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THE RICCATI INTEGRAL EQUATION ARISING IN OPTIMAL

CONTROL OF DELAY DIFFERENTIAL EQUATIONS[*]

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In this paper we discuss the linear regulator problem associated with delay-differential equations. Formulas for the optimal feedback control and optimal trajectory are derived; this naturally leads to a Riccati integral equation in the state space of the delay equation. Finally, the linear regulator problem for the delay equation is approximated by sequences of regulator problems associated with ordinary differential equations in Euclidean space.

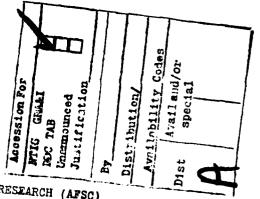DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE

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Abstract:

In this paper we discuss the linear regulator problem associated with delay-differential equations. Formulas for the optimal feedback control and optimal trajectory are derived; this naturally leads to a Riccati integral equation in the state space of the delay equation. Finally, the linear regulator problem for the delay equation is approximated by sequences of regulator problems associated with ordinary differential equations in Euclidean space.

Introduction and Notation.

We study the linear-quadratic optimal control problem for a certain class of delay-differential equations. In Section 2 an analytical solution in terms of a Riccati integral equation in the state space of the delay equation is presented. Subsequently, in the third section, we develop a general scheme for the approximation of this optimal control problem. The averaging approximation scheme and the spline schemes are seen to be special cases.

As usual $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with norm $|\cdot|$ and inner product $(\cdot,\cdot)_{\mathbb{R}^n}$. For the interval $(a,b) \subseteq (-\infty,\infty)$ the Hilbert space of equivalence classes of measurable functions $x:(a,b) \to \mathbb{R}^n$ with $\int_a^b |x(s)|^2 ds < \infty$ is denoted by $L^2(a,b;\mathbb{R}^n)$ and is endowed with the usual inner product and norm. For $L^2(-r,0;\mathbb{R}^n)$, $r > 0$, we simply write L^2. The state space for the presentation will be $Z = \mathbb{R}^n \times L^2$ which in an obvious way becomes a Hilbert space with inner product $\langle\cdot,\cdot\rangle_Z$. Generally, norms of elements will be denoted by $|\cdot|$ and operator norms by $\|\cdot\|$. The set of all bounded linear operators between Hilbert spaces X and Y is $\mathscr{L}(X,Y)$ and A^* denotes the Hilbert space adjoint of the operator A. Finally, for $x:[-r,\alpha) \to \mathbb{R}^n$, $\alpha > 0$, the symbol x_t, $0 \leq t < \alpha$, stands for $x_t(s) = x(t+s)$ for $s \in [-r,0]$.

2. The Optimal Control Problem.

We are concerned with functional differential equations of the type

$$
\left.
\begin{aligned}
\dot{x}(t) &= \sum_{i=0}^{\ell} A_i x(t-r_i) + \int_{-r}^0 A_{-1}(s)x(t+s)ds, \quad \text{for} \quad t \geq 0 \\
\\
(x(0),x_0) &= (\eta,\varphi) \in Z
\end{aligned}
\right\} \qquad (2.1)
$$

where A_i, $i=0,\ldots,\ell$ are real $n \times n$ matrices, A_{-1} is an $n \times n$-matrix of L^2-valued functions and $0 = r_0 < \ldots < r_\ell = r$. Although we restrict ourselves to the autonomous case here, the results are remain true in the case where A_i depends on t. For convenience we introduce the operator L given by

$$L\varphi = \sum_{i=0}^{\ell} A_i \varphi(-r_i) + \int_{-r}^{0} A_{-1}(s) \varphi(s)ds.$$ It is quite well known that (global) solutions $x = x(\cdot;\eta,\varphi)$ of (2.1) exist and that they do not depend on the representation of an equivalence class $\varphi \in L^2$ [7]; here $x(\cdot;\eta,\varphi)$ is called the solution of (2.1) if $x(s;\eta,\varphi) = \varphi(s)$ almost everywhere on $[-r,0]$ and

$$x(t;\eta,\varphi) = \eta + \int_0^t Lx_s(\cdot;\eta,\varphi)ds \quad \text{for} \quad t \geq 0.$$ By $T(t):Z \to Z$ we denote the family of solution operators associated with (2.1) via $T(t)(\eta,\varphi) = (x(t;\eta,\varphi),x_t(\cdot;\eta,\varphi))$. It is well known that $T(t)$ is a linear C_0-semigroup of bounded linear operators for which $\|T(t)\| \leq M \exp(\omega t)$ holds for some M and ω in $\mathbb{R}$; see, e.g., [1].

We can now state the following optimal control problem associated with (2.1):

$$(P) \begin{cases} \text{Find} \quad u \in L^2(0,t^*;\mathbb{R}^m) \quad \text{which minimizes} \\[2mm] J_0(\eta,\varphi,u) = (Fx(t^*),x(t^*))_{\mathbb{R}^n} + \int_0^{t^*}(Dx(t),x(t))_{\mathbb{R}^n}dt + \int_0^{t^*}(Cu(t),u(t))_{\mathbb{R}^m}dt \\[2mm] \text{subject to} \\[2mm] \dot{x}(t) = Lx_t + Bu(t) \quad \text{for} \quad t \in [0,t^*], \quad \text{and} \\[2mm] (x(0),x_0) = (\eta,\varphi), \quad \text{where} \quad t^* > 0 \quad \text{and} \quad (\eta,\varphi) \in Z \quad \text{are given.} \end{cases}$$

The $n \times n$-matrices F and D, the $m \times m$-matrix C and the $n \times m$-matrix B, which are assumed to be independent of time for the sake of a simpler presentation, satisfy: F,D and C and are symmetric and $F \geq 0$, $D \geq 0$, $C > 0$. Of course, (P)

is an extension to delay-differential equations of the unrestricted finite terminal time linear regulator problem, which is well known for ordinary differential equations. But, as a consequence of the fact that the statespace of delay-differential equations is a function space over the delay interval $[-r,0]$ (in our case Z), the analysis of the problem becomes much more difficult than for ordinary differential equations. Many an earlier investigation was directed towards studying (P) and we shall mention some of them further below. Here we first reformulate (P) in the state space Z.

$$(\mathscr{P}) \begin{cases} \text{Find } u \in L^2(0,t^*;\mathbb{R}^m) \text{ which minimizes} \\ J_0(\eta,\varphi,u) = \langle \mathscr{F}z(t^*),z(t^*)\rangle_Z + \int_0^{t^*} \langle \mathscr{D}z(t),z(t)\rangle_Z dt + \int_0^{t^*} (Cu(t),u(t))_{\mathbb{R}^m} dt \\ \text{subject to} \\ z(t) = T(t)(\eta,\varphi) + \int_0^t T(t-s)\mathscr{B}u(s)ds \quad \text{for} \quad t \in [0,t^*] \qquad (2.2) \\ \text{for given } t^* > 0 \text{ and } (\eta,\varphi) \in Z. \end{cases}$$

We used the notation $\mathscr{F}(\theta,\psi) = (F\theta,0)$, $\mathscr{D}(\theta,\psi) = (D\theta,0)$ and $\mathscr{B}(\theta,\psi) = (B\theta,0)$. In [1] it is proved that the solution $z(t) = z(t;u)$ is related to the solutions $x(t;\eta,\varphi,u)$ of

$$\dot{x}(t) = Lx_t + Bu(t)$$

$$(x(0),x_0) = (\eta,\varphi)$$

via $z(t;u) = (x(t;\eta,\varphi,u), x_t(\cdot;\eta,\varphi,u))$ for all $(\eta,\varphi) \in Z$. Therefore, $(\mathscr{P})$ and (P) are equivalent in the sense that u^* with associated trajectory $x^*(\cdot;\eta,\varphi,u^*)$ is a solution of (P) if and only if u^* with associated trajectory $t \to (x(t;\eta,\varphi,u^*), x_t(\cdot;\eta,\varphi,u^*)$ is a solution of $(\mathscr{P})$. The usual

solution of (P) for the ordinary differential equation case (or even in the case of functional differential equations) involves a Riccati differential equation of one appearance or another [3, pp.130; 4, 13]. In our approach the place of the Riccati differential equation will be taken by an appropriate Riccati integral equation, which we shall explain next. First, note that ($\mathscr{P}$) obviously has a unique solution u*; this is a consequence of the fact that $J_0(\eta,\varphi,\cdot):L^2 \to \mathbb{R}^+$ is a strictly convex, continuous and radially unbounded functional [9]. The optimal control u* can be found from the equation

$$J_0'(\eta,\varphi,u)v = 0 \qquad \text{for all} \quad v \in L^2(0,t*;\mathbb{R}^m), \qquad (2.3)$$

where $J_0'(\eta,\varphi,u)v$ denotes the Fréchet derivative at u in direction of v. After some calculation one finds that u* is given by

$$u*(t) = -((V_0)^{-1}W_0(\eta,\varphi))(t) \qquad \text{for almost every} \quad t \in [0,t*],$$

where $V_0 \in \mathscr{L}(L^2(0,t*;\mathbb{R}^m),L^2(0,t*;\mathbb{R}^m))$ and $W_0 \in \mathscr{L}(Z,L^2(0,t*;\mathbb{R}^m))$ are given by

$$V_0 = C + \mathscr{B}^*\mathscr{T}_0^*\mathscr{D}\,\mathscr{T}_0\mathscr{B} + \mathscr{B}^*\mathscr{T}_0^*\mathscr{G}\,\mathscr{T}_0\mathscr{B}$$

and

$$W_0 = \mathscr{B}^*\,\mathscr{T}_0^*\mathscr{D}T_0 + \mathscr{B}^*\mathscr{T}_0^*\mathscr{G}T(t*).$$

Here $\mathscr{T}_0 \in \mathscr{L}(L^2(0,t*;Z),L^2(0,t*;Z))$, $\mathscr{F}_0 \in \mathscr{L}(L^2(0,t*;Z),Z)$ and $T_0 \in \mathscr{L}(Z,L^2(0,t*;Z))$ are defined by

$$(\mathscr{T}_0\varphi)(s) = \int_0^s T(s-\eta)\varphi(\eta)d\eta \quad \text{for} \quad \varphi \in L^2(0,t*;Z)$$

$$(\mathscr{F}_0\varphi) = (\mathscr{T}_0\varphi)(t*), (T_0\tilde{z})(t) = T(t)\tilde{z} \quad \text{for} \quad \tilde{z} \in Z,$$

$$(\mathscr{T}_0^*\varphi)(t) = \int_0^{t*} T^*(\eta-t)\varphi(\eta)d\eta, (\mathscr{F}_0^*\tilde{z})(t) = T^*(t*-t)\tilde{z}.$$

Now for $s \in [0,t*]$ let us consider the optimal control problem $(P)_s$, which is defined by replacing '0' by 's' in (P). Then, similar to the way we found $u*$, one can derive the optimal control $(u*)_s$ of $(P)_s$ which is given by $(u*)_s(t) = ((V_s)^{-1}W_s(\eta,\varphi))(t)$ almost everywhere in $[s,t*]$ with $V_s \in \mathscr{L}(L^2(s,t*;\mathbb{R}^m), L^2(s,t;\mathbb{R}^m))$ and $W_s \in \mathscr{L}(Z,L^2(s,t*;\mathbb{R}^m))$ defined analogously to V_0 and W_0. For $(\eta,\varphi) \in Z$ the optimal trajectory corresponding to $(P)_s$ is therefore given by

$$S(t,s)(\eta,\varphi) = T(t-s)(\eta,\varphi) - \int_s^t T(t-\sigma)\mathscr{B}((V_s)^{-1}W_s(\eta,\varphi))(\sigma)d\sigma. \tag{2.4}$$

It can be shown [6] that $\{S(t,s)|0 \le s, t \le t*\}$ is an evolution operator on Z. Taking a slightly different route in the calculations after (2.3) we find that $u*$ is also given by

$$u*(t) = -C^{-1}\mathscr{B}^*\mathbb{K}(t)S(t)(\eta,\varphi) \quad \text{almost everywhere}, \tag{2.5}$$

with $\Pi(t)(\eta,\varphi) = T^*(t^*-t)\mathscr{F}S(t^*,t)(\eta,\varphi) + \int_t^{t^*} T^*(\sigma-t)\mathscr{D}S(\sigma,t)(\eta,\varphi)d\sigma$,   for

$t \in [0,t^*]$. Moreover $\Pi(t) \in \mathscr{L}(Z,Z)$ is found to be nonnegative, self-adjoint and is a solution of the following Riccati integral equation in Z;

$$\Pi(t)(\eta,\varphi) = T^*(t^*-t)\mathscr{F}T(t^*-t)(\eta,\varphi) +$$

$$+ \int_t^{t^*} T^*(\sigma-t)[\mathscr{D} - \Pi(\sigma)\mathscr{B}C^{-1}\mathscr{B}^*\Pi(\sigma)]T(\sigma-t)(\eta,\varphi)d\sigma \qquad (2.6)$$

for $t \in [0,t^*]$. Much of the above development is greatly facilitated by recent results [6] on the regulator problem in a general Hilbert space; the details for the case of delay-differential equations are given in [9]. Formal differentiation of (2.6) leads to

$$\frac{d}{dt}\Pi(t)z = -A^*\Pi(t)z - \Pi(t)Az + \Pi(t)\mathscr{B}C^{-1}\mathscr{B}^*\Pi(t)z - \mathscr{D}z \quad \text{for} \quad t \in [0,t^*]$$
$$\Pi(t^*) = \mathscr{F} \qquad (2.7)$$

where $z = (\eta,\varphi)$ and A and A^* are the infinitesimal generators of the semigroups $T(t)$ and $T^*(t)$, respectively. Of course, (2.7) resembles the well-known Riccati differential equation arising in optimal control theory for the linear ordinary differential equation $\dot{x}(t) = A_0 x(t)$, and many former investigations have been directed towards finding the "correct" form of the analogous Riccati equation for delay equations. This is not an easy task, since the operators A and A^* are differential operators with $\text{Dom}(A) = \{(\varphi(0),\varphi)|\varphi \text{ absolutely continuous and } \dot{\varphi} \in L^2\}$, $A(\varphi(0),\varphi) = (L\varphi,\dot{\varphi})$; for a characterization of A^* we refer to [14]. For one discrete delay

$(\ell = 1, A_{-1} \equiv 0)$ coupled systems of nonlinear partial differential equations of Riccati-type in Euclidean space have been derived in [5,10,12,13]. For the more general functional L it is shown in [4] that (2.7) can be properly studied in the densely injected triple of spaces $\mathscr{W}^{1,2} \subset Z \subset (\mathscr{W}^{1,2})^*$, where $\mathscr{W}^{1,2} = \{(\varphi(0),\varphi) \in Z | \ \varphi \text{ absolutely continuous and } \dot{\varphi} \in L^2\}$ and $(\mathscr{W}^{1,2})^*$ is the dual of $\mathscr{W}^{1,2}$. The Riccati integral equation (2.6) avoids many of the technical difficulties arising in the development of [4] and is also more appropriate for our next goal, the discussion of approximation schemes for (P). Finally, we mention that the equations derived in this section hold for more general right-hand sides than those included in L.

3. Approximation Schemes.

Recent results on the approximation of linear delay-differential equations allow discussing approximation methods for (P) within a general framework, including a number of examples of specific schemes. The objective is to approximate (P), or equivalently $(\mathscr{P})$, by a sequence of optimal control problems for ordinary differential equations in Euclidean space (linear regulator problems) for which computer algorithms are readily available. We shall make use of the following definitions and hypotheses. With $\{Z^N\}$, $N = 1,2,\ldots$ we denote a sequence of linear, finite dimensional subspaces of Z, the canonical projections from Z onto Z^N are denoted by P^N, and $Q_0: Z \to Z$ is the operator given by $Q_0(\eta,\varphi) = (\eta,0)$.

(H1) There exists a family of semigroups $T^N(t): Z \to Z$, for $N = 1,\ldots$ and $t \in [0,t^*]$, such that

(i) $\|T^N(t)\| \leq \bar{M} \exp(\bar{\omega}t)$ for some $\bar{M} > 0$, $\bar{\omega} \in R$ and $t \in [0,t*]$

(ii) $T^N(t)Z^N \subset Z^N$ for all $t \in [0,t*]$

(iii) $|T(t)z - T^N(t)z| \leq \bar{\rho}(N,z)$ for some real valued mapping $\bar{\rho}$.

Of course, in the examples that we have in mind $\bar{\rho}$ will tend to 0 at a certain rate, as N goes to ∞.

(H2) $\lim_{N \to \infty} P^N z = z$ for all $z \in Z$.

(H3) There exists a sequence of linear operators $Q^N : \mathbb{R}^n \to Z^N$, $N = 1,2,\ldots$ such that

(i) $\|Q^N - Q_0\|_{\mathscr{L}(\mathbb{R}^n;Z)} = \rho_Q(N)$ for some real valued mapping ρ_Q with $\lim_{N \to \infty} \rho_Q(N) = 0$,

(ii) $\|Q^N\|_{\mathscr{L}(\mathbb{R}^n,Z)} \leq q$ for some $q \geq 1$ independently of N.

Conditions (H1) and (H2) are satisfied if Z^N are chosen as the subspaces arising from averaging approximations [1] or spline functions [2]; in these cases $\lim_N \bar{\rho}(N,z) = 0$ for all $z \in Z$. (H3) will be of importance in the approximation of (2.2), which can be written as

$$z(t) = T(t)z + \int_0^t T(t-s)Q_0 Bu(s)ds, \text{ with } z = (\eta,\varphi).$$

The obvious approach to approximate (2.2) is to study $z^N(t) = T^N(t)P^N z + \int_0^t T^N(t-s)P^N Q_0 Bu(s)ds$ as N tends to ∞. In [9] it is shown that although this leads to satisfactory convergence results, rate of convergence results for the optimal control, payoff and trajectory cannot be expected in general. Thus we are led to study an approximate optimal control problem of a more elaborate form:

$$
(\mathscr{P}^{N,\mu}) \begin{cases}
\text{Minimize} \\[4pt]
J_0(P^{N+\mu}z,u) = \langle P^{N+\mu}\mathscr{F}P^{N+\mu}z^{N,\mu}(t*),z^{N,\mu}(t*)\rangle_Z + \\[6pt]
\quad + \int_0^{t*}(\langle P^{N+\mu}\mathscr{D}P^{N+\mu}z^{N,\mu}(t),z^{N,\mu}(t)\rangle_Z + (Cu(t),u(t))_{\mathbb{R}^m})dt \\[8pt]
\text{over} \quad u \in L^2(0,t*;\mathbb{R}^m), \quad \text{subject to} \\[6pt]
z^{N,\mu}(t) = T^{N,\mu}(t)P^{N+\mu}z + \int_0^t T^{N+\mu}(t-\sigma)Q^N Bu(\sigma)d\sigma, \quad \text{for} \quad t \in [0,t*].
\end{cases}
$$

It is well known that estimates on the approximation of a real valued function by piecewise polynomial functions depend on powers of the derivative of the approximated function [11]; this explains, roughly speaking, the problem for convergence rates for (2.2), which contains the "jump operator" Q_0. In $(\mathscr{P}^{N,\mu})$ we try to avoid this difficulty by introducing the family of operators Q^N. In certain cases Q^N will act as "smoothing operators" by mapping $\mathbb{R}^n$ in the subspace Z^N consisting of smoother functions than Z. In practice Q^N could be chosen to be $P^N Q_0$, or also as some interpolating operator appropriate for a previously chosen Z^N. It is quite clear that $(\mathscr{P}^{N,\mu})$ can be solved analytically in the same fashion as $(\mathscr{P})$ and we denote the optimal controls and optimal trajectories of $(\mathscr{P}^{N,\mu})$ by $\tilde{u}^{N,\mu}$ and $z^{N,\mu}(t;\tilde{u}^{N,\mu})$, respectively. In the convergence results below we shall use $\rho(N,z) = \bar{\rho}(N,z) + \bar{M} \exp(\bar{\omega}t*)|P^N z - z|$ for $N = 1,2,\dots$ and $z \in Z$, and $\tilde{\rho}(N+\mu,Q^N) = \max_{j=1,\dots,N} \bar{\rho}(N+\mu,[Q^N]_j)$, where $[Q^N]$ is the $n \times n$-matrix representation of $Q^N: \mathbb{R}^n \to Z^N$, whose column vectors $[Q^N]_j$ are elements of Z. Employing (2.4)–(2.6) one can derive the following convergence results:

Theorem 3.1. Assume that (H1)-(H3) hold. Then for each $\varepsilon > 0$ there exists an index N_0 and a constant K_1 such that

$$|\tilde{u}^{N_0,\mu} - u^*|_{L^2(0,t^*;\mathbb{R}^m)} \leq \varepsilon + K_1(\tilde{\rho}(N_0+\mu,Q^{N_0}) + \rho(N_0+\mu,z)).$$

In the above theorem $\varepsilon = K_2(\rho_Q(N) + \|P^{N+\mu}Q_0 - Q_0\|)$; the constants K_1 and K_2, and the index N_0 can be calculated explicitly in terms of the parameters of (P). Theorem 3.1 therefore establishes that the rate of convergence of the approximating optimal controls $\tilde{u}^{N,\mu}$ of $(\mathscr{P}^{N,\mu})$ into any specified ε-neighborhood of u^* is determined by the rate of convergence of the unperturbed semigroups $T^N(t)$ to $T(t)$ when acting on $[Q^N]_j$ and z. Of course, for $\mu = 0$ Theorem 3.1 asserts convergence of the optimal controls $\tilde{u}^{N,0}$ to u^* if only $\overline{\rho}(N,z)$ and $\overline{\rho}(N,[Q^N]_j)$ tend to 0 as N goes to infinity.

Theorem 3.2. Let (H1)-(H3) hold. Then

(i) $|z(t;u^*) - z^{N,\mu}(t;\tilde{u}^{N,\mu})| \leq K_3[\rho(N+\mu,z) + \tilde{\rho}(N+\mu,Q^N) + \rho_Q(N) + |u^* - u^{N,\mu}|_{L^2(0,t^*;\mathbb{R}^m)}]$

(ii) $|\langle \Pi(t)z - \Pi^{N,\mu}(t)P^{N+\mu}z, y \rangle_Z| \leq K_4\{\overline{\rho}(N+\mu,y)|z| + |y|[\rho(N+\mu,z) + |z|\tilde{\rho}(N+\mu,Q^N) +$

$+ |u^* - \tilde{u}^{N,\mu}|_{L^2(0,t^*;\mathbb{R}^m)} + |z|\,\rho_Q(N) + |z|\,\|Q_0 - P^{N+\mu}Q_0\|]\}$,

for all y and z in Z; the constants K_3 and K_4 can be calculated explicitly and K_4 does not depend on y or z.

It is simple to see that Theorems 3.1 and 3.2 imply an estimate on the rate of convergence of $J_0(P^{N+\mu}z, \tilde{u}^{N,\mu})$ to $J_0(z,u*)$, see [9].

Finally, we also have the following result on convergence of $\tilde{u}^{N,0}$ in the supremum norm:

Theorem 3.3. If (H1)-(H3) hold and if $\lim_{N\to\infty} \bar{\rho}(N,\tilde{z}) = 0$ for all $\tilde{z} \in Z$, then
$$\lim_{N\to\infty} \sup_{t \in [0,t*]} |\tilde{u}^{N,C}(t) - u*(t)| = 0.$$

To use the above theorems for actual computations we choose bases $\langle \beta_1^N, \ldots, \beta_{k_N}^N \rangle$ of the k_N-dimensional spaces Z^N. The matrix representations of the infinitesimal generators $T^N(t)$ and $T^N(t)*$ with respect to the chosen basis in Z^N are denoted by A^N and A_*^N, respectively. Similarly, we use $[F^N]$ and $[D^N]$ for the matrix representation of $P^N \mathcal{F} P^N$ and $P^N \mathcal{D} P^N$. For any pair of integers N and μ the coordinate vector of the representation of $z^{N,\mu}(t; u^{N,\mu})$ in $Z^{N+\mu}$ is denoted by $w^{N,\mu}(t)$ and $\pi^{N,\mu}(t)$ stands for the matrix representation of $\Pi^{N,\mu}(t)$. Due to the fact that Q^N maps $\mathbb{R}^n$ into Z^N we need an additional, but not restrictive, hypothesis:

(H4) For each $N = 1,2,\ldots$ there exists an integer $\mu > 0$ such that
$$Z^N \subset Z^{N+\mu}.$$

For $N = 1,2,\ldots$ we assume that $\mu = \mu(N)$ is chosen according to (H4) and we let $[Q^N]_{N+\mu}$ and $[(Q^N)^*]_{N+\mu}$ denote the matrix representations of $Q^N: \mathbb{R}^n \to Z^{N+\mu}$ and $(Q^N)^*: Z^{N+\mu} \to \mathbb{R}^n$ with respect to the bases $\langle \beta_1^N, \ldots, \beta_{k_N}^N \rangle$ of Z^N. We are now prepared to present the solution of $(\mathcal{P}^{N,\mu})$

in terms of an ordinary differential equation in $\mathbb{R}^{n \times k_{N+\mu}}$, coupled with a matrix-Riccati differential equation and a feedback control law.

$$
\left.
\begin{aligned}
\dot{w}^{N,\mu}(t) &= A^{N+\mu} w^{N,\mu}(t) + [Q^N]_{N+\mu} B(t) u^{N,\mu}(t), \quad \text{for} \quad 0 \leq t \leq t*, \\
w^{N,\mu}(0) &= w_0^{N,\mu} \\
\frac{d}{dt} \pi^{N,\mu}(t) &= -A_*^{N+\mu} \pi^{N,\mu}(t) - \pi^{N,\mu}(t) A^{N+\mu} - [D^{N+\mu}] + \\
&\quad + \pi^{N,\mu}(t) [Q^N]_{N+\mu} B(t) C(t) B*(t) [(Q^N)*]_{N+\mu} \pi^{N,\mu}(t) \\
\pi^{N,\mu}(t*) &= [F^{N+\mu}] \\
u^{N,\mu}(t) &= -C^{-1} B* [(Q^N)^*]_{N+\mu} \pi^{N,\mu}(t) w^{N,\mu}(t)
\end{aligned}
\right\} \quad (3.1)
$$

where $w_0^{N,\mu}$ is given by $P^{N+\mu} z = \sum_{i=1}^{k_{N+\mu}} \beta_i^{N+\mu} (w_0^{N,\mu})_i$. The system (3.1) is a consequence of (2.4)-(2.6), where we replaced all operators by their respective approximating operators and z by $P^{N+\mu} z$; since $T^{N+\mu} z^{N,\mu} \subset Z^{N+\mu}$ and by (H3) and (H4) these equations are then actually equations in the finite dimensional space $Z^{N+\mu}$.

If according to Theorem 3.1 we want to determine N_0 for a specified ϵ and subsequently take the limit as μ goes to infinity, then (H4) has to be replaced by

(H4*)   for each $N = 1,2,\ldots$ there exists a nontrivial sequence $\mu_k = \mu_k(N)$ such that $Z^N \subset Z^{N+\mu_k(N)}$, for $k = 1,2,\ldots$

For subspaces of averaging approximations and also for subspaces of spline functions many of the details that are necessary to determine (3.1) as well as estimates on the rate of convergence can be found in [9].

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