AD-A088 089 NAVAL RESEARCH LAB WASHINGTON DC F/G 8/5
ELASTIC DEFORMATIONS OF A ROTATING SPHEROIDAL EARTH DUE TO SURF--ETC(U)
JUN 80 P LANZANO, J C DALEY
UNCLASSIFIED NRL-8410 NL

1 OF 1
AD
A088089

END
DATE
FILMED
9-80
DTIC

NRL Report 8410

Elastic Deformations of a Rotating Spheroidal Earth Due to Surface Loads

PAOLO LANZANO AND JOHN C. DALEY

Space Systems Division

June 18, 1980

NAVAL RESEARCH LABORATORY
Washington, D.C.

12 139

16 RR03303

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: NRL Report 8410	2. GOVT ACCESSION NO.: AD-A088089
3. RECIPIENT'S CATALOG NUMBER: 9	
4. TITLE (and Subtitle): 6 ELASTIC DEFORMATIONS OF A ROTATING SPHEROIDAL EARTH DUE TO SURFACE LOADS	5. TYPE OF REPORT & PERIOD COVERED: Final report ~~on one phase of a continuing NRL problem.~~
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): 10 Paolo Lanzano [redacted] John C. Daley	8. CONTRACT OR GRANT NUMBER(s): 17
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Research Laboratory, Washington, DC 20375	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61153N-33; RR033 03 41; 79-0739-0-0
11. CONTROLLING OFFICE NAME AND ADDRESS: Department of the Navy, Office of Naval Research, Arlington, VA 22217	12. REPORT DATE: 11 18 Jun[redacted]80
	13. NUMBER OF PAGES: 38
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 14 NRL-8410	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

DTIC
ELECTE
AUG 20 1980
S D
B

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

- Earth density model
- Spheroidal deformations
- Toroidal deformations
- Load tide
- Geoid
- Navier-Stokes equations
- Clairaut equation
- Discontinuous solutions
- Legendre polynomials

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We have developed a differential system of equations for representing the deformations of a rotating elastic Earth which has a spheroidal shape and which is hydrostatically prestressed. Our system is meant to represent a more sophisticated model for the Earth's load tide than the ones hereto studied because of the inclusion of rotational terms. In this study, we have assumed that the rotational axis is fixed with respect to the spheroid and that the rotational velocity is constant. We have reached a linearized version of the Navier-Stokes equations consisting of six equations which simultaneously

(Continues)

251950 JW

20. Abstract (Continued)

relate three orders of harmonics. We have a boundary-value problem whose solutions must be regular functions of the radial distance in the neighborhood of the center of the configuration and which must also satisfy three other conditions at the free surface varying according to loading conditions.

Numerical integration of this differential system requires the knowledge of an Earth model consisting of a density profile and of the elastic parameters as functions of the radial distance. Because of the vanishing of the rigidity, the differential system valid for the liquid outer core shrinks to a system of only four equations; discontinuity of some of the variables are to be entertained at the interfaces between the liquid outer core and the solid inner core and/or the solid mantle. We briefly discuss the proposed method of numerical solution.

CONTENTS

ELASTIC DEFORMATIONS OF A ROTATING SPHEROIDAL EARTH DUE TO SURFACE LOADS

INTRODUCTION

The study of solid Earth dynamics and more particularly of the elastic deformations of the Earth has been in the foreground of investigation for a couple of decades and has important ramifications into geodesy, geophysics, and astronomy. Before examining some of the major problems related to an elastic Earth, it is appropriate to furnish a few definitions and basic facts.

The Earth's gravity field is due to two forces: the gravitational attraction exerted by the whole Earth at each point according to Newton's law and the centrifugal force due to the Earth's rotation. The magnitude of the vector field is the intensity of gravity, and its direction provides the direction of the vertical at that point.

Next in order of importance is the tidal potential generated by the Moon and Sun (also referred to as the luni-solar potential). This potential varies with time because of the relative motion of these two bodies with respect to the Earth. In particular, the true orbit of the Moon is quite elaborate and differs considerably from a Keplerian ellipse. Various perturbing terms must be taken into account; foremost among them is the evection term.

The most important consequence of these tidal perturbations is the ocean tide, whereby the free surface of the ocean conforms to the changing field and remains an equipotential surface. Another consequence of this tidal potential is the Earth's tide or the elastic deformation of the Earth, which was first considered by Darwin in 1883 [1] and ascertained to be about a third of the ocean tide.

Our knowledge of the interior of the Earth is provided by seismic studies, which studies have been responsible for ascertaining the velocity of propagation of the seismic waves and for measuring the spheroidal and toroidal oscillations experienced by the Earth subsequent to a seismic event. From this knowledge one can ultimately determine a rheological profile of the Earth comprising the variation of its density and of the Lamé elastic parameters along various layers. Love introduced in 1909 [2] special parameters which bear his name and which are related to the elastic deformation of a homogeneous spherical Earth. Like the moments of inertia, these Love numbers represent intrinsic properties of the planet and provide estimates for the maximum rigidity of the solid inner core, which seismology cannot furnish, since no transverse wave can penetrate the liquid outer core.

Henceforth in this report, by Earth tides we shall signify the elastic deformations of the Earth due to the luni-solar potential. Let us now examine a number of phenomena which are related to and can be better explained through the concept of an elastic Earth:

Manuscript submitted March 31, 1980.

- Earth tides produce a periodic variation in the direction of the vertical which is reflected in the variation of latitude for an astronomical observatory and for satellite tracking stations. Thus Earth tides have implications to observational astronomy and the orbit determination for artificial satellites.

- Long-period Earth tides perturb the Earth's axis having the largest moment of inertia. To conserve angular momentum, the Earth reacts by changing its rotational rate. First predicted by Jeffreys, this phenomenon was later confirmed by the use of atomic standards. The present treatment of the problem is still based on the assumption of a rigid Earth [3].

- The dissipation of energy due to the tidal motion contributes to the secular retardation of the Earth's rotation.

- The luni-solar tidal force acts on the spheroidal Earth by exerting a torque about an equatorial axis. The rotating Earth reacts as a gyroscope, giving rise to an angular motion which can be decomposed into a precession of the Earth's axis about the normal to the ecliptic augmented by a minor oscillation about the instantaneous axis known as a nutational motion. The most recent treatment of this phenomenon is due to Kinoshita [4] and also assumes a rigid Earth.

- The presence of the liquid outer core within the Earth produces a resonance for oncoming waves having a tesseral harmonic character. Poincaré [5] first demonstrated that a fluid ellipsoid contained within a rotating mantle exhibits a free nutational mode. The most advanced theory, formulated by Molodenskii [6], ascertains this resonance, but it still is far from constituting a final body of results on this subject (see Melchior [7]).

- Amplitudes of the Earth tides reach as much as 0.3 to 0.4 m and cannot be neglected with regard to lunar and satellite laser measurements, which have accuracies good to few centimeters.

- The tidal perturbations of the Earth's potential on a typical satellite orbit can amount to 50 m and must be taken into account.

- A better determination of the surface strains produced by the Earth's tides will increase the accuracy that can be reached in long-base interferometer measurements as well as by other instrumentation which depends on Earth-fixed length standards.

- Cubic expansion of the Earth's crust will produce oscillations in water wells and can have a bearing in ascertaining the storage capacity and porosity of an aquifer. In general, the tidal variation of underwater fluids can play a factor in tectonic processes and in their diagnosis.

Considering the variety of problems where the influence of the tidal potential plays a significant role, it is appropriate that more precise and sophisticated methods be developed to ascertain both the ocean and Earth tides. The present knowledge of the global ocean tides is based on the numerical integration of the Laplace tidal equations. However, the simplifying assumptions of a rigid Earth and of a homogeneous incompressible ocean in hydrostatic equilibrium which are used to solve those equations seem to vitiate the results and fail to give close agreement with tidal observations at midocean islands. The problem should be formulated as a boundary-value problem by considering the continental boundary of the ocean basin and considering the ocean bottom with friction losses. Furthermore, the attraction of the ocean mass should also be taken into account, and this will transform the Laplace tidal equations into a system of integro-differential equations.

The Earth tides are due primarily to two effects. One is the yielding of the elastic Earth to the gravitational attraction of the Moon and the Sun. This direct effect goes by the name of body tide and is rather well understood, because astronomical observations have furnished precise information on the forcing function. There is also an indirect effect due to the yielding of the Earth to the load of the harmonically varying ocean tide upon the continents; this is also known as load tide. The corresponding forcing function is, however, not well known, not only because of our scanty information of the ocean tides but also because we still do not know the influence played by the structural heterogeneities in the Earth's crust and upper mantle. However, it is evident that load tides and ocean tides are interrelated phenomena.

The direct and indirect Earth tides are both ultimately derived from the same astronomical input; consequently they will have the same frequencies. An eventual lag of these tides with respect to the acting potential should give information about the viscosity of the Earth. However, the spatial behavior of these two tides is quite different: the body tide varies smoothly over the Earth, whereas the load tide presents irregularities because of the discontinuity in the forcing function at the coastline and also because the ocean tide exhibits a circulation motion around the amphidromes.

Crustal loading constitutes a phenomenon whose frequencies are intermediate between those of seismic-wave transmission, which is an elastic event, and tectonic deformations, which have an anelastic character. Thus one might expect that the study of loading deformations are useful in determining the threshold of anelasticity for the crust and upper mantle layers.

Ocean load effects are responsible in part for the variation of the vertical component of the gravity field (also referred to as tidal gravity), for surface strains (strain tide), and for the deflection of the vertical (tilt tide). As a consequence, if for a certain region the Earth model obtainable from seismic evidence is better known than the cotidal charts of the ocean tide, then one could expect [8] that accurate tidal gravity measurements on adjacent lands (which now can reach an accuracy of 1 microgal) can be used to improve our knowledge of the ocean tide. Conversely, should the reverse be true, the ocean tide can provide some information on the elastic properties of the crust.

The purpose of this report and possibly of future reports and papers is to provide a comprehensive study of the load tides for a rotating spheroidal Earth. The basic work which constitutes our theoretical background goes back to Longman [9,10] and Farrell [11]. These two authors have provided the basic equations and boundary conditions for the load tide of an elastic nonrotating spherical Earth. More recently, Smith [12] obtained a model for the elastic deformations of a rotating spheroidal Earth. His formulation is very general and mathematically correct; however, it was found to be rather cumbersome to attack from a computational point of view due to his use of a special set of harmonic functions.

For future geophysical developments, the effect of the Earth's rotation on the Earth tide should be studied and ascertained from a numerical point of view. We have therefore approached here the same problem of determining the elastic deformations of a rotating spheroidal Earth and have been able to provide a much simpler analytic representation of such deformations. Our success is primarily due to our having been able to write the equations pertaining to the nth-order harmonic based on a simple decomposition of the product of Legendre polynomials and their derivatives into spheroidal components. This simple representation of the governing equations augurs well for their numerical solution. We present in this report the theoretical work that leads to the representation of the equations valid for the deformation of a rotating spheroidal Earth and also give an insight on future numerical work. The full expansion of this numerical endeavor will, however, constitute material for a future report.

LINEARIZED NAVIER-STOKES EQUATIONS

With respect to an inertial frame $(O; x_1, x_2, x_3)$, with the origin O coinciding with the centroid of the planet, the equations representing the displacement field $\vec{u}$ from the initial reference state can be written

$$\rho \frac{d^2 u_i}{dt^2} = \rho \frac{\partial V}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{1}$$

where the summation convention applies to repeated indices. Here the quantities u_i are the components of the displacement field $\vec{u}$ and the quantity ρ is the mass density and is the sum of ρ_0, the mass density of the reference state, and ρ_1, the change in mass density due to the displacement field, with ρ_1 and the components u_i being related by the continuity equation

$$\rho_1 = -\frac{\partial}{\partial x_i}(\rho_0 u_i) . \tag{2}$$

Further in Eq. (1), V is the total potential and is the sum of V_0, the potential of the reference state, and V_1, the potential due to the deformation. V_0 consists of the gravitational potential and the rotational potential; it satisfies the Poisson equation

$$\nabla^2 V_0 = -4\pi G \rho_0 + 2\omega^2 , \tag{3}$$

where G is the gravitational constant and ω is the Earth's angular velocity. V_1 satisfies also a Poisson equation

$$\nabla^2 V_1 = -4\pi G \rho_1 . \tag{4}$$

The components of gravity are then

$$g_{0,i} = \frac{\partial V_0}{\partial x_i}, \quad g_{1,i} = \frac{\partial V_1}{\partial x_i} . \tag{5}$$

The gravitational attraction is given by

$$\gamma_0(r) = \frac{4\pi G}{r^2} \int_0^r \rho_0(r) r^2 \, dr = \frac{4\pi G}{3} r \bar{\rho}_0(r) , \tag{6}$$

where r is the radial distance measured from the centroid and

$$\bar{\rho}_0(r) = \frac{3}{r^3} \int_0^r \rho_0(r) r^2 \, dr \tag{7}$$

is the mean density. The derivative of the gravitational attraction with respect to r satisfies the relation

$$\gamma_0'(r) = -\frac{2}{r} \gamma_0 + 4\pi G \rho_0 \, . \tag{8}$$

Finally in Eq. (1), σ_{ij} are the components of the total stress field. This total field consists of two fields. The first is the initial stress field, which we assume to be hydrostatic equilibrium and whose components T_{ij} satisfy the conditions

$$T_{ij} = -p_0 \delta_{ij} \, , \qquad \frac{\partial p_0}{\partial x_i} = \rho_0 g_{0,i} \, , \tag{9}$$

where p_0 is the hydrostatic pressure and the quantities δ_{ij} are the Kronecker deltas. The second is the stress field valid for a perfectly elastic and isotropic medium, whose components τ_{ij} are related to the Lamé elastic parameters λ and μ by the relation

$$\tau_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{10}$$

Since we are assuming infinitesimal elastic deformations, the above relationship applies to the coordinates that the points of the medium had before the occurrence of the deformation. The same assumption, however, cannot be made for the initial stress field. As a consequence, the total stress at the undeformed points of the medium must be written as

$$\sigma_{ij} = T_{ij} - \frac{\partial T_{ij}}{\partial x_k} u_k + \tau_{ij} \, . \tag{11}$$

By differentiating Eq. (9), we can easily get

$$\frac{\partial T_{ij}}{\partial x_k} = -\frac{\partial p_0}{\partial x_k} \delta_{ij} = -\rho_0 g_{0,k} \delta_{ij} \, , \tag{12}$$

and this can be used to rewrite the fundamental equations of the displacement (Eq. (1)) as

$$\rho \frac{d^2 u_i}{dt^2} = \rho_0 g_{1,i} + \rho_1 g_{0,i} + \frac{\partial}{\partial x_i} (\rho_0 g_{0,k} u_k) + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{13}$$

where we have neglected the product $\rho_1 g_{1,i}$ because it is of the second order in the displacements.

In a previous work [13], we gave details on how to write the previous equation in vectorial form, and we obtained

$$\rho \frac{d^2\vec{u}}{dt^2} = \rho_0\vec{g}_1 + \rho_1\vec{g}_0 + \nabla(\rho_0\vec{u}\cdot\vec{g}_0) + \vec{E}(\lambda, \mu), \tag{14}$$

where

$$\vec{E}(\lambda, \mu) \equiv (\lambda + 2\mu)\nabla(\nabla\cdot\vec{u}) - \mu\nabla\times\nabla\times\vec{u} + (\nabla\lambda + \nabla\mu)(\nabla\cdot\vec{u})$$
$$+\nabla(\vec{u}\cdot\nabla\mu) + \nabla\times(\vec{u}\times\nabla\mu) - \vec{u}\nabla^2\mu \tag{15}$$

stands for the agglomerate of terms due to the elastic parameters λ and μ.

The first three terms on the right-hand side of Eq. (14) represent respectively the action of the perturbed gravity field on the initial density profile, the action of the initial gravity field on the deformed density, and the contribution due to the work performed by the elastic displacement vector against the initial hydrostatic field. Equation (14) is essentially a linearized version of the Navier-Stokes equation when allowance is made for the variation of the material's elastic parameters and quadratic terms in the displacements are dropped.

The acceleration of $\vec{u}$ with respect to the inertial frame is given by

$$\frac{d^2\vec{u}}{dt^2} = \frac{\partial^2\vec{u}}{\partial t^2} + 2\vec{\omega}\times\frac{\partial\vec{u}}{\partial t} + \frac{d\vec{\omega}}{dt}\times\vec{u} + (\vec{\omega}\cdot\vec{u})\vec{\omega} - \omega^2\vec{u}, \tag{16}$$

where the partial-derivative symbol refers to the variation of a vector with respect to the rotating frame.

In this report, we are interested in ascertaining permanent deformations of a rotating Earth, so that the first and second time derivatives of $\vec{u}$ with respect to the rotating frame must vanish. Also, we are assuming a constant angular rotation and no oscillation for the position of the rotational axis, which is equivalent to saying that the time derivative of $\vec{\omega}$ is also vanishing. We are therefore left with the last two terms of Eq. (16), which pertain to the centrifugal acceleration.

Reverting to Eq. (14), we shall write

$$\rho_0[(\vec{\omega}\cdot\vec{u})\vec{\omega} - \omega^2\vec{u}] = \rho_0\vec{g}_1 + \rho_1\vec{g}_0 + \nabla(\rho_0\vec{u}\cdot\vec{g}_0) + \vec{E}(\lambda, \mu), \tag{17}$$

where the term $\rho_1\vec{u}$ has been neglected, being of the second order in the displacements. This is the fundamental equation of motion, and we shall seek solutions to this equation in the form of spheroidal and toroidal deformations.

In the sequel we shall assume rotational symmetry, whereby the displacement vector $\vec{u}$ depends only on the radial distance r and the colatitude θ of the position. The spherical harmonics $Y_n(\theta, \phi)$ will then reduce to the Legendre polynomials $P_n(\cos\theta)$. We use spherical coordinates (r, θ, ϕ) with the origin at the centroid, θ being the colatitude and ϕ the longitude. Accordingly, the rotational vector is representable as

$$\vec{\omega} \equiv \{\omega \cos\theta \,; \quad -\omega \sin\theta \,; \quad 0\} \tag{18}$$

and the unperturbed gravity vector as

$$\vec{g}_0 \equiv \nabla V_0(r, \theta) \equiv \left\{\frac{\partial V_0}{\partial r} \,; \quad \frac{1}{r}\frac{\partial V_0}{\partial \theta} \,; \quad 0\right\} . \tag{19}$$

SPHEROIDAL DEFORMATIONS

The spheroidal displacement vector $\vec{u}$ has components of the form

$$\begin{aligned} u &= \sum_{n=0}^{\infty} U_n(r) P_n(\cos\theta) \,, \\ v &= \sum_{n=1}^{\infty} V_n(r) \frac{\partial P_n}{\partial \theta} \,, \\ w &= 0 \end{aligned} \tag{20}$$

and has the property that the radial component of its curl vanishes. We shall determine the two sequences of unknown functions $U_n(r)$ and $V_n(r)$ by imposing the conditions that the two series in Eq. (20) satisfy the fundamental equations of motion: Eq. (17).

Another quantity of physical significance related to our problem is the deformed potential V_1, which we assume to be of the form

$$V_1(r, \theta) \equiv \sum_{n=0}^{\infty} R_n(r) P_n(\cos\theta) \,. \tag{21}$$

This introduces another sequence $R_n(r)$ of unknown functions. It follows that $\vec{g}_1$, being the gradient of V_1, is given by

$$\vec{g}_1 \equiv \nabla V_1 \equiv \left\{ R_n' P_n \,; \quad \frac{R_n}{r}\frac{\partial P_n}{\partial \theta} \,; \quad 0 \right\} . \tag{22}$$

Henceforth primes shall denote derivatives with respect to the radial distance r.

From Eq. (20), we obtain that the dilatation Δ can be written as

$$\Delta = \nabla \cdot \vec{u} = \sum_{n=0}^{\infty} X_n(r) P_n(\cos\theta) \,, \tag{23}$$

where

$$X_n = U_n' + \frac{2}{r} U_n - n(n+1) \frac{V_n}{r}, \qquad (24$$

and that the perturbed density, as given by the continuity equation, is

$$\rho_1 = -\nabla \cdot (\rho_0 \vec{u}) = -\sum_{n=0}^{\infty} (U_n \rho_0' + X_n \rho_0) P_n(\cos\theta). \qquad (25$$

The last of the inertia terms appearing in Eq. (17) can be developed as

$$\nabla(\rho_0 \vec{u} \cdot \vec{g}_0) = (\vec{u} \cdot \vec{g}_0) \nabla \rho_0 + \rho_0 \nabla(\vec{u} \cdot \vec{g}_0), \qquad (26)$$

where ρ_0 is a given function of r. The above expression implies the knowledge of the partial derivatives of V_0 up to and including the second order. V_0 refers to the initial hydrostatic regime. In Appendix A we have expressed these derivatives in terms of the rotational deformation obtainable from the Clairaut equation.

To complete the analytical derivation of the equations of motion, one first must eliminate the trigonometric terms via the Legendre polynomials. The required identities simply reduce to

$$\begin{aligned} \sin^2\theta &= \frac{2}{3}(1 - P_2), \\ \cos^2\theta &= \frac{1}{3}(1 + 2P_2), \\ \sin\theta\cos\theta &= -\frac{1}{3}\frac{\partial P_2}{\partial\theta}. \end{aligned} \qquad (27)$$

Then one must express the spheroidal components of the products $P_2 P_n$, $P_2 \partial P_n/\partial\theta$, $P_n \partial P_2/\partial\theta$, and $(\partial P_2/\partial\theta) \cdot (\partial P_n/\partial\theta)$ for a generic value of n. This is equivalent to writing these products as linear combinations of P_n and $\partial P_n/\partial\theta$.

This operation has been accomplished in Appendix B by elaborating on certain well-known relationships valid for the Legendre polynomials. Let us also note that the second derivatives of the P's can be eliminated by means of the fundamental equation of definition

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial P_n}{\partial\theta}\right) = -n(n+1)\sin\theta\, P_n. \qquad (28)$$

The elastic terms $\vec{E}(\lambda, \mu)$ which are given by Eq. (15) have also been expanded and found to agree with the results of Alterman et al. [14] and of Longman [9]. We therefore shall refrain from discussing their derivation in this report.

The radial component of Eq. (17) was found to be as follows:

$$\gamma_0\rho_0(4 - 3D_0)U_n - n(n+1)\gamma_0\rho_0 V_n + r\rho_0 R_n' + r(\lambda X_n + 2\mu U_n')'$$
$$+\frac{\mu}{r}[4rU_n' - 4U_n + n(n+1)(3V_n - rV_n' - U_n)]$$
$$+r\rho_0\omega^2(A_1U_{n-2} + A_2U_n + A_3U_{n+2} + A_4V_{n-2} + A_5V_n + A_6V_{n+2}$$
$$+A_7rV_{n-2}' + A_8rV_n' + A_9rV_{n+2}') = 0 . \qquad (29)$$

The two transversal components of the same equation give identical contributions, which amount to

$$-\gamma_0\rho_0U_n + \rho_0R_n + \lambda X_n + r\left[\mu\left(V_n' - \frac{V_n}{r} + \frac{U_n}{r}\right)\right]'$$
$$+\frac{\mu}{r}[5U_n - V_n - 2n(n+1)V_n + 3rV_n']$$
$$+r\rho_0\omega^2(B_1U_{n-2} + B_2U_n + B_3U_{n+2} + B_4V_{n-2}$$
$$+B_5V_n + B_6V_{n+2} + B_7rU_{n-2}' + B_8rU_n' + B_9rU_{n+2}') = 0 . \qquad (30)$$

Here $\gamma_0(r)$ is the gravitational attraction given by Eq. (6) and $D_0 = \rho_0/\bar{\rho}_0$ is the ratio between density and mean density as defined by Eq. (7). The quantities A_n and B_n ($n = 1, 2, \ldots, 9$) are two sets of known functions which depend on the radial distance r, the density ρ_0, the hydrostatic deformation f_{21}, and their derivatives ρ_0' and f_{21}'. They are related to the quantities A^*, B^*, and C^* obtained in Appendix A. It is premature at this stage to explicitly write down these 18 A and B functions, simply because for purposes of numerical evaluation it is more expedient to deal with particular linear combinations of the A and B functions, and these will be furnished later by Tables 1 and 2.

We need an extra equation to complete the determination of the three sets U_n, V_n, and R_n of unknown functions; this is provided by the Poisson equation (Eq. (4)), which explicitly becomes

$$r^2R_n'' + 2rR_n' - n(n+1)R_n = 4\pi Gr^2(\rho_0'U_n + \rho_0X_n) . \qquad (31)$$

The terms appearing within Eqs. (29) and (30) and which are due to the rotation ω relate to the coefficients of the harmonics of order $n - 2$, n, and $n + 2$. Thus we have an infinite set of equations relating three harmonics.

Related to these three sets of unknowns are the Love numbers, which are nondimensional quantities defined as

$$1 + k_n(r) = R_n(r)/r\gamma_0(r) ,$$
$$h_n(r) = U_n(r)/r , \tag{32}$$
$$\ell_n(r) = V_n(r)/r .$$

They depend on the order n and the radial distance r.

REDUCTION TO NORMAL FORM

The numerical evaluation of the second-order differential Eqs. (29), (30), and (31) requires that they be reduced to their normal form, that is, to the first-order system $y' = F(x, y)$, which is amenable to programming on a computer. This reduction can be achieved by introducing in addition to the original set of variables

$$\begin{aligned} &U_n(r) \quad \text{(radial displacement)}, \\ &V_n(r) \quad \text{(transversal displacement)}, \\ &R_n(r) \quad \text{(change in gravitational potential)} \end{aligned} \tag{33}$$

a new set of three variables. The first new variable is

$$\begin{aligned} E_n(r) &= \lambda X_n + 2\mu U_n' \\ &= (\lambda + 2\mu) U_n + \frac{2\lambda}{r} U_n - n(n+1) \frac{\lambda}{r} V_n , \end{aligned} \tag{34}$$

which appears in the expansion of the radial stress

$$\tau_{rr} = \sum_{n=0}^{\infty} E_n(r) P_n(\cos\theta) . \tag{35}$$

The second new variable is

$$F_n(r) = \mu \left(V_n' - \frac{1}{r} V_n + \frac{1}{r} U_n \right), \tag{36}$$

which is connected with the transversal stress

$$\tau_{r\theta} = \sum_{n=1}^{\infty} F_n(r) \frac{\partial P_n}{\partial \theta} . \tag{37}$$

The third new variable is

$$H_n(r) = R_n' - 4\pi G \rho_0 U_n , \tag{38}$$

which represents the change in gravitational flux.

The linearized Navier-Stokes equations representing the spheroidal deformations can now be written down as the following differential system of six equations:

$$
\begin{aligned}
U_n' &= -\frac{2\lambda}{\lambda + 2\mu}\frac{1}{r}U_n + n(n+1)\frac{\lambda}{\lambda + 2\mu}\frac{1}{r}V_n + \frac{1}{\lambda + 2\mu}E_n, \\
V_n' &= -\frac{1}{r}U_n + \frac{1}{r}V_n + \frac{1}{\mu}F_n, \\
R_n' &= 4\pi G\rho_0 U_n + H_n, \\
H_n' &= -4\pi G\rho_0\frac{n(n+1)}{r}V_n + \frac{n(n+1)}{r^2}R_n - \frac{2}{r}H_n, \\
E_n' &= 4\left[\frac{\gamma_0\rho_0}{r} - \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right)\frac{\mu}{r^2}\right]U_n \\
&\quad + n(n+1)\left[\frac{\gamma_0\rho_0}{r} - \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right)\frac{2\mu}{r^2}\right]V_n \\
&\quad - \frac{4\mu}{\lambda + 2\mu}\frac{1}{r}E_n + \frac{n(n+1)}{r}F_n - \rho_0 H_n \\
&\quad + \rho_0\omega^2[C_1U_{n-2} + C_2U_n + C_3U_{n+2} + C_4V_{n-2} + C_5V_n + C_6V_{n+2} \\
&\quad + \frac{r}{\mu}(C_7F_{n-2} + C_8F_n + C_9F_{n+2})], \\
F_n' &= \left[\frac{\gamma_0\rho_0}{r} - \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right)\frac{2\mu}{r^2}\right]U_n \\
&\quad + \left[\frac{(2n^2 + 2n - 1)\lambda + 2(n^2 + n - 1)\mu}{\lambda + 2\mu}\right]\frac{2\mu}{r^2}V_n \\
&\quad - \frac{\rho_0}{r}R_n - \frac{\lambda}{\lambda + 2\mu}\frac{1}{r}E_n - \frac{3}{r}F_n \\
&\quad + \rho_0\omega^2[D_1U_{n-2} + D_2U_n + D_3U_{n+2} + D_4V_{n-2} + D_5V_n + D_6V_{n+2} \\
&\quad + \frac{r}{\lambda + 2\mu}(D_7E_{n-2} + D_8E_n + D_9E_{n+2})].
\end{aligned}
\tag{39}
$$

Here primes denote as usual derivatives with respect to the radial distance r. The two sets of dimensionless functions C and D are explicitly given in Tables 1 and 2 respectively in

Table 1 — The C Functions

$$C_1(n, r) = \frac{(n-1)n}{(2n-3)(2n-1)} \left[1 - (n-2)A^* + B^* + \frac{1}{2} C^*\right]$$

$$C_2(n, r) = \frac{1}{3} \frac{n(n+1)}{(2n-1)(2n+3)} (2 + 3A^* + 2B^* + C^*)$$

$$C_3(n, r) = \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \left[1 + (n+3)A^* + B^* + \frac{1}{2} C^*\right]$$

$$C_4(n, r) = -\frac{(n-2)(n-1)n}{(2n-3)(2n-1)} \left[1 - \frac{r\rho_0'}{\rho_0} A^* + \frac{n-3}{2} B^*\right]$$

$$C_5(n, r) = -n(n+1) \left\{\frac{2}{3} + \frac{1}{(2n-1)(2n+3)} \left[-1 + \frac{r\rho_0'}{\rho_0} A^* + \frac{n^2+n+3}{3} B^*\right]\right\}$$

$$C_6(n, r) = \frac{(n+1)(n+2)(n+3)}{(2n+3)(2n+5)} \left(1 - \frac{r\rho_0'}{\rho_0} A^* + \frac{n+4}{2} B^*\right)$$

$$C_7(n, r) = \frac{(n-2)(n-1)n}{(2n-3)(2n-1)} A^*$$

$$C_8(n, r) = -\frac{n(n+1)}{(2n-1)(2n+3)} A^*$$

$$C_9(n, r) = -\frac{(n+1)(n+2)(n+3)}{(2n+3)(2n+5)} A^* .$$

Table 2 — The D Functions

$$D_1(n, r) = \frac{(n-1)(1-\delta_{0n})}{(2n-3)(2n-1)} \left[1 + \left(\frac{r\rho_0'}{\rho_0} + \frac{4\mu}{\lambda + 2\mu}\right) A^* - \frac{n}{2} B^*\right]$$

$$D_2(n, r) = -\frac{2}{3} + \frac{1}{(2n-1)(2n-3)} \left[1 + \left(\frac{r\rho_0'}{\rho_0} + \frac{4\mu}{\lambda + 2\mu}\right) A^* - \frac{n(n+1)}{3} B^*\right]$$

$$D_3(n, r) = -\frac{n+2}{(2n+3)(2n+5)} \left[1 + \left(\frac{r\rho_0'}{\rho_0} + \frac{4\mu}{\lambda + 2\mu}\right) A^* + \frac{n+1}{2} B^*\right]$$

$$D_4(n, r) = -\frac{(n-2)(n-1)(1-\delta_{0n})}{(2n-3)(2n-1)} \left(1 - \frac{n\lambda + 2\mu}{\lambda + 2\mu} A^*\right)$$

$$D_5(n, r) = -\frac{1}{3} - \frac{1}{(2n-1)(2n+3)} \left[\frac{2}{3}(n^2 + n - 3) + n(n+1)\frac{4\mu}{\lambda + 2\mu} A^*\right]$$

$$D_6(n, r) = -\frac{(n+2)(n+3)}{(2n+3)(2n+5)} \left[1 + \frac{(n+1)\lambda - 2\mu}{\lambda + 2\mu} A^*\right]$$

$$D_7(n, r) = \frac{(n-1)(1-\delta_{0n})}{(2n-3)(2n-1)} A^*$$

$$D_8(n, r) = \frac{1}{(2n-1)(2n+3)} A^*$$

$$D_9(n, r) = -\frac{n+2}{(2n+3)(2n+5)} A^*$$

δ_{0n} = Kronecker delta

= 1 when $n = 0$

= 0 when $n = 1, 2, 3, \ldots$

terms of the A^*, B^*, and C^* quantities that were obtained in Appendix A. They ultimately depend on the solutions of the Clairaut equation.

Before proceeding to the numerical solutions of Eqs. (39), we must:

- Determine the conditions that must be satisfied in order that these equations have regular solutions at the center of the configuration ($r = 0$), that is, solutions which are expressible as infinite power series about the center;

- Ascertain the limiting form of these equations for the case $n = 0$ (zero-order harmonic), since the nth-order harmonic depends on the harmonic of order $n - 2$;

- Ascertain the limiting form of these equations which is valid within the liquid outer core where $\mu = 0$; and

- Determine the conditions that the system must satisfy at the free surface ($r = a_1$) depending on the loading conditions.

REGULARITY CONDITIONS FOR THE SOLUTIONS AT THE CENTER WHEN $\mu \neq 0$ AND $n \neq 0$

We assume power-series expansions in the neighborhood of $r = 0$ for each of the six variables

$$\begin{aligned} U_n(r) &= U_{n0} + U_{n1} r + U_{n2} r^2 + \cdots, \\ &\cdots, \\ H_n(r) &= H_{n0} + H_{n1} r + H_{n2} r^2 + \cdots \end{aligned} \tag{40}$$

and replace them within both sides of Eqs. (39). From a physical point of view, we must also require that the components of the displacement be zero at the origin:

$$U_{n0} = 0, \quad V_{n0} = 0. \tag{41}$$

Since the right-hand sides of the equations contain terms in $1/r^2$ and $1/r$, the regularity conditions will be obtained by imposing the vanishing of the coefficients of the two negative powers of r and by equating the constant terms appearing on both sides. All other positive powers of r do not yield any extra conditions, since we deal with the limit when r approaches zero.

Let us remark that from Eq. (6) it follows that

$$\lim_{r \to 0} \gamma_0(r)/r = \frac{4\pi G}{3} \bar{\rho}_0(0) \tag{42}$$

is finite and also that the rotational terms will not give any contribution, since they are formed with the U and V terms.

Each equation can give rise to at most three conditions. We ultimately arrive at ten conditions among the coefficients of the variables. Five of these conditions can be immediately analyzed:

$$
\begin{aligned}
R_{n0} &= 0 , \\
R_{n1} &= H_{n0} , \\
(n-1)(n+2)H_{n0} &= 0 , \\
U_{n1} &= \frac{1}{\mu} F_{n0} , \\
R_{n2} &= \frac{3}{n(n+1)} H_{n1} + 4\pi G \rho_0 V_{n1} .
\end{aligned}
\tag{43}
$$

From the second and third conditions, we see that when $n \neq 1$, we must have

$$H_{n0} = R_{n1} = 0 , \quad n \neq 1 , \tag{44}$$

whereas when $n = 1$, one can only say

$$R_{11} = H_{10} . \tag{45}$$

Next, we have three linear homogeneous relations among F_{n0}, E_{n0}, and V_{n1}. The determinant made with the coefficients of the unknowns vanishes for $n = 2$. Thus in this case we reach solutions of the form

$$V_{21} = \frac{1}{2\mu} F_{20} , \quad E_{20} = 2F_{20} , \tag{46}$$

whereas when $n \neq 2$, there is only the solution

$$V_{n1} = E_{n0} = F_{n0} = 0 , \tag{47}$$

and this leads to $U_{n1} = 0$.

The last two conditions of the set relate the six quantities U_{n2}, V_{n2}, E_{n1}, F_{n1}, R_{n1}, H_{n0}. In the case $n \neq 1$, when $H_{n0} = R_{n1} = 0$, these conditions can be used to ascertain U_{n2} and V_{n2} in terms of E_{n1} and F_{n1} according to the formulas

$$
\begin{aligned}
2\mu(n-1)(n+2)V_{n2} &= 3E_{n1} - (n^2 + n - 8)F_{n1} , \\
2\mu(3\lambda + 2\mu)(n-1)(n+2)U_{n2} &= [(5n^2 + 5n - 1)\lambda + 6(n^2 + n - 1)\mu]E_{n1} \\
&\quad -n(n+1)[(2n^2 + 2n - 13)\lambda + 2(n^2 + n - 5)\mu]F_{n1} .
\end{aligned}
\tag{48}
$$

When $n = 1$, the same conditions can be used to ascertain E_{11} and F_{11} in terms of U_{12}, V_{12}, and $H_{10} = R_{11}$ according to

$$\begin{aligned} E_{11} &= 2\mu\left(\frac{3\lambda + 2\mu}{\lambda + 4\mu}\right)(U_{12} - V_{12}) - \frac{\lambda + 2\mu}{\lambda + 4\mu}\rho_0 H_{10}\,, \\ F_{11} &= -\mu\left(\frac{3\lambda + 2\mu}{\lambda + 4\mu}\right)(U_{12} - V_{12}) - \frac{\mu}{\lambda + 4\mu}\rho_0 H_{10}\,. \end{aligned} \tag{49}$$

Consequently, for $n = 1$ the series expansions are

$$\begin{aligned} U_1 &= U_{12}r^2 + \cdots, \\ V_1 &= V_{12}r^2 + \cdots, \\ R_1 &= H_{10}r + (3/2)H_{11}r^2 + \cdots, \\ E_1 &= E_{11}r + \cdots, \\ F_1 &= F_{11}r + \cdots, \\ H_1 &= H_{10} + H_{11}r + \cdots, \end{aligned} \tag{50}$$

with H_{10}, H_{11}, H_{12}, and V_{12} arbitrarily chosen and with E_{11} and F_{11} given by Eqs. (49). When $n = 2$, the series expansions are

$$\begin{aligned} U_2 &= (F_{20}/\mu)r + U_{22}r^2 + \cdots, \\ V_2 &= (F_{20}/2\mu)r + V_{22}r^2 + \cdots, \\ R_2 &= (1/2)\left(H_{21} + \frac{4\pi G\rho_0}{\mu}F_{20}\right)r^2 + \cdots, \\ E_2 &= 2F_{20} + E_{21}r + \cdots, \\ F_2 &= F_{20} + F_{21}r + \cdots, \\ H_2 &= H_{21}r + \cdots, \end{aligned} \tag{51}$$

with arbitrary values for F_{20}, E_{21}, F_{21}, and H_{21} and with U_{22} and V_{22} given by Eqs. (48). Finally, when $n \geqslant 3$, we can write expansions of the sort

$$\begin{aligned} U_n &= U_{n2}r^2 + \cdots, \\ V_n &= V_{n2}r^2 + \cdots, \\ R_n &= [3/n(n+1)]H_{n1}r^2 + \cdots, \\ E_n &= E_{n1}r + \cdots, \\ F_n &= F_{n1}r + \cdots, \\ H_n &= H_{n1}r + \cdots, \end{aligned} \tag{52}$$

with E_{n1}, F_{n1}, and H_{n1} arbitrarily chosen and with U_{n2} and V_{n2} also determined by Eqs. (48).

FUNDAMENTAL EQUATIONS OF MOTION FOR THE ZERO-ORDER HARMONIC

When $n = 0$, there is no transverse spheroidal component for the displacement and for the stress, since $\partial P_0/\partial\theta \equiv 0$. This translates into the fact that there exists no equation for the θ component: $V_0(r)$ is arbitrary and so is $F_0(r)$, which depends on $V_0(r)$. The system of equations in its normal form cannot contain any equation pertaining to V_0 and F_0; these two variables do not appear within the remaining four equations relating U_0, R_0, E_0, and H_0. From

$$H_0' = -(2/r)H_0 \tag{53}$$

we solve and get $H_0(r) \equiv h/r^2$. Regularity at $r = 0$, however, requires that h must be taken equal to zero; thus $H_0 \equiv 0$. The remaining equations are

$$\begin{aligned} U_0' &= -\frac{2\lambda}{\lambda + 2\mu}\frac{1}{r} U_0 + \frac{1}{\lambda + 2\mu} E_0 , \\ R_0' &= 4\pi G\rho_0 U_0 , \\ E_0' &= 4\left[-\frac{\gamma_0\rho_0}{r} + \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right)\frac{\mu}{r^2}\right] U_0 - \frac{4\mu}{\lambda + 2\mu}\frac{1}{r} E_0 \\ &\quad +\rho_0\omega^2 \left[C_3(0)U_2 + C_6(0)V_2 + \frac{r}{\mu} C_9(0)F_2\right]. \end{aligned} \tag{54}$$

Regularity at $r = 0$ imposes the conditions

$$U_{00} = R_{01} = 0 \tag{55}$$

and

$$\begin{aligned} E_{00} &= (3\lambda + 2\mu)U_{01} , \\ E_{01} &= 4\mu\left(\frac{3\lambda + 2\mu}{\lambda + 6\mu}\right) U_{02} . \end{aligned} \tag{56}$$

Thus the series expansions in the neighborhood of $r = 0$ can be written as

$$\begin{aligned} U_0 &= U_{01}r + U_{02}r^2 + \cdots, \\ R_0 &= R_{00} + R_{02}r^2 + \cdots, \\ E_0 &= E_{00} + E_{01}r + \cdots, \end{aligned} \tag{57}$$

where U_{01}, U_{02}, R_{00}, and R_{02} can be chosen arbitrarily and E_{00} and E_{01} are obtainable from Eqs. (56).

Note that $V_0(r)$ and $F_0(r)$ are arbitrary functions. They do not appear in Eq. (54) pertaining to $n = 0$; neither will they appear in the other possible case, that is, among the rotational terms of Eqs. (39) when $n = 2$. This is so because one can verify from Tables 1 and 2 that $C_4(2) = C_7(2) = D_4(2) = 0$.

EQUATIONS VALID FOR THE OUTER CORE

To investigate the equations of motion valid within the liquid outer core, one must evaluate the limit reached by the system of Eqs. (39) when the rigidity μ approaches zero.

The U equation becomes simply

$$U_n' = -\frac{2}{r} U_n + \frac{n(n+1)}{r} V_n + \frac{1}{\lambda} E_n . \tag{58}$$

The V equation, when written as

$$rF_n = \lim_{\mu \to 0} \mu(rV_n' + U_n - V_n) , \tag{59}$$

reveals that for the values of $r \neq 0$ it must be $F_n = 0$, whereas the terms appearing within the parenthesis could be arbitrarily chosen.

The R and H equations do not contain the parameter μ and do not have to be modified. The E equation yields

$$\begin{aligned} E_n' = & -4 \frac{\gamma_0 \rho_0}{r} U_n + n(n+1) \frac{\gamma_0 \rho_0}{r} V_n - \rho_0 H_n \\ & + \rho_0 \omega^2 [C_1 U_{n-2} + C_2 U_n + C_3 U_{n+2} + C_4 V_{n-2} + C_5 V_n \\ & + C_6 V_{n+2} + C_7 (rV_{n-2}' + U_{n-2} - V_{n-2}) \\ & + C_8 (rV_n' + U_n - V_n) + C_9 (rV_{n+2}' + U_{n+2} - V_{n+2})] . \end{aligned} \tag{60}$$

Here use has been made of Eq. (59) to eliminate the terms $(r/\mu)F_p$ in favor of the expression $rV_p' + U_p - V_p$ (where p takes on the three values $n - 2$, n, and $n + 2$), which is not an indeterminate form.

The F equation degenerates into an algebraic equation and can be used as a checkout condition:

$$\begin{aligned} E_n = & \gamma_0 \rho_0 U_n - \rho_0 R_n + r\rho_0 \omega^2 \Big[D_1 U_{n-2} + D_2 U_n \\ & + D_3 U_{n+2} + D_4 V_{n-2} + D_5 V_n + D_6 V_{n+2} \\ & + \frac{r}{\lambda} (D_7 E_{n-2} + D_8 E_n + D_9 E_{n+2}) \Big] . \end{aligned} \tag{61}$$

Equation (60) is a differential equation not in its normal form, since both E_n' and $\rho_0 \omega^2 V_p'$ ($p = n - 2, n, n + 2$) appear in it. One could use Eqs. (60) and (61) for a simultaneous determination of E_n and V_n. Computationally this procedure will not be convenient, since it is not straightforward. It will be numerically more expedient and still physically plausible to assume $V_n \equiv 0$ throughout the whole interior of the outer liquid core (to assume the vanishing of the tangential displacement), since V_n has been left arbitrary in the previous considerations. If so, and for the case $n \neq 0$, we get the following system:

$$
\begin{aligned}
V_n &= 0, \quad F_n = 0, \\
U_n' &= -\frac{2}{r} U_n + \frac{1}{\lambda} E_n, \\
R_n' &= 4\pi G \rho_0 U_n + H_n, \\
H_n' &= \frac{n(n+1)}{r^2} R_n - \frac{2}{r} H_n, \\
E_n' &= -4 \frac{\gamma_0 \rho_0}{r} U_n - \rho_0 H_n + \rho_0 \omega^2 [(C_1 + C_7) U_{n-2} \\
&\quad + (C_2 + C_8) U_n + (C_3 + C_9) U_{n+2}], \\
E_n &= \gamma_0 \rho_0 U_n - \rho_0 R_n + r \rho_0 \omega^2 \Big[D_1 U_{n-2} + D_2 U_n + D_3 U_{n+2} \\
&\quad + \frac{r}{\lambda} (D_7 E_{n-2} + D_8 E_n + D_9 E_{n+2}) \Big].
\end{aligned}
\tag{62}
$$

No series expansions are required for these equations, because they are valid within the outer core, and their initial conditions at a value of $r \neq 0$ are furnished by the final values of the variables at the inner core interface.

Let us now consider the more particular case when both μ and n are zero. We should then ascertain the limit of Eqs. (54) when the rigidity approaches zero and eliminate the term $(r/\mu) F_2$ according to Eq. (59). We should also take $V_2 = 0$ in agreement with Eq. (62). The two functions F_0 and V_0, which in Eqs. (54) were left arbitrary, can now be chosen as both vanishing. We are left then with the system

$$
\begin{aligned}
U_0' &= -\frac{2}{r} U_0 + \frac{1}{\lambda} E_0, \\
R_0' &= 4\pi G \rho_0 U_0, \\
E_0' &= -4 \frac{\gamma_0 \rho_0}{r} U_0 + \rho_0 \omega^2 [C_3(0) + C_9(0)] U_2, \\
V_0 &\equiv F_0 \equiv H_0 \equiv 0.
\end{aligned}
\tag{63}
$$

If we differentiate the last equation of Eqs. (62) with respect to r and combine it with the equation appearing prior to it in the same set, we get, after performing a number of simplifications, a relationship which symbolically can be abbreviated as

$$\left(\frac{\gamma_0 \rho_0}{\lambda} + \frac{\rho_0'}{\rho_0}\right) E_n = \rho_0 \omega^2 \sum_p \left(U_p ; \frac{r}{\lambda} E_p ; \frac{r^2}{\lambda} E_p'\right), \quad p = n - 2, n, n + 2. \tag{64}$$

The vanishing of the term

$$\frac{\gamma_0 \rho_0}{\lambda} + \frac{\rho_0'}{\rho_0} = 0 \tag{65}$$

has been known as the Adams-Williamson condition [15]. It was first connected to this problem by Longman [10] and was afterward extensively studied by Pekeris and Accad [16]. In the absence of rotational terms, two possibilities were available: either the Adams-Williamson condition should have been valid within the outer core, and this was found to be unsatisfactory from a physical point of view, or one should have $E_n = 0$, which according to Eqs. (34) and (23) amounts to zero dilatation. This latter condition should be discarded, because it restricts the number of parameters available to match the boundary conditions. Thus, the inclusion of rotational terms in the study of elastic deformation of the Earth not only gives rise to a more realistic physical model but also avoids the use of the Adams-Williamson condition, which use has been controversial.

Let us finally note that the Brunt-Väisälä frequency N for a stratified liquid is given by

$$N^2 = -\gamma_0 \left(\frac{\gamma_0 \rho_0}{\lambda} + \frac{\rho_0'}{\rho_0}\right), \tag{66}$$

and it is related to the stability in the circulation motion of the liquid (see, for example, Ref. 17). Equation (64) can then be interpreted as providing a connection between the rotation of the Earth and the stability of the induced motion within its liquid layer. For this reason the quantity N, which does not vanish everywhere, should be considered a more appropriate parameter than the older Adams-Williamson condition in studying the interface between the liquid core and the mantle.

BOUNDARY CONDITIONS

We must consider three types of boundary conditions:

- At the center of the configuration, where $r = 0$, the solutions must be regular, that is, representable by means of infinite power series in the radial distance. These series can then be used as first approximations to initiate integration. In particular, for physical reasons, we must have $U(0) = V(0) = 0$.

- At any surface representing a discontinuity between two layers, the variables U, R, E, and H must be continuous across them; the variables V and F can be discontinuous. This situation applies in particular to the liquid-solid interface; however, between two solid regions separated by a discontinuity we could admit V to be continuous.

• At the outer surface ($r = a_1$) we shall consider two models. One model consists of assuming a constant atmospheric pressure (≈ 1 bar) at $r = a_1$ to be matched by the normal stress E. The harmonic expansion for this type of condition is as follows:

$$\begin{aligned} E_0(a_1) &\approx 10^{-3} \text{ kbar} = 1 \text{ atm}, \\ E_n(a_1) &= 0, \quad n = 1, 2, 3, \ldots, \\ F_n(a_1) &= 0, \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{67}$$

A second model for the boundary conditions is to neglect both atmospheric and oceanic layers and consider a localized loading generated by the total mass m_0 uniformly distributed atop a spherical cap of semiaperture θ_0. This generates a mass density

$$\delta(a_1, \theta_0) = m_0/2\pi a_1^2(1 - \cos\theta_0). \tag{68}$$

The spherical harmonic representation of this mass distribution is given for $n = 0$ by

$$\delta_0(a_1, \theta_0) = \frac{1}{2}\int_{\cos\theta_0}^{1} \delta(a_1, \theta_0)\, d\nu = \frac{m_0}{4\pi a_1^2}, \tag{69}$$

and is given for $n \geqslant 1$ by

$$\begin{aligned} \delta_n(a_1, \theta_0) &= \frac{2n+1}{2}\int_{\cos\theta_0}^{1} \delta(a_1, \theta_0) P_n(\nu)\, d\nu \\ &= \frac{(2n+1)m_0}{4\pi a_1^2(1-\cos\theta_0)}\int_{\cos\theta_0}^{1} P_n(\nu)\, d\nu \\ &= \frac{m_0}{4\pi a_1^2(1-\cos\theta_0)}\left[P_{n-1}(\cos\theta_0) - P_{n+1}(\cos\theta_0)\right]. \end{aligned} \tag{70}$$

Here we have used the facts that

$$(2n+1)P_n(\nu) = \frac{dP_{n+1}}{d\nu} - \frac{dP_{n-1}}{d\nu}, \quad n = 1, 2, 3, \ldots \tag{71}$$

and

$$P_n(1) = 1, \quad n = 0, 1, 2, \ldots, \tag{72}$$

as given on pages 17 and 33 of Ref. 18.

When the spherical cap shrinks to a point ($\theta_0 \to 0$), we get the case of a concentrated mass load m_0. We can handle the ensuing indeterminate form by means of the l'Hospital rule and by using the two above-mentioned relationships to obtain

$$\lim_{\nu \to 1} \frac{P_{n-1}(\nu) - P_{n+1}(\nu)}{1 - \nu} = \lim_{\nu \to 1} \left(\frac{dP_{n+1}}{d\nu} - \frac{dP_{n-1}}{d\nu} \right)$$

$$= (2n + 1) P_n(1) = 2n + 1 \, . \tag{73}$$

Thus

$$\lim_{\theta_0 \to 0} \delta_n(a_1, \theta_0) = \frac{(2n + 1) m_0}{4\pi a_1^2} = \delta_n(a_1) \tag{74}$$

is the representation of the loading in the limiting case (concentrated loading), and this is valid even for $n = 0$ (see Eq. (69)).

At the surface of the Earth, the radial and transversal components of the stress must then satisfy the conditions

$$\begin{aligned} E_n(a_1) &= -\gamma_0(a_1) \delta_n(a_1) = -\frac{(2n + 1) m_0}{4\pi a_1^2} \gamma_0(a_1) , \quad n = 0, 1, 2, \ldots, \\ F_n(a_1) &= 0 , \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{75}$$

which have been expressed only in the first-order approximation and with the assumption of a purely normal loading.

Another boundary condition holds for the deformed potential. This is obtainable by considering, first, the continuity of the deformed potential at the boundary (that is, $(R_n)_e = R_n$, where the subscript e refers to the exterior space) and, second, the discontinuity suffered by the normal derivative of the potential because of the presence of a surface mass distribution (this is the Gauss theorem of potential theory):

$$\left(\frac{\partial R_n}{\partial n} \right)_e - \frac{\partial R_n}{\partial n} = -4\pi G \left(\frac{dm}{dS} \right)_{a = a_1}$$

$$= -4\pi G [\rho_0(a_1) U_n(a_1) + \delta_n(a_1)] \, . \tag{76}$$

In first-order approximation, $\vec{n}$ coincides with the radial direction. Also, since the exterior potential is an infinite series in $(1/r)^{n+1}$, we can write

$$R_e' = -\frac{n + 1}{r} R_e \, . \tag{77}$$

Thus, at $r = a_1$ we have

$$\left(\frac{\partial R_n}{\partial n} \right)_e = (R_n')_e = -\frac{n + 1}{r} (R_n)_e = -\frac{n + 1}{r} R_n \, , \tag{78}$$

since $R_e(a_1) = R(a_1)$. We get

$$R_n'(a_1) + \frac{n+1}{a_1} R_n(a_1) = 4\pi G \left[\rho_0(a_1) U_n(a_1) + \frac{(2n+1)m_0}{4\pi a_1^2} \right]. \tag{79}$$

In terms of the normalized variables, we can write

$$H_n(a_1) + \frac{n+1}{a_1} R_n(a_1) = (2n+1) \frac{Gm_0}{a_1^2} = (2n+1)\gamma_0(a_1). \tag{80}$$

Thus, the conditions that hold at the surface ($r = a_1$) are expressed by Eqs. (67) and (80) or by Eqs. (75) and (80).

TOROIDAL DEFORMATIONS

When rotational symmetry is assumed, as has been the case throughout this report, the toroidal displacement vector $\vec{u}$ has components of the form

$$u = 0, \quad v = 0, \quad w = -\sum_{n=1}^{\infty} W_n(r) \partial P_n / \partial \theta. \tag{81}$$

The functions W_n constitute a set of unknown functions that will be determined from the equations of motion.

One can immediately verify that for this type of deformation both the dilatation Δ and the perturbed density ρ_1 vanish. As a consequence, the Poisson equation becomes $\nabla^2 V_1 = -4\pi G \rho_1 = 0$ and yields

$$r^2 R_n'' + 2r R_n' - n(n+1) R_n = 0. \tag{82}$$

The general solution is $R_n = r^n$. However, a potential function must vanish at infinity, which means that we must have $R_n \equiv 0$. This leads to the vanishing of V_1 and of the components of its gradient $\vec{g}_1$. The Poisson equation is identically satisfied and does not furnish any condition.

By straightforward calculations, it is easy to realize that only the longitude-dependent component (the ϕ component) of the equations of motion gives a contribution, and it amounts to the equation

$$\mu \left(W_n'' + \frac{2}{r} W_n' \right) + \mu' \left(W_n' - \frac{1}{r} W_n \right) + \left[\rho_0 \omega^2 - \mu \frac{n(n+1)}{r^2} \right] W_n = 0. \tag{83}$$

We introduce

$$Z_n = \mu \left(W_n' - \frac{1}{r} W_n \right) \tag{84}$$

as a new auxiliary variable. It represents the radial factor of the transversal component $\tau_{r\phi}$ of the stress. The normal form of Eqs. (83) can then be written as the system

$$\begin{aligned} W_n' &= \frac{1}{r} W_n + \frac{1}{\mu} Z_n , \\ Z_n' &= \left[(n+2)(n-1) \frac{\mu}{r^2} - \rho_0 \omega^2 \right] W_n - \frac{3}{r} Z_n . \end{aligned} \tag{85}$$

The boundary conditions are

$$Z_n(a_1) = 0 , \tag{86}$$

denoting the vanishing of the transversal stress at the free surface, and

$$Z_n(b_1) = 0 , \tag{87}$$

where b_1 is the outer core radius. When $\mu = 0$, one can see from Eqs. (85) that $W_n \equiv 0$ and $Z_n \equiv 0$.

Regularity conditions at the origin can be easily obtained. The series expansions for $n \neq 1$ are

$$\begin{aligned} W_n &= \qquad r^2 W_{n2} + \cdots , \\ Z_n &= r Z_{n1} + \cdots \end{aligned} \tag{88}$$

with arbitrary W_{n2} and

$$Z_{n1} = \frac{1}{4} (n+2)(n-1) \mu W_{n2} . \tag{89}$$

For $n = 1$ one gets

$$\begin{aligned} W_1 &= r W_{11} + \cdots , \\ Z_1 &= \qquad r^2 Z_{11} + \cdots , \end{aligned} \tag{90}$$

where both coefficients are arbitrary parameters.

THE RESULTS AND FUTURE NUMERICAL WORK

We shall briefly summarize and discuss the results that have been obtained in this report. We shall also express some views on the extensive numerical work which we plan to undertake as a followup of this theoretical work.

We start by considering a rotating spheroidal Earth which is in hydrostatic equilibrium under the influence of its self-gravitation and its rotational acceleration. We further assume that the rotational axis is fixed with respect to the spheroid and that the angular velocity is constant. We next consider linear perturbing effects due to the elastic property of the various layers comprising the interior of our planet. The dilatation of the elastic material yielding to the gravity force and centrifugal force gives rise to a perturbed density distribution within the Earth which is obtainable from the continuity equation and which generates a perturbed potential through the related Poisson equation.

Elaborating on the equation of angular momentum, we ultimately reach a linearized version of the Navier-Stokes equation and look for solutions in the form of spheroidal and toroidal deformations. Seismological evidence has corroborated the existence of such permanent deformations. We have developed all the variables in series of Legendre polynomials, and we have ignored the terms due to the Coriolis acceleration, since we are assuming that there is no relative motion with respect to the rotating spheroid. We are limiting ourselves to studying the centrifugal acceleration effects. Further studies of the oscillatory motion of the Earth will, however, require the retention of the Coriolis acceleration.

The toroidal deformations are the easier of the two cases to handle, because they are not associated with any dilation. Their study depends on one second-order differential equation which is related to the longitudinal component of the Navier-Stokes equation. The treatment of the spheroidal deformations on the other side gives rise to three second-order differential equations: two of them depend on the radial and transversal components of the Navier-Stokes equation; the third one depends on the Poisson equation. The three unknowns are the radial and transversal components of the displacement vector field and the perturbed potential. The reduction of the said second-order system to its normal form, a necessary feature when one wants to perform any numerical evaluation, introduces the components of the stress and the gravitational flux as auxiliary variables.

The retention of the centrifugal acceleration in the equations of motion produces terms which consist of the product of Legendre polynomials and of their derivatives. The representation of the spheroidal components of such products has constituted one of our major mathematical problems. In Appendix B we elaborate on the well-known Adams-Neumann product formula of the Legendre polynomials and succeed in expressing the spheroidal components of the product of the mixed terms and of the derivative terms without introducing any new set of harmonic functions.

Another problem of an analytical character which was encountered in the development of the equations of motion was to provide the partial derivatives of the first and second order for the unperturbed potential with respect to the radial distance and the colatitude. This task was achieved quite simply in Appendix A by making use of previously obtained results pertaining to the equilibrium configuration of a rotating planet which had been published in the astrophysical literature. Our results depend on the solution of the Clairaut equation and have been expressed up to the first power of a rotational parameter which characterizes the deformation of the original sphere into a spheroid.

Of the six differential equations comprising our normalized system, only two, the equations expressing the derivatives of the stress, contain rotational terms. Because of these rotational terms, each equation representing the nth order harmonic of the stress contains also the harmonics of order $n - 2$ and $n + 2$ of some of the other variables. Thus we have to deal with a system simultaneously relating three different orders of harmonics; this will present some difficulty in its numerical treatment, since some assumption has to be made regarding the harmonics of order higher than the order we are solving or have solved for.

Our next task will be to numerically solve the system of six differential equations. For this purpose we plan to use a Runge-Kutta integrator software package, since this approach was found satisfactory by Longman [9,10] and Farrell [11] in dealing with the case of a nonrotating Earth. Most likely we shall use DVERK, which is a fifth-order integrator authored by Verner [19] and which has been discussed by Jackson et al. [20]. We shall be dealing with a boundary-value problem with conditions to be satisfied both at the center of the configuration and at the free surface, and where certain discontinuities of the variables can be allowed at the interface between layers. Different Earth models will be used. However, we plan to devote more time to the 1066 model which was developed by Gilbert and Dziewonski [21]. It consists of about 160 data points for the density and the elastic parameters, and it is considered to be the most sophisticated of the existing models.

At the center of the configuration, the displacement vector shall vanish, and all of the six variables must be regular functions of the radial distance. We have, therefore, obtained power-series solutions valid in the neighborhood of the origin and which satisfy the given differential system of equations. Some of the coefficients of the lowest powers appearing in these series are free parameters and must be used in initiating the numerical solution and in attempting to satisfy the other conditions at the free surface. Another parameter that can play a useful role is the initial distance from the origin where the function evaluations are being performed for the first time.

The presence of the liquid outer core where the rigidity vanishes presents considerable complications. In evaluating the limit of the general equations when the rigidity approaches zero, we find that the transversal component of the stress must vanish and, at the same time, that the transversal component of the displacement can be chosen arbitrarily. Thus these two variables can suffer discontinuities at the interface between the inner and outer cores and also at the transition between the outer core and the mantle. Further analysis reveals that we are left with four differential equations and one algebraic relationship.

Our fundamental result in this connection is that, due to the adoption of a rotating Earth model, we are not compelled to fall back to the Adams-Williamson condition pertaining to the density distribution within the outer core, which was never completely accepted in the past. We tend to agree with other researchers to point out that a better parameter to be used at this interface is the Brunt-Väisälä frequency for the circulation motion of the fluid. Another approach which we shall also consider, and which might have more realistic physical implications, is the adoption of Molodenskii's analytic theory [6] to represent the motion within the liquid layer.

At the other surfaces of discontinuity within the mantle, we shall suppose that all six variables are continuous unless numerical considerations will otherwise impose upon us a different outlook. At the free surface, we have three conditions. One is obtained by applying Gauss theorem to the discontinuity suffered by the mass distribution. The second condition is the vanishing of the tangential stress. The third condition is the matching of the radial stress with the loading conditions. We are assuming a concentrated load and have followed Longman's approach in obtaining its analytic representation as a limit of a load uniformly distributed upon a small spheroidal cap when the aperture of this cap goes to zero.

Starting from the fundamental system of equations corresponding to the zero-order harmonic, we can recursively obtain the Love numbers and the load numbers of any order. It will be interesting at this stage to study the variation of these two sets of numbers with the particular Earth model used. Speculation by both Farrell [11] and Varga [22] has been that the Love numbers of large orders are not much influenced by the structure of the core but depend essentially on the values that the variables assume at the base of the mantle.

Once the load numbers have been ascertained up to a certain finite order, one has to consider the sum of the infinite Legendre polynomial expansions to obtain the spheroidal deformations and other pertinent information about the tilt and tidal gravity. For this purpose, we shall elaborate upon certain procedures followed by Farrell in Ref. 11 that deal with the nonrotating Earth. These procedures entail: the use of certain algorithms to improve the rapidity of convergence of the series involved, the development of an asymptotic theory to ascertain the appropriate location where the truncation of the series should take place, the sum of the Poisson series of Legendre polynomials for values of the argument less than 1 to provide the value of the deformation at a certain angular distance from the location of the applied load, and the use of the well-known Boussinesq [23] solution for an elastic flat plate to approximate the values that the variables assume at points in the immediate neighborhood of the applied load.

REFERENCES

1. G. H. Darwin, *Scientific Papers*, Cambridge Univ. Press, Vol. 1, 1907.
2. A. E. H. Love, Proc. Royal Soc. Lond. A **82**, 73 (1909).
3. W. H. Munk and G. J. F. McDonald, *The Rotation of the Earth*, Cambridge Univ. Press, 1960.
4. H. Kinoshita, Celestial Mech. **15**, 277 (1977).
5. H. Poincaré, Bull. Astr. 27, 321 (1910).
6. M. S. Molodenskii, Comm. Obs. Roy. Belgique **288**, 25 (1961).
7. P. Melchior, *The Tides of the Planet Earth*, Pergamon Press, Oxford, 1978.
8. J. T. Kuo and R. C. Jachens, Ann. de Géophys. 33, 73 (1977).
9. I. M. Longman, J. Geophys. Res. **67**, 845 (1962).
10. I. M. Longman, J. Geophys. Res. **68**, 485 (1963).
11. W. E. Farrell, Rev. Geophys. Space Phys. **10**, 761 (1972).
12. M. L. Smith, Geophys. J. Roy. Astron. Soc. **37**, 491 (1974).
13. P. Lanzano and J. C. Daley, "Rotational Deformation of the Earth and Major Planets," NRL Report 8252, Aug. 1978.
14. Z. Alterman, H. Jarosch, and C. L. Pekeris, Proc. Roy. Soc. Lond. A **252**, 80 (1959).
15. L. H. Adams and E. D. Williamson, J. Wash. Acad. Sci. **13**, 418 (1923).
16. C. L. Pekeris and Y. Accad, Phil. Trans. Roy. Soc. Lond. A **273**, 237 (1972).
17. J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, New York, 1979, p. 553.
18. E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea, New York, 1955.
19. J. H. Verner, Soc. Indust. and Applied Math. (SIAM) J. on Numerical Analysis **15**, 772 (1978).
20. K. R. Jackson, W. H. Enright, and T. E. Hull, Soc. Indust. and Applied Math. (SIAM) J. on Numerical Analysis **15**, 618 (1978).
21. F. Gilbert and A. M. Dziewonski, Phil. Trans. Roy. Soc. Lond. A **278**, 187 (1975).
22. P. Varga, Pageoph. **112**, 777 (1974).
23. J. Boussinesq, *Application des Potentiels à L'Etude de L'Equilibre et du Mouvement des Solides Elastiques*, Gauthier-Villars, Paris, 1885.

Appendix A

DERIVATIVES OF THE UNPERTURBED POTENTIAL

We must obtain the partial derivatives of the unperturbed potential $V_0(r, \theta)$ pertaining to hydrostatic equilibrium with respect to both variables r and θ. For this purpose, we refer to formula (27) in Ref. A1, in which reference the said potential was determined in terms of the mean radius a which characterizes an equipotential surface. More specifically, let

$$r = a[1 + qf_{21}(a)P_2(\cos\theta)] + \cdots \tag{A1}$$

be the equation of an equipotential surface to first-order terms in the rotational parameter

$$q = \frac{\omega^2 a_1^3}{3Gm_1} = \frac{\omega^2}{4\pi G\bar{\rho}_0(a_1)}, \tag{A2}$$

where a_1 is the mean radius of the outermost surface and where the nondimensional function f_{21}, a solution of the Clairaut equation, represents the spheroidal deformation. In Ref. A1 the potential $V_0(r, \theta)$ was determined along an equipotential surface $r(a, \theta)$ as a function of a:

$$V_0[r(a, \theta);\ \theta] \equiv \psi_0(a). \tag{A3}$$

We use this function $\psi_0(a)$ to calculate the required derivatives. This can be achieved most expeditiously by recalling a well-known expansion theorem due to Lagrange (see, for example, Ref. A2), which states: if the variables a and r are implicitly related according to the relation

$$a = r + q\phi(a),$$

then

$$F(a) = F(r) + q\phi(r)\frac{\partial F(r)}{\partial r} + \cdots, \tag{A4}$$

where F stands for a generic function. Equation (A1) reveals that

$$\phi(a) \equiv -af_{21}(a)P_2(\cos\theta), \tag{A5}$$

and this leads to

$$\begin{aligned}\psi_0(a) &= \psi_0(r) + q[-rf_{21}(r)P_2(\cos\theta)]\frac{\partial\psi_0}{\partial r} + \cdots \\ &= 4\pi G\left[\frac{1}{3}r^2\bar{\rho}_0(r) + \int_r^{r_1}\rho_0(r)r\,dr\right] \\ &\quad + \frac{1}{3}\omega^2 r^2 A^*(r)P_2(\cos\theta) + \frac{1}{3}\omega^2 r^2 + \cdots.\end{aligned} \tag{A6}$$

Here we have set

$$A^*(r) \equiv \frac{\bar{\rho}_0(r)}{\bar{\rho}_0(r_1)} f_{21}(r), \tag{A7}$$

a nondimensional function of r, and have used the formula

$$\bar{\rho}_0' = \frac{3}{r}(\rho_0 - \bar{\rho}_0), \tag{A8}$$

where primes as usual denote derivatives with respect to r. By differentiation of Eq. (A6) with respect to r and θ, we get

$$\begin{aligned}
\frac{\partial V_0}{\partial r} &= -\frac{4}{3}\pi G r\bar{\rho}_0(r) + \frac{2}{3}\omega^2 r + \frac{1}{3}\omega^2 r B^*(r) P_2(\cos\theta), \\
\frac{\partial V_0}{\partial \theta} &= \frac{1}{3}\omega^2 r^2 A^*(r)\frac{\partial P_2}{\partial \theta}, \\
\frac{\partial^2 V_0}{\partial r^2} &= \frac{4}{3}\pi G(2\bar{\rho}_0 - 3\rho_0) + \frac{2}{3}\omega^2 - \frac{1}{3}\omega^2 C^*(r) P_2(\cos\theta), \\
\frac{\partial^2 V_0}{\partial r\partial \theta} &= \frac{1}{3}\omega^2 r B^*(r)\frac{\partial P_2}{\partial \theta}, \\
\frac{\partial^2 V_0}{\partial \theta^2} &= \frac{1}{3}\omega^2 r^2 A^*(r)\frac{\partial^2 P_2}{\partial \theta^2}.
\end{aligned} \tag{A9}$$

The following additional notation has been used

$$\begin{aligned}
B^*(r) &\equiv \frac{\bar{\rho}_0(r)}{\bar{\rho}_0(r_1)}(3D_0 + \eta_{21} - 1) f_{21}(r), \\
C^*(r) &\equiv \frac{\bar{\rho}_0(r)}{\bar{\rho}_0(r_1)}\left(6D_0 + 2\eta_{21} - 8 - 3\frac{r\rho_0'}{\bar{\rho}_0}\right) f_{21}(r).
\end{aligned} \tag{A10}$$

These are nondimensional expressions which depend on the solution of the Clairaut equation. We have also set

$$D_0 = \rho_0/\bar{\rho}_0, \quad \eta = rf'/f \tag{A11}$$

and have eliminated the terms containing η' by means of the Radau equation

$$r\eta' = 6 + \eta(1-\eta) - 6D_0(1+\eta) \tag{A12}$$

(see Ref. A3). It is easy to verify that

$$\begin{aligned} B^* &= 2A^* + r(A^*)', \\ -C^* &= B^* + r(B^*)'. \end{aligned} \tag{A13}$$

These relations can be used to generate B^* and C^* if one wants to avoid numerically differentiating ρ_0. For computational purposes, it is appropriate to note that the limit of the three functions $A^*(r)$, $B^*(r)$, and $C^*(r)$ when r approaches zero is also zero.

REFERENCES

A1. P. Lanzano, Astrophys. Space Sci. 37, 173 (1975).

A2. W. M. Smart, *Celestial Mechanics*, Longmans, Green, London, 1953, p. 27.

A3. P. Lanzano, Astrophys. Space Sci. 29, 161 (1974), p. 170.

Appendix B

SPHEROIDAL COMPONENTS FOR THE PRODUCT OF TWO LEGENDRE POLYNOMIALS AND THEIR DERIVATIVES

We start from the well-known Adams-Neumann formula which was proved independently by each of these authors in 1878 [B1,B2] (see Refs. B3, B4, and 18). The said formula expresses the product of two Legendre polynomials as a sum of the same polynomials:

$$P_p(\cos\theta)P_q(\cos\theta) = \sum_{r=0}^{q} B(p,q;r)P_{p+q-2r}(\cos\theta). \tag{B1}$$

Here, $p \geqslant q$ are positive integers, and the B coefficients are given by

$$B(p,q;r) = \frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}} \frac{2p+2q-4r+1}{2p+2q-2r+1}, \tag{B2}$$

with

$$\begin{aligned} A_r &= \frac{1\cdot 3\cdot 5\ \ldots\ (2r-1)}{r!}, \quad r = 1, 2, \ldots, \\ A_0 &= 1. \end{aligned} \tag{B3}$$

Note that p and q appear symmetrically within B.

We differentiate Eq. (B1) twice with respect to θ and eliminate the second derivatives by using the well-known result (see Eq. (28))

$$\frac{\partial^2 P_p}{\partial\theta^2} = -(\cot\theta)\frac{\partial P_p}{\partial\theta} - p(p+1)P_p. \tag{B4}$$

After some reordering of the terms, we get

$$2\frac{\partial P_p}{\partial\theta}\frac{\partial P_q}{\partial\theta} - (\cot\theta)\left[\frac{\partial}{\partial\theta}(P_pP_q) - \sum_{r=0}^{q} B\frac{\partial P_{p+q-2r}}{\partial\theta}\right] = [p(p+1) + q(q+1)]P_pP_q - \sum_{r=0}^{q}(p+q-2r)(p+q-2r+1)BP_{p+q-2r}. \tag{B5}$$

The bracketed term which appears in the left-hand side vanishes, because it is a differential consequence of Eq. (B1). We next proceed to transform the product $P_p P_q$ in the right-hand side by using again Eq. (B1) and get

$$2\,\frac{\partial P_p}{\partial\theta}\,\frac{\partial P_q}{\partial\theta} = {\sum_{r=0}^{q}}' \,[p(p+1) + q(q+1) - (p+q-2r)(p+q-2r+1)]BP_{p+q-2r}\,. \tag{B6}$$

We can then write

$$\frac{\partial P_p}{\partial\theta}\,\frac{\partial P_q}{\partial\theta} = {\sum_{r=0}^{q}}' \,C(p,q;r)P_{p+q-2r}\,, \tag{B7}$$

where

$$C(p,q;r) = [r(2p+2q-2r+1) - pq]B(p,q;r)\,. \tag{B8}$$

Note that the coefficient C is symmetric with respect to p and q. Let us now turn to the product $P\,\partial P/\partial\theta$ and assume that it can be represented as

$$\frac{\partial P_p}{\partial\theta}\,P_q = \sum_{r=0}^{q} D(p,q;r)\,\frac{\partial P_{p+q-2r}}{\partial\theta}\,. \tag{B9}$$

It is a matter then of determining an algebraic expression for D. We differentiate Eq. (B9) once with respect to θ and eliminate the second derivatives through Eq. (B4) and write

$$\begin{aligned}&{\sum_{r=0}^{q}}' \,(p+q-2r)(p+q-2r+1)DP_{p+q-2r}\\ &\quad = p(p+1)P_pP_q - \frac{\partial P_p}{\partial\theta}\,\frac{\partial P_q}{\partial\theta} + (\cot\theta)\left[\frac{\partial P_p}{\partial\theta}\,P_q - \sum_{r=0}^{q} D\,\frac{\partial P_{p+q-2r}}{\partial\theta}\right].\end{aligned} \tag{B10}$$

The term within brackets vanishes because of Eq. (B9). We next replace both products in the right-hand side by means of Eqs. (B1) and (B7). We equate the coefficients of the polynomials of like order and get

$$(p+q-2r)(p+q-2r+1)D = p(p+1)B - C\,. \tag{B11}$$

Recalling the expression for C given by Eq. (B8), we can finally write

$$D(p,q;r) = \frac{p(p+1) + pq - r(2p+2q-2r+1)}{(p+q-2r)(p+q-2r+1)}\,B(p,q;r)\,. \tag{B12}$$

This expression is not symmetric in p and q. Naturally

$$P_p \frac{\partial P_q}{\partial \theta} = \sum_{r=0}^{q} D(q, p; r) \frac{\partial P_{p+q-2r}}{\partial \theta}. \tag{B13}$$

Also, since

$$\frac{\partial P_p}{\partial \theta} P_q + P_p \frac{\partial P_q}{\partial \theta} = \frac{\partial}{\partial \theta} (P_p P_q), \tag{B14}$$

we must have

$$D(p, q; r) + D(q, p; r) = B(p, q; r), \tag{B15}$$

and this can be directly verified. When $q = 0$, the summations appearing in Eqs. (B1), (B7), and (B9) reduce to only one term, which is due to $r = 0$. The limiting values for the coefficients are

$$\begin{aligned} B(p, 0; 0) &= 1, \quad C(p, 0; 0) = 0, \\ D(p, 0; 0) &= 1, \quad D(0, p; 0) = 0. \end{aligned} \tag{B16}$$

We have used the previously obtained formulas to evaluate the products

$$P_n P_2, \quad \frac{\partial P_n}{\partial \theta} \frac{\partial P_2}{\partial \theta}, \quad \frac{\partial P_n}{\partial \theta} P_2, \quad P_n \frac{\partial P_2}{\partial \theta} \tag{B17}$$

which appear in the fundamental equations of motion for a generic value of n. The results are presented in Table B1. The terms multiplied by the factor $(1 - \delta_{1n})$, where δ is the Kronecker delta, will not appear when $n = 1$.

Table B1 — Resolution of Products of Legendre Polynomials Into Spheroidal Components

$$2P_n P_2 = \frac{3(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} + \frac{2n(n+1)}{(2n-1)(2n+3)} P_n + \frac{3(n-1)n}{(2n-1)(2n+1)} P_{n-2}$$

$$\frac{1}{3} \frac{\partial P_n}{\partial \theta} \frac{\partial P_2}{\partial \theta} = -\frac{n(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} + \frac{n(n+1)}{(2n-1)(2n+3)} P_n + \frac{n(n-1)(n+1)}{(2n-1)(2n+1)} P_{n-2}$$

$$2 \frac{\partial P_n}{\partial \theta} P_2 = \frac{3n(n+1)}{(2n+1)(2n+3)} \frac{\partial P_{n+2}}{\partial \theta} + \frac{2(n^2+n-3)}{(2n-1)(2n+3)} \frac{\partial P_n}{\partial \theta} + \frac{3n(n+1)(1-\delta_{1n})}{(2n-1)(2n+1)} \frac{\partial P_{n-2}}{\partial \theta}$$

$$\frac{1}{3} P_n \frac{\partial P_2}{\partial \theta} = \frac{n+1}{(2n+1)(2n+3)} \frac{\partial P_{n+2}}{\partial \theta} + \frac{1}{(2n-1)(2n+3)} \frac{\partial P_n}{\partial \theta} - \frac{n(1-\delta_{1n})}{(2n-1)(2n+1)} \frac{\partial P_{n-2}}{\partial \theta}$$

REFERENCES

B1. J. C. Adams, Proc. Roy. Soc. 27, 86 (1878).

B2. F. E. Neumann, *Beiträge Zur Theorie der Kugelfunctionen*, Vol. 2, Leipzig, 1878.

B3. W. N. Bailey, Proc. Camb. Phil. Soc. 29, 173 (1933).

B4. E. T. Whittaker and G. N. Watson, *Modern Analysis*, Camb. Univ. Press, 1952.