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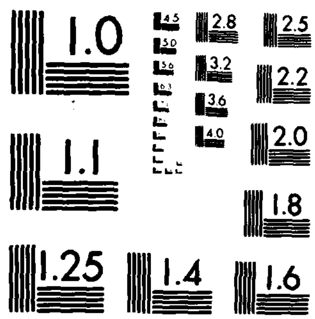

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MATHEMATICAL MODEL FOR DYNAMICS OF PACK ICE IN THE ARCTIC OCEAN AND ITS SURROUNDING SEAS

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ABSTRACT

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I. Introduction

For the research of the dynamics of the pack ice in the Arctic Ocean and its surrounding seas, one of the main purposes is to coordinate theoretical and field investigations such that the forecasting of ice motion (Wittmann-MacDonell, 1964) or knowledge of the general circulation in the Arctic Ocean and its surrounding seas (Campbell, 1965), may be improved. Considerable theoretical research works have been carried out (AIDJEX Bulletins No. 2 and No. 3). But our knowledge on the dynamics of pack ise is still meager and more theoretical analyses are needed.

In the Arctic Ocean and its surrounding seas, an enormous number of ice floes of various sizes and shapes are floating on the surface. Most of these ice floes, under the influence of winds, ocean currents and tidal waves are in constant but chaotic motion and are continuously being deformed. However, for many practical reasons, we are interested in the average motion of sea ice over a large area such as several tens, or hundreds kilometers radius or larger, and for an extended period of time, say several days, months or years. For such large spatial areas and long period, the individual properties of ice floes become less important and the pack ice may be considered as a continuum. In this report, a new mathematical model for the dynamics of pack ice in Arctic Ocean and its surrounding seas is proposed so that the pack ice may be treated as a continuum.

The mathematical model of dynamics of pack ice may be divided into two classes:

In the first class which is used by the majority of the research workers in this field, one considers the pack ice as a pseudo-fluid (Campbell, 1965; Fel'zenbaum, 1958; Doronin, 1970), while in the second class, one considers the pack ice as an elastic-plastic material (Coon et al, 1974). Most of the computation of ice drift in Arctic Ocean have been carried out by the first class of mathematical model. Our model is a most advanced mathematical model of the first class.

Our main concept of the mathematical model is the two-phase flow approach. We consider the Arctic pack ice as a mixture of solid (ice) and a fluid (sea water). Let the depth of the ice be $H(x,y,t)$ which is in general a function of x and y, the two spatial coordinates on the surface of the ocean, and the time t. The maximum thickness of the ice H_o is of the order of ten meters. We choose a typical length of the ocean L to represent the dimension of the surface of the ocean which we are interested in such as $x \sim L$ and $y \sim L$. The representative length L may be of the order of 1,000 kilometers. Hence L is much larger than H_o. We may divide the ice packed ocean into two layers. In the upper layer, i.e., $z \leq H_o$, where z is the coordinate perpendicular to the surface of the ocean measured from the surface of the ice and positive downward, we have a mixture of ice floes and sea water. Since $L \gg H_0$, the ice floes may be considered as small solid particles in a fluid (sea water). In our mathematical model, we use the concept of fluidization of solid particles in a fluid (Pai, 1971). Hence these ice floes may be considered as a pseudo-fluid. For $z \leq H_0$, we have a mixture of two fluids; one is the sea water and the other is the pseudo-fluid of ice floes. We may extend the theory of two-phase flows of a mixture of small solid particles and a fluid to study the fundamental equations of ice dynamics for the pack ice. We derive rigorously the fundamental equations of ice dynamics from the two-phase flow theory (Pai, 1971), particularly the equation of continuity of pack ice in which the source term depends on the freezing and melting of ice floes and which is coupled with the energy equation.

It should be noted that the three-dimensional fundamental equations just mentioned are not the fundamental equations of ice circulation used by many authors in the literature (Campbell, 1965; Fel'zenbaum, 1958; Doronin, 1970). The fundamental equations used in literature by these authors are the integrated forms of the two-phase flow equations of three dimensions. Because of the fact that $L \gg H_0$, we may integrate the three-dimensional equations of ice dynamics with respect to z, the vertical coordinate, with the limits from $z = 0$ to $z = H(x, y, t)$. The resultant equations depend only on two spatial coordinates x and y and time, t, with the thickness of ice $H(x, y, t)$ as an unknown parameter to be determined. This procedure is similar to the integral method which has been extensively used in boundary layer flow (Pai, 1956). Even though these integrated equations have been generally used, it seems to us that no one has previously presented the fundamental equations of the effective two-dimensional ice flow in this manner.

The integrated equations of motion of ice floes are of a form similar to the equations of motion for a single floe used by Campbell (Campbell, 1965) and others (Fel'zenbaum, 1958; Doronin, 1970; Rothrock, 1970). But the integrated equations give us much more information. In the integrated equations of motion, the shearing stresses on the ice surfaces $z = 0$ and $z = H$ occur such that at $z = 0$, the shearing stresses are equal to the wind stresses τ_a while at $z = H$, the shearing stresses are equal to the water current stresses τ_w. Conceptually, the integrated forms of the fundamental equations are consistent with the mathematical model of pack ice. We cannot justify the derivation of the equations of motion of a continuum model of pack ice by using the equations of motion of a single ice floe. From the equations of motion for a single ice floe, we cannot obtain the effect of compactness. Furthermore, we have a better understanding of various terms in the integrated forms of the fundamental equations such as the source terms and the interaction terms. We also expect some new information about the effective two-dimensional equations such as the interaction forces between ice floes and the effects due to the heat transfer and mass transfer.

For $z \geq H$, our equations of two-phase flow will be reduced to those of ocean water where the compactness is zero. Hence we need only to derive the fundamental equations for the upper layer, $z \leq H$. Those in the lower layer where $z > H$ are a special case of those of the upper layer with zero compactness.

We shall apply the modern technique of fluid mechanics from both the macroscopic (the continuum theory) and the microscopic (the kinetic theory) point of view to derive the fundamental equations of two-phase flow of a mixture of ice and water. Special attention will be given to the effect of heat transfer and mass transfer between ice, water and air.

From the kinetic theory of gases, the fundamental equations of fluid dynamics may be derived as the transfer equations of the Boltzmann equation of a gas and some insights of various terms by these equations may be obtained. The continuum theory of fluid dynamics cannot give us any insights of these terms. On the other hand, because of the mathematical and physical complexity, we can only deal successfully with Boltzmann equation for simple kinetic picture and for very simple configurations. For practical problems, we have to use the macroscopic variables and the continuum theory. We postulate the fundamental equations of fluid-particle system based on the conservation laws of mass, momentum and energy. These fundamental equations should be consistent with those derived from the kinetic theory. We are especially interested in those new terms of interaction between the fluid and solid particles, which may be obtained by the kinetic theory but may be ignored in the ordinary derivation of the fundamental equations by the continuum theory. Since the complete kinetic theory of the fluid-particle system has not yet been developed, it is appropriate to use both the macroscopic and microscopic approaches to derive a general set of the three-dimensional fundamental equations of ice dynamics. From these three dimensional equations, the effective two-dimensional equations of pack ice may be obtained.

With these effective two-dimensional equations of pack ice, we calculate the ice circulation in the Polar Ocean and compare these results with those in the literature and with the field observations. We also compare our model with AIDJEX's model for the similarities and the differences, and briefly discuss how to use AIDJEX's experimental data to verify the essential points of our model.

II. Microscopic Approach: Kinetic Theory of Ice Dynamics.

Since the kinetic theory of gases gives us the Navier-Stokes equations of a viscous fluid as a first approximation, we would expect that the kinetic theory of a gas-particle system

(Pai, 1971; Kuentzmann, 1973) would give the fundamental equations of ice dynamics (a mixture of ice floes and sea water) as a first approximation. We consider an ideal case of a gas (which may represent the water in our final result) and many small solid particles (which may represent the elementary ice particles as we will define later). The kinetic theory of this gas-particle system may be treated by using a single particle distribution function for the gas, f_g, and that for the solid particles, f_p. Each of these distribution functions is governed by a Boltzmann equation.

The distribution function for the solid particles, f_p, is different from that of gas molecules, f_g, because we have to consider the different sizes, shapes and physical properties of the solid particles. In general, the shapes of the solid particles are arbitrary. However, if we were to consider the arbitrary shapes of the solid particles, the results would be too complicated and too detailed. Hence we will assume that the shapes of these solid particles are similar. Thus, we need to use one of its dimensions as its size. With these approximations, the distribution function of the solid particles may be defined as follows:

The number density of solid particles, dn_p, with the size range h_i and $h_i + dh_i$, in the volume x_j and $x_j + dx_j$, having the particle instantaneous velocity, c_j, in the range of c_j and $c_j + dc_j$, and the instantaneous temperature Θ of the particle in the range of Θ and $\Theta + d\Theta$ is

$$dn_p = f_p(x_j, c_j, \Theta, h_i, t)\, d^3c_j\, d\Theta\, dh_i \tag{1}$$

where n_p is the number density of the solid particles, x_j is the jth spatial coordinate of the point considered, c_j is the jth component of the instantaneous velocity of the solid particle in the direction of x_j, d^3c_j is the elementary volume of the velocity space, Θ is the instantaneous temperature of a solid particle which may be varied from particle to particle, and h_i is the length scale of the particle which represents the size of the particle and which may be different for different particles. For ice dynamics, we should choose the scale of h_i such that an elementary ice particle would have uniform physical properties such as salinity, porosity, etc. Hence the elementary ice particle in the microscopic analysis is much smaller than an ice floe. In fact, an ice floe may be considered as a group of a large number of our elementary solid ice particles. The rate of change of the size of our elementary ice particle depends on the physical and mechanical processes such as evaporation, condensation, freezing, melting, disintegration, hummocking, etc. Hence h_i is a random variable in the microscopic analysis.

The distribution function of the gas, f_g, may be considered as a special case of f_p, in which the variation of the distribution function with size parameter h_i, and with temperature Θ

may be dropped. Hence from now on, we shall consider only the distribution function, f_p, except where specified otherwise.

The distribution function, f_p, is governed by the Boltzmann equation which shows that the total rate of change of f_p with time must equal the change of the number of particles per unit volume by collision effects in the range of variation considered, i.e.,

$$\frac{\partial f_p}{\partial t} + \frac{\partial c_j f_p}{\partial x_j} + \frac{1}{m_p} \frac{\partial \phi_j f_p}{\partial c_j} + \frac{\partial}{\partial \Theta}\left(\frac{Q_h f_p}{m_p c_s}\right) + \frac{\partial}{\partial h_i}\left(\frac{M_p f_p}{k m_p}\right) = \left(\frac{\delta f_p}{\delta t}\right)_p + \left(\frac{\delta f_p}{\delta t}\right)_g \qquad (2)$$

where ϕ_j is the jth component of the body force on a solid particle of size h_i, and mass m_p, which is a function of h_i; Q_h is the heat transfer rate from a particle of size h_i; c_s is the specific heat of the particle; M_p is the mass transfer rate of the particle of size h_i which may be due to the thermal and mechanical effects, and k is a constant depending on the shape of the particle. If the particle is of spherical shape and h_i is its radius, $k = 3$.

The right-hand side terms of (2) are the collision terms. The first term is the collision between solid particles and the second term is the collision between gas molecules and a solid particle of the size h_i. We shall study these two terms in detail because they have not been previously examined thoroughly. We are especially interested in the study of collisions between ice floes such as hummocking and rafting. The first collision term between solid particles has been briefly studied by Kuentzmann (1973). But for ice dynamics, many additional effects such as hummocking, etc., should be included.

The collision terms for gas molecules and solid particle may be split into two parts: one is due to the mean flow of the gas molecule, i.e., the fluid is considered as a continuum, and the other is due to the random motion of the gas molecules. Thus

$$\left(\frac{\delta f_p}{\delta t}\right)_g = \left(\frac{\delta f_p}{\delta t}\right)_{gm} + \left(\frac{\delta f_p}{\delta t}\right)_{gr} \qquad (3)$$

the first term $(\delta f_p / \delta t)_{gm}$, may be expressed as a body force on the solid particle, ϕ_j. Hence we should use the continuum concept to find out the proper body force which contain a drag force similar to the Stokes formula, but, also including other effects such as volume fraction, Reynolds number, pressure gradient, and other effects. (Pai-Hsieh, 1973). This is the area where simultaneous studies of the microscopic and the macroscopic point of view would yield useful information.

The other use of (2) is to derive the fundamental equations for the macroscopic approach from the transfer equations. Before we derive the macroscopic fundamental equations, we must define the macroscopic variables in terms of the distribution function, f_p, as follows:

The macroscopic variables are the moments of the distribution function. We have

(i) For zeroth moment, we obtain the number density and the mass density.

The number density, n_p, of the solid particles is

$$n_p = \iiint f_p(x_j, c_j, \Theta, h_i, t)\, d^3c_j\, d\Theta\, dh_i \tag{4}$$

and the mass density, $\bar{\rho}_p$, of the pseudo-fluid of solid particles is

$$\bar{\rho}_p = \iiint m_p(h_i)\, f_p(x_j, c_j, \Theta, h_i, t)\, d^3c_j\, d\Theta\, dh_i$$

$$= \bar{m}_p n_p = \rho_{sp} Z \tag{5}$$

where $\bar{\rho}_p = \rho_{sp} Z$ is the partial density of the pseudo-fluid of the solid particles in the mixture, $m_p(h_i)$ is the mass of a particle of size h_i, $\bar{m}_p$ is the mean mass of solid particles in the pseudo-fluid, ρ_{sp} is the species density of the particle, i.e., the density of the particle itself without considering it as a part of the mixture, and Z is the volume fraction of the solid particles in the mixture. In two-phase flow, the partial density should be used as we shall show later.

(ii) For the first moment of the distribution function, we have the following macroscopic variables:

The temperature of the solid particle, T_p, is

$$T_p = \frac{1}{\bar{m}_p n_p} \iiint m_p(h_i)\, \Theta\, f_p\, d^3c_j\, d\Theta\, dh_i \tag{6}$$

The jth component of the flow velocity of the solid particles is

$$v_j = \frac{1}{\bar{m}_p n_p} \iiint m_p(h_i)\, c_j\, f_p\, d^3c_j\, d\Theta\, dh_i \tag{7}$$

(iii) For the second moment, we have the jkth component of the stress tensor of the pseudo-fluid of the solid particles:

$$S_{pjk} = \iiint m_p(h_i)(c_j - v_j)(c_k - v_k)\, f_p\, d^3c_j\, d\Theta\, dh_i \tag{8}$$

The stress tensor S_{pjk} consists of the partial pressure p_p of the pseudo-fluid of solid particles and the viscous stress tensor of the pseudo-fluid of solid particles due to the random montion such that

$$S_{pjk} = p_p \delta_{jk} - \tau_{pjk} \tag{9}$$

where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$. The jkth component of the viscous stress tensor is τ_{pjk}.

Similarly, we may define other macroscopic variables such as the heat flux due to conduction, the interaction forces exerted upon solid particles by the pseudo-fluid, work done on the fluid by the entire particle cloud, etc., as shown in reference by Pai (1973).

The equation of continuity of a species in the mixture may be obtained by taking the zeroth moment of the Boltzmann equation. We multiply (2) by m_p (h_i) and integrate the resultant equation with respect to d^3c_j over the whole velocity space, with respect to Θ over the whole temperature space, and with respect to h_i over the whole size space. We may write the resultant equation of continuity of the pseudo-fluid of solid particles (pack ice) as follows:

$$\frac{\partial \bar{\rho}_i}{\partial t} + \frac{\partial \bar{\rho}_i u_i}{\partial x} + \frac{\partial \bar{\rho}_i v_i}{\partial y} + \frac{\partial \bar{\rho}_i w_i}{\partial z} = \sigma i \qquad (10)$$

where $\bar{\rho}_p = \bar{\rho}_i = \bar{m}_i n_i = \rho_i Z_i$, ρ_i is the species density of the pack ice, and $\bar{\rho}_i$ is the partial density of the pack ice in the mixture of ice and sea water. We have two different definitions of density for each species in the mixture. Let $V = V_i + V_w$ be the elementary volume of the mixture where V_i is the volume occupied by the ice, and V_w is the volume occupied by the sea water. Let $M = M_i + M_w$ be the total mass of the mixture of ice and water in V where M_i is the mass of ice in V and M_w is the mass of water in V. The species density of ice is

$$\rho_i = \frac{M_i}{V_i} = \rho_i\,(T_i, p_i, S_i, \varepsilon) \qquad (11)$$

where ρ_i is in general a function of ice temperature T_i, ice pressure p_i, ice salinity S_i, and ice porosity ε. The partial density of the ice, $\bar{\rho}_i$, in the mixture, which is the value of density used in the two-phase flow theory, e.g., in (10), is

$$\bar{\rho}_i = \frac{M_i}{V} = \frac{M_i}{V_i}\,\frac{V_i}{V} = \rho_i Z_i \qquad (12)$$

where

$$Z_i = \frac{V_i}{V} = \text{volume fraction of ice in the mixture} \qquad (13)$$

The source term, σ_i, depends on the mass transfer M_p and the integral of the collision terms. We shall study this term in detail in the near future.

The equation of motion is associated with the first moment of the Boltzmann equation. We multiply (2) by $m_p c_j$ and integrate the resultant equation with respect to $d^3c_j\, d\Theta\, dh_i$ and obtain the equation of motion of the pseudo-fluid of the pack ice as follows:

$$\frac{\partial Z_i \rho_i u_i}{\partial t} + \frac{\partial Z_i \rho_i u_i^2}{\partial x} + \frac{\partial Z_i \rho_i u_i v_i}{\partial y} + \frac{\partial Z_i \rho_i u_i w_i}{\partial z} - Z_i \rho_i f v_i +$$

$$+ 2 Z_i \rho_i w_i \Omega \cos\phi = - Z_i \frac{\partial p}{\partial x} + \frac{\partial \tau_{ixy}}{\partial x} + \frac{\partial \tau_{ixy}}{\partial y} + \frac{\partial \tau_{ixz}}{\partial z} +$$

$$+ \sigma_i F_{ui} + K(Z_i)(u_w - u_i) \tag{14}$$

where $f = 2\Omega \sin\phi$ = coriolis force factor, ϕ is the latitude, and Ω is the angular velocity of the Earth.

The factor $K(Z_i)$ is the friction coefficient of ice floes in water which is a function of the compactness or volume fraction Z_i. We should study it in detail. For a first approximation, we may write

$$K(Z_i) = CZ_i \tag{15}$$

where C is a constant.

The factor F_{ui} is the momentum transfer per unit mass due to the source term, which may be obtained by studying the mass transfer term and collision terms in the Boltzmann equation. For a first approximation, we may assume that $F_{ui} = u_i$, i. e., new ice has the same velocity as the local flow velocity of the pseudo-fluid of pack ice.

In a similar form, we have the equations of motion in the y- and z-directions.

The stress tensor of the pack ice with component τ_{ixx}, etc., [as defined in (9) and (10)] is due to the random motion of the ice particles. Since the determination of these stress tensors is very difficult or nearly impossible from Boltzmann equation, particularly for the turbulent flow, we have to postulate the expression of such stress in terms of macroscopic variables such as velocity gradient, etc. For instance, Doronin (1970) used the form similar to the conventional expression of turbulent stress. We may also assume that the pseudo-fluid of pack ice is a non-Newtonian fluid or even as an elastic-plastic material as AIDJEX modeling group did (Coon et al, 1974). The final decision should be made by checking with experimental results as those will be obtained by AIDJEX experiment. We may use various expressions for the stress tensor to calculate the ice circulation in the Arctic Ocean to find out the difference due to these expressions and check them with field observations.

If we multiply (2) by $m_p c_s \Theta$ and integrate over the spaces $d^3c_j \, d\Theta \, dh_i$, we obtain the temperature or energy equation of the pseudo-fluid of solid particles. This temperature equation with the kinetic temperature equation of the particles gives us the energy equation (Pai, 1973) of the pack ice. We will study this energy equation in detail in the near future.

III. Macroscopic Approach: Effective Two-Dimensional Equation of Ice Dynamics.

In the macroscopic approach, we consider the mixture of solid particles and a fluid as a mixture of two fluids: one is the pseudo-fluid of solid particles and the other is the real fluid, gas or liquid. For each species r in the mixture, we would like to know its velocity vector $\vec{q}_r$ with components, u_r, v_r and w_r, its temperature T_r, its partial pressure p_r and its partial density $\bar{\rho}_r$. Hence for each species, we have six fundamental equations, i.e., one equation of state, one equation of continuity, three equations of motion and one equation of energy. These fundamental equations may be derived from thermodynamics (for equation of state), and conservation laws of mass, momentum and energy (for the other five equations). These fundamental equations should be consistent with those derived from the microscopic approach, i.e., the various transfer equations of the Boltzmann equation (2). This is the point where we may compare the results of the microscopic and the macroscopic approaches.

We have shown that for ice movement in the polar region, because two dimensions along the surface of the ocean are much larger than the thickness of the ice, it is convenient to treat the ice movement as an effective two-dimensional problem. As shown in reference by Pai-Li (1971), the correct way to derive the fundamental equations of these effective two-dimensional equations is to integrate the three-dimensional equations such as (11) and (14) etc., with respect to the vertical coordinate z from z = O to z = H(x, y, t), where H is the average thickness of the pack ice over an elementary area at the point (x, y, t).

We are going to show some details for such derivations which will give us new information for ice dynamics.

(a) Equation of continuity of pack ice - compactness of pack ice, N.

Equation (10) may be written as follows:

$$\frac{\partial Z_i \rho_i}{\partial t} + \frac{\partial Z_i \rho_i u_i}{\partial x} + \frac{\partial Z_i \rho_i v_i}{\partial y} + \frac{\partial Z_i \rho_i w_i}{\partial z} = \sigma_i \qquad (10a)$$

Now we integrate (10a) with respect to z from z = O to z = H(x, y, t) and have the following equation:

$$\int_O^H \frac{\partial Z_i \rho_i}{\partial t} dz + \int_O^H \frac{\partial Z_i \rho_i u_i}{\partial x} dz + \int_O^H \frac{\partial Z_i \rho_i v_i}{\partial y} dz + \int_O^H \frac{\partial Z_i \rho_i w_i}{\partial z} dz = \int_O^H \sigma_i dz \qquad (16)$$

Since the ice remains on the top layer of the ocean, we may assume that at z = O and z = H, w_i = O or, if we consider freezing and melting ice, the rate of increase or decrease at the surface z = O and z = H are approximately equal, then the fourth term on the left-hand side of (16) vanishes. We have also the following relation:

$$\frac{\partial}{\partial t}\int_0^H Z_i \rho_i \, dz = Z_i \rho_i \frac{\partial H}{\partial t} + \int_0^H \frac{\partial Z_i \rho_i}{\partial t} \, dz \tag{17}$$

and similar relations for the second and the third terms on the left-hand side of (16). Furthermore, since H(x,y,t) is small with respect to the horizontal dimension L, we may assume that the variations of ρ_i, Z_i, u_i and v_i with respect to z are negligibly small and we may replace their values by the average values of these variables between the limits z = O to z = H. Hence, we have the following relations:

$$\int_0^H Z_i \rho_i \, dz = H N \rho_i \tag{18a}$$

$$\int_0^H Z_i \rho_i u_i \, dz = H N \rho_i u_i \tag{18b}$$

$$\int_0^H Z_i \rho_i v_i \, dz = H N \rho_i v_i \tag{18c}$$

$$\int_0^H \sigma_i \, dz = H \sigma_i = E_i \tag{18d}$$

where N is the compactness of the pack ice which is the average volume fraction of the pack ice between z = O to z = H(x,y,t) at any point on the surface of the ocean (x,y) and at a given time t.

Substituting (17) and (18) into (16), we have the equation of continuity of pack ice in the effective two-dimensional flow of ice dynamics in polar ocean as follows:

$$\frac{\partial NH\rho_i}{\partial t} + \frac{\partial NH\rho_i u_i}{\partial x} + \frac{\partial NH\rho_i v_i}{\partial y} = N\rho_i \frac{D_i H}{Dt} + E_i \tag{19}$$

where

$$\frac{D_i}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x} + v_i \frac{\partial}{\partial y} \tag{20}$$

Equation (19) may be written as an equation for compactness N as follows:

$$\rho_i H \frac{D_i N}{Dt} = - \rho_i NH\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) - NH \frac{D_i \rho_i}{Dt} + E_i \tag{21}$$

It is interesting to compare our equation of compactness N, (21) with those equations of compactness usually found in literature.

Rothrock (1970) gave the following equation for compactness:

$$\rho_i N \frac{D_i N}{Dt} = - \rho_i NH\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) - \rho_i NH \psi + \phi_2 \tag{22}$$

The differences between (21) and (22) are tri-fold: (i) on the left-hand side of these equations, H in (21) replaces N on Rothrock's equation (22). It is evident that N in (22) must be a printing error because the dimensions of this term are not the same as those of all the other terms in (22). (ii) $NH(D_i \rho_i/Dt)$ of (21) replaces $NH \rho_i \psi$ of (22). Rothrock gave little information about his function ψ. We have no problem in explaining our term, $NH(D_i \rho_i/Dt)$. (iii) There are some basic differences in concept between (21) and (22). In general, the velocities of ice (u_i and v_i) are different from those of water (u_w and v_w) in the top layer, $0 \leq z \leq H$. The distinction between these velocities should be noted, and in (21), the use of the operator (D_i/Dt) should be kept in mind. In Rothrock's analysis, such a distinction was not mentioned. Rothrock separated the source term into two parts, and in (22), only one part of the source term, i.e., the rate of formation of ice by freezing open water is included. In Rothrock's analysis, if one makes the distinction between the freezing at the surface $z = 0$ and freezing at $0 < z \leq H$, one cannot use the effective two-dimensional approach. For an effective two-dimensional approach, the source term should be the average source due to freezing or melting at the point (x,y). As a result, our $E_i = H\sigma_i$ is different from Rothrock's ϕ_2.

We see that the thickness H occurs in every term of (21) if we write $E_i = H\sigma_i$, and the compactness equation is independent of the thickness of the ice. This result is essential so that the pack ice may be considered as a two-dimensional continuum.

Various Russian authors have used simplified equations for compactness. We are going to compare their simplified equations with our exact equation of compactness, (21).

Drogaitsev (1956) considered the conservation of the mass of ice per unit area and obtained the following equation for the two-dimensional ice dynamics.

$$\frac{\partial N \rho_i H}{\partial t} + \frac{\partial NH\rho_i u_i}{\partial x} + \frac{\partial NH\rho_i v_i}{\partial y} = 0 \qquad (23)$$

Comparing (23) with our exact (19), we see that Drogaitsev neglected the source term E_i. Furthermore, he missed the important term due to the variation of the thickness of ice, H. This is the main difference between the <u>effective</u> two-dimensional treatment of ice dynamics and the <u>ordinary</u> two-dimensional approach obtained by simply dropping the variation with z-coordinate. As we shall show later, if we simply drop the z-variation in the three-dimensional equations of motion, we will never get the terms due to the air and water stresses on the surface of the ice. Thus, such a simple equation of continuity (23) is not consistent with the correct two-dimensional equations of motion of ice dynamics. Therefore, (23) is correct only if we neglect the source term and if we assume that the thickness of the ice, H, is a constant.

Nikoforov (1957) and Doronin (1970) used the following equation for compactness:

$$\frac{\partial N}{\partial t} = -\left(\frac{\partial N u_i}{\partial x} + \frac{\partial N v_i}{\partial y}\right) \qquad (24)$$

If we compare (24) with our (21), we see that both Nikiforov and Doronin neglected the source term σ_i, and the variation of the species density of ice, ρ_i. Thus (24) is valid only if we assume that there is no source term and that the variation of the species density of ice, ρ_i, is negligible.

Similarly, the three-dimensional equation of continuity of the water in the mixture is

$$\frac{\partial (1-Z_i)\rho_w}{\partial t} + \frac{\partial (1-Z_i)\rho_w u_w}{\partial x} + \frac{\partial (1-Z_i)\rho_w v_w}{\partial y} + \frac{\partial (1-Z_i)\rho_w w_w}{\partial z} = \tau_w = -\tau_i \qquad (25)$$

Where the subscript w refers to the value of water in the mixture, and we assume $\sigma_w = \sigma_i$, i.e., we neglect the evaporation or condensation of water vapor in the mixture.

If we integrate (25) with respect to z from z = O to z = H, and use the same arguments as in the case of ice and similar notations, we have the equation of continuity of water in the effective two-dimensional flow of ice dynamics as follows:

$$\rho_w H \frac{D_w N}{Dt} = (1-N)\rho_w H\left(\frac{\partial u_w}{\partial x} + \frac{\partial v_w}{\partial y}\right) + (1-N) H \frac{D_w \rho_w}{Dt} + H\sigma_i + (1-N)(\rho_w w_w)_H \qquad (26)$$

where

$$\frac{D_w}{Dt} = \frac{\partial}{\partial t} + u_w \frac{\partial}{\partial x} + v_w \frac{\partial}{\partial y}$$

and $(\rho_w w_w)_H$ is the value of $(\rho_w w_w)$ at z = H which may or may not be negligible. The value of $(\rho_w w_w)$ at z = O is negligible in our case. The difference between (D_i/Dt) and (D_w/Dt) in (21) and (25) should be noted. In general, u_i is not equal to u_w and v_i is not equal to v_w. The source term $E_i = H\sigma_i$ in (21) and (26) are the same. Equation (26) shows that the vertical component of the water at the bottom of the ice, z = H may affect the compactness of the ice. This is a point which has not been extensively investigated.

(b) Equations of motion of pack ice: Interaction forces between ice floes.

Similar to the analysis of the equation of continuity, we may obtain the three-dimensional equations of motion of ice dynamics from the first moment of the Boltzmann equation (2). The complete three-dimensional equations of motion including heat and mass transfer, and the general collision terms have not been studied. Some simple models of two-phase flow, however, have been worked out (Pai, 1973; Kuentzmann, 1973). We shall study the general three-dimensional equations of motion of ice dynamics in the near future. We will now derive the effective two-dimensional equations from the simple three-dimensional equations of motion for the two-phase flow of ice dynamics by integrating the three-dimensional equations of ice dynamics (Pai, 1973; Kuentzmann, 1973) with respect to z from z = O to z = H. We are especially interested in the new information obtained in this manner as compared with the derivation of the effective two-dimensional equations of motion from the force balance of a single ice floe.

The three-dimensional equations of motion of the pseudo-fluid of pack ice in the mixture of ice and sea water are given in (14).

New we will derive the equations of motion of the effective two-dimensional dynamics of pack ice by integrating (14) with respect to z from z = O to z = H. With the same arguments as in the case of derivation of the continuity equation (19), we have the following effective two-dimensional equations of motion for the pseudo-fluid of pack ice:

$$\frac{D_i u_i}{Dt} - f v_i = -\frac{1}{\rho_i}\frac{\partial p}{\partial x} + \frac{1}{N\rho_i}\left(\frac{\partial \tau_{ixx}}{\partial x} + \frac{\partial \tau_{ixy}}{\partial y}\right) - \frac{\partial \tau_{ixzo}}{NH\rho_i} + $$

$$+ \frac{\tau_{ixzH}}{NH\rho_i} + \frac{K(N)}{N\rho_i}(u_w - u_i) \tag{27}$$

$$\frac{D_i v_i}{Dt} + f u_i = -\frac{1}{\rho_i}\frac{\partial p}{\partial y} + \frac{1}{N\rho_i}\left(\frac{\partial \tau_{ixy}}{\partial x} + \frac{\partial \tau_{iyy}}{\partial y}\right) - \frac{\tau_{iyzo}}{NH\rho_i} + $$

$$+ \frac{\tau_{iyzH}}{NH\rho_i} + \frac{K(N)}{N\rho_i}(v_w - v_i) \tag{28}$$

The source terms disappear because we assume $F_{ui} = u_i$, etc. The terms τ_{ixzo} and τ_{iyzo} are respectively the corresponding wind stresses on the top of the surface of ice floes, z = O, in the direction of x and y. The terms τ_{ixzH} and τ_{iyzH} are respectively the water stresses acting on the lower surface of the ice, z = H, in the direction of x and y.

There are two types of interaction forces in (27) and (28): (i) one is the friction force due to the difference in velocities between ice and water, i.e.,

$$\vec{F}_{i1} = \frac{K(N)}{N\rho_i}(\vec{q}_w - \vec{q}_i) \tag{29}$$

where $\vec{q}_w$ and $\vec{q}_i$ are respectively the velocity vector of water and ice in the mixture. This is the drag force of the ice floes moving in water, which is mainly due to the collision between the mean water force and the ice floes as we mentioned in II. This force was first proposed by Sverdrup (1928) but neglected in much of the literature (Campbell, 1965; Doronin, 1970;

Fel'zenbaum, 1961). In Sverdrup's original proposal, the water velocity $\vec{q}_w$ (u_w, v_w) has been neglected. In our general mathematical model, the water velocity components (u_w, v_w) are retained, which may or may not be negligible to ice velocity components (u_i, v_i). If the ice motion is mainly due to the wind stress, the velocity of the ice (u_i, and/or v_i) may be much larger than the water velocity (u_w and/or v_w). This is the reason why Sverdrup neglected the water velocity in his formula. On the other hand, if we study the tidal effect of ice, the water velocity due to the tidal waves may be much larger than that part of velocity of ice due to the tidal force. Thus, we have to include the water velocity in the formula of the friction forces. (ii) The other interaction force is the internal stresses of ice floes due to their random motion. One of the main objectives of AIDJEX is to find the proper expression of these internal stresses. According to Doronin (1970), we may write

$$\frac{\partial \tau_{ixx}}{\partial x} + \frac{\partial \tau_{ixv}}{\partial y} = \frac{\partial}{\partial x}\left(\varepsilon_t N\rho_i \frac{\partial u_i}{\partial x} \right) + \frac{\partial}{\partial y}\left(\varepsilon_t N\rho_i \frac{\partial u_i}{\partial y} \right) \tag{30a}$$

$$\frac{\partial \tau_{ixy}}{\partial x} + \frac{\partial \tau_{iyy}}{\partial y} = \frac{\partial}{\partial x}\left(\varepsilon_t N\rho_i \frac{\partial v_i}{\partial x} \right) + \frac{\partial}{\partial y}\left(\varepsilon_t N\rho_i \frac{\partial v_i}{\partial y} \right) \tag{30b}$$

where ε_t is the turbulent exchange coefficient to be determined by experiment or estimated by current knowledge on turbulent flow. One of the new features of Doronin's formula is that he includes the effect of compactness. His formula is consistent with the two-phase approach because in two-phase flow analysis, the partial density $N\rho_i$ should be used in all equations instead of the species density ρ_i. We think that (30) may be used in (27) and (28) as a first approximation, but there is a difference between our equations and those of Doronin. In our equations (27) and (28), the stress terms (30) are divided by $N\rho_i$, while in Doronin's equations of motion, they are divided by ρ_i only. When we consider the mixture of ice and water, we should use the partial density $N\rho_i$ in the equations of motion. Hence, Doronin's equations of motion were not correct by using the species density ρ_i in the inertial terms.

Similarly, we may derive the effective two-dimensional equations of motion of water from two-phase flow, three-dimensional equations.

We shall also study the three-dimensional equations of energy for both water and ice from the Boltzmann equation and their effective two-dimensional form for a complete study of ice dynamics. We may also derive equations for salinity for the two-phase analysis. The equation of salinity of water is the diffusion equation of salt in water. The salinity of ice depends on the salinity of water from which the ice is formed and other processes (Zubov, 1945), such as the rate of ice formation, the state of sea during the ice formation, and the age of the ice.

IV. Comparison of our mathematical model with other models.

Now we compare our model with those other models in the literature (Campbell, 1965; Doronin, 1970; Fel'zenbaum, 1961; Coon, et al, 1974). In general, there are two different types of mathematical models for pack ice. One is the fluidized model in which the pack ice is considered as a pseudo-fluid. Our model is essentially a very generalized version of this model. Those models in references by Campbell (1965), Doronin (1970), Fel'zenbaum (1961) and Sverdrup (1928) belong to this type. We are going to discuss the relationship between our model and those other fluidized models in detail in the following sections. The other type is the elastic-plastic model in which the pack ice is considered as an elastic-plastic material known as AIDJEX's model (Coon, et al, 1974). We shall discuss the similarities and differences of our model and the AIDJEX's model later.

a. Fluidized Model.

In our two-phase flow model, we consider the mixture of ice and water as a mixture of two fluids that have the following variables:

$$\vec{q}_r,\ T_r,\ p_r,\ \bar{\rho}_r,\ S_r \qquad \text{(I)}$$

The species r may be the pseudo-fluid of pack ice ($r = i$) or the water ($r = w$); $\bar{q}$ is the mean flow velocity vector with component u, v and w; hence there are six velocity components (u_i, v_i, w_i, u_w, v_w, w_w). There are two temperature (T_i and T_w), and two pressures (p_i and p_w) which may be expressed in terms of the total pressure of the mixture p and the volume fraction Z_i, i.e., $p_i = Z_i p$ and $p_w = (1 - Z_i)\, p$. There are two partial density $\bar{\rho}_i$ and $\bar{\rho}_w$, which are respectively $\bar{\rho}_i = Z_i \rho_i$ and $\bar{\rho}_w = (1 - Z_i)\, \rho_w$. The species density ρ_r (ρ_i or ρ_w) is given by the equation of state, i.e.,

$$\rho_r = \rho_r\,(T_r,\ p_r,\ S_r) \qquad (31)$$

where S_r is the salinity of the species r.

In the most general case, we should consider the 14 variables represented by (I). But because of the fact that the dimensions along the ocean surface, L, are much larger than the thickness of the ice, we may consider the effective two-dimensional flow of ice dynamics as shown in Section III. In this effective two-dimensional flow, we consider the total pressure p as a known function given by the atmospheric pressure above the pack ice and the inclination of the ocean and the vertical velocity component w_r, which is much smaller than the horizontal velocity components u_r and/or v_r, may be eliminated by taking the mean values across the thickness of the ice layer. Thus, we have to deal only with 11 variables, i.e.,

$$u_i,\ v_i,\ u_w,\ v_w,\ T_i,\ T_w,\ \rho_i,\ \rho_w,\ N,\ S_i,\ S_w \qquad \text{(II)}$$

Our model (II) is governed by 11 fundamental equations for the 11 variables (II). These fundamental equations are:

(i) Equation of compactness (21)

(ii) Four equations of motion (27) and (28)

(iii) Two equations of state (31)

(iv) Two equations of energy (T_i and T_w)

and (v) Two equations of salinity (S_i and S_w)

The authors believe that such a complete analysis has not been studied yet. It is our intention to study such a complete model eventually.

We may simplify our complete model (II) by various approximations. Since the salinity of ice depends on the salinity of sea water from which the ice is formed, the rate of ice formation, the state of sea during the ice formation and the age of ice, no simple relationship for the determination of ice salinity has been found in connection with pack ice circulation. Hence, in most analysis, we may assume S_i and S_w as constants or simple functions of ρ_r and T_r. Then we have only nine variables:

$$u_i,\ v_i,\ u_w,\ v_w,\ T_i,\ T_w,\ \rho_i,\ \rho_w,\ N \qquad \text{(III)}$$

We feel that model (III) should give good results for ice circulation in the polar ocean by solving the equation of compactness, four equations of motion, two equations of energy, and two equations of state together with the equations for air above the ice and the equations of water below the ice.

Most of the analyses in the literature are much simpler than our model (III). Our model may be reduced to these simplified models as follows:

(i) <u>The simplest fluid model.</u>

The simplest fluid model considers only the four variables:

$$u_i,\ v_i,\ u_w,\ v_w \qquad \text{(IV)}$$

In references by Campbell (1965), Fel'zenbaum (1961) and Sverdrup (1928), this model has been used. In order to reduce our model (III) to the simplest model (IV), we have to assume that the temperatures T_r and the density ρ_r are constant. We also assume N = constant. For instance, in references by Campbell (1965), Fel'zenbaum (1961) and Sverdrup (1928), the compactness is implicitly assumed to be unity. Furthermore in these references, different types of equations of motion have been used. In reference by Campbell (1965), only the viscous stresses (with $N = 1$) of (30) are used but not the interaction force (29), while in reference by Sverdrup (1928), the interaction force (29) with $\bar{q}_w = O$ but not the viscous stresses (30) have been used. In reference by Fel'zenbaum (1961), both interaction force (29) and the viscous stresses (30) are neglected. In our simplest model, we would like to use both interaction force (29) and the internal stresses (30) with $N = 1$.

(ii) Doronin Model.

In reference by Doronin (1970), the following variables are considered:

$$u_i,\ v_i,\ u_w,\ v_w,\ N \qquad \text{(V)}$$

Doronin added an equation of compactness (24) to the equations of motion. Furthermore, Doronin implicitly assumed in his model that the velocity components of ice and water in the layer $O \leqq z \leqq H$ are equal, i.e., $u_i = u_w$, $v_i = v_w$ so that the equations of motion of ice (27) and (28) are solved with the equation of compactness (24). Because of the assumption of $u_i = u_w$ and $v_i = v_w$, the friction force (29) is identically equal to zero. Doronin also considered the thermal effects of compactness and the thickness of ice in an empirical manner by assuming that the temperature is known and given by the temperature of the atmosphere at a certain height. After he obtained the dynamic effects from (24), (27) and (28), he added the thermal *effects to his results by some empirical relationships.*

We may improve Doronin's model (V) by considering the following variables:

$$u_i,\ v_i,\ u_w,\ v_w,\ T,\ N \qquad \text{(VI)}$$

where $T = T_i = T_w$ which may be different from the atmospheric temperature and which is given by the energy equation of the mixture. Furthermore, we may calculate the case $u_i \neq u_w$ and $v_i \neq v_w$ in the top layer $O \leqq z \leqq H$.

b. Comparison of Pai-Li Model with AIDJEX's Model.

The AIDJEX's dynamic model and its relationship to the atmospheric and ocean global models are shown in Figure 1, Figure 6 of reference by Untersteiner (1974). Our mathematical model is shown in Figure 2. There are some similarities and some differences between these two models.

The similarities of these two models are as follows:

In both models, the essential efforts are given to the fundamental equations which govern the properties of the pack ice as a continuum, such as the ice flow velocity, ice stresses, ice thickness, etc. The fundamental equations of the ice layer should be solved with the fluid dynamics of the atmosphere above the ice and that of the ocean current below the ice. Thus,

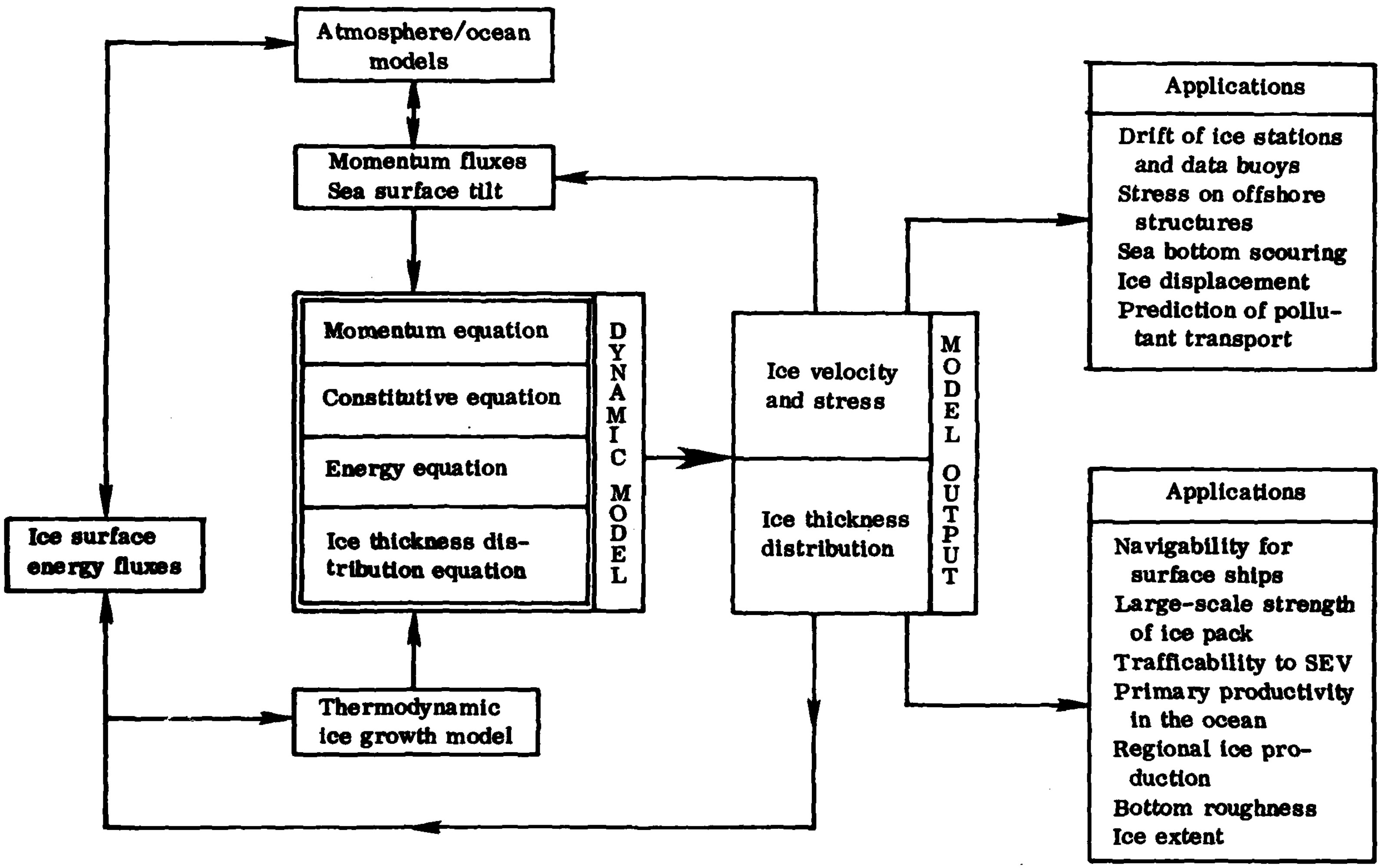

Figure 1. The AIDJEX dynamic ice model and its relationship to the atmosphere and the ocean global models. The nature of the expected model output and potential applications to practical problems are also indicated.

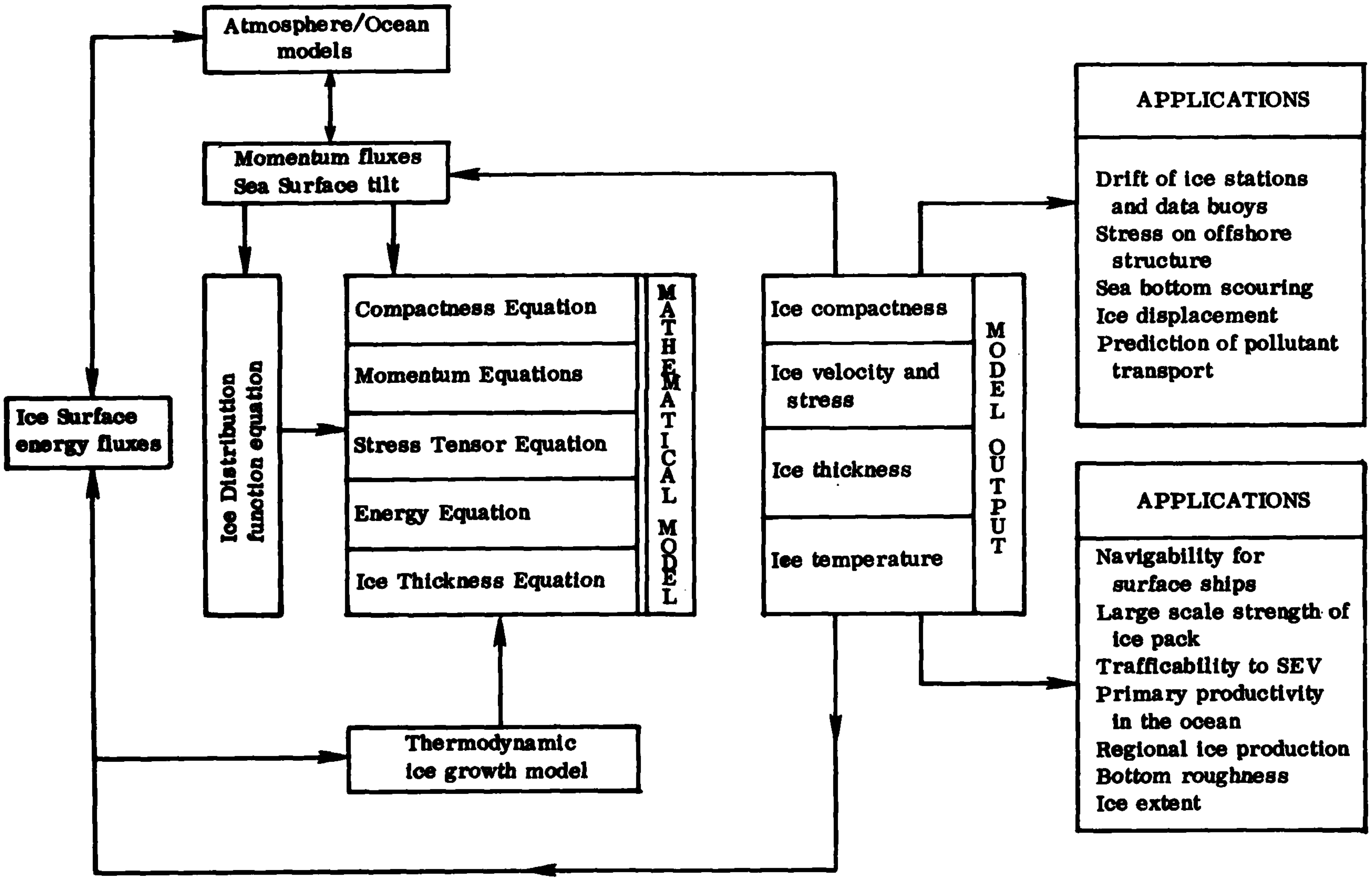

Figure 2. Pai-Li mathematical model of pack ice and its relationship to the atmosphere and the ocean global models

in Figures 1 and 2, only the dynamic model part is different.

The primary differences of these two models are as follows:

(i) The distribution functions in these two models are of entirely different concept. In our model, the distribution function is connected with the microscopic analysis while in AIDJEX's model, the distribution function is connected with the macroscopic approach. We use the Boltzmann equation to determine the distribution function. In AIDJEX's model, the equation of ice distribution is solved simultaneously with the macroscopic equation of ice flow velocity.

(ii) In the Pai-Li model, the pack ice is considered as a mixture of ice and water, and the compactness of pack ice is considered as a dependent variable. In the AIDJEX model, the pack ice is considered as an elastic-plastic material. The compactness is not considered as one of the dependent variables.

(iii) In the AIDJEX model, the constitutive equation is assumed according to the elastic-plastic material. In Pai-Li's model, the consititutive equation is that of a viscous or turbulent fluid. For instance, a constitutive equation similar to that of Doronin (30) may be used. Furthermore, in the Pai-Li model, no limitation on the constitutive equation is imposed. If we assume that the pack ice may be considered as a non-Newtonian fluid, a rather general consitutive equation including that of an elastic-plastic material may be used to calculate the ice circulation in the Polar Ocean.

(iv) In the AIDJEX model, the thermodynamic effect is included by assuming that the temperature of ice is known. In Pai-Li's model, the temperature of the pack ice will be calculated from the energy equation of the pack ice using the atmospheric and oceanic data as boundary conditions.

Since there are several significant differences in these two models, numerical solutions using both models should be carried out and compared with each other and with observational data.

V. Summary and Conclusions.

The following are the summary and conclusions of our mathematical model:

(1) Based on two-phase flow theory (a mixture of a fluid and small solid particles), a new mathematical model of the ice dynamics in the Artic Ocean is formulated.

(2) This mathematical model is developed from both the microscopic and the macroscopic approaches.

(3) In the microscopic approach, the distribution functions for the pack ice and for sea water are studied and so are the Boltzmann equations which govern these distribution functions.

The relations between the distribution function and the macroscopic variables are given. The derivations of the macroscopic equations from the transfer equations of Boltzmann equation are discussed.

(4) In the macroscopic approach, the derivations of the effective two-dimensional equations of pack ice are discussed in detail, particularly the equation of compactness, the equations of motion and the interaction forces between ice and water.

(5) We compare our mathematical model with various models used in literature and show that our model is the most general one for the fluidized model.

(6) We also show that under various simplified approximations, our model may be reduced to those models in literature. We also discuss the improvement of those simplified models in literature based on our mathematical model.

VI. Recommendations.

The following problems are suggested for further development and application of our mathematical model.

(1) We will apply some of our simplified models such as (V) to reinvestigate the ice circulation in the central portion of Arctic Basin. The main point is to show the improvements of our model (V) over those by Campbell (1965) and Fel'zenbaum (1961) where the effect of compactness is not included. We may also show the effects of various interaction forces such as given by (29) and (30).

(2) We will apply our model (V) to study the ice movement near the coastal region or in the marginal zone so that the effect of tidal wave will be included. This problem is very important from practical point of view but the numerical calculation has not been carried out.

(3) We shall study the energy equation of the effective two-dimensional ice dynamics in detail. After we obtain the energy equation, we shall apply our model (VI) to study the ice circulation in the central portion of Arctic Ocean and compare with Doronin's results (1970) with ours.

(4) We will apply our model (VI) to the coastal regions. We may contribute to the improvement of long range ice prediction by considering the thermal effect and the dynamic effect simultaneously.

(5) From the energy equation of two-phase flow, we will study unsteady ice dynamic problems such as the growth and decay of the young ice and one-year ice, particularly those near the coastal region which is a very important practical problem.

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