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ELASTIC PLATE VIBRATIONS BY BOUNDARY INTEGRAL EQUATIONS. PART 1--ETC(U)
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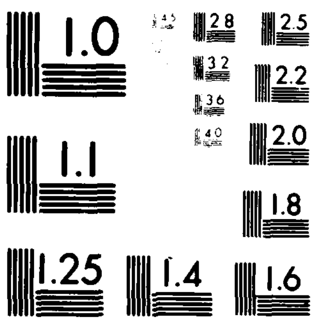
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ELASTIC PLATE VIBRATIONS BY BOUNDARY INTEGRAL EQUATIONS:
Part 1: Infinite Plates

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BRIEF ABSTRACT

One of the prime difficulties in developing two dimensional dynamic elastic plate theories from the three dimensional equations of elasticity is the choice of functional dependence on the thickness coordinate. This difficulty may be circumvented by formulating the problem first as a boundary integral equation; then the dependence on the independent variable through the plate thickness follows form as a direct quadrature with no assumptions of functional form required.

In particular, the examination of separate symmetric and antisymmetric modes allows the boundary integral equation to be written with unknowns evaluated on a single surface.

Consider an infinite elastic plate of thickness h. The dispersion equation for elastic vibrations of such a plate is well known, e.g. Ewing, Jardetsky and Press (1957). The first step in this research is to duplicate such a dispersion equation by means of a boundary integral equation approach. The purpose of this development is not to demonstrate an alternative derivation procedure (in fact, this approach would appear clumsy compared to the classical one), but rather to illustrate an approach which may have much wider application. In particular, the boundary integral equation method allows solution of the exact, three dimensional equations of linear elasticity, but on two dimensional surfaces. This accomplishes the primary goal of plate theory, i.e. the reduction of the dimensionality of the governing equations, without paying the corresponding price of a physical approximation. Furthermore, if approximations are made based on the physical property that the plate thickness is small compared to the wave lengths involved, their effect can be followed directly in the governing integral equations which may themselves be solved through an expansion in orders of plate thickness.

Define the upper and lower surfaces, at $y = +h/2$ and $-h/2$ respectively, as S_U and S_L. These shall be taken to be stress free. The problem may be treated as two dimensional in that waves are taken to propagate only in one horizontal direction (x) with no dependence on the second horizontal direction (z normal to x and y). This allows the introduction of two scalar displacement potentials, ϕ and ψ , e.g.

$$\bar{u} = \nabla\phi + \nabla x(\psi\hat{k})$$

$$u_x = \phi,_x + \psi,_y$$

$$u_y = \phi,_y - \psi,_x$$

These displacement potentials both satisfy wave equations. Assuming time harmonic dependence exp (-iwt), these equations are then Helmholtz equations,

$$\nabla^2\phi + k_D^2\,\phi = 0$$

$$\nabla^2\psi + k_T^2\,\psi = 0$$

where the dilatational and transverse wave numbers, k_D and k_T, are $w/\sqrt{(\lambda + 2\mu)/\rho}$ and $w/\sqrt{\mu/\rho}$ respectively with Lamé constants λ and μ.

The stresses are related to the displacement potentials by

$$\sigma_{xx} = \lambda[\phi,_{xx} + \phi,_{yy}] + 2\mu[\phi,_{xx} + \psi,_{xy}]$$

$$\sigma_{yy} = \lambda[\phi,_{xx} + \phi,_{yy}] + 2\mu[\phi,_{yy} - \psi,_{xy}]$$

$$\sigma_{xy} = \mu[2\phi,_{xy} + \psi,_{yy} - \psi,_{xx}]$$

Boundary conditions for stress free surfaces at constant y require σ_{xy} and σ_{yy} to be zero or

$$\phi,_{xy} = \psi,_{xx} + k_T^2\,\psi/2$$

$$\psi,_{xy} = -k_T^2\,\phi/2 - \phi,_{xx}$$

The governing two dimensional Helmholtz equation may be rewritten as an integral (Weber) equation, e.g. Sneddon (1957) by use of Green's theorem and a fundamental solution. (Alternative formulations are available which treat stress directly rather than through a displacement potential, e.g. Cruse and Rizzo (1968)). For an infinite elastic plate, the Weber equations are

$$\varepsilon\phi(\bar{r}) = \frac{1}{4i}\int_S \left[\phi\left(\bar{r}_o\right)\frac{\partial H_o^{(1)}\left(k_D R\right)}{\partial\, n_o} - H_o^{(1)}\left(k_D R\right)\frac{\partial\phi\left(\bar{r}_o\right)}{\partial\, n_o}\right] dS_o$$

$$\varepsilon\psi(\bar{r}) = \frac{1}{4i}\int_S \left[\psi\left(\bar{r}_o\right)\frac{\partial H_o^{(1)}\left(k_T R\right)}{\partial\, n_o} - H_o^{(1)}\left(k_T R\right)\frac{\partial\psi\left(\bar{r}_o\right)}{\partial\, n_o}\right] dS_o$$

where S represents both the upper and lower surfaces, $S_U(y = +h/2)$, $S_L(y = -h/2)$. n_o is the outward (from the solid) normal to these surfaces, $\bar{r}$ is a field point, $\bar{r}_o$ is an integration variable, R is the distance between $\bar{r}$ and $\bar{r}_o$ and ε is (0, 1/2, 1) depending on whether $\bar{r}$ is exterior to, on the surface of, or interior to the elastic body respectively. When $\bar{r}$ is allowed to approach either surface, these equations reduce the dimensionality of the problem, i.e. they involve only surface values of the dependent variables. The variation of these variables through the thickness of the plate is prescribed by using the same equations with the field point interior to the plate; here, however, once the surface values have been obtained by their own system of equations, the interior values are found through a direct quadrature.

Two basic forms of motion are possible; the symmetric form in which $\phi_U \equiv +\phi_L$, $\psi_U \equiv -\psi_L$ and the antisymmetric form in which $\phi_U \equiv -\phi_L$ and $\psi_U \equiv +\psi_L$ where U, L refer to upper and lower surfaces respectively. A periodic dependence on x in the form exp. $(i\kappa x)$ will be assumed; this form could also be obtained through the integral equations.

Consider the field point to lie on the upper surface. By symmetry, only ϕ and ψ values on this surface are required for the complete problem. Define

$$\phi(x,h/2) = \phi_U(x) = \pm\phi_L(x) = A \exp\,(i\kappa x)$$

$$\psi(x,h/2) = \psi_U(x) = \mp\psi_L(x) = B \exp\,(i\kappa x)$$

$$\frac{\partial\phi}{\partial y}\,(x,h/2) = \frac{\partial\phi_U(x)}{\partial y} = \mp\frac{\partial\phi_L}{\partial y}\,(x) = C \exp\,(i\kappa x)$$

$$\frac{\partial\psi}{\partial y}\,(x,h/2) = \frac{\partial\psi_U}{\partial y}\,(x) = \pm\frac{\partial\psi_L}{\partial y}\,(x) = D \exp\,(i\kappa x)$$

where the two signs refer to symmetric and antisymmetric modes respectively, and the two subscripts U and L refer to uppper and lower surfaces respectively.

The equation then becomes

$$(1/2)\phi_U(x) = (1/4i) \int_{-\infty}^{\infty} \left[-H_o^{(1)}\left(k_D R_U\right)\frac{\partial \phi_U}{\partial y_o}\left(x_o\right)\right]_{y_o = y = h/2} d\,x_o$$

$$+ (1/4i) \int_{-\infty}^{\infty} \left[\pm H_o^{(1)}\left(k_D R_L\right)\frac{\partial \phi_U\left(x_o\right)}{\partial y_o}\right.$$

$$\left.\mp \phi_U\left(x_o\right) \frac{\partial}{\partial y_o}\left[H_o^{(1)}\left(k_D R_L\right)\right]\right]_{y = h/2 = -y_o} d\,x_o$$

with a similar equation on ψ with k_T replacing k_D and the signs on the lower surface integral reversed. Here $R_U = \left|x - x_o\right| = |\xi|$, $R_L = \left[\left(x - x_o\right)^2 + h^2\right]^{1/2}$ and $\xi = x_o - x$; where $\partial R_U / \partial y_o$ vanishes on the upper surface.

The boundary conditions provide

$$C = i\kappa B - \left(i\, k_T^2 / 2\kappa\right) B$$

$$D = -i\kappa A + \left(i\, k_T^2 / 2\kappa\right) A$$

Then

$$(1/2)A \exp\,(i\kappa x) = (1/4i)\int_{-\infty}^{\infty}\left[-C \exp\,\left(i\kappa x_o\right) H_o^{(1)}\left(k_D R_U\right)\right]_{y = y_o = h/2} d\,x_o$$

$$\mp(1/4i)\int_{-\infty}^{\infty}\left|-A\left(k_D h / R_L\right) H_1^{(1)}\left(k_D R_L\right) + C\, H_o^{(1)}\left(k_D R_L\right)\right|_{y = h/2 = -y_o} \exp\left(i\kappa x_o\right) d\,x_o$$

and

$$(1/2)B \exp(i\kappa x) = (1/4i)\int_{-\infty}^{\infty}\left|-D \exp\,\left(i\kappa x_o\right) H_o^{(1)}\left(k_T R_U\right)\right|_{y = y_o = h/2} d\,x_o$$

$$\pm(1/4i)\int_{-\infty}^{\infty}\left[-B\cdot\left(k_T h / R_L\right) H_1^{(1)}\left(k_T R_L\right) + D\, H_o^{(1)}\left(k_T R_L\right)\right]_{y = h/2 = -y_o} \exp\left(i\kappa x_o\right) d\,x_o$$

These can be shifted to $\xi = x_o - x$, thus removing the x dependence from the left hand side of the equation. These are clearly Fourier Transforms and may be evaluated using the Bateman tables, Erdelyi, et al. (1954), e.g.:

$$\int_{-\infty}^{\infty} \exp(i\kappa\xi) H_o^{(1)}(k|\xi|)\, d\xi$$

$$= 2\int_{o}^{\infty} \cos(\kappa\xi) H_o^{(1)}(\kappa\xi)\, d\xi$$

$$= 2 \begin{cases} (k^2 - \kappa^2)^{-1/2}; & 0 < \kappa < k \\ -i(\kappa^2 - k^2)^{-1/2}; & k < \kappa < \infty \end{cases}$$

$$\int_{-\infty}^{\infty} \exp(i\kappa\xi) H_o^{(1)}(k[\xi^2 + h^2]^{1/2})\, d\xi$$

$$= 2\int_{o}^{\infty} \cos(\kappa\xi) H_o^{(1)}(k[\xi^2 + h^2]^{1/2})\, d\xi$$

$$= 2 \begin{cases} +\ (k^2 - \kappa^2)^{-1/2} \exp[ih(k^2 - \kappa^2)^{1/2}]; & 0 < \kappa < k \\ -i(\kappa^2 - k^2)^{-1/2} \exp[-h(\kappa^2 - k^2)^{1/2}]; & k < \kappa < \infty \end{cases}$$

$$\int_{-\infty}^{\infty} (-kh) \exp(i\kappa\xi) H_1^{(1)}(k[\xi^2 + h^2]^{1/2}) \cdot (\xi^2 + h^2)^{-1/2}\, d\xi$$

$$= 2\int_{o}^{\infty} (-kh) \cos(\kappa\xi) H_1^{(1)}(k[\xi^2 + h^2]^{1/2}) \cdot (\xi^2 + h^2)^{-1/2}\, d\xi$$

$$= 2 \begin{cases} +i \exp\ [ih(k^2 - \kappa^2)^{1/2}]; & 0 < \kappa < k \\ +i \exp\ [-h(\kappa^2 - k^2)^{1/2}]; & k < \kappa < \infty \end{cases}$$

There are four homogeneous equations on the four unknown coefficients which then must define a dispersion relationship.

$$(1/2)A = (1/4i)\left[-2C\begin{Bmatrix}\left(k_D^2-\kappa^2\right)^{-1/2}\\ -i\left(\kappa^2-k_D^2\right)^{-1/2}\end{Bmatrix} \pm A\cdot 2i\cdot\begin{Bmatrix}\exp\left[ih\left(k_D^2-\kappa^2\right)^{1/2}\right]\\ +\exp\left[-h\left(\kappa^2-k_D^2\right)^{1/2}\right]\end{Bmatrix}\right.$$

$$\left.\mp C\cdot 2\cdot\begin{Bmatrix}\left(k_D^2-\kappa^2\right)^{-1/2}\exp\left[ih\left(k_D^2-\kappa^2\right)^{1/2}\right]\\ -i\left(\kappa^2-k_D^2\right)^{-1/2}\exp\left[-h\left(\kappa^2-k_D^2\right)^{1/2}\right]\end{Bmatrix}\right]; \quad \begin{matrix}0<\kappa<k_D\\ k_D<\kappa<\infty\end{matrix}$$

$$(1/2)B = (1/4i)\left[-2D\begin{Bmatrix}\left(k_T^2-\kappa^2\right)^{-1/2}\\ -i\left(\kappa^2-k_T^2\right)^{-1/2}\end{Bmatrix} \mp B\cdot 2i\cdot\begin{Bmatrix}\exp\left[ih\left(k_T^2-\kappa^2\right)^{1/2}\right]\\ +\exp\left[-h\left(\kappa^2-k_T^2\right)^{1/2}\right]\end{Bmatrix}\right.$$

$$\left.\pm 2\cdot D\cdot\begin{Bmatrix}\left(k_T^2-\kappa^2\right)^{-1/2}\exp\left[ih\left(k_T^2-\kappa^2\right)^{1/2}\right]\\ -i\left(\kappa^2-k_T^2\right)^{-1/2}\exp\left[-h\left(\kappa^2-k_T^2\right)^{1/2}\right]\end{Bmatrix}\right]; \quad \begin{matrix}0<\kappa<k_T\\ k_T<\kappa<\infty\end{matrix}$$

$$C = \left[i\kappa - ik_T^2/2\kappa\right]B$$

$$D = -\left[i\kappa - ik_T^2/2\kappa\right]A$$

Since $k_D < k_T$, there are three distinct regions of solution:

I : $0 < \kappa < k_D < k_T$

II : $0 < k_D < \kappa < k_T$

III: $0 < k_D < k_T < \kappa$

in addition to the two basic modes (symmetric and antisymmetric). Only case (III) will be considered since only long waves are anticipated. Consider the case (III) - symmetric: the governing equations lead directly to the well-known dispersion equation, for general elastic, e.g. Ewing, Jardetsky and Press 6-12 (1957), with $\nu^2 = \kappa^2 - k_D^2$ and $\nu'^2 = \kappa^2 - k_T^2$,

$$\frac{\tanh\,[h\nu'/2]}{\tanh\,[h\nu/2]} = \frac{4\nu\nu'\,\kappa^2}{(\kappa^2 + \nu'^2)^2}$$

For h very small, this reduces to

$$4\left(1 - k_D^2\big/k_T^2\right)\left(\kappa^2\big/k_T^2\right) = 1$$

which is the well-known elastic plate dispersion equation, e.g. Ewing, Jardetsky and Press, 6-16, (1957).

Case III - antisymmetric follows in much the same way, resulting in the dispersion equation

$$\frac{\tanh\,[h\nu/2]}{\tanh\,[h\nu'/2]} = \frac{4\nu\nu'\,\kappa^2}{(\kappa^2 + \nu'^2)^2}$$

as found as equation 6-11 in Ewing, Jardetsky and Press (1957).

Interior values of ϕ and ψ can be found once the surface values are known (at least to within a constant multiplier). Placing the field point at (x,y) where $-h/2 < y < +h/2$ yields

$$\phi(x,y) = (1/4i)\int_{-\infty}^{\infty}\Big[\phi_U\left(x_o\right)\frac{\partial}{\partial y_o}\,H_o^{(1)}\left(k_D R_U\right) - H_o^{(1)}\left(k_D R_U\right)\frac{\partial}{\partial y_o}\,\phi_U\left(x_o\right)\Big|\,d\,x_o\Big|_{y_o = h/2}$$
$$+\frac{1}{4i}\int_{-\infty}^{\infty}\left[\pm\phi_U\left(x_o\right)\frac{\partial}{\partial y_o}\,H_o^{(1)}\left(k_D R_L\right) \pm H_o^{(1)}\left(k_D R_L\right)\frac{\partial\phi_U}{\partial y_o}\left(x_o\right)\right]d\,x_o\Bigg|_{y_o = -h/2}$$

and a similar equation on ψ. Using slight modifications of the previous Fourier transforms leads directly to interior values, e.g. the symmetric solution for ϕ for case III is

$$\phi(x,y) = -\exp(i\kappa x) \cdot \exp\left[-h\left(\kappa^2 - k_D^2\right)^{1/2} 2\right]$$

$$\cdot \left[A - C \Big/ \left(\kappa^2 - k_D^2\right)^{1/2}\right] \cdot \left\{\begin{matrix}\cosh \\ \sinh\end{matrix}\right. \left[y\left(\kappa^2 - k_D^2\right)^{1/2}\right]$$

which again agrees with equation 6-13 of Ewing, Jardetsky and Press.

REFERENCES

Cruse, T. and F. Rizzo; "A Direct Formulation and Numerical Solution of the General Transient Elastrodynamic Problem," Jour. Math. Anal. App. Vol. 22, pp. 341 - 355, 1968.

Erdelyi, A., et al.; Bateman Manuscript Project, Tables of Integral Transforms, Vol. 1, McGraw Hill, 1954.

Ewing, M., W. Jardetsky and F. Press; Elastic Waves in Layered Media, McGraw Hill, 1957

Sneddon, I.; Elements of Partial Differential Equations, McGraw Hill, 1957.

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One of the prime difficulties in developing two dimensional dynamic elastic plate theories from the three dimensional equations of elasticity is the choice of functional dependence on the thickness coordinate. This difficulty may be circumvented by formulating the problem first as a boundary integral equation; then the dependence on the independent variable through the plate thickness follows form as a direct quadrature with no assumptions of functional form required. (turn over)

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In particular, the examination of separate symmetric and antisymmetric modes allows the boundary integral equation to be written with unknowns evaluated on a single surface.

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