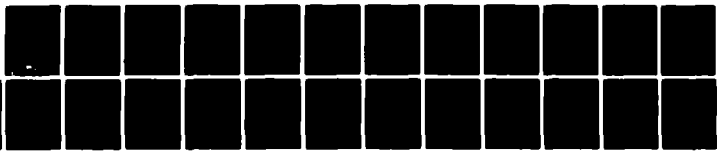


AD-A088 760

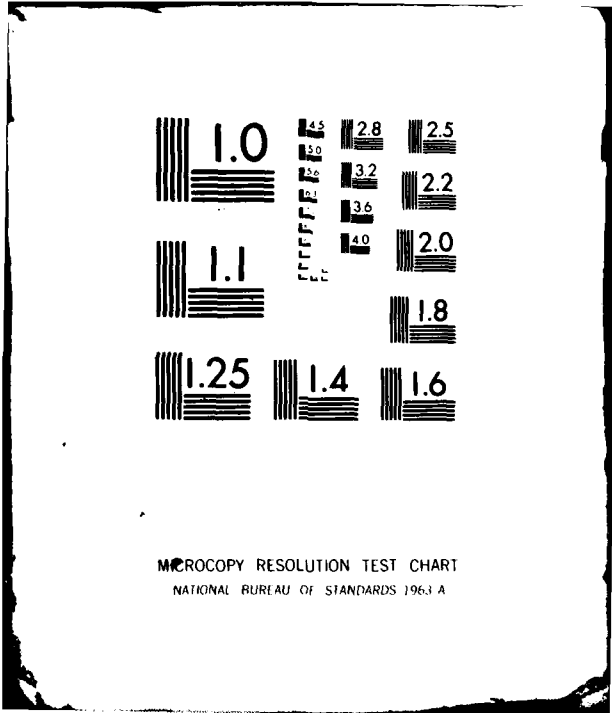
CARNEGIE-MELLON UNIV PITTSBURGH PA MANAGEMENT SCIENC--ETC F/6 12/1  
ON THE CONVERGENCE OF A BLOCK SUCCESSIVE OVERRELAXATION METHOD --ETC(U)  
JUL 80 R W COTTLE, J S PANG N00014-75-C-0621  
MSRR-462 NL

UNCLASSIFIED

1-1  
2 APR 75



END  
DATE  
FILMED  
10 80  
DTIC

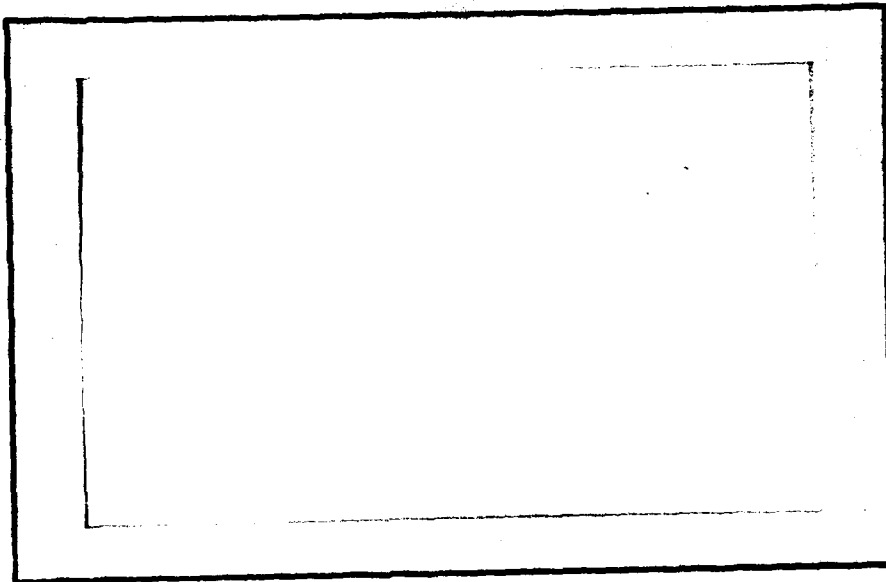


MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS 1963-A

**LEVEL II**

(D)

AD A 088760



**Carnegie-Mellon University**

PITTSBURGH, PENNSYLVANIA 15213

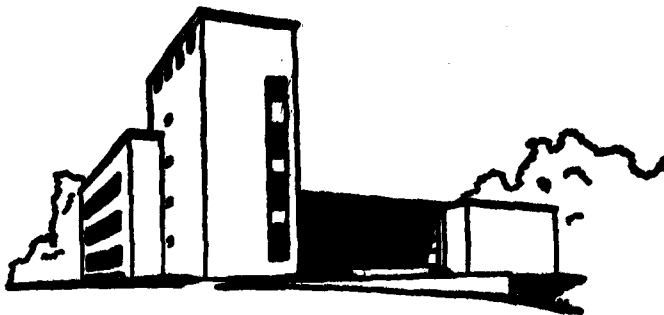
DTIC  
ELECTRONIC  
S AUG 28 1980 D

**GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION**

WILLIAM LARIMER MELLON, FOUNDER

A

DOC FILE COPY



DISTRIBUTION STATEMENT A  
Approved for public release;  
Distribution Unlimited

80 8 25 152

12) 272

9 Management Science Research Report, No. 462

6 ON THE CONVERGENCE OF A BLOCK SUCCESSIVE OVERRELAXATION METHOD FOR A CLASS OF LINEAR COMPLEMENTARITY PROBLEMS.

11) Jul 80

by R. W. Cottle & J.S. Pang<sup>2</sup>

14) MSRR-462

July, 1980

15) N00014-75-C-0624,  
NSF-ECS 79-26320



TECHNICAL REPORT SOL 80-17

1. Stanford University. Research of this author and reproduction of this report were partially supported by the Department of Energy Contract DE-AC03-76SF00326, PA# DE-AT03-76ER72018 and the Office of Naval Research Contract N00014-75-C-0267.
2. Carnegie-Mellon University. Research of this author was supported by the Office of Naval Research under Contract N00014-75-C-0621 NR 047-048 and National Science Foundation Grant ECS-7926320.

Research in whole or in part is permitted for any purposes of the United States Government.

Management Sciences Research Group  
Graduate School of Industrial Administration  
Carnegie-Mellon University  
Pittsburgh, Pennsylvania 15213

slh

DISTRIBUTION STATEMENT A  
Approved for public release;  
Distribution Unlimited

403426

ON THE CONVERGENCE OF A BLOCK SUCCESSIVE  
OVERRELAXATION METHOD FOR A CLASS OF  
LINEAR COMPLEMENTARITY PROBLEMS

by

R.W. Cottle and J.S. Pang

**Abstract.** This paper develops a reduced block successive over-relaxation method for solving a class of (large-scale) linear complementarity problems. The main new feature of the method is that it contains certain reduction operations at each iteration. Such reductions are needed in order to ensure the boundedness (and therefore the existence of accumulation points) of the sequence of iterates produced by the algorithm. Convergence of the method is established by using a theorem due to Zangwill.

**Key Words:** Convergence, block successive overrelaxation algorithms, linear complementarity problem, quadratic programming, compactness, level sets.

**Abbreviated title:** A block SOR method

**Correspondence address:** Prof. Jong-Shi Pang  
Graduate School of Industrial Administration  
Carnegie-Mellon University  
Pittsburgh, PA 15213  
U.S.A.

Accession For	
NTIS	<input checked="" type="checkbox"/>
DDC	<input type="checkbox"/>
Unassessable	<input type="checkbox"/>
Unassessable	<input type="checkbox"/>
Date	
A	

ON THE CONVERGENCE OF A BLOCK SUCCESSIVE  
OVERRELAXATION METHOD FOR A CLASS OF  
LINEAR COMPLEMENTARITY PROBLEMS

R.W. Cottle and J.S. Pang

1. Introduction. The present research is motivated by an investigation (still in progress) of methods for solving a certain class of "capacitated quadratic transportation problems". One of these calls for the application of the block successive overrelaxation (BSOR) method [4] to the dual of the given problem. However, a technical problem is engendered by the unboundedness of the level sets of the dual objective function and the consequent breakdown of the convergence proof used in [4]. At issue is the existence of an accumulation point of the sequence of iterates produced by the algorithm. Fortunately, the structure of the problem permits a modification of the algorithm that leads to a remedy for this complication. Applying a simple transformation to the iterates forces the new points to lie in a compact set. Convergence of the algorithm can then be established by invoking a theorem of Zangwill [8].

Our purpose in this paper is to establish the convergence of the modified BSOR for a class of problems somewhat larger than that under consideration in the aforementioned study. To be precise, we concentrate our attention on a (large-scale) linear complementarity problem of the form: Find  $y, v \in R^N$  such that

$$v = f + FAc + FAF'y \geq 0, \quad y \geq 0, \quad v'y = 0. \quad (1)$$

The following blanket assumptions will be maintained throughout this paper:

(A1) The matrix  $A \in R^{p \times p}$  is symmetric and positive semi-definite;

(A2) there exists a vector  $x$  such that

$$FAx \leq f; \quad (2)$$

(A3) there exists an index set  $\alpha$  such that for any  $y$  satisfying

$$\begin{aligned} AF'y &= 0 \\ f'y &= 0 \\ 0 \neq y &\geq 0 \end{aligned} \quad (3)$$

it follows that  $y_j > 0$  if and only if  $j \in \alpha$ .

Remarks. (A3) holds vacuously if (3) has no solution. In fact, the nonexistence of a solution to (3) is equivalent to the so-called Slater condition, i.e. the consistency of the linear inequality system  $FAx < f$ .

If (A3) holds nonvacuously, the index set  $\alpha$  must be nonempty, though its cardinality could be as low as 1. In the capacitated quadratic transportation problem mentioned earlier, (3) does have a solution. There, the introduction of a seemingly mild additional hypothesis on the capacities ensures the validity of (A3), and the index set  $\alpha$  is easily identified. It corresponds to the supply and demand constraints of the problem. See Section 6 for further discussion of this application.

Under assumption (A1), the linear complementarity problem (1) is the set of Karush-Kuhn-Tucker conditions for the convex quadratic program

$$\begin{aligned} \text{minimize } \psi(y) &= (f + FAc)'y + \frac{1}{2}y'FAF'y \\ \text{subject to } y &\geq 0. \end{aligned} \quad (4)$$

By a proof like that of Proposition 2.1 below, (A1) and (A2) imply the existence of a solution to (1) or, equivalently, (4).

If the matrix A is in fact positive definite, then (4) is essentially the dual of the strictly convex quadratic program

$$\begin{aligned} &\text{minimize} && \phi(x) = c'x + \frac{1}{2}x'A^{-1}x \\ &\text{subject to} && Fx \leq f. \end{aligned} \tag{5}$$

Note that (A2) implies the feasibility of (5) and thus the existence of an optimal solution.

In proving the convergence of iterative procedures for nonlinear programming, it is customary to require that the iterates lie in a compact set. The set in question is often a level set of the function being minimized. In the context of the quadratic program (4), the *minimand* is  $\psi$ . As we shall show later, the level sets of  $\psi$  are not bounded if the system (3) is consistent.

In the present paper, we shall show how the BSOR method described in [4] can be modified in such a way that the possible unboundedness of level sets will not affect the convergence of the method for solving (4) - or, equivalently, (1). Our analysis provides a unified treatment for both bounded and unbounded level sets. In particular, the analysis includes, as a special case, the recent study of Mangasarian [5] who treats the quadratic program (5) under a Slater condition.



2. Preliminary discussion. Throughout the paper we denote the linear complementarity problem (LCP)

$$w = q + Mz \geq 0, \quad z \geq 0, \quad w'z = 0$$

by the pair  $(q, M)$ . For a given  $M$ , let  $K(M)$  be the set of all vectors  $q$  for which  $(q, M)$  has a solution. We recall that if  $M$  is positive semi-definite, then  $q \in K(M)$  if and only if the inequalities

$$q + Mz \geq 0, \quad z \geq 0$$

are consistent. (See [1].)

In the next three results, we present some properties of the LCP (1).

Proposition 2.1. For all vectors  $a$ , the linear complementarity problem

$$(f + FAa, \quad FAF') \tag{6}$$

has a solution.

Proof. As  $FAF'$  is (symmetric and) positive semi-definite, only consistency need be verified. If the LCP  $(f + FAa, FAF')$  is inconsistent, there must exist a vector  $u$  such that

$$u'(f + FAa) < 0, \quad u'FAF' \leq 0, \quad u \geq 0.$$

In the presence of (A1), the latter implies

$$u'f < 0, \quad u'FA = 0, \quad u \geq 0. \tag{7}$$

But (A2) implies the existence of a vector  $x$  such that (2) holds.

Clearly (2) and (7) cannot both hold simultaneously, so (6) must be consistent.  $\square$

The theorem below characterizes the boundedness (and, consequently, the compactness) of the level sets of certain convex quadratic functions  $\psi$  of interest in the present study. Part of its proof relies on a much more general result of Rockafellar.

Theorem 2.1. Suppose (A1) and (A2) are satisfied. For any  $a$ , the following statements are equivalent for the quadratic function

$$\psi(y) := (f + FAa)'y + \frac{1}{2}y'FAF'y.$$

(1°) For each  $\lambda \in \mathbb{R}$ , the set

$$Y(\lambda) := \{y \geq 0 : \psi(y) \leq \lambda\}$$

is compact;

(2°)  $f + FAa \in \text{int } K(FAF')$ ;

(3°) there exists no vector  $y$  such that

$$AF'y = 0, \quad f'y = 0, \quad 0 \neq y \geq 0; \tag{8}$$

(4°) there exists a vector  $\tilde{x}$  such that

$$FA\tilde{x} < f;$$

(5°) the LCP  $(f + FAa, FAF')$  has a bounded solution set.

Proof. Define

$$\Psi(y) = \begin{cases} \psi(y) & \text{if } y \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Then (1°) holds if and only if the level sets of  $\Psi$  are bounded. By [7, Corollary 14.2.2], this is so if and only if  $0 \in \text{int dom } \Psi^*$  where  $\Psi^*$  denotes the convex conjugate of  $\Psi$ . Now for any  $y^*$  we have

$$\begin{aligned}
\psi^*(y^*) &= \sup \{y'y^* - \psi(y) : y \text{ arbitrary}\} \\
&= \sup \{y'y^* - \psi(y) : y \geq 0\} \\
&= -\inf \{y'(q - y^*) + \frac{1}{2}y'My : y \geq 0\} \\
&= \begin{cases} \frac{1}{2}\bar{y}'M\bar{y} & \text{if } \bar{y} \text{ solves } (q-y^*, M) \\ -\infty & \text{if } (q-y^*, M) \text{ has no solution} \end{cases}
\end{aligned}$$

where  $q = f + FAa$  and  $M = FAF'$ . It therefore follows that

$$\text{dom } \psi^* = \{y^* : q - y^* \in K(M)\}$$

From this it is apparent that  $0 \in \text{int dom } \psi^*$  if and only if  $q \in \text{int } K(M)$ . Thus (1°) is equivalent to (2°). It is known from [2] that (2°) is equivalent to the condition that  $q \in K(M)$  and the only solution of the system

$$u'q = 0, \quad u'M = 0, \quad u \geq 0 \tag{9}$$

is the zero vector. By Proposition 2.1,  $q \in K(M)$  is implied by (A1) and (A2) which are in force here. By the definitions of  $q$  and  $M$  and the assumed properties of  $A$ , (9) becomes

$$u'f = 0, \quad u'FA = 0, \quad u \geq 0.$$

Hence the equivalence of (2°) and (3°) follows. Combined with (A1) and (A2), condition (3°) is equivalent to the fact that

$$u'f \leq 0, \quad u'FA = 0, \quad u \geq 0$$

has only the zero solution. By an alternative theorem, this is equivalent to (4°). The equivalence of (4°) and (5°) is a direct consequence of [2, Theorem 3.1].  $\square$

Theorem 2.1 has much in common with some characterizations obtained by Mangasarian [6]. Our work along these lines was done independently, however.

Theorem 2.1 implies among other things that if the system in (3°) has a non-zero solution, then the level sets of the function  $\psi$  are unbounded. In fact, the following stronger result obviously holds.

Proposition 2.2. Let  $y^*$  be any solution of the system

$$AF'y = 0, \quad f'y = 0.$$

Then

$$\psi(y + \theta y^*) = \psi(y)$$

for all  $y$  and  $\theta$ .  $\square$

To describe the BSOR method, we let the rows of the matrix  $F$  be partitioned into blocks  $F_i$  ( $i = 1, \dots, m$ ). This induces a partitioning of  $M = FAF'$  into submatrices  $M_{ij} = F_i AF_j'$ . Let the vector  $f$  be partitioned accordingly. Let  $J_i$  denote the set of indices of the rows in  $F_i$  (and  $f_i$ ). Let  $n_i$  denote the cardinality of  $J_i$ , and finally (referring to (A3)) let

$$\alpha_i = \alpha \cap J_i \quad i = 1, \dots, m.$$

Obviously, the following implication holds:

$$\left. \begin{array}{l} AF_i'y_i = 0 \\ f_i'y_i = 0 \\ 0 \neq y_i \geq 0 \end{array} \right\} \Rightarrow (y_i)_j > 0 \quad \text{if and only if } j \in \alpha_i \quad (10)$$

Once the partitioning above is introduced, then for  $i = 1, \dots, m$  Proposition 2.1, Theorem 2.1, and Proposition 2.2 apply to the subproblems  $(f_i + F_i A a, F_i A F_i')$  and quadratic functions

$$\psi_i(y_i) = (f_i + F_i A a)' y_i + \frac{1}{2} y_i' F_i A F_i' y_i .$$

In particular, as we shall show (Proposition 2.4), assumption (A1) implies that at most one of these subproblems can have an unbounded solution set. Before proving this, we give a geometrical interpretation of the assumption. Let  $C$  denote the set of all vectors  $y$  satisfying the system (3) and also containing the zero vector. The next result shows that assumption (A3) holds if and only if the set  $C$  is a ray emerging from the origin.

Proposition 2.3. Assumption (A3) holds if and only if there exists a non-negative vector  $y^*$  such that

$$C = \{y : y = \lambda y^* \text{ for some } \lambda \geq 0\} .$$

Proof. Suppose  $C$  is of this form. If the system (3) is inconsistent, there is nothing to prove, so suppose it is consistent. This implies that the vector  $y^*$  must be nonzero. Let  $\alpha$  be the set of indices which correspond to the nonzero components of  $y^*$  (i.e., its support). Obviously, if  $y \in C \setminus \{0\}$ , then  $y_j > 0$  if and only if  $j \in \alpha$ .

Conversely, suppose that assumption (A3) holds. If (3) is inconsistent it suffices to let  $y^*$  be the zero vector. On the other hand, if (3) is consistent, let  $y^*$  be any one of its solutions. Let  $y \in C \setminus \{0\}$ . Consider the vector  $y - \lambda y^*$ . For suitable  $\lambda \geq 0$ , the vector  $y - \lambda y^*$  will belong to  $C$  and have at least one zero component, say the  $j$ -th one with  $j \in \alpha$ . By (A3) this is impossible unless  $y - \lambda y^*$  is the zero vector. This proves the proposition.  $\square$

Proposition 2.4. Let assumption (A3) hold. Then for any partitioning of the rows of  $F$ , there can exist at most one index  $i$  for which the system

$$AF_i'y_i = 0, \quad f_i'y_i = 0, \quad 0 \neq y_i \geq 0 \quad (11)_i$$

is consistent.

Proof. Indeed, if there are indices  $i_1 \neq i_2$  for which  $(11)_{i_1}$  and  $(11)_{i_2}$  are consistent, let  $y_{i_1}^*$  and  $y_{i_2}^*$  be solutions of these systems, respectively. Obviously, the vectors  $y^1 = (y_\ell^1)$  and  $y^2 = (y_\ell^2)$  with

$$y_\ell^1 = \begin{cases} 0 & \text{if } \ell \neq i_1 \\ y_{i_1}^* & \text{if } \ell = i_1 \end{cases} \quad y_\ell^2 = \begin{cases} 0 & \text{if } \ell \neq i_2 \\ y_{i_2}^* & \text{if } \ell = i_2 \end{cases}$$

satisfy the system (3). By (A3), we must have  $\alpha \subset J_{i_1} \cap J_{i_2} = \emptyset$ . This contradiction establishes the proposition.  $\square$

3. Closedness of the component maps. The main tool used in our convergence result for the modified BSOR method is Convergence Theorem A of Zangwill [8]. To apply the theorem, it is necessary to show that the "algorithmic map" involved is closed. In this section, we establish some preliminary results useful for this purpose.

The total number of rows in the matrix  $F$  is  $N = \sum_1^m n_i$ . For each  $i = 1, \dots, m$  let  $y_i^* \in R^{n_i}$  be either a fixed vector satisfying the system  $(11)_i$  if the system is consistent or the zero vector if it is inconsistent. Let  $y^* = (y_1^*, \dots, y_m^*)$ . Then obviously,  $y^*$  is either 0 or it satisfies the system (3).

A vector  $y_i \in R_+^{n_i}$  is said to be reduced (with respect to the index set  $\alpha_i$ ) if at least one component in the subvector  $(y_i)_{\alpha_i}$  is equal to zero. Then by  $(11)_i$ ,  $y_i^*$  is either zero or not reduced. Let  $S_i$  denote the set of all reduced vectors in  $R_+^{n_i}$ . The  $i$ -th reduction map  $R_i : R_+^{n_i} \rightarrow S_i$  is defined as follows:

$$R_i(y_i) = y_i - \rho_i y_i^*$$

where

$$\rho_i = \rho_i(y_i) = \begin{cases} \min \{(y_i)_j / (y_i^*)_j : j \in \alpha_i\} & \text{if } y_i^* \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

The reduction map  $R_i$  is well defined and continuous. If  $y_i^* = 0$ , then  $\alpha_i$  is just the identity map. Similarly, by dropping the subscript  $i$ , we may define the reduced vectors in  $R_+^N$  as well as the complete reduction map  $R : R_+^N \rightarrow S$ , where  $S$  is the set of all reduced vectors in  $R_+^N$ .

We define the i-th complementarity map  $C_i : R_+^N \rightarrow R_+^N \times S_i$  as follows. Given  $y = (y_1, \dots, y_m) \in R_+^N$ ,  $C_i(y)$  denotes the set of all points  $(y, R_i(\bar{y}_i))$  where  $\bar{y}_i$  solves the LCP  $(f_i + F_i A a, F_i A F_i')$  where

$$a = c + \sum_{\ell \neq i} F_\ell' y_\ell .$$

In general,  $C_i$  is a point-to-set map. By Proposition 2.1,  $C_i(y)$  is nonempty for each  $i$ .

Note that if the subproblem has a unique solution, then by Theorem 2.1,  $R_i(\bar{y}_i) = \bar{y}_i$  so that the  $i$ -th reduction is unnecessary. Roughly speaking, the motivation for including the reduction step in defining the map  $C_i$  is to ensure that  $C_i$  maps bounded sets into bounded sets.

Let  $\omega^* < 2$  be a given positive scalar. Define the  $i$ -th relaxation map  $P_i : R_+^N \times R_+^{n_i} \rightarrow R_+^N$  as follows. For  $(y, \hat{y}_i) \in R_+^N \times R_+^{n_i}$ , the set  $P_i(y, \hat{y}_i)$  consists of all vectors of the form

$$(y_1, \dots, y_{i-1}, \tilde{y}_i, y_{i+1}, \dots, y_m)$$

where  $\tilde{y}_i = y_i + \tilde{\omega}(\hat{y}_i - y_i)$  for some  $\tilde{\omega}$  such that

$$\min\{\omega^*, 1\} \leq \tilde{\omega} \leq \omega^* \quad \text{and} \quad \tilde{y}_i \geq 0.$$

The relaxation step in [4] is a particular realization of the relaxation map where  $\tilde{\omega}$  is chosen as the largest possible value of  $\omega$  for which

$$\omega \leq \omega^* \quad \text{and} \quad y_i + \omega(\hat{y}_i - y_i) \geq 0.$$

A point to set map  $M : U \rightarrow V$  is bounded if for every subset  $T \subset U$ , the image  $U(M(t) : t \in T)$  is a bounded subset of  $V$ .



Let  $B_i = P_i \circ C_i$  denote the composition of the  $i$ -th complementarity and relaxation maps. In what follows, we show that  $B_i$  is a closed and bounded map from  $R_+^N$  into itself. We first prove this for  $C_i$ .

For each index  $i$  and vector  $a \in R^D$  let  $X_i(a)$  denote the set of all solutions of the LCP  $(f_i + FAa, F_iAF_i')$ .

Lemma 3.1.  $R_i(X_i(a)) = \{y_i \in X_i(a) : \prod_{j \in \Omega_i} (y_i)_j = 0\}$ .

Proof. For brevity, let  $T_i$  be the set on the right. Since  $\rho_i(y_i) = 0$  for each  $y_i \in T_i$ , it follows that  $T_i \subset R_i(X_i(a))$ . Conversely, let

$\hat{y}_i = R_i(\bar{y}_i)$  where  $\bar{y}_i \in X_i(a)$ . Then obviously,  $\prod_{j \in \Omega_i} (\hat{y}_i)_j = 0$ .

It can easily be shown that  $\hat{y}_i$  also solves the LCP  $(f_i + FAa, F_iAF_i')$ .  $\square$

Proposition 3.1. The  $i$ -th complementarity map  $C_i$  is both closed and bounded.

Proof. To show that  $C_i$  is closed, let

$$y^k \rightarrow y, \quad y^k \in R_+^N$$

$$z^k \rightarrow z = (y, \hat{y}_i), \quad z^k = (y^k, R_i(\bar{y}_i^k)) \in C_i(y^k).$$

As  $\hat{y}_i$  is the limit of a sequence of reduced vectors  $\bar{y}_i^k = R_i(\bar{y}_i^k)$ , it is itself reduced. It therefore suffices to prove that  $\hat{y}_i \in X_i(a)$  where

$a = c + \sum_{j \neq i} F_j' y_j$ . Lemma 3.1 implies that for each  $k$

$$\hat{y}_i^k \geq 0, \quad \hat{v}_i^k = f_i + F_i A a^k + F_i A F_i' y_i^k \geq 0, \quad (\hat{v}_i^k)' (\hat{y}_i^k) = 0$$

where  $a^k = c + \sum_{l \neq i} F_l' y_l^k$ . Passing to the limit as  $k \rightarrow \infty$ , we obtain

$$\hat{y}_i \geq 0, \quad \hat{v}_i = f_i + F_i A a + F_i A F_i' \hat{y}_i \geq 0, \quad (\hat{v}_i)' (\hat{y}_i) = 0.$$

This establishes the closedness of  $C_i$ . It is also bounded, for suppose the contrary. Then there exists a bounded subset  $T \subset R_+^N$  such that

$U(C_i(t) : t \in T)$  is unbounded. Hence there exists sequences

$\{y^k\} \in T$  and  $\{z^k\} = \{(y^k, R_i(\bar{y}_i^k))\}$  with  $z^k \in C_i(y^k)$  such that  $\|z^k\| \rightarrow \infty$ .

Since  $\{y^k\}$  is bounded, it has a convergent subsequence  $\{y^{k_v}\}$  tending to

some vector  $y \in R_+^N$ . Let  $\hat{y}_i^k = R_i(\bar{y}_i^k)$ . Since  $\|z^k\| \rightarrow \infty$  and  $\{y^k\}$  is

bounded, we must have  $\|\hat{y}_i^k\| \rightarrow \infty$ . However, the normalized sequence

$\{\hat{y}_i^{k_v} / \|\hat{y}_i^{k_v}\|\}$  has a limit point,  $\hat{y}_i$ , and clearly  $\hat{y}_i$  is reduced. Without

loss of generality, we may assume  $\hat{y}_i^{k_v} / \|\hat{y}_i^{k_v}\| \rightarrow \hat{y}_i$ . For each  $k_v$  we

have by Lemma 3.1

$$(\hat{y}_i^{k_v})' (f_i + F_i A a^{k_v} + F_i A F_i' \hat{y}_i^{k_v}) = 0$$

where  $a^{k_v} = c + \sum_{l \neq i} F_l' y_l^{k_v}$ . Dividing the above equation by  $\|\hat{y}_i^{k_v}\|^2$  and

passing to the limit we obtain

$$(\hat{y}_i)' F_i A F_i' \hat{y}_i = 0.$$

By (A1) it follows that

$$AF_i' \hat{y}_i = 0. \quad (12)$$

Furthermore, we have

$$0 \geq (\hat{y}_i^{k_v})' (f_i + F_i A a^{k_v})$$

Dividing by  $\|\hat{y}_i^{k_v}\|$  and passing to the limit as  $k_v \rightarrow \infty$ , we obtain (in view of (12))

$$(\hat{y}_i)' f_i \leq 0.$$

By Proposition 2.1, we conclude that  $\hat{y}_i$  satisfies

$$0 \neq \hat{y}_i \geq 0, \quad (\hat{y}_i)' f_i = 0, \quad AF_i' \hat{y}_i = 0.$$

Consequently, it follows that  $(\hat{y}_i)_{\alpha_i} > 0$ . But this contradicts the fact that  $\hat{y}_i$  is reduced.  $\square$

**Proposition 3.2.** The  $i$ -th relaxation map  $P_i$  is both closed and bounded.

**Proof.** The boundedness is obvious. To show that  $P_i$  is closed,

let  $(y^k, \hat{y}_i^k) \rightarrow (y, \hat{y}_i)$  and  $z^k \rightarrow z$  where

$$z^k = (y_1^k, \dots, y_{i-1}^k, \hat{y}_i^k, y_{i+1}^k, \dots, y_m^k) \in P_i(y^k, \hat{y}_i^k)$$

and

$$z = (y_1, \dots, y_{i-1}, \hat{y}_i, y_{i+1}, \dots, y_m).$$

It suffices to show that there exists a scalar  $\tilde{\omega}$  with  $\min\{\omega^*, 1\} \leq \tilde{\omega} \leq \omega^*$  such that  $\tilde{y}_i = y_i + \tilde{\omega}(\hat{y}_i - y_i)$ . But for each  $k$ , there exists a scalar  $\tilde{\omega}^k \in [\min\{\omega^*, 1\}, \omega^*]$  such that  $\tilde{y}_i^k = y_i^k + \tilde{\omega}^k(\hat{y}_i^k - y_i^k) \geq 0$ . Since the  $\tilde{\omega}^k$  lie in a compact interval they have a limit point  $\tilde{\omega}$ . This  $\tilde{\omega}$  will do.  $\square$

Lemma 3.2. The composition of two bounded (point-to-set) maps is bounded.

Proof. Indeed, if  $M_1 : U \rightarrow V$  and  $M_2 : V \rightarrow W$  are two bounded (point-to-set) maps and  $T$  is a bounded subset of  $U$ , then the set

$$M_2 \circ M_1(T) = \cup \{M_2(s) : s \in M_1(T)\}$$

is obviously bounded.  $\square$

Combining these results, we obtain immediately

Proposition 3.3. The map  $B_i$  is both closed and bounded.

Proof. This follows from Propositions 3.1 and 3.2 by applying Lemma 3.2 and [8, Lemma 4.2].  $\square$

Remark. The boundedness of the complementarity map  $C_i$  is crucial in order to apply Lemma 4.2 in [8] to deduce that  $B_i$  is closed. For the same reason, the boundedness of  $B_i$  is important in proving the closedness of the algorithmic map to be given later. The role played by the reduction maps  $R_i$  in these deductions should now be very transparent.

We point out that a vector  $z \in B_i(y)$  might not be reduced with respect to  $\alpha_i$ . This is because the relaxation map  $P_i$  does not necessarily preserve "reducedness".

4. The Reduced BSOR Algorithm. In its simplest form, the modified version of the BSOR method for solving the LCP (1) can be described by its associated algorithmic map

$$A = R \circ B_m \circ \dots \circ B_1 \tag{13}$$

More precisely, given an arbitrary non-negative vector  $\hat{y}^0$ , the algorithm generates a sequence  $\{\hat{y}^k\}$  of vectors as follows. If  $\hat{y}^k$  solves the problem (1), stop; Otherwise pick a vector  $\hat{y}^{k+1} \in A(\hat{y}^k)$  and repeat. For an obvious reason, we call this the Reduced BSOR Algorithm. It is clear that any fixed point of the map  $A$  solves the LCP (1).

There are essentially two new features in this Reduced BSOR Algorithm. First, a (possibly unnecessary) reduction is performed after each linear complementarity subproblem is solved. (The precise manner in which these subproblems are solved is optional.) Second, at the end of each iteration, a complete reduction (defined by the map  $R$ ) is performed. We have seen how reductions of the first kind are useful. Basically, the second kind of reduction is needed for a similar reason; namely, to ensure the boundedness of the sequence  $\{\hat{y}^k\}$  generated by the algorithm.

Our principal convergence result for the Reduced BSOR Algorithm is

Theorem 4.1 Applied to the LCP (1) for which (A1), (A2) and (A3) are satisfied, the Reduced BSOR Algorithm either terminates with a

solution or else the sequence of iterates contains an accumulation point which solves the problem.

We first establish three preliminary results. The first one extends Theorem 1 in [4].

Lemma 4.1 Let

$$\phi(x_1, x_2) = \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix}' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where  $M_{11}$ ,  $M_{12} = M_{21}'$ ,  $M_{22}$ ,  $r$ , and  $s$  are given,  $M_{11}$  is symmetric and positive semi-definite, and  $x_1$ ,  $x_2$  are vector variables.

Let  $\bar{x}_1$  solve the LCP  $(r + M_{12}\bar{x}_2, M_{11})$  for some vector  $\bar{x}_2$ . Then for all  $x_1 \geq 0$  and all  $\omega \in (0, 2)$

$$\phi(x_1 + \omega(\bar{x}_1 - x_1), \bar{x}_2) \leq \phi(x_1, \bar{x}_2)$$

with equality if and only if  $x_1$  also solves the LCP  $(r + M_{12}\bar{x}_2, M_{11})$ .

Proof. Let  $\delta = (\phi(x_1 + \omega(\bar{x}_1 - x_1), \bar{x}_2) - \phi(x_1, \bar{x}_2))/\omega$ .

By the proof of Theorem 1 in [4], we have

$$\begin{aligned} \delta &= (\bar{x}_1 - x_1)'(r + M_{12}\bar{x}_2) + (\bar{x}_1 - x_1)'M_{11}x_1 + \frac{\omega}{2}(\bar{x}_1 - x_1)'M_{11}(\bar{x}_1 - x_1) \\ &\leq (\bar{x}_1 - x_1)'(r + M_{12}\bar{x}_2) + (\bar{x}_1 - x_1)'M_{11}x_1 + (\bar{x}_1 - x_1)'M_{11}(\bar{x}_1 - x_1) \\ &= \bar{x}_1'(r + M_{11}\bar{x}_1 + M_{12}\bar{x}_2) - x_1'(r + M_{11}\bar{x}_1 + M_{12}\bar{x}_2) \leq 0. \end{aligned}$$

If  $\delta = 0$ , then

$$(\bar{x}_1 - x_1)'M_{11}(\bar{x}_1 - x_1) = x_1'(r + M_{11}\bar{x}_1 + M_{12}\bar{x}_2) = 0$$

Since  $M_{11}$  is symmetric and positive semi-definite,  $M_{11}x_1 = M_{11}\bar{x}_1$  and hence,  $x_1$  solves the linear complementarity problem  $(r + M_{12}\bar{x}_2, M_{11})$  as well.  $\square$

Corollary 4.1. For any  $y \in R_+^N$  and  $z \in A(y)$ ,

$$\psi(z) \leq \psi(y)$$

with equality if and only if  $y$  solves (1).

Proof. This follows easily from the definition of  $A$  and repeated use of Proposition 2.2 and Lemmas 3.1 and 4.1.  $\square$

Lemma 4.2. The sequence  $\{\hat{y}^k\}$  of iterates generated by the Reduced BSOR algorithm is bounded.

Proof. By Corollary 4.1, we have for each  $k$

$$\psi(\hat{y}^k) \leq \lambda = \psi(\hat{y}^0)$$

The remainder of the proof resembles that of Proposition 4.1 and is omitted.  $\square$

Proof of Theorem 4.1. By repeated use of Proposition 3.3 and [8, Lemma 4.2] one can easily show that the algorithmic map is closed. The desired conclusion now follows from Lemma 4.2, Corollary 4.1 and Convergence Theorem A in [8].

5. An extension. It is rather easy to extend the reduced BSOR algorithm to treat the following generalization of the quadratic program (4):

Find a vector  $y \in R^N$  to

$$\text{minimize } \psi(y) = q'y + \frac{1}{2}y'My \text{ subject to } y_i \in Y_i \quad i = 1, \dots, m. \quad (14)$$

Here the vector  $y$  is partitioned into subvectors  $y_i \in R^{n_i}$  and each  $Y_i$  is a nonempty polyhedral set in  $R^{n_i}$ :

$$Y_i = \{y_i \in R^{n_i} : B_i y_i \leq b_i \text{ and } E_i y_i = f_i\}$$

where  $B_i$  and  $E_i$  are arbitrary matrices and  $b_i$  and  $f_i$  are arbitrary vectors. The matrix  $M$  in (14) is symmetric and positive semi-definite and is partitioned into submatrices  $M_{ij}$  ( $i, j = 1, \dots, m$ ) where each  $M_{ij}$  is  $n_i$  by  $n_j$ . The vector  $q$  is partitioned accordingly.

Without repeating many of the details, we shall in what follows simply present the generalized version of (A3), define the component maps and state the main theorem of convergence for the algorithm. We point out that the program (14) includes as a special case the one treated in [3]. In the latter program, each  $Y_i$  is a closed interval of  $R$  and the matrix  $M$  is symmetric and positive definite.

For  $i = 1, \dots, m$ , let  $J_i$  denote the set of indices in the subvector  $y_i$  and let  $O^+Y_i$  denote the recession cone [7] of the set  $Y_i$ , i.e.,

$$O^+Y_i = \{d_i \in R^{n_i} : B_i d_i \leq 0 \text{ and } E_i d_i = 0\}.$$

Define

$$C_i = \{d_i \in R^{n_i} : q_i'd_i = 0, M_{ij}d_i = 0\} \cap O^+Y_i$$

and let

$$C = \{d \in R^N : q'd = 0, Md = 0\} \cap \prod_{i=1}^m O^+Y_i.$$



Let  $B$  denote the block diagonal matrix whose diagonal blocks are the  $B_i$ .

Finally, let  $Y = \prod_{i=1}^m Y_i$  be the feasible set of the program (14). We

state the generalized version of the assumption (A3):

(A4) There exists a nonempty index set  $\alpha$  such that for any vector  $d \in C \setminus \{0\}$ , it follows that  $(Bd)_j < 0$  if and only if  $j \in \alpha$ .

For each  $i = 1, \dots, m$ , let  $\alpha_i = \alpha \cap J_i$ . Let  $d_i^*$  be a vector in  $C_i \setminus \{0\}$  if  $C_i \neq \{0\}$  or the zero vector if  $C_i = \{0\}$ . Let  $d^* = (d_1^*, \dots, d_m^*)$ .

A vector  $y_i \in Y_i$  is said to be reduced (with respect to the index set  $\alpha_i$ ) if at least one component in the subvector  $(b_i - B_i y_i)_{\alpha_i}$  is zero.

Let  $S_i$  denote the set of reduced vectors in  $Y_i$ . The reduction map  $R_i: Y_i \rightarrow S_i$  is defined as follows:

$$R_i(y_i) = y_i - \rho_i d_i^*$$

where

$$\rho_i = \rho_i(y_i) = \begin{cases} \min \{ (B_i y_i - b_i)_j / (B_i d_i^*)_j : j \in \alpha_i \} & \text{if } d_i^* \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, by dropping the subscript  $i$ , we may define reduced vectors in  $Y$  and the complete reduction map  $R: Y \rightarrow S$ , where  $S$  is the set of reduced vectors in  $Y$ .

Extending the  $i$ -th complementarity map, we define the  $i$ -th sub-program map  $S_i: Y \rightarrow Y \times S_i$ . Given  $y = (y_1, \dots, y_m) \in Y$ ,  $S_i(y)$  denotes the set of all points  $(y, R_i(\bar{y}))$  where  $\bar{y}_i$  solves the quadratic program

$$\text{minimize } (q_i + \sum_{j \neq i} M_{ij} y_j)' z_i + \frac{1}{2} z_i' M_{ii} z_i \quad \text{subject to } z_i \in Y_i. \quad (15)_i$$

It is important to note that the set  $Y_i$  is included as the feasible region of the  $i$ -th subprogram.

Finally, the  $i$ -th relaxation map  $P_i: Y \times Y_i \rightarrow Y$  is defined as follows. Let  $\omega^* < 2$  be a given positive scalar. For  $(y, \bar{y}_i) \in Y \times Y_i$ , the set  $P_i(y, \bar{y}_i)$  consists of all vectors of the form

$$(y_1, \dots, y_{i-1}, \tilde{y}_i, y_{i+1}, \dots, y_m)$$

where  $\tilde{y}_i = y_i + \bar{\omega}(\bar{y}_i - y_i)$  for some  $\bar{\omega}$  such that  $\min\{\omega^*, 1\} \leq \bar{\omega} \leq \omega^*$  and  $\tilde{y}_i \in Y_i$ .

Let  $B_i = P_i \circ S_i$  be the  $i$ -th component map. The algorithmic map  $A$  is defined by (13). The main convergence theorem is the following.

Theorem 5.1. Suppose that the quadratic program (14) has an optimal solution and that assumption (A4) holds. Then, provided that the initial vector  $\hat{y}^0$  is feasible, the same conclusion of Theorem 4.1 holds for the reduced BSOR algorithm applied to the program (14).

Remark. The assumption that the program (14) has an optimal solution is not crucial for the applicability of the algorithm. In fact, without the assumption, the algorithm can still be applied but may terminate at a situation where a certain subprogram  $(15)_i$  has an unbounded objective function value. It is easy to show that if this happens, then the original program (14) must have an unbounded objective as well.

6. Concluding Remarks. This paper is intended to provide the theoretical foundations for the Reduced BSOR method which is one of the algorithms being considered in our investigation of computational procedures for solving the capacitated quadratic transportation problem. One of the possible formulations of the latter problem leads to a natural partitioning (of  $F$ ) with  $m = 4$ . It also has the property that  $A$  is positive definite and diagonal.

Preliminary computational experience with problems of considerable size (e.g.  $N \approx 5000$ ) suggests that the Reduced BSOR method may prove efficient in this application and possibly others as well. We plan to report on our computational results elsewhere.

## REFERENCES

1. R.W. Cottle, "Note on a fundamental theorem in quadratic programming", Journal of the Society for Industrial and Applied Mathematics 12 (1964) 663-665.
2. R.W. Cottle, "Solution rays for a class of complementarity problems", Mathematical Programming Study 1 (1974) 59-70.
3. R.W. Cottle and M.S. Goheen, "A special class of large quadratic programs", in: O.L. Mangasarian, R.R. Meyer, and S.M. Robinson, eds., Nonlinear Programming 3 (Academic Press, New York, 1978) pp. 361-389.
4. R.W. Cottle, G.H. Golub and R.S. Sacher, "On the solution of large, structured linear complementarity problems: The block partitioned case", Applied Mathematics and Optimization 4 (1978) 347-363.
5. O.L. Mangasarian, "Iterative solution of linear programs", Technical Report #327, Computer Sciences Department, University of Wisconsin, Madison, June 1979.
6. O.L. Mangasarian, "Characterization of bounded solutions of linear complementarity problems", Technical Report #359, Computer Sciences Department, University of Wisconsin, Madison, December 1979.
7. R.T. Rockafellar, Convex Analysis (Princeton University Press, Princeton, 1970).
8. W.I. Zangwill, Nonlinear Programming: A Unified Approach (Prentice-Hall, Englewood Cliffs, NJ, 1969).

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER MSRR #462	2. GOVT ACCESSION NO. AD-A088	3. RECIPIENT'S CATALOG NUMBER 760
4. TITLE (and Subtitle) ON THE CONVERGENCE OF A BLOCK SUCCESSIVE OVER- RELAXATION METHOD FOR A CLASS OF LINEAR COMPLEMENTARITY PROBLEMS		5. TYPE OF REPORT & PERIOD COVERED Technical Report 1980
7. AUTHOR(s) R. W. Cottle J. S. Pang		6. PERFORMING ORG. REPORT NUMBER MSRR #462
8. CONTRACT OR GRANT NUMBER(s) NSF ECS-7926320 N00014-75-C-0267		9. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS DE-AC03-76SF00326 PA# DE-AT03-76ER72018 N00014-75-C-0621
10. PERFORMING ORGANIZATION NAME AND ADDRESS Graduate School of Industrial Administration Carnegie-Mellon University Pittsburgh, Pa. 15213		11. REPORT DATE July 1980
12. CONTROLLING OFFICE NAME AND ADDRESS Personnel and Training Research Programs Office of Naval Research (Code 458) Arlington, Virginia 22217		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited		17. SECURITY CLASS. (of this report) Unclassified
18. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
19. SUPPLEMENTARY NOTES		
20. KEY WORDS (Continue on reverse side if necessary and identify by block number) Convergence, Block successive overrelaxation algorithms, Linear complementarity problem, Quadratic programming, Compactness, Level sets.		
21. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper develops a reduced block successive overrelaxation method for solving a class of (large-scale) linear complementarity problems. The main new feature of the method is that it contains certain reduction operations at each iteration. Such reductions are needed in order to ensure the boundedness (and therefore the existence of accumulation points) of the sequence of iterates produced by the algorithm. Convergence of the method is established by using a theorem due to Zangwill.		

DD FORM 1473  
1 JAN 73

EDITION OF 1 NOV 68 IS OBSOLETE  
S/N 0102-914-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)