [\dot F E + F \dot E] \\
&= E^T F^T \dot F E + E^T \dot E \,,
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
\Delta^I &= E^T G^T Q^T [\dot Q G E + Q \dot G E + Q G \dot E] \\
&= E^T G^T Q^T \dot Q G E + E^T G^T \dot G E + E^T \dot E \,.
\end{aligned} \tag{35}
$$

The matrices E, F, and G are defined explicitly by Equations (25), (26), and (27), respectively, so the only matrices appearing in Equations (34) and (35) that are not either available or easily determined are Q^T and $\dot Q$. It will be shown in the next step that we do not need to define Q explicitly; thus Step (3) is finished.

The transformation of Ψ^I to Σ_P is straightforward. It yields

$$
\Psi^P = FE\Psi^I E^T F^T \,, \tag{36}
$$

or, using Equation (34) in Equation (36),

$$\Psi^P = \dot{F}F^T + F\dot{E}E^TF^T . \tag{37}$$

Since Ψ^P is skew-symmetric, it may be written in the form,

$$\Psi^P = \begin{pmatrix} 0 & \psi_3^P & -\psi_2^P \\ -\psi_3^P & 0 & \psi_1^P \\ \psi_2^P & -\psi_1^P & 0 \end{pmatrix} . \tag{38}$$

Performing the matrix multiplication indicated in Equation (37) and equating the result with the right-hand side of Equation (38) we see that

$$\left.\begin{aligned} \psi_1^P &= (\dot{\lambda} + \Omega_e)\sin\phi , \\ \psi_2^P &= -\dot{\phi} , \\ \psi_3^P &= (\dot{\lambda} + \Omega_e)\cos\phi . \end{aligned}\right\} \tag{39}$$

The transformation of Δ^I to Σ_P is not so straightforward since the strictly mechanical approach would lead to an erroneous result. Instead, we must think carefully about how a local level INS works and modify our approach accordingly. Before discussing the INS, however, it is helpful to transform Δ^I to Σ_c. This yields

$$\Delta^c = GE\Delta^I E^T G^T , \tag{40}$$

or, using Equation (35) in Equation (40),

$$\Delta^c = Q^T\dot{Q} + \dot{G}G^T + G\dot{E}E^TG^T . \tag{41}$$

Now, at this point the mechanical application of the techniques developed in Section II would lead us to transform Δ^c to Σ_m by the equation,

$$\Delta^m = Q\Delta^c Q^T , \tag{i}$$

which, by use of Equation (41), becomes

$$\Delta^m = \dot{Q}Q^T + Q[\dot{G}G^T + G\dot{E}E^TG^T]Q^T \tag{ii}$$

and then transforms Δ^m to Σ_P by use of the equation,

$$\Delta^P = H^T\Delta^m H . \tag{42}$$

Although Equation (ii) is mathematically correct, it does not represent the behavior of the actual local level INS. The expression contained in brackets in Equation (ii), i.e., $[\dot{G}G^T + G\dot{E}E^TG^T]$, is the matrix associated with the angular velocity operator, $\mathcal{O}[I, c; c]$. The INS computer tries to maintain the actual platform axes in the same orientation as Σ_c by computing torquing commands that are sent to the platform hardware. These torquing commands are just the components of the angular velocity vector associated with the operator given above in brackets. Thus, they represent rates about the axes of Σ_c.

Since the INS computer has no knowledge of the misorientation between Σ_c and Σ_m, it must assume that the two frames are coincident. It follows that the commanded rates are computed for one set of axes, Σ_c, and applied to a different set of axes, Σ_m. This, of course, is not correct and is one source of the total platform orientation error.

The INS behavior just described is equivalent, in terms of the notation used in this report, to assuming that Q equals the identity matrix *when transforming the bracketed expression from* Σ_c *to* Σ_m. Thus, the correct form for Δ^m is

$$\Delta^m = \dot{Q}Q^T + [\dot{G}G^T + G\dot{E}E^TG^T] . \tag{43}$$

Notice that the first term on the right-hand side of Equation (41) has been transformed under the assumption that Q *does not* equal the identity. This yields the first term on the right-hand side of Equation (43) since $Q[Q^T\dot{Q}]Q^T = \dot{Q}Q^T$.

Before Equation (43) can be evaluated, something must be done about the term "$\dot{Q}Q^T$." This term is just the matrix associated with the angular velocity operator, $\mathcal{O}[c, m; m]$. This angular velocity arises because of the gyro drift rates, which must be determined emperically. Since $\dot{Q}Q^T$ is skew-symmetric, we may write

$$N \equiv \dot{Q}Q^T = \begin{pmatrix} 0 & \epsilon_3 & -\epsilon_2 \\ -\epsilon_3 & 0 & \epsilon_1 \\ \epsilon_2 & -\epsilon_1 & 0 \end{pmatrix} . \tag{44}$$

It may be assumed that the gyro drift rates ϵ_1, ϵ_2, and ϵ_3 are given, and so N may be assumed to be known. Using this notation, Equation (43) becomes

$$\Delta^m = N + [\dot{G}G^T + G\dot{E}E^TG^T] . \tag{45}$$

Letting

$$\Delta^m = \begin{pmatrix} 0 & \delta_3^m & -\delta_2^m \\ -\delta_3^m & 0 & \delta_1^m \\ \delta_2^m & -\delta_1^m & 0 \end{pmatrix}, \tag{46}$$

we have, after equating the right-hand sides of Equations (45) and (46),

$$\left.\begin{aligned} \delta_1^M &= (\dot{\lambda}^s + \Omega_e) \sin \phi^s + \epsilon_1 , \\ \delta_2^M &= -\dot{\phi}^s + \epsilon_2 , \\ \delta_3^M &= (\dot{\lambda}^s + \Omega_e) \cos \phi^s + \epsilon_3 . \end{aligned}\right\} \tag{47}$$

Substituting the expression for Δ^M given by Equations (46) and (47) into Equation (42) yields Δ^P. Since Ψ^P is already available from Equations (38) and (39), Γ^P may now be determined from the equation,

$$\Gamma^P = \Delta^P - \Psi^P = H^T \Delta^M H - \Psi^P . \tag{48}$$

Letting

$$\Gamma^P = \begin{pmatrix} 0 & \gamma_3^P & -\gamma_2^P \\ -\gamma_3^P & 0 & \gamma_1^P \\ \gamma_2^P & -\gamma_1^P & 0 \end{pmatrix} \tag{49}$$

and equating the right-hand sides of Equations (48) and (49) yield

$$\begin{aligned} \gamma_1^P = {} & \cos\theta_2 \cos\theta_3 [(\dot{\lambda}^s + \Omega_e) \sin \phi^s + \epsilon_1] \\ & - \cos\theta_2 \sin\theta_3 [-\dot{\phi}^s + \epsilon_2] \\ & + \sin\theta_2 [(\dot{\lambda}^s + \Omega_e) \cos\phi^s + \epsilon_3] \\ & -(\dot{\lambda} + \Omega_e) \sin\phi , \end{aligned} \tag{50a}$$

$$\begin{aligned} \gamma_2^P = {} & (\sin\theta_1 \sin\theta_2 \cos\theta_3 + \cos\theta_1 \sin\theta_3)[(\dot{\lambda}^s + \Omega_e) \sin\phi^s + \epsilon_1] \\ & -(\sin\theta_1 \sin\theta_2 \sin\theta_3 - \cos\theta_1 \cos\theta_3)[-\dot{\phi}^s + \epsilon_2] \\ & - \sin\theta_1 \cos\theta_2 [(\dot{\lambda}^s + \Omega_e) \cos\phi^s + \epsilon_3] \\ & +\dot{\phi} , \end{aligned} \tag{50b}$$

$$\begin{aligned} \gamma_3^P = {} & (\sin\theta_1 \sin\theta_3 - \cos\theta_1 \sin\theta_2 \cos\theta_3)[(\dot{\lambda}^s + \Omega_e) \sin\phi^s + \epsilon_1] \\ & +(\cos\theta_1 \sin\theta_2 \sin\theta_3 + \sin\theta_1 \cos\theta_3)[-\dot{\phi}^s + \epsilon_2] \\ & + \cos\theta_1 \cos\theta_2 [(\dot{\lambda}^s + \Omega_e) \cos\phi^s + \epsilon_3] \\ & -(\dot{\lambda} + \Omega_e) \cos\phi . \end{aligned} \tag{50c}$$

This completes Step (4) of the solution outline. (It should be noted that these equations involve both indicated and true values of latitude and longitude.)

The fifth and final step is accomplished by substituting the expressions for γ_1^P, γ_2^P, and γ_3^P, given in Equations (50a), (50b), and (50c), into Equation (31) to obtain

$$\begin{aligned}\dot{\theta}_1 = {} & [(\dot{\lambda}^s + \Omega_e) \sin \phi^s + \epsilon_1] \sec \theta_2 \cos \theta_3 \\ & -[-\dot{\phi}^s + \epsilon_2] \sec \theta_2 \sin \theta_3 \\ & +\dot{\phi} \tan \theta_2 \sin \theta_1 \\ & +(\dot{\lambda} + \dot{\Omega}_e) [\cos \phi \tan \theta_2 \cos \theta_1 - \sin \phi] ,\end{aligned} \tag{51a}$$

$$\begin{aligned}\dot{\theta}_2 = {} & [(\dot{\lambda}^s + \Omega_e) \sin \phi^s + \epsilon_1] \sin \theta_3 \\ & +[-\dot{\phi}^s + \epsilon_2] \cos \theta_3 \\ & +\dot{\phi} \cos \theta_1 \\ & -(\dot{\lambda} + \Omega_e) \cos \phi \sin \theta_1 ,\end{aligned} \tag{51b}$$

$$\begin{aligned}\dot{\theta}_3 = {} & - [(\dot{\lambda}^s + \Omega_e) \sin \phi^s + \epsilon_1] \tan \theta_2 \cos \theta_3 \\ & +[-\dot{\phi}^s + \epsilon_2] \tan \theta_2 \sin \theta_3 \\ & +[(\dot{\lambda}^s + \Omega_e) \cos \phi^s + \epsilon_3] \\ & -(\dot{\lambda} + \Omega_e) \cos \phi \cos \theta_1 \sec \theta_2 \\ & -\dot{\phi} \sin \theta_1 \sec \theta_2 .\end{aligned} \tag{51c}$$

These are the desired nonlinear differential equations for the Euler angles that relate the ideal platform axes to the actual platform axes. This completes the derivation.

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