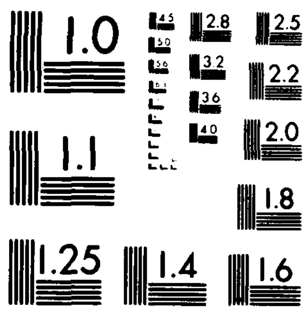
AD-A088 224 TENNESSEE UNIV KNOXVILLE DEPT OF MATHEMATICS F/G 12/1
SEMISTABLE LAWS ON TOPOLOGICAL VECTOR SPACES.(U)
SEP 79 D M CHUNG, B S RAJPUT, A TORTRAT N00014-78-C-0468
UNCLASSIFIED TR-7901 NL

1 OF 1
AD
A088224

END
DATE
FILMED
9-80
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

SEMISTABLE LAWS ON TOPOLOGICAL VECTOR SPACES

by

Dong M. Chung
Sogang University
Seoul, Korea

Balram S. Rajput
The University of Tennessee
Knoxville, USA

and

Albert Tortrat
The University of Paris, France

SEMISTABLE LAWS ON TOPOLOGICAL VECTOR SPACES

by

Dong M. Chung
Sogang University
Seoul, Korea

Balram S. Rajput
The University of Tennessee
Knoxville, USA

and

Albert Tortrat
The University of Paris, France

Accession For

SEMISTABLE LAWS ON

TOPOLOGICAL VECTOR SPACES

by

Dong M. Chung
Sogang University
Seoul, Korea

Balram S. Rajput*
The University of Tennessee
Knoxville, USA

and

Albert Tortrat
The University of Paris, France

A

ABSTRACT

In this paper we discuss three types of results: Firstly, we present two Lévy-Khinchin type representations of Poisson type infinitely divisible (i.d.) laws on certain topological vector (TV) spaces; one of these complements a known representation due to Dettweiler. Secondly, we define and characterize r-semistable laws on locally convex TV spaces and also obtain good representation of their characteristic functions. Finally, we discuss the existence and the continuity of the semigroup $\{\mu^t : t > 0\}$ of i.d. laws μ on locally convex TV spaces. These complement similar known results of Siebert.

* The research of this author is partially supported by the Office of Naval Research under contract No. N0014-78-C-0468.

D

I. INFINITELY DIVISIBLE LAWS

Let X and Y be two real vector spaces in duality, with Y separating points of X ; and let μ be a weakly Radon law on X with Lévy measure F . We recall that F is a measure on $X\setminus\{\theta\}$ such that F is finite and weakly Radon outside every nbd. V of θ (for the weak topology σ) and that the relation

$$(1) \qquad \operatorname{Log} \hat{\mu}_s(y) = \int [\cos y(x) - 1] d\tilde{F} ,$$

where $y \in Y$ and $\tilde{F}(dx) = F(dx) + F(-dx)$, defines the characteristic (ch.) function $\hat{\mu}_s$ of μ_s , the symmetrization of μ . More precisely the family $\{\mu_{sV}\}$ of the laws, defined by (1) by reducing the integral in (1) to $X\setminus V \equiv V^c$, is tight and converges to μ_s as the filter $\{V\} \downarrow \{\theta\}$. We recall also that μ and μ_s are infinitely divisible (i.d.): $\mu^{1/n}$ and $\mu_s{}^{1/n}$ exist and are weakly Radon for every positive integer n . The measure $\mu_s{}^{1/n}$ is obtained by replacing $\tilde{F}$ by $\tilde{F}/n$ in (1). We denote the class of these measures by T and say that $\mu \in T$ is of "Poisson type" if μ has no Gaussian component.

<u>Lemma 1</u>. Let G be a (bounded) measure and let

$e(G)$ ($\equiv e^{-g} \sum_{n=0}^{\infty} \frac{G^n}{n!}$, $g = G(X)$) is a proper factor of μ ; i.e. $\mu = e(G)\nu$ with $\nu \in T$ (ν being Radon for σ). Denote by $\delta(\varepsilon)$ the inverse function of $\varepsilon(\delta) = \delta e^{-2\delta}$, with $2\delta \leq 1$. If $\varepsilon = \mu_s^t(A^c) \leq \frac{1}{2}e$, then one has, for every positive rational t ,

$$(1') \qquad tG\{(A - A)^c\} \leq \delta(\mu_s^t(A^c)) ;$$

in particular,

$$(1'') \qquad tG(A^c) \leq \delta(\mu_s^t(A^c/2)) ,$$

if A is a.c. (absolutely convex).

<u>Proof</u>. Denote by G_t' the restriction of tG to $(A - A)^c$ and by g_t the total mass of tG. Since by hypothesis ν^t exists, it follows that $\rho \equiv e(G_t')$ is a factor of μ^t and hence also of μ_s^t. This implies that there exists an a such that $\rho(A - a) \geq 1 - \varepsilon$; which in turn implies that $\rho_s(A - A) \geq 1 - 2\varepsilon$. Thus, we have $2\varepsilon \geq e^{-2g_t} 2g_t$; implying $g_t \leq \delta(\varepsilon)$, completing the proof.

<u>Remark 1</u>. This lemma (among its other applications) is, in fact, used to define F_{V^c} , the restriction of F to V^c, provided that μ is known to have finite projections μ_I of the form $e(F_I)$ (: $\mu_I = \pi_I(\mu)$, $\pi_I(x) = \pi_I y_i(x)$, for I a finite subset of Y). To see this, one takes $t = 1$ and $\mu_s K^c \leq \frac{1}{2}e$, for an appropriate weakly compact set K , and applies Prokhorov's theorem to μ_I , F_{I,V^c} , with V open cylinder sets; this assures the σ-Radon extension of F_{I,V^c} to a bounded measure F_{V^c} which assigns zero mass to V .

Also, if μ is Radon for a topology C finer (than σ) having the dual Y , then F_V's and $F = \sup F_V$ are C-Radon: Let U be a given C-nbd. of θ ; choose a symmetric C-nbd. U' of θ with $U' + U' \subseteq U$ and a positive integer n_0 such that $\mu_s^{1/n_0}(U') > 1 - \frac{1}{2}e$ (this is possible from (1) and the observation that the family $\{\mu_s^{1/n}\}$

is $\mathcal{C}$-tight); then for $A = U'$, (1') yields

$$G(U^c) \subseteq G(U' + U')^c \leq n_0 \delta(\mu_s^{1/n_0}(U'^c)) .$$

This proves the fact mentioned in the beginning for $(\mathcal{X}, \mathcal{C})$ (with dual $\mathcal{Y}$) assuming the only hypothesis that μ_I are of Poisson type (see [5], for the existence of $\mu^{1/n}$).

Theorem 1. Let $\mathcal{X}$ be a topological vector space (TVS) such that the dual $\mathcal{Y}$ of $\mathcal{X}$ separates points of $\mathcal{X}$ and let μ' be a weakly Radon Poisson type law on $\mathcal{X}$ with Lévy measure F. Assume that for some a.c. weakly compact set K one has

(2) $$F(k^c) < \infty , \text{ and } \int_k |x|^\ell \, dF < \infty , \quad \ell = 1, 2,$$

where $|x| = |x|_K = \inf \{t > 0 : x \in tK\}$. Then the formula

(3) $$\text{Log } \hat{\mu}_\eta(y) = \int_{\{\eta<|x|\}} [e^{iy(x)} - 1 - iy(x) \, I_k(x)] dF \quad \text{or} \quad \int_{\{\eta<|x|\}} (e^{iy(x)} - 1) dF ,$$

if $\ell = 1$, defines a Radon law μ_η, for each $\eta \in [0, 1]$; further, the family $\{\mu_\eta : 0 < \eta \leq 1\}$ is weakly tight and converges to μ ($= \mu_0$), which is a translate of μ'.

Proof.* Let $p(y) = p_{K^0}(y)$ (the Minkowski functional of K^0, the polar of K) and $\varepsilon_N \equiv \mu_s'(NK^c/2)$; using Lemma 1, we have $F(NK^c) \leq \delta(\varepsilon_N)$.

To begin with we assume that F is bounded outside every ηK. This assures us that the law μ_η, defined by (3), is Radon, for every $\eta > 0$. Let ε_N be as above and $\varepsilon_N' > 0$ be such that $\sum \varepsilon_N/\varepsilon_N' < 1$;

* The method of proof here, as well as that of Theorem 4, is similar to that of [6].

then, as $\mu_{\eta s}$ and therefore μ_η are factors of μ_s, using a well known method [4, 9], we can choose $a_\eta \in X$ such that $\{\mu'_\eta = \mu_\eta \delta(a_\eta)\}$ is tight; in fact, $\mu'_\eta(NK^c) \leq \varepsilon'_N$. Now in $\{\eta \leq |x| \leq 1\}$, the absolute value of the integral in (3) is dominated by $p^\ell(y) \int_{\{\eta < |x| \leq 1\}} \frac{|x|^\ell}{\ell!} dF$;

and since $p \leq 1$, $p^\ell(y) \leq p(y)$. In $\{1 < |x| \leq N\}$, clearly,

$$\left|\int_{\{1 < |x| \leq N\}} (e^{iy(x)} - 1)dF\right| \leq p(y) \int_{\{1 < |x| \leq N\}} |x|^\ell dF .$$

Thus, we have

$$|\mathrm{Log}\ \hat{\mu}_\eta(y)| \leq 2\delta(\varepsilon_N) + C_N p(y) , \tag{4}$$

where $C_N = \int_{NK} |x|^\ell dF < \infty$, as F is bounded outside K. But, clearly,

$$|\mu'_\eta(y) - 1| \leq 2\varepsilon'_N + Np(y) . \tag{4'}$$

Thus $|\hat{\mu}_\eta(y) - 1|$ and $|\hat{\mu}'_\eta(y) - \hat{\mu}_\eta(y)|$ are both uniformly small with $p(y)$; therefore, as $\hat{\mu}'_\eta(y)/\hat{\mu}_\eta(y) = e^{iy(a_\eta)}$, $|e^{iy(a_\eta)} - 1|$ and hence $y(a_\eta)$ are uniformly small for all $y \in tK^0$ (for small t). Thus $a_\eta \in t^{-1}K$, for all η. This proves $\{\mu_\eta\}$ is tight.

Now we show that (3) defines a Radon law when $F(\eta K^c)$ is not necessarily finite: The preceeding arguments, applied to a net $\{G_\alpha\}$ of finite measures with $G_\alpha \uparrow F_\eta \equiv F/\{x: |x| > \eta\}$, η fixed, show that (4) holds with the same constants C_N and $\delta(\varepsilon_N)$ for the law $\mu_{\eta\alpha}$ defined by (3) with F replaced by G_α. Similar arguments apply to (4'). Thus μ_η, being the limit of the tight net $\{\mu_{\eta\alpha}\}$ of Radon laws, is Radon.

The following two corollaries are immediate from the theorem.

Corollary 1. Let F satisfy the hypothesis (2), $\{F_\alpha\}$ an increasing filter with $F_\alpha \uparrow F$ and let μ_α be the Radon law defined by (3) with $\eta = 0$ and F_η replaced by F_α. Then $\{\mu_\alpha\}$ is tight and $\mu_\alpha \to \mu$, where μ is as in the theorem.

Corollary 2. Let (X,C) be a TVS with the dual Y separating points of X. If μ' is C-Radon and F satisfies (2) for some C-compact set K, then the conclusions of the theorem hold in the sense that tightness as well as the convergence are in the sense of C.

The hypothesis $F(K^c) < \infty$ (which always holds for a suitable set K, by Lemma 1) does not necessarily imply that $\int_K |x|^\ell dF < \infty$, for $\mu \in T$. We give another theorem which assumes the hypothesis:

(5) (X,C) is locally convex (l.c.) and complete.

This hypothesis has the drawback in that it is not applicable when $C = \sigma$. We begin with a lemma.

Lemma 2. Let (X,C) be a complete l.c. TVS.

(i) If $\{\mu_\alpha\}$ is a convergent (following a certain filter) tight family, then the ch. functions $\hat{\mu}_\alpha(y)$ of μ_α converge uniformly on the polar U^0 of every C-nbd. U of θ.

(ii) If the family $\{\mu_\alpha * \delta(a_\alpha)\}$ of the translates of μ_α is tight and if the family $\{\hat{\mu}_\alpha\}$ of ch. functions converge (following a certain filter) to a function ϕ uniformly on the polar of every nbd. belonging to a C-nbd. base of θ, then there exists a unique C-Radon law μ such that $\mu_\alpha \to \mu$ and $\hat{\mu} = \phi$.

Proofs of these are given in [1]. Using a contrapositive argument, a simpler proof of (i) follows easily by observing that U^0 is compact for σ, that the topology C^0 of uniform convergence on C-compact subsets of X coincides with σ on U^0 and that the condition of C-tightness of $\{\mu_\alpha\}$ implies the equicontinuity of $\{\hat{\mu}_\alpha\}$ on U^0 for $\sigma = C^0$.

For the proof of (ii) see ([1] p. 293): One shows, as in the proof of Theorem 2 below, that $\{e^{iy(a_\eta)}\}$ forms a Cauchy filter uniformly on suitable sets K^0 which assures, as K^0 absorbs every U^0, that $\{a_\eta\}$ is C-Cauchy; thus the necessity of the completeness of (X,C).

In the following theorem the method of proof is that of [1]; it provides another expression obtained in Theorem 2.5 of [1] which gives a Lévy-Khintchin formula.

<u>Theorem 2</u>. Let μ' be a Poisson type C-Radon law with Lévy measure F on (X,C) ((X,C) l.c. and complete) and let $\{F_\alpha\}$ be an increasing net of Radon laws on $X\setminus\{\theta\}$ with $F_\alpha \uparrow F$. Then for every bounded Borel set B with $F(B^c) < \infty$, the formula

$$\text{Log}\ \hat{\mu}_\alpha(y) = \int_B [e^{iy(x)} - 1 - iy(x)]I(x)dF_\alpha \tag{5}$$

defines a Radon law μ_α; further if F_α in (5) is replaced by F, then (5) defines a Radon law μ which is a translate of μ' and the net $\{\mu_\alpha\}$ converges (in the sense of C) to μ.

Proof. To begin with we argue by assuming that (5) defines the Radon laws. The laws $\mu_{\alpha s}$ form a tight family (see Lemma 2 of [4]), which converges to μ_s . Moreover μ_s is continuous in $(\mathcal{Y}, C^0)$, which assures that $|\hat{\mu}_s(y)| \geq 1/2$ on the polar of a suitable compact set K and hence also on $\frac{1}{n} U^0$, for every C-nbd. U of θ and large enough n , depending on U . Then, form (i) of Lemma 2, we have

$$\int_B [1 - \cos y(x)] d(F - F_\alpha) \to 0 , \tag{6}$$

uniformly in each $\frac{1}{n} U^0$; hence also

$$\int_B y^2(x)\ d(F - F_\alpha) \to 0 , \tag{7}$$

uniformly on $\frac{1}{n'} U^0$ as long as $2B \subseteq n'U$ and $n' \geq n$, because $|t| \leq 1/2$ implies $1 - \cos t \geq t^2/3$. We thus have

$$\int_B [e^{iy(x)} - 1 - iy(x)] d(F - F_\alpha) \to 0 , \tag{7'}$$

uniformly on $\frac{1}{n'} U^0$. Thus, since the laws defined by

$\operatorname{Log} \hat{\mu}_\alpha(y) = \int_B [e^{iy(x)} - 1 - iy(x)] dF_\alpha$ are factors of μ (see below), Lemma 2 (ii) applies and the result follows (we do not affirm that $\{\mu_\alpha\}$ is tight).

To see that (5) indeed defines Radon laws; let, for fixed F_α , $\{G_{\alpha\beta}\}$ be an $\uparrow$ net of bounded measures with $G_{\alpha\beta} \uparrow F_\alpha$. The preceeding estimates and arguments a fortiori hold, proving that μ_α , being the limit of the Radon laws $\mu_{\alpha\beta}$ which are defined by (5) and $G_{\alpha\beta}$, is Radon.

Note that this applies to $F'_\alpha = F - F_\alpha$; therefore μ_α is indeed a factor of μ (the cofactor μ'_α is defined by (5) and F'_α).

II. SEMISTABLE LAWS ON TVS

<u>Lemma 3</u>. Let $(\mathcal{X},\mathcal{C})$ be a TVS, whose dual $\mathcal{Y}$ separates points of $\mathcal{X}$. If μ is weakley τ-regular law on $\mathcal{X}$, then there exists an element a in $\mathcal{X}$ such that $\mu_{-a} = \mu\delta(-a)$ is supported by the space $E \equiv \{x: y(x) = 0$, for every y for which μ_y is degenerate$\}$.

<u>Proof</u>. Let $S = \{x: y(x) = a_y$, if $y = a_y$ a.s. $\mu\}$; then S is weakley closed and $\mu(S) = 1$. Now it is clear that for any a in S, we have $E = S - a$ with $\mu_{-a}(E) = 1$.

The following theorem defines and characterizes r-semistable laws:

<u>Theorem 3</u>. Let $(\mathcal{X},\mathcal{C})$ be l.c. TVS. Let μ, ν be two $\mathcal{C}$-Radon laws (with μ convexly tight*) and let $0 < r < 1$. Then (8) and (9) below are equivalent:

(8) $\left\{\begin{array}{l}\text{There exist sequences } \{a_n\},\ a_n > 0,\ \{b_n\} \subseteq \mathcal{X} \text{ and a} \\ \text{strictly increasing seuqence } \{k_n\} \text{ of positive integers such} \\ \text{that } {}^{**}(a_n \cdot \nu^{k_n})\delta(b_n) \to \mu,\quad \dfrac{k_{n+1}}{k_n} \to \dfrac{1}{r},\end{array}\right\}$

(9) $\left\{\begin{array}{l}\mu \text{ is (i.d.) and } \mu^r = (a\cdot\mu)\delta(b), \text{ for some } b \in E \text{ and } a > 0, \\ \text{where } E \text{ is as in Lemma 3.}\end{array}\right\}$

(If μ satisfies (8), then μ is called <u>r-semistable</u>.)

---

* μ is tight following compact convex sets. Note any $\mathcal{C}$- tight law is convexly tight if $(\mathcal{X},\mathcal{C})$ is quasi-complete; same remark applies for σ if $\mathcal{X}$ is complete for the Mackey topology.

** If ν is the law of a random vector X and $a \neq 0$, then $a\cdot\nu$ denotes the law of aX.

<u>Proof</u>. If $y \in E'$ (the dual of E), $y \neq \theta$, then $\mu_y \equiv \mu \circ y^{-1}$ and hence ν_y is non-degenerate, and one has

$$(a_n \cdot \nu_y^{k_n})\delta(y(b_n)) \to \mu_y ,$$

as $n \to \infty$. This implies that μ is i.d. and μ^r exists (see [5], p. 320). Let ϕ and ψ denote, respectively, the ch. functions of ν and μ. Then (8) implies

$$e^{ity(b_n)}\phi^{k_n}(a_n ty) \to \psi(ty) ,$$

and

$$[e^{ity(b_{n+1})}\phi^{k_{n+1}}(a_{n+1}ty)]^{\frac{k_n}{k_{n+1}}} \to \psi^r(ty) \tag{10}$$

(uniformly on bounded subsets of reals), as $n \to \infty$. But left side of (10) can be written as

$$e^{ity(b_n)}\phi^{k_n}(\frac{a_{n+1}}{a_n} a_n ty)e^{ity(\frac{k_n}{k_{n+1}} b_{n+1} - b_n)} ;$$

hence, using the types of convergence theorem on R, the reals, we have $\frac{a_{n+1}}{a_n} \to a > 0$ $y(\frac{k_n}{k_{n+1}} b_{n+1} - b_n) \to b_y$ (with a independent of y), and

$$\mu_y^r = (a \cdot \mu_y)\delta(b_y) . \tag{10'}$$

But (10') and Corollary 1 of Lemma 2 of [4] imply that there exists a $b \in E$ such that $b_y = y(b)$ and that $\mu^r = (a \cdot \mu)\delta(b)$.

Conversely, let k_n be the integral part of r^{-n} $(r < 1)$, then $r^n k_n \to 1$. Then (9) and continuity of $\{\mu^s\}$ (see Section III) gives

$$\mu^{k_n r^n} = (a_r^n \cdot \mu^{k_n})\delta(k_n b_{r^n}) \to \mu ,$$

where $b_{r^n} \in E$. Hence (9) holds with $\frac{k_{n+1}}{k_n} \to \frac{1}{r}$.

Lemma 4. Let μ be as in Theorem 3 with (X,C) l.c. TVS and let $H = \{t > 0 : \mu^t = (a_t \cdot \mu)\delta(b_t)$, for some $a_t > 0$, $b_t \in X\}$ $(H \neq \phi$, for μ satisfying (9)). Then H is a closed multiplicative subgroup of $R^+ \equiv \{r : r > 0\}$. Thus either $H = R^+$ (then μ is stable) or H is generated by r_0 the largest element in H less than 1 (then μ is r_0-semistable). In particular, one has $\mu^{r^n} = (a_r^n \cdot \mu)\delta(b_{r^n})$, $a(r)a(r') = a(rr')$, let $a_r = r^\lambda$.

Proof. It is clear than $t^{-1} \in H$, whenever $t \in H$. On the other hand if t , $t' \in H$, then $\psi^t(y) = \psi(a_t y)e^{iy(b_t)}$, and $\psi^{tt'}(y) = (\psi^t(y))^{t'} = \psi(a_{t'} a_t y)e^{i(b_{t'} + t'b_t)}$ showing $tt' \in H$. Finally, let $\{t_n\} \subseteq H$ with $t_n \to t$; then $\mu^{t_n} = (a_{t_n} \cdot \mu)\delta(b_n)$ for each n . Now by continuity of $\{\mu^s\}$, we have $\lim_n (a_{t_n} \cdot \mu)\delta(b_n) = \lim_n \mu^{t_n} = \mu^t$. This and the types of convergence theorem (after projecting to R) shows that $\mu_y^t = (a \cdot \mu)\delta(b_y)$, for every $y \in E'$, with a independent of y , as in Theorem 3. Hence from ([4], p. 304), we have that $t \in H$.

Lemma 5. Let μ and (X,C) be as above. If $\lambda \neq 1$, then μ can be centered: for some $m \in X$, $\nu \equiv \mu_{-m}$ satisfies $\nu^t = t^\lambda \cdot \nu$, for all $t \in H$.

Proof. Let X_r and X be vectors with laws μ^r $(r \neq 1)$ and μ , respectively, then one has

(11) $$m = \frac{b_r}{r - ra_r} \Rightarrow \text{(11')} \qquad X_r - rm \overset{\text{law}}{=} a_r(X - m) .$$

Now if H is generated by r_0 define m by r_0, then $\nu^t = t^{\lambda} \cdot \nu$ holds for all $t \in H$, by (11) and iteration. For the case $H = R^+$, μ is stable, and m is independent of t because m_y 's are (see Lemma 3, p. 178 and Note 5 of [6]).

Lemma 6. Under the hypothesis of Theorem 3, (8) for $r = 1$ is equivalent to (9) for all r (with $a_r = r^{\lambda}$, $\lambda \geq 1/2$ (from Theorem 4)); i.e. μ is stable of exponent $\alpha = 1/\lambda$.

Proof. Taking $r = 1/n$, we obtain from (9), $\mu^{1/n} = (a_n \cdot \mu)\delta(b_n)$ which yields $\mu = (a_n \cdot \mu^n)\delta(nb_n)$. Showing μ satisfies (8) with $k_n = n$. Conversely, assume μ satisfies (8) with $r = 1$. Let now $r \in (0,1)$ one can choose a sequence $\lambda_n = k_n/k_{n'}$ with $n' > n$, such that $\lambda_n \to r$. Now, for $y \in E'$, it follows, from (8), that

$$\phi^{k_{n'}}(a_{n'}ty)e^{iy(tb_{n'})} \to \psi(ty)$$

and

$$[\phi^{k_{n'}}(a_{n'}ty)e^{iy(tb_{n'})}]^{k_n/k_{n'}} \to \psi^r(ty)$$

uniformly on bounded subsets of R. Then repeating the arguments used in the proof of Theorem 3 we see that (9) holds.

Theorem 4. Let μ be a C-Radon convexly tight r-semistable law on (X,C) (with (X,C) l.c.). Assume that $\alpha = \frac{1}{\lambda} < 2$ (for $\alpha = 2$ and μ centered, $\psi^r(yt) = \psi(\sqrt{r}\, ty)$ gives, $\hat{\mu}_y(t) = \hat{\mu}_y^{t^2}(1)$ and so μ is Gaussian), then we have:

(i) The Lévy measure F of μ satisfies

$$r^{n\lambda} \cdot F = r^n F \tag{12}$$

for all $n = \pm 1, \pm 2, \ldots$. For every μ-positive a.c. bounded closed set B , F is bounded outside every set ηB $(\eta > 0)$ and is supported by the space $E_B = \cup \eta B$. Further, $|x|^\gamma$ is F-integrable on $\{|x| \leq \eta\}$ if and only if $\gamma > \alpha$ and on $\{|x| \geq \eta\}$ if and only if $\gamma < \alpha$, for all $\eta > 0$, where $|x| = p_B(x) = \inf\{t > 0 : x \in tB\}$.

(ii) The ch. function of the centered law μ_c of μ for $\alpha \neq 1$, has the form

$$\text{Log } \hat{\mu}_c(y) = \int (e^{iy(x)} - 1) dF , \tag{13}$$

if $\alpha < 1$, and

$$\text{Log } \hat{\mu}_c(y) = \int [e^{iy(x)} - 1 - iy(x)] dF , \tag{13'}$$

if $1 < \alpha < 2$. For every B (compact for σ or C), (13) and (13') signify that if μ_η denotes the law corresponding to the ch. function obtained by truncating the integrals to $\{|x| > \eta\}$, then $\{\mu_\eta\}$ is tight and converge to μ_c as $\eta \to 0$. For $\alpha = 1$, one has the same result for a translate μ' of μ (μ' in general is not centered if F is not symmetric) with

$$\text{Log } \hat{\mu}'(y) = \int [e^{iy(x)} - 1 - iy(x) I_B(x)] dF \tag{13''}$$

for an appropriately chosen compact set B .

(iii) If (X,C) is, in addition, complete and metrizable, these properties of the families $\{\mu_\eta\}$ and $\{\mu'_\eta\}$ hold for all μ-positive a.c. bounded closed sets B . If (X,C) is only l.c. complete (and not metrizable), the convergence property of the measures still holds, but we no longer affirm that the family is tight.

Proof of (i). Since $\mu_s^r = r^{\lambda}\cdot\mu_s$, and Log $\hat{\mu}_s(y) = \int [\cos y(x) -1]d\tilde{F}$, we have

$$\int[\cos y(x) -1]rd\tilde{F} = \int[\cos r^{\lambda}y(x) - 1]d\tilde{F} = \int[\cos y(x) - 1]d(r^{\lambda}\cdot F) \ ;$$

thus (12) holds for $\tilde{F}$. But in finite dimension the same thing holds modulo the translation with $\int(e^{iy(x)} - 1)dF$; thus F_y satisfies (12); therefore so does F .

Fix B , then, since $\mu_s^t = t^{\lambda}\cdot\mu_s$, for all $t = r^n$, and $\mu(B) > 0$, we have

$$\mu_s^t(\frac{B}{2}) = \mu_s(\frac{B}{2t^{\lambda}}) \uparrow 1$$

as $n \to \infty$, by the 0-1 law (see [10]) and by the fact that $\mu_s(2B^c) \le (\mu(B^c))^2$. Lemma 1, therefore assures us that F is bounded outside each set ηB . Thus, by (12), $r^n F(B^c) = F(B^c/r^{n\lambda}) \downarrow 0$ as $n \to \infty$, showing E_B supports F .

Now let $\eta = r^{\lambda n_0}$ (by the choice of n_0 , we can make η arbitrarily small or large), then this same formula (12) gives:

(a)
$$\infty > \int_{\{|t|\le 1\}} t^2 dF_y(t) \ge \sum_0^{\infty} r^{2\lambda(n+1)} F_y\{r^{\lambda(n+1)} < |t| \le r^{\lambda n}\}$$

$$= (1 - r)F_y\{|t| > 1\} \sum_0^{\infty} r^{(n+1)(2\lambda -1)}$$

(showing $\frac{1}{\lambda} = \alpha < 2$),

(b)
$$\int_{\{|x| \le r^{\lambda n_0}\}} |x|^{\gamma} dF = \sum_{n_0}^{\infty} r^{\gamma n\lambda} F\{r^{\lambda(n+1)} < |x| \le r^{n\lambda}\}$$

$$= \frac{1 - r}{r} F(B^c) \sum_{n_0}^{\infty} r^{n(\gamma\lambda-1)} < \infty$$

if $\gamma > \alpha$; and

$$\text{(c)} \qquad \int_{\{|x| > r^{\lambda n_0}\}} |x|^\gamma dF \le \sum_{-n_0}^{\infty} r^{-\gamma\lambda(n+1)} F\{r^{-\lambda n} < |x| \le r^{-\lambda(n+1)}\}$$

$$= \left(\frac{1-r}{r^{\gamma\lambda}}\right) F(B^c) \sum_{-n_0}^{\infty} r^{n(1-\gamma\lambda)} < \infty ,$$

$$\text{if} \quad \gamma < \alpha .$$

The same method proves that in (b) and in (c) γ cannot be $\le \alpha$, respectively, $\ge \alpha$.

Proof of (ii). Using (i), we apply Theorem 1 by taking an appropriate compact set $K = B$ and $\ell = 1$, if $\alpha < 1$, and $\ell = 2$ if $\alpha \ge 1$. Clearly in the first case one obtains (13) and the proof of the assertion (ii), the same in the second case after translation by $-\int_{B^c} x dF$. For the case $\alpha = 1$ and F non-symmetric one uses (c) to obtain (13") (as in the case of stable law of exponent 1 in R).

Proof of (iii). Apply Theorem 2 with F_η equal to the restriction of F to $\{x: |x| > \eta\}$ and completing by the translation $-\int_B x dF$, if $\alpha < 1$, and by $\int_{B^c} x dF$, if $\alpha \ge 1$. That the families $\{\mu_\eta\}$ and $\{\mu'_\eta\}$ are tight, under the hypothesis that X is complete metric space follows from a result of Topsøe ([3], p. 43). One shows, in fact, that $\{\mu_\eta\}$ is relatively compact (in the set of C-Radon laws), for $0 < \eta \le 1$.

III. THE SEMIGROUP μ^t on R^+

Let μ be i.d. and Radon on (X,C) with (X,C) l.c., then μ^t is defined for all t (rational) > 0. Siebert has shown ([2], p. 243) that μ^t can be defined for all $t > 0$; in fact, he proved the following:

Theorem 5. (Siebert) Let μ be i.d. and C-Radon and convexly tight on (X,C) with (X,C) l.c. Then the semigroup $\{\mu^t : t > 0\}$ is uniquely defined and continuous, and $\{\mu^t : t \in (0,T]\}$ is tight, for every $T > 0$.

Remark 2. Theorem 4 of Siebert ([2], p. 243] gives probably the simplest method to obtain the above result which is nearly the most general result of this type. In the following we give another presentation of this problem. Even though our hypotheses (both on μ and X) are weaker, we are able to affirm the continuity of $\{\mu^r\}$ only over the set of rationals. The definition and continuity of μ^t, for all real $t > 0$, is also proved but this is done under more restrictive conditions. We being with a lemma:

Lemma 7. Let X be a Banach space and μ an i.d. Radon law on X. Then a (unique) continuous semigroup $\{\mu^t : t > 0\}$ always exists; further, $\{\mu^t : t \in (0,T]\}$ is tight, for every $T > 0$.

Proof. It has been proved more than twelve years ago by Parathasarathy* for the space $C[0,1]$. This applies for the case considered here, since $\mu^{1/n}$, $n = 1, 2, \ldots$, are all supported by a separable subspace of X which can be isometrically embedded in $X = C[0,1]$. (Note that

---

* Preprint; the proof is similar to that of Siebert: $\{\mu^t\delta(a_t) : t > 0\}$ being tight, one shows that so is $\{\delta(a_t)\}$.

$\mathcal{X}$ supports μ^t, because every $y \in Y \equiv X'$ which is zero a.s. μ is also zero a.s. μ^t, for fixed t, and μ^t is τ-regular in X and hence in $\mathcal{X}$). We provide a direct simple proof of this lemma; let μ' be the law with

$$\operatorname{Log} \hat{\mu}'^t(y) = \int_B [e^{iy(x)} - 1 - iy(x)]t\,dF ,$$

where $B = \{x : \|x\| \leq 1\}$, and t rational in $[0,1]$. The inequality

$$t \sup_{\|y\| \leq \delta} \int_B y^2 dF \leq 3t \sup_{\|y\| \leq \delta} \int_B [1 - \cos y(x)]dF \leq \text{Const. } t$$

(note $|\hat{\mu}_s'(y)| \geq \eta > 0$ in $\{y : \|y\| \leq \delta\}$, for some $\delta > 0$) gives $|\operatorname{Log} \hat{\mu}'^t(y)| \leq t \sup_{\|y\| \leq 1} \int_B y^2(x)dF \leq \text{Const. } t$. This and Lemma 2 assure the definition of μ^t by continuity, for all $t > 0$. Thus the continuity of the semigroup on R^+ is proved; that $\{\mu^t : t \in (0,T]\}$ is tight is proved as in (iii) of Theorem 4.

Theorem 6. Let $(X,\mathcal{C})$ be l.c. and let μ be a weakley τ-regular i.d. law on the $\mathcal{C}$-Borel σ-algebra of $\mathcal{X}$. Then the semigroup $\{\mu^r : r > 0 ,\ r \text{ rational}\}$ is continuous. If, in addition, $\mathcal{X}$ is metrizable and complete then a unique continuous extension of the above semigroup to R^+ exists.

Proof. Let $X_0 = \{x : y(x) = 0 \text{ if } y \equiv 0 \text{ a.s. } \mu\}$; then since $\mu^{1/n}$, $n = 1, 2, \ldots$, are weakley τ-regular and $y \equiv 0$ a.s. μ implies $y \equiv 0$ a.s. $\mu^{1/n}$, we have $\mu^{1/n} X_0 = 1$, for all $n = 1, 2, \ldots$; we thus can replace $\mathcal{X}$ by X_0, for the proof. For a closed a.c. $\mathcal{C}$-nbd. U of X_0, denote by X_U the quotient space

$\mathcal{X}_0 / p_U^{-1}\{\theta\}$, where p_U is the Minkowski functional of U , then $\mathcal{X}_U$ is a separable normed space under the norm $|x_U| = p_U(x)$, where x_U is the image of x under the natural projection (see [7], p. 135-136). Now in $\mathcal{X}_0$, $U = \{x : p_U(x) \leq 1\}$ and $\mu^r(U) = \mu_U^r\{x_U : |x_U| \leq 1\} = \tilde{\mu}_U^r\{\tilde{x} \in \tilde{\mathcal{X}}_U : |\tilde{x}| \leq 1\}$, where μ_U^r and $\tilde{\mu}_U^r$ denote, respectively, the images under the obvious maps of μ^r in $\mathcal{X}_U$ and $\tilde{\mathcal{X}}_U$, the completion of $\mathcal{X}_U$. Since $\tilde{\mu}_U^r\{|\tilde{x}| \leq \eta\} \to 1$ as $r \to 0$ (r rational), for all $\eta > 0$ (from Lemma 7), $\mu^r(\eta U) \to 1$, for all ηU , proving the continuity of μ^r for all r (i.e. as $r_n \to r$, $\mu^{|r_n - r|} \to \delta(\theta)$), finishing the proof of the first part. The last part under additional conditions on $\mathcal{X}$ now follows easily: Let μ^t be the law of $\sum X_n$ with X_n's independent $\mathcal{X}$-valued r.v.'s where law of $X_n = \mu^{r_{n+1} - r_n}$ and $r_n \uparrow t$; then μ^t is uniquely defined. It is continuous in all t , since $\mu^t(U) \geq \limsup_{r_n \uparrow t} \mu^{r_n}(U)$ assures that $\mu^t \to \delta(\theta)$, as $t \to 0$.

<u>Corollary 3</u>. Let $(\mathcal{X}, \mathcal{C})$ be a separable metric l.c. space and μ an i.d. law on $\mathcal{C}$-Borel σ-algebra of $\mathcal{X}$. Then the semigroup $\{\mu^r : r \text{ (rational)} > 0\}$ is continuous in the sense of $\mathcal{C}$ and also in every topology of a separable normed space E which is continuously embedded in $\mathcal{X}$ and which supports μ . Further, a unique continuous extension in R^+ of the semigroup exists in the completion $\tilde{E}$ of the space E .

<u>Proof</u>. Let $E \hookrightarrow \mathcal{X}$ be such a space (for the existence see [8]). From Theorem 1 of [8], one can suppose that E is dense in $\mathcal{X}$ and that μ

is defined on the Borel σ-algebra of E (which is the trace of the Borel σ-algebra of X), it follows μ is i.d. on E . Hence, as in Theorem 6, this proves that $\{\mu^r\}$ is continuous on E . The last part is immediate from the Theorem 6.

BIBLIOGRAPHY

[1] Dettweiler, E., Grenzwertsätze für Wahrscheinlichkeitsmasse auf Badrikianschen Räumen, Z. W., 34 (1976) 285-311.

[2] Siebert, E., Einbettung unendlich teilbarer Wahrscheinlichkeitsmasse auf topologischen gruppen, Z. W., 28 (1974) 227-247.

[3] Topsøe, F., Topology and measure, Lecture notes, No. 133, Springar, New York.

[4] Tortrat, A., Structure des lois indéfiniment divisibles dans un E. V. T., Lecture notes, No. 31, Springer-Verlag, New York.

[5] ________., Sur la structure des lois indéfiniment divisibles dans un E. V. T., Z. W., 11 (1969) 311-326.

[6] ________., Lois e(λ) dans les espaces vectoriels et lois stables, Z. W., 37 (1975) 175-182.

[7] ________., τ-régularité des lois, séparation au sens de A. Tulcea et propriété de Radon-Nikodym, Ann. H. Poincaré, XII - 2 (1976) 313-150.

[8] ________., Sur les probabilités dans les E. V. T., Proceedings of the symposium to honour Jerzy Neyman, (1974) 319-325.

[9] ________., Lois de probabilité sur un espace topologique complétement régulier et produits infinis à termes indépendants dan un groupe topologique, Ann. H. Poincaré, 1 (1965) 217-237.

[10] Tortrat, A., Rajput, B.S. and Louie, D., A 0-1 law for a class of measures on groups, preprint.

(14) TR-79Ø1

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TECHNICAL REPORT #7901	2. GOVT ACCESSION NO. AD-A088224	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) Semistable Laws on Topological Vector Spaces.		5. TYPE OF REPORT & PERIOD COVERED (9) INTERIM rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) [redacted]M./Chung (Sogang University, Seoul, Korea) Balram S. Rajput (University of Tennessee) [redacted] Tortrat (University of Paris, France)		8. CONTRACT OR GRANT NUMBER(s) (15) N00014-78-C-0468
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Department University of Tennessee, Knoxville, TN 37916		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 042-400
11. CONTROLLING OFFICE NAME AND ADDRESS Statistics and Probability Program Office of Naval Research Arlington, Virginia 22217		12. REPORT DATE (11) Sep~~tember, 1~~979
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 24		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

SEMISTABLE LAWS, STABLE AND GAUSSIAN LAWS

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper ~~we discuss~~ three types of results WERE DISCUSSED: Firstly, ~~we present~~ two Lévy-Khinchin type representations of Poisson type infinitely divisible (i.d.) laws on certain topological vector (TV) spaces; one of these complements a known representation due to Dettweiler. Secondly, ~~we define and characterize~~ THE r-semistable laws on locally convex TV spaces and also obtain

over

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

411894

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. good representation of their characteristic functions. Finally, ~~we discuss~~ the existence and the continuity of the semigroup $\{\mu^t : t > 0\}$ of i.d. laws μ on locally convex TV spaces. These complement similar known results of Siebert.

nu

mu superscript t where t greater than 0

S N 0102-LF-014-6601

END

DATE
FILMED
9-80

DTIC