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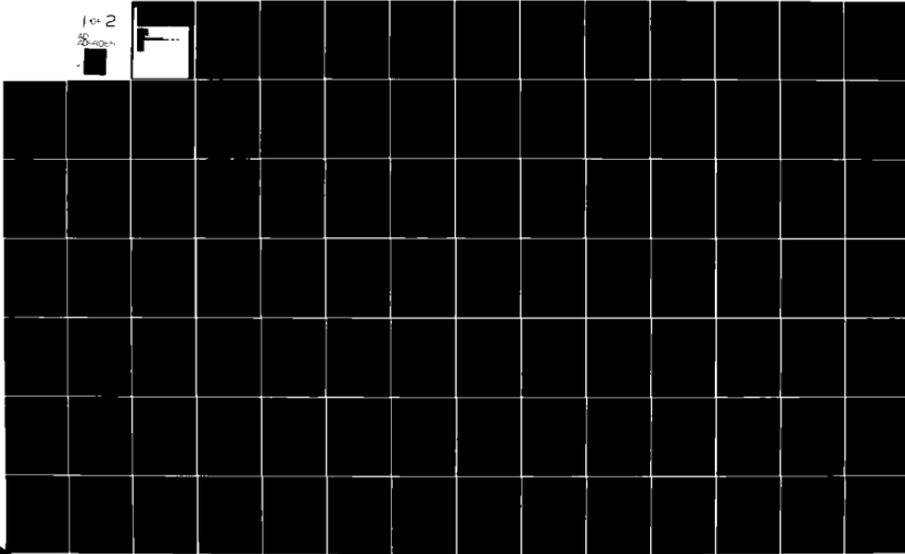
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RAPID SOLUTION OF FINITE ELEMENT EQUATIONS
ON LOCALLY REFINED GRIDS BY MULTI-LEVEL METHODS

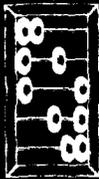by

John R. Van Rosendale

May 1980

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DEPARTMENT OF COMPUTER SCIENCE
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This thesis is concerned with the use of multi-level methods to solve the
linear systems arising from finite element discretizations of elliptic
equations. In all, three multi-level methods are considered. The first
of these is applicable only to quasi-uniform grids, but is simpler than
other algorithms considered in previous theoretical work. The other two
algorithms are applicable to both quasi-uniform grids, and locally refined
grids, those grids on which the size of the largest and smallest elements

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20. may differ by an arbitrarily large factor. All three algorithms are asymptotically optimal, producing good solutions in $O(N)$ operations on a finite element grid with N elements. These asymptotically optimal complexity bounds for the last two algorithms are the first such bounds for multi-level methods on locally refined grids. One of these algorithms achieves this $O(N)$ complexity bound under weaker than expected conditions on the dimensions of the finite element spaces used by the algorithm.

The multi-level convergence results for locally refined grids shown here are based on a new approximation result given in this thesis. This approximation result is of interest for several reasons, the main one being that it is completely local, making no use of global properties such as the regularity of the problem. In consequence, it provides an independent demonstration of the asymptotically optimal complexity of multi-level algorithms on non-convex domains, shown previously by Bank and Dupont. It also permits one to determine explicit upper bounds on the rate of convergence of multi-level methods on irregular finite element grids using only local properties of the finite element spaces involved.

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1. INTRODUCTION

1.1. Scope of Thesis

Multi-level, or multi-grid, methods have existed for more than fifteen years, but have only recently become popular. Thus, there remain many unanswered theoretical questions. Several of these are addressed in this thesis, the major one being the computational complexity of multi-level methods for locally refined finite element grids. These are grids for which the ratio of diameters of the largest and smallest elements is very large, becoming unbounded on the family of discretizations of interest. The principal result of this thesis is that multi-level methods can approximately solve the linear algebraic systems for such finite element discretizations in $O(N)$ operations for a grid with N elements. Such asymptotically optimal complexity bounds have been given previously for multi-level methods on non-locally-refined grids, but not for locally refined grids. This result provides theoretical justification for the use of multi-level methods on locally refined grids, which is becoming increasingly popular. The proof of this result, which occupies almost half of this thesis, is also of interest for several reasons. Perhaps the most significant of these is that the rate of convergence shown for the multi-level iteration used depends only on local properties of the finite element spaces involved. Global properties, including elliptic regularity, are never invoked. As a consequence this result provides an independent verification of the $O(N)$ complexity of multi-level methods for non-convex domains, established previously by Bank and Dupont (1978). It also opens the door to easily computable bounds on the convergence rate of multi-level methods on irregular finite element grids.

In all, three multi-level algorithms are considered in detail
in this thesis. The first, analyzed in chapter three, is similar to
methods considered by Nicolaides (1977) and by Bank and Dupont (1978),
but is simpler than these others. It is shown to converge in $O(N)$
operations for N-point quasi-uniform finite element grids on convex
domains, through a convergence proof similar to those given by the above
authors. This proof is given here since it is less complex than an
earlier one based on Fourier analysis of the truncation error. The
algorithm for quasi-uniform grids is treated here partly as background
for the analysis of algorithms for locally refined grids in chapter four.
It is also of independent interest that an $O(N)$ bound can be shown for
this algorithm even though it is simpler than those in the literature.

Two algorithms are considered in chapter four, both applicable
to locally refined grids on arbitrary polygonal domains. The first of
these is quite similar to the method used in the adaptive finite element
code of Bank and Sherman (1978). It is the closest algorithm to theirs
that could be analyzed by the theoretical techniques here. This method
is shown to converge in $O(N)$ operations subject to a condition on the
rate of growth of the dimensions of the finite element spaces used. The
necessary growth condition requires that the finer finite element spaces
used by the multi-level algorithm have far more elements than the
coarser spaces used. Such a condition is also imposed in the Bank and
Sherman code though it is less stringent there due to their technique
of "level compression," Bank and Sherman (1978).

Since these growth conditions can be quite restrictive, a
second algorithm is also considered in chapter four. It is a modification

of the first, and it will be shown to converge in O(N) operations under a considerably weaker growth condition on the dimensions of the finite element spaces used. Though the theoretical interest in this modification is apparent, the author is at present unsure of the practical utility of this second algorithm. One pays a large penalty for the weaker growth conditions through an increase in the hidden constant in the O(N) work bound.

1.2. History of Multi-Level Methods

Multi-level or multi-grid methods are usually thought of as originating in the work of Fedorenko (1961, 1964), although related ideas had been put forth earlier. The idea of using several discretizations in the solution of a PDE (partial differential equation) is actually quite old. Southwell (1940) suggested using the solution of an elliptic problem on a coarse finite difference grid to obtain a starting value for relaxation on a finer grid. He also recommended a technique called "block relaxation" which is closely related to multi-level methods. In this technique the person doing the relaxation calculation (in the era before computers) would find a region of the finite difference grid where the residuals were all of the same sign. A constant would then be added to the value of the trial solution throughout this region to reduce the average residual on this region as much as possible. As a consequence, after this correction the residuals would oscillate in sign on this region. Southwell observed that as long as there were no large areas where the residual did not alternate in sign, relaxation converged rapidly. Of course, though quite effective, such a heuristic approach was not of much help in the early days of computers, so simpler methods like SOR took over.

The idea of making block corrections essentially died out until Fedorenko (1961) observed that rather than making corrections in isolated blocks as suggested by Southwell, one could make a global correction by solving a so-called coarse grid residual problem. For example, suppose that in iteratively solving Poisson's equation on a grid with mesh size h one has at the n-th iteration a trial vector $\underline{u}_n^h \in G_h$ and residual $\underline{r}_n^h \in G_h$ where G_h is the vector space of mesh functions on the grid. If Δ_h is the discrete Laplacian being used,

$$\underline{r}_n^h = \underline{f}^h - \Delta_h \underline{u}_n^h$$

where $\underline{f}^h$ is the data vector. Now let G_{2h} be the vector space of mesh functions on a coextensive coarser grid with mesh size 2h and suppose we have an interpolation mapping

$$I_{2h}^h: \quad G_{2h} \to G_h .$$

One can ask for a correction vector $\underline{v}^{2h} \in G_{2h}$ such that the improved solution

$$\underline{u}_{n+1}^h = \underline{u}_n^h + I_{2h}^h \underline{v}^{2h}$$

would have smaller residual. This idea has been considered independently by others, Wachspress (1974). It is attractive since $\underline{v}^{2h}$ lies in the lower dimensional space G_{2h} and so should be relatively easy to compute. Fedorenko's fundamental contribution was to note that $\underline{v}^{2h}$ could be chosen as

(1.2.1) $$\Delta_{2h} \underline{v}^{2h} = I_h^{2h} \underline{r}_n^h$$

where $I_h^{2h}: G_h \to G_{2h}$ is an interpolation mapping and Δ_{2h} is a discrete Poisson operator on G_{2h}. Since (1.2.1) is just a discretization of Poisson's equation with data $I_h^{2h} \underline{r}_n^h$, he realized that (1.2.1) could be

approximately solved by relaxation and the recursive use of corrections on a still coarser grid G_{4h}.

Fedorenko's theoretical work showed that for a model problem multi-level iteration reduced the convergence error by an amount per iteration independent of the mesh size. Thus, reducing the convergence error to a magnitude comparable to the truncation error required $O(N \log N)$ operations on an N point grid, the same as ADI. Shortly thereafter Bakhvalov (1966) considered beginning the solution process on a very coarse grid, using the solution there as a starting value for multi-level iteration on a slightly finer grid, and so on, gradually descending down to the level of refinement desired. This idea, which was probably old when Southwell was young, changes the work bound from $O(N \log N)$ to $O(N)$. No other current method, either direct or iterative, achieves this asymptotically optimal work bound.

Besides improving the asymptotic work bound on multi-level iteration, Bakhvalov extended the method to a large class of PDEs on a square finite difference grid. After his work, almost no work seems to have been done on multi-level methods until this decade. Then several people became interested in this method including Brandt, Hackbusch and Nicolaides. Brandt in particular is largely responsible for popularizing this method and developing it into a practical computational tool. The other two people mentioned have been working more on the theoretical questions involved in this method and on generalization to finite element grids.

In the last few years there has been a rapid growth of interest in multi-level methods, and there is considerable ongoing research. This

research is directed both at the theoretical problems, Bank and Dupont
(1978), Hackbusch (1978), Nicolaides (1977), and at developing algorithms
for the many applications areas. Representative work of the latter
type may be found in Bank and Sherman (1978), Brandt (1977), Brandt,
Dendy and Ruppel (1978), Nicolaides (1975), and Poling (1977). If
the author is not mistaken, we should see continued research in this
area for many years and may eventually see multi-level methods as
the method of choice for most parabolic and elliptic problems.

1.3. The Finite Element Approach

Design and analysis of multi-level methods can be done from
either a finite difference or finite element point of view. In one
sense these two approaches are almost the same; finite element methods
are a special class of finite difference method. Nevertheless, the way
people think about finite element methods and the kinds of grids they
use for these methods are so different from the usual way people
understand and use finite difference methods that we may think of these
classes as completely separate. This thesis is devoted to analysis
of multi-level methods from the finite element point of view. Several
of the many reasons the author had for adopting this point of view will
be described in this section.

While the multi-level method originated as a means of solving

finite difference equations and was, until recently, solidly wedded to

the finite difference approach, much of the current work on multi-level

methods is being done in the finite element context. Starting with

Hackbusch (1977) and Nicolaides (1977), who seem to have been the first

to consider multi-level methods as a means of solving finite element

equations, there has been increasing interest in this approach. The

reasons for this are not complicated. Finite element theory is in many

respects superior to finite difference theory. A great many problems

in the finite difference approach either do not arise in the finite

element approach or are easily handled there. Questions of stability,

the use of irregular grids, treatment of corner singularities, treatment

of curved boundaries and construction of error estimates are all much

simpler in the finite element context.

Aside from the obvious (to the author) superiority of the

finite element method, there are special reasons unique to multi-level

algorithms for doing their theory in the finite element setting. The

first of these is that the finite element error estimates are based on

approximation results rather than on asymptotic expansions. What this

means for multi-level methods is that, unlike some of the early finite

difference arguments where rapid convergence of the iteration could be

shown only for sufficiently small size, no such proviso is ever

needed in the finite element context. A second, more important reason

is that multi-level algorithms always involve more than one

discretization. Thus one requires mappings from the space of grid

functions or finite element functions of each discretization to those of

other discretizations. In the finite difference context such mappings
are done by interpolations which require separate error analyses. In
the finite element context the grids can be designed so that the
finite element space on a coarser grid will be a subspace of the finite
element spaces on finer grids. Then the mapping from a coarser grid to
a finer grid will be just the natural injection. To go the other way,
some kind of projection is needed, but many natural ones are available.
The fact that different finite element discretizations can be related
in this way yields a great simplification of the theory of multi-level
methods. Unfortunately only the theory is simplified, not the
programming. Rather than an interpolation formula, one has a similar
formula relating the basis functions of one finite element space to
those of another. It is interesting that in many cases these change
of basis formulas arising in the finite element setting are exactly
the same as the interpolation formulas already in use in the finite
difference setting.

There is another reason why studying multi-level convergence
is important for this thesis. In chapter four we treat multi-level
methods for locally refined grids. While it may be possible to analyze
multi-level methods for locally refined finite difference grids in
special cases, it seems unlikely that an analysis as general as that
given here would be possible outside finite element theory. For
locally refined grids, finite element theory also helps in coding.
One never needs to choose special formulas for points where the mesh
size changes abruptly or worry about truncation error. There are
also beginning to be reliable a posteriori error estimates for finite

elements enabling one to design robust adaptive algorithms, Babushka and Rheinboldt (1978). In short, multi-level methods, adaptive refinement, and finite element theory mesh together quite nicely.

The main limitation of the finite element approach is that it is far more natural and powerful for self-adjoint problems than for the general case. Neither finite difference nor finite element methods work very well for non-self-adjoint problems, but it is frequently easier to see what to do on regular finite difference grids than on general finite element grids. While it would have been possible to extend many of the results of this thesis to the non-self-adjoint case by adding various restrictions, that was a more ambitious project than the author was prepared to attempt. If one takes the results here as indicative of the kind of results that could be proved in the general non-self-adjoint case, one should not go far wrong.

2. PRELIMINARIES

In this chapter we collect notations and basic results from finite element theory. Since this material is completely standard, the reader familiar with finite elements can probably skip this chapter entirely, referring back to it later as required. The reader less familiar with the finite element approach to partial differential equations may find this chapter helpful.

2.1. Elliptic Equations

Throughout this thesis we will be concerned with solving self-adjoint second order boundary-value problems. The problem of most interest will be the Dirichlet problem

$$(2.1.1a) \qquad Lu = f \text{ on } \Omega$$

$$(2.1.1b) \qquad u = 0 \text{ on } \partial\Omega ,$$

where Ω is a bounded open domain in the plane $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. L may be any self-adjoint second order operator, for example the Laplacian, or more generally any operator in divergence form

$$Lu = -\nabla \cdot (a \nabla u) + bu$$

where a, b are C^∞ functions on Ω. In order for this operator to be elliptic on Ω, we require that there be constants $\underline{a}$, $\bar{a}$, $\bar{b}$ such that

$$\underline{a} \leq a \leq \bar{a}$$

$$0 \leq b \leq \bar{b}$$

hold throughout Ω. We will also be considering the Neumann problem

$$(2.1.2a) \qquad Lu = f \text{ on } \Omega$$

$$(2.1.2b) \qquad u_n = 0 \text{ on } \partial\Omega ,$$

where u_n is the derivative of u in the direction normal to the boundary.

In this case ellipticity requires that b be positive on some neighborhood in Ω. Bank and Dupont consider also the singular Neumann problem, $b \equiv 0$, but we exclude it here.

The finite element approach to boundary-value problems is based in part on the functional analytic theory of partial differential equations, in which the boundedness or invertibility of a differential operator is expressed in terms of Sobolev norms. These norms are a natural generalization of the L^2 norm, defined for non-negative integers ℓ by

$$\|u\|_\ell = \left(\sum_{0 \le |\alpha| \le \ell} \|D^\alpha u\|_{L^2}^2 \right)^{1/2} ,$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index with $\alpha_1, \alpha_2 \ge 0$ and

$$|\alpha| = \alpha_1 + \alpha_2$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} .$$

This definition makes sense for all functions on Ω whose distribution derivatives up to order ℓ exist and lie in L^2. The completion of $C^\infty(\Omega)$ in $\|\cdot\|_\ell$ is denoted $H^\ell(\Omega)$ and is a Hilbert space relative to the inner product

$$(u,v)_\ell = \int_\Omega \left(\sum_{0 \le |\alpha| \le \ell} (D^\alpha u)(D^\alpha v) \right) .$$

If ℓ is a negative integer $\|\cdot\|_\ell$ is defined by duality,

$$\|u\|_\ell = \sup_{v \in \bar{H}^{-\ell}(\Omega)} \frac{(u,v)}{\|v\|_\ell} .$$

Here and throughout, $(\cdot, \cdot)$ is the L^2 inner product while $\|\cdot\|$ with no subscript denotes the L^2 norm. Sobolev spaces can also be defined for arbitrary real ℓ either through the interpolation theory of Hilbert

spaces or by Fourier transforms. Details may be found in Oden and Reddy (1976).

Equations (2.1.1) and (2.1.2) implicitly require the solution u to be twice differentiable, which is overly restrictive for many purposes. Consider the problem

(2.1.3a) $Lu = f$ on Ω

(2.1.3b) $u = 0$ on Γ_1

(2.1.3c) $u_n = 0$ on Γ_2 ,

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$, which includes both (2.1.1) and (2.1.2) as special cases. Let $H_E^1(\Omega)$ be the subspace of $H^1(\Omega)$ satisfying the (essential) Dirichlet boundary condition (2.1.3b) but otherwise unrestricted. If (2.1.3) is satisfied then

$$(Lu,v) = (f,v), \qquad v \in H_E^1(\Omega) \ .$$

The notation here means that equality holds for all test functions v in $H_E^1(\Omega)$. Defining the bilinear form

$$a(u,v) = (Lu,v) \ ,$$

we can ask for the element u in $H_E^1(\Omega)$ such that

(2.1.4) $a(u,v) = (f,v), \qquad v \in H_E^1(\Omega) \ .$

It is only necessary that u lie in H_E^1 since Green's theorem can be used to shift one derivative from u to v. A function u satisfying (2.1.4) is called a weak solution of (2.1.3), and (2.1.4) is called the weak form of the equation. The advantage here is that the weak solution exists quite generally and agrees with the classical solution satisfying equation (2.1.3) pointwise when the latter exists.

The condition for existence and uniqueness of the solution of (2.1.4) is that the problem be elliptic. This means that the energy norm $||| \cdot |||$ defined by

$$||| u |||^2 = a(u,u)$$

for $u \in H^1_E(\Omega)$ is equivalent to the first Sobolev norm $|| \cdot ||_1$. That is,

$$\sigma_1 || u ||_1 \leq ||| u ||| \leq \sigma_2 || u ||_1 , \qquad u \in H^1_E(\Omega) ,$$

for fixed σ_1, $\sigma_2 > 0$ independent of u.

If the problem (2.1.3) is elliptic, a simple calculation shows that for ℓ a non-negative integer

$$|| u ||_{\ell+2} \geq c || Lu ||_\ell , \qquad u \in H^{\ell+2}(\Omega) ,$$

for some $c > 0$. The principal results in elliptic theory go the other way. For $\ell > -\frac{3}{2}$ and any $f \in H^\ell(\Omega)$ the weak solutions of (2.1.1) or (2.1.2) satisfy the regularity estimate

(2.1.5)
$$|| u ||_{\ell+2} \leq c || f ||_\ell$$

for some $c > 0$, assuming that the domain is smooth and the coefficients of L are C^∞. For the technical definition of smooth domains see Babushka and Aziz (1972) or Oden and Reddy (1976).

The regularity estimate (2.1.5) implies existence, uniqueness, and that the solution has two more derivatives in the mean square sense than its data. In particular, if f is infinitely differentiable, u will be as well. These results hold only for smooth domains. For domains that are only piecewise smooth the corners introduce singularities that limit the regularity of the problem. On a convex polygonal domain the solution will lie in H^2 but not necessarily in H^ℓ for $\ell > 2$. In this case we have the regularity estimate

(2.1.6)
$$\|u\|_{\ell+2} \le c \|f\|_{\ell}$$

for $-\frac{3}{2} < \ell \le 0$. For non-convex polygonal domains (2.1.6) will not hold for $\ell = 0$ because the reentrant corners, that is corners with interior angles exceeding π, introduce severe singularities. In the worst case of a slit domain, ℓ must be restricted to $(-\frac{3}{2}, -\frac{1}{2})$. The solution has less than one and a half derivatives in the mean square sense.

2.2. Finite Element Spaces

The finite element discretization of (2.1.3) begins with the selection of a finite dimensional subspace M of $H^1_E(\Omega)$. The discrete solution to (2.1.3) is taken as that function $\tilde{u} \in M$ satisfying the finite element equation

(2.2.1)
$$a(\tilde{u},\phi) = (f,\phi), \quad \phi \in M .$$

By the Ritz theorem, holding for self-adjoint problems, $\tilde{u}$ is the function in M nearest to the true solution u in energy norm.

Finite element spaces are ordinarily constructed by dividing the domain into a finite family of elements, usually triangles or rectangles. Once the elements are chosen M may be taken as a finite dimensional space of piecewise polynomial functions that are polynomial on each element. Though it is not strictly necessary for multi-level methods, throughout we will require that M be conforming, that is that it lie in $H^1_E(\Omega)$. This translates into the requirement that the functions in M be continuous and satisfy the essential boundary condition (2.1.3b) exactly. We also require that M be of degree at least one in order to obtain convergence. For a finite element space of piecewise polynomial functions the degree is the integer k such that any polynomial of degree

k can be exactly represented on every element away from the boundary. Finally it is necessary to bound the highest degree polynomial on any element and require that M have a local basis. This is in order that the basis be uniformly linearly independent. The question of the linear independence of the basis, or conditioning of the Gram matrix will be taken up in the next section.

Approximation questions are dealt with in finite element theory in terms of a sequence of finite element spaces of increasing dimension $\{M_j\}_{j=1}^{\infty}$, which eventually become dense in the sense that for any neighborhood N in $H_E^1(\Omega)$ there exists J such that

$$N \cap M_j \neq 0 \text{ for } j > J \ .$$

The approximation properties of each of these spaces M_j depend on the size of the elements $\{e\}$ of M_j and the element geometry, as well as the degree of the finite element space. Element geometry matters only to the extent that it must not degenerate as j increases. For standard finite elements it suffices that all elements in M_j be convex and have interior angles bounded away from 0 and π uniformly in j.

The size of an element e is measured by its diameter, d_e, defined as

$$d_e = \sup_{x,y \in e} \|x-y\|_{\ell^2} \ ,$$

where $\|\cdot\|_{\ell^2}$ denotes the Euclidean norm. The family of spaces $\{M_j\}$ is said to be quasi-uniform if there is a corresponding family of parameters $\{h_j\}$ and $\sigma > 1$ such that

(2.2.2)
$$\frac{1}{\sigma} h_j \leq d_e \leq \sigma h_j$$

for all elements e of M_j and all j. If not, that is if the constant σ

must depend on j, the family of spaces $\{M_j\}$ is said to be locally refined.

While strictly speaking these concepts apply only to an infinite family

of finite element spaces, we will talk about an individual finite element

space being quasi-uniform or locally refined. This usage is extremely

convenient though it should be born in mind that this really makes sense

only with respect to membership in an infinite family of spaces $\{M_j\}$.

This definition of the mesh parameter h_j associated with a

quasi-uniform finite element space M_j is not completely standard. Often

h_j is taken as the maximum of the element diameters. However, the

extra freedom allowed by (2.2.2) makes no difference in the error

estimates since σ is a fixed constant, and is quite convenient. In

particular this freedom makes it easy to have a quasi-uniform family

$\{M_h\}_{h\in(0,1)}$ parameterized by h rather than j. In this case the diameters

of elements in M_h are required to satisfy

$$\frac{1}{\sigma} h \leq d_e \leq \sigma h .$$

The standard finite element error estimates give the rate

of convergence in any Sobolev norm of the finite element approximations

in a quasi-uniform space M_h in terms of the mesh parameter h and the

degree of M_h. For smooth solutions the convergence rate may be made

arbitrarily high by taking finite element spaces $\{M_h\}$ of sufficiently

high degree. But on a polygonal domain the corner singularities limit

the rate of convergence attainable by any family of piecewise

polynomial finite element spaces.

The most important error estimate in the next chapter will be

the L^2 error estimate

(2.2.3)
$$\|u-u^h\|_{L^2} \leq Ch^2 \|f\|_{L^2} ,$$

which holds for quasi-uniform spaces $\{M_h\}$ of degree at least one when

the partial differential equation is H^2 regular, that is, when (2.1.6)

holds with $\ell = 0$. Thus, this bound applies to the Dirichlet problem

(2.1.1) and the Neumann problem (2.1.2) on any convex polygonal domain.

The convergence in energy for these two problems is of first

order on any convex polygonal domain. On non-convex domains it is of

less than first order. For a domain with a slit, the worst case

ordinarily arising, it is of order one-half.

Much of the utility of the finite element method comes from

the fact that these meager convergence rates attained on polygonal

domains are easily improved through the addition of singular functions

or by doing local refinement to cope with the corner singularities. In

chapter four the use of multi-level methods on locally refined finite

element grids will be considered. A simple error estimate showing

the quality of approximation of finite element methods on such grids

will be given there.

2.3. Linear Equations

By selecting a basis for the finite element space M, the

finite element equation (2.2.1) is translated into a linear system.

Let M be a finite element space for problem (2.1.3) and let $\{\phi_i\}_{i=1}^N$

be a basis for M. Any function $u \in M$ can be expressed in this basis as

$$u = \sum_{i=1}^N \underline{u}_i \, \psi_i ,$$

where $\underline{u}$ is the coordinate vector of u relative to this basis. The convention of using underbars for vectors will be adhered to throughout.

Given the basis $\{\phi_i\}$, we can define the mass and stiffness matrices M and K with elements

$$m_{ij} = (\phi_i, \phi_j)$$

$$k_{ij} = a(\phi_i, \phi_j)$$

respectively. M is just the Gram matrix of the basis relative to the L^2 inner product while K is the Gram matrix in the energy inner product. To form the linear system corresponding to the Ritz equation (2.2.1), let $f \in L^2(\Omega)$ and let $\underline{F}$ be given by

$$\underline{F}_i = (f, \phi_i), \quad i = 1,\ldots, N .$$

Then (2.2.1) is equivalent to the linear system

(2.3.1) $$K\underline{\tilde{u}} = \underline{F}$$

where $\underline{\tilde{u}}$ is the coordinate vector of the finite element solution.

Numerical solution of the linear system (2.3.1) is aided by several properties the matrices K and M have. First, M will always be symmetric positive definite and since L is self-adjoint K will be symmetric positive definite as well. Second, for the usual finite element bases M and K will be quite sparse. Finally, both M and K are usually well conditioned. The common finite element bases are largely motivated by these last two considerations which are of great importance for the numerical inversion of (2.3.1) or (2.3.2) whether this is done directly or iteratively.

To make these considerations precise, consider a family of finite element spaces $\{M_j\}_{j=1}^{\infty}$ of increasing dimension. We require that each space M_j, $1 \leq j < \infty$, have a basis $\{\phi_i^{(j)}\}_{i=1}^{N_j}$ that is local in the

sense that the number of non-zero elements in each row of the corresponding
mass and stiffness matrices is bounded uniformly in j. This will be true
if the number of basis functions in $\{\phi_i^{(j)}\}_{i=1}^{N_j}$ whose supports intersect
the supports of any given basis element $\phi_i^{(j)}$ is bounded uniformly in j.

Usually, under these conditions the condition number $\kappa(M_j)$ of
the mass matrix M_j, $1 \leq j < \infty$, will be uniformly bounded. For this to
hold the proper scaling must be chosen and the condition numbers of
the element mass matrices must be uniformly bounded. This is true for
the common finite element bases as long as the element geometry does
not degenerate. Then if the basis elements are scaled as

$$\|\phi_i^{(j)}\| = 1, \qquad 1 \leq i \leq N_j , \qquad 1 \leq j < \infty ,$$

which we assume throughout, it is immediate that the eigenvalues of
M_j will lie in an interval $[\frac{1}{\sigma}, \sigma]$ for $\sigma > 1$ independent of j.

Since the stiffness matrices $\{K_j\}$ are exactly analogous to
the matrices arising in finite difference discretizations, their
condition numbers cannot be uniformly bounded. Just as in the finite
difference case the maximum eigenvalue of K_j satisfies

(2.3.2) $$\lambda_{max}(K_j) \leq ch_{min}^{-2}$$

under ordinary circumstances, where h_{min} is the diameter of the
smallest element of M_j. The minimum eigenvalue of K_j can be bounded
below by a simple argument. We have

$$\lambda_1(K_j) = \min \frac{\underline{x}^T K_j \underline{x}}{\underline{x}^T \underline{x}} \geq \min \frac{\underline{x}^T K_j \underline{x}}{\underline{x}^T M_j \underline{x}} \min \frac{\underline{x}^T M_j \underline{x}}{\underline{x}^T \underline{x}}$$

$$= \lambda_1(M_j^{-1} K_j) \lambda_1(M_j)$$

$$\geq \lambda_1(M_j^{-1} K_j)/\sigma .$$

The operator $M_j^{-1} K_j$ is the linear operator corresponding to the finite element equation (2.2.1). Since the finite element eigenvalues approximate the eigenvalues of the PDE from above, $\lambda_1(K_j)$ must be bounded below uniformly in j. Thus one ordinarily has

(2.3.3)
$$\kappa(K_j) \leq ch_{min}^{-2}$$

or for a quasi-uniform spaces M_h.

(2.3.4)
$$\kappa(K_h) \leq ch^{-2} .$$

The difficulty in iteratively inverting the linear system (2.3.1) arises directly from this growth of the condition of the stiffness matrix as h decreases.

These bounds on the condition of M and K are proved in Fried (1971) and may be found also in Strang and Fix (1973).

3. QUASI-UNIFORM GRIDS

In this chapter a multi-level algorithm simpler than those considered in previous theoretical work will be given. It will be shown to be of optimal order, producing a good solution on a quasi-uniform finite element grid in O(N) operations on an N point grid.

3.1. Introduction

In this chapter we look at a multi-level algorithm for self-adjoint elliptic boundary value problems on quasi-uniform finite element grids. This algorithm is related to algorithms considered by Nicolaides, Bank and Dupont, Hackbusch and others, but is less complex than these other algorithms. Like many of those considered by others, the algorithm here is of optimal order, producing a second order accurate solution in O(N) operations on an N point grid.

The optimal order convergence of the algorithm in this chapter was originally shown for finite difference discretizations by a cumbersome Fourier technique, and then for finite element discretizations in the same way. Here an easier proof, following the ideas of the above authors, will be given. The proof is based particularly on the work of Bank and Dupont but employs several minor simplifications. Throughout we follow, for the most part, their notation. It is convenient here since little reference to algebraic quantities such as vectors or matrices will be made.

3.2. L^2 Convergence

In this section and the next we examine the convergence of a simple multi-level algorithm. This algorithm can be used to obtain finite element approximations to the solutions of the elliptic problems (2.1.1) and (2.1.2) on a convex polygonal domain Ω. The restriction to convex domains is necessary since we need to use the L^2 error estimate (2.2.3) which requires H^2 regularity. Using curved elements as in Bank and Dupont [1978] the convergence results given here extend immediately to piecewise smooth domains free of reentrant corners. For details on the treatment of curved elements, the interested reader is referred to their paper.

The algorithm considered here uses a nested family of quasi-uniform finite element spaces $\{M_j\}_{j=1}^{\infty}$, nested in the sense that

$$M_j \subset M_{j+1} , \qquad 1 \le j < \infty .$$

To approximately solve the finite element equations for one of these spaces M_k, $k \ge 2$, this algorithm uses a simple simultaneous displacement smoothing iteration applied to the equations for M_k plus approximate solution of related problems in M_{k-1}. If $k - 1 \ge 2$ these approximate solutions are obtained recursively by applying the same algorithm to the equations for M_{k-1}. The recursion continues, finally requiring approximate solutions to the equations for M_1. These can be found directly or by some convenient iteration.

The multi-level method considered here is related to algorithms for finite elements by Nicolaides, Hackbusch, and Bank and Dupont and to methods for finite difference equations considered by Brandt, Hackbusch, Nicolaides, Bakhvalov, Fedorenko and others. The algorithm here differs

from algorithms described by others in that the solution to the equations for M_k requires only two approximate solutions to the equations for M_{k-1}. In other multi-level algorithms, to solve the equations for M_k approximate solution of related problems in M_{k-1} is used in two ways. First it is used as a means of obtaining a good starting value for iteration on M_k. Subsequently it is used as an acceleration procedure for iteration on M_k.

The first approximate solution of the equations for M_{k-1} is used to obtain a starting value for iteration on M_k, and considerable accuracy is required. It is necessary to reduce the convergence error of the problem on M_{k-1} so it is comparable to the truncation error on M_{k-1}. In subsequent solutions of related problems in M_{k-1} used to accelerate convergence of the iteration in M_k less accuracy is required. For example, in one of the proofs given by Bank and Dupont and in the related proof of theorem 4.1 in chapter four of this thesis it suffices to solve the related problem with a relative accuracy of 33%.

The point of this chapter is that if all of the related problems in M_{k-1} are solved to the same accuracy, namely the error in the approximate solutions must be comparable to the truncation error in M_{k-1}, then only two approximate solutions in M_{k-1} are required. To approximately solve the elliptic problem (2.1.1) or (2.1.2) in M_k, $k \geq 2$, the algorithm begins by obtaining a good approximate solution to the finite element equations for M_{k-1}. This approximate solution is used as a starting value for a smoothing iteration on M_k. Though this iteration could be continued until adequate convergence was attained this would be inefficient. Instead, after a fixed number of iterations, m, independent of k, we terminate the iteration.

The resulting approximate solution is then corrected by subtracting from it the projection of the convergence error onto M_{k-1}. This corrected approximate solution in M_k is then taken as the accepted solution.

The reason this is effective goes back to the work of Southwell. He observed that at least initially relaxation rapidly reduces the residual, while the error itself may be only slowly reduced. At the end of our algorithm's smoothing iteration the residual will be small relative to the convergence error. This means that the convergence error is quite smooth. In this case the convergence error can be well approximated in the coarser space M_{k-1} and the final correction will remove most of it.

Brandt describes this same phenomenon in terms of the reduction of Fourier components of the convergence error. Most convergence proofs, including ours, follow Fedorenko who considered the reduction of the error components in the direction of eigenfunctions of the discrete partial differential operator. Fedorenko refers to "good" meaning smooth eigenfunctions and "bad" meaning oscillatory eigenfunctions.

For practical purposes many possible smoothing iterations would be acceptable. Jacobi, Gauss-Seidel, and symmetric SOR are most often used. The author expects that improvement could be made through the use of Chebyshev acceleration, although in numerical tests so far changing to more subtle iterative schemes has had little effect for the well behaved self-adjoint problems considered here (Bank and Sherman [1979], Brandt [1977]).

In this section the smoothing iteration will be taken as a simple but computationally unattractive simultaneous displacement iteration considered also by Bank and Dupont. The advantage of this

iteration is that the resulting convergence is easy to analyze and is basis independent. In the next section we will show that this smoothing iteration can be replaced by a more computationally attractive iteration without altering the convergence result. In this case the algorithm will be shown to be of optimal order, producing a second order accurate solution in $O(N)$ operations on a finite element grid with N elements.

Let $\{M_j\}_{j=1}^{\infty}$ be a nested family of quasi-uniform finite element spaces, and suppose we have associated mesh parameters $\{h_j\}$ satisfying

$$(3.2.1) \qquad h_j = \rho^{1-j} h_1 , \qquad j \geq 1 ,$$

for $\rho > 1$ a fixed constant. Let $u^{(j)} \in M_j$, $j \geq 1$, be corresponding finite element solutions of problem (2.1.1) or (2.1.2). That is, $u^{(j)} \in M_j$ are functions satisfying

$$(3.2.2) \qquad a(u^{(j)},\phi) = (f,\phi) , \qquad \phi \in M_j .$$

From the error estimate (2.2.3)

$$(3.2.3) \qquad \|u-u^{(j)}\| \leq c \, h_j^2 \, \|f\|$$

for fixed $c > 0$ where u is the true solution. In the space M_j associated with the finite element equation (3.2.2) we have eigenfunctions $\{\psi_i^{(j)}\}_{i=1}^{N_j}$, where N_j is the dimension of M_j, and associated eigenvalues $\{\lambda_i^{(j)}\}_{i=1}^{N_j}$ with

$$0 < \lambda_1^{(j)} \leq \lambda_2^{(j)} \leq \ldots \leq \lambda_{N_j}^{(j)} .$$

That is

$$a(\psi_i^{(j)}, \phi) = \lambda_i^{(j)}(\psi_i^{(j)}, \phi) , \qquad \phi \in M_j ,$$

for $1 \leq i \leq N_j$. From the eigenvalue bound (2.3.2) and the fact that the eigenvalues of M_j, the mass matrix corresponding to M_j, lie in $[\frac{1}{\sigma}, \sigma]$ for $\sigma > 1$ independent of j, we have the bound

(3.2.4)
$$\lambda_{N_j} \leq c \, h_j^{-2} .$$

To solve the finite element equation in M_k, $k \geq 2$, the algorithm to be considered produces a sequence of iterates u_1, u_2,...., u_{m+1} in M_k beginning with an approximate solution u_0 to the finite element equations in M_{k-1}. For some fixed positive integer m, the approximate solution u_{m+1} will be taken as the final answer. The iterates u_ℓ, $1 \leq \ell \leq m$, are produced by a simple smoothing iteration while the last one, which will be the accepted solution, is the result of a correction involving the approximate solution of a related problem in M_{k-1}. For a fixed constant $c_0 > 0$ and data f in $L^2(\Omega)$ the algorithm is as follows:

1. Let $u_0 \in M_{k-1}$ be an approximate solution to (3.2.2) satisfying

(3.2.5)
$$\|u_0 - u^{(k-1)}\| \leq c_0 \, (h_{k-1})^2 \|f\|$$

where $u^{(k-1)}$ is the solution of (3.2.2).

2. Let u_ℓ, $1 \leq \ell \leq m$ be the element of M_k satisfying

(3.2.6)
$$(u_\ell - u_{\ell-1}, \phi) = \frac{-1}{\lambda_{N_k}^{(k)}} (a(u_{\ell-1}, \phi) - (f, \phi)) , \qquad \phi \in M_k .$$

3. Let $\bar{v} \in M_{k-1}$ be the solution of

$$a(\bar{v}, \phi) = a(u_m, \phi) - (f, \phi) , \qquad \phi \in M_{k-1} .$$

That is, $\bar{v}$ is given by

$$a(\bar{v},\phi) = -(r,\phi) , \qquad \phi \in M_{k-1}$$

where $r \in M_k$ is defined by

$$(r,\phi) = (f,\phi) - a(u_m,\phi) , \qquad \phi \in M_k .$$

Let v be an approximation to $\bar{v}$ satisfying

(3.2.7)
$$\|v-\bar{v}\| \le c_0 \, h_{k-1}^2 \, \|r\|$$

and let u_{m+1} be given by

(3.2.8)
$$u_{m+1} = u_m - v$$

The corrected approximate solution u_{m+1} is taken as the final answer.

If we can show that the approximate solution, u_{m+1}, produced is accurate enough uniformly in $k \ge 2$, the use of this algorithm recursively to solve the subproblems in steps one and three will be immediately justified. This is the content of the following theorem.

Theorem 3.1. For any $c_0 > 0$ there is a positive integer m, independent of k, such that the approximate solution u_{m+1} produced by algorithm (3.2.5)-(3.2.8) satisfies

(3.2.9)
$$\|u_{m+1} - u^{(k)}\| \le c_0 \, h_k^2 \, \|f\| .$$

Proof. For convenience we drop the superscript k on eigenfunctions and eigenvalues and the dimension N_k of M_k. From (3.2.3) and the triangle inequality

$$\|u^{(k)} - u^{(k-1)}\| \le c(h_k^2 + h_{k-1}^2) \, \|f\|$$

$$\le c(1 + \rho^2) \, h_k^2 \, \|f\| .$$

From this and (3.2.5)

$$(3.2.10) \qquad \| u_0 - u^{(k)} \| \le (c(1 + \rho^2) + c_0\,\rho^2)\,h_k^2\,\|f\| \,.$$

Now, for $\ell = 0, 1,\ldots, m+1$ let e_ℓ be the convergence error

$$e_\ell = u_\ell - u^{(k)} \,.$$

Expanding e_0 in eigenfunctions

$$e_0 = \sum_{i=1}^{N} \alpha_i\,\psi_i$$

Then by (3.2.6)

$$e_m = \sum_{i=1}^{N} \alpha_i (1 - \frac{\lambda_i}{\lambda_N})^m\,\psi_i$$

It remains to estimate the result of the correction in step 3.

We have

$$r = \sum_{i=1}^{N} \alpha_i\,\lambda_i (1 - \frac{\lambda_i}{\lambda_N})^m\,\psi_i$$

and by the error estimate (2.2.3)

$$\| \bar{v} - e_m \| \le c\,h_{k-1}^2\,\|r\| \,;$$

so by (3.2.7)

$$\| e_{m+1} \| = \| e_m - v \| \le \| e_m - \bar{v} \| + \| \bar{v} - v \|$$

$$\le c\,h_{k-1}^2\,\|r\| + c_0\,h_{k-1}^2\,\|r\|$$

$$\le (c + c_0)\,\rho^2\,h_k^2\,\|r\|$$

$$= (c + c_0)\,\rho^2\,h_k^2\,\| \sum_{i=1}^{N} \alpha_i\,\lambda_i (1 - \frac{\lambda_i}{\lambda_N})^m\,\psi_i \|$$

$$\le (c + c_0)\,\rho^2\,h_k^2\,\|e_0\| \max_{0 \le \lambda \le \lambda_N} (\lambda(1 - \frac{\lambda}{\lambda_N})^m)$$

where the last step depends on the L^2 orthogonality of the eigenfunctions $\{\psi_i\}$. By a simple calculation

$$\max_{0 \leq \lambda \leq \lambda_N} \left(\lambda(1 - \frac{\lambda}{\lambda_N})^m\right) \leq \frac{\lambda_N}{m+1} .$$

Then from the eigenvalue bound (3.2.4)

$$\|e_{m+1}\| \leq c \, \rho^2 \, \frac{c + c_0}{m+1} \, \|e_0\| .$$

It follows from this and (3.2.10) that

$$\|e_{m+1}\| \leq c \, \rho^2 \, \frac{c + c_0}{m+1} \, (c(1 + \rho^2) + c_0 \, \rho^2) \, \|f\| .$$

Choose m large enough that

$$(3.2.11) \qquad m+1 \geq \frac{c \, \rho^2}{c_0} \, (c + c_0) \, (c(1 + \rho^2) + c_0 \, \rho^2) ,$$

and the result follows. $\square$

In this proof we have carried the constants ρ and c_0 explicitly to show the dependence of m on these quantities. From (3.2.11) m apparently goes as $\frac{\rho^4}{c_0}$. This sharp dependence of the number of iterations m on the mesh ratio is typical of multi-level algorithms, but the dependence on c_0 is not. For more common multi-level algorithms, such as that considered by Nicolaides [1977] and that considered in chapter 4 for locally refined grids, m depends on c_0 only as $|\log(c_0)|$. Thus if one wishes to reduce the convergence error well below the truncation error, the algorithm here cannot be recommended. However, in the usual case where it suffices to have convergence error comparable to the truncation error this algorithm should perform about as well as other multi-level methods.

3.3. Computational Cost

While iteration (3.2.6) used in step two of algorithm (3.2.5)-
(3.2.8) was convenient in the last section since it made the analysis
there basis independent, it is impractical. Each of the iterations,
(3.2.6), required the solution of a linear system involving the mass
matrix because of the implicit definition on the left side. To avoid
this, we will show in this section that the smoothing iteration (3.2.6)
can be replaced by a simple simultaneous displacement iteration without
impairing the convergence result. This analysis follows Bank and
Dupont [1978]. With this replacement the overall algorithm is of optimal
order, producing a second order accurate solution in O(N) operations for
a finite element grid with N elements.

Given a basis $\{\phi_i\}_{i=1}^{N}$ for M_k, for each $u \in M_k$ there corresponds
a coordinate vector $\underline{u} \in \mathbb{R}^N$. As in the last section, we have temporarily
dropped the subscript k on the dimension N of M_k and on eigenvectors and
eigenvalues. Let $b(\cdot, \cdot)$ denote the bilinear form

$$b(u,v) = \underline{u}^T \underline{v}$$

The norm induced by this bilinear form is the same as the Euclidean norm
on corresponding vectors,

$$\|u\|_b^2 = b(u,u) = \|\underline{u}\|_{\ell^2}^2 .$$

Now let $\{\tilde{\underline{\psi}}_i\}$ be the eigenvectors of the stiffness matrix K with
corresponding eigenvalues

$$0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_N .$$

The eigenvectors $\{\tilde{\underline{\psi}}_i\}$ are coordinate vectors for eigenfunctions $\{\tilde{\psi}_i\}$
satisfying

$$(3.3.1) \qquad a(\tilde{\psi}_i, \phi) = \tilde{\lambda}_i \, b(\tilde{\psi}_i, \phi) \, , \qquad \phi \in M_k$$

and the orthogonality relations

$$a(\tilde{\psi}_i, \tilde{\psi}_j) = b(\tilde{\psi}_i, \tilde{\psi}_j) = 0 \text{ for } i \neq j \, .$$

We wish to consider algorithm (3.2.5)-(3.2.8) of the last section but with the smoothing iteration (3.2.6) replaced by the simultaneous displacement iteration

$$(3.3.2) \quad b(u_\ell - u_{\ell-1}, \phi) = -\frac{1}{\tilde{\lambda}_N} \, (a(u_{\ell-1}, \phi) - (f, \phi)) \, , \qquad \phi \in M_k \, .$$

The use of $\tilde{\lambda}_N$ here is only for convenience. It could be replaced by a convenient upperbound on the eigenvalues of the stiffness matrix, such as (2.3.2), without altering the convergence result.

The convergence of this algorithm is shown by

Theorem 3.2. For any $c_0 > 0$ there is a positive integer m independent of $k \geq 2$ such that the approximate solution $u_{m+1}^{(k)}$ produced by algorithms (3.2.5(-(3.2.8) with (3.2.6) replaced by (3.2.2) satisfies

$$\| u_{m+1} - u^{(k)} \| \leq c_0 \, h_j^2 \, \| f \| \, .$$

Proof. The proof is a slight modification of the proof of Theorem 3.1. Expanding e_0 in the eigenfunctions $\{\tilde{\psi}_i\}$ instead of in the eigenfunctions $\{\psi_i\}$ we have

$$e_0 = \sum_{i=1}^{N} \alpha_i \, \tilde{\psi}_i \, .$$

Then, as before,

$$e_m = \sum_{i=1}^{N} \alpha_i (1 - \frac{\tilde{\lambda}_i}{\tilde{\lambda}_{N_k}})^m \, \tilde{\psi}_i \, .$$

Let $r \in M_k$ be given by

$$(r,\phi) = - a(e_m, \phi) , \qquad \phi \in M_k$$

and $\tilde{r} \in M_k$ by

$$b(\tilde{r},\phi) = - a(e_m, \phi) , \qquad \phi \in M_k .$$

Then

$$\underline{\tilde{r}} = M\underline{r} ;$$

so we have

$$\|r\|^2 = \underline{r}^T M\underline{r} = (M^{-1}\underline{\tilde{r}})^T M(M^{-1}\underline{\tilde{r}})$$

$$= \underline{\tilde{r}}^T M^{-1} \underline{\tilde{r}}$$

$$\leq \sigma \|\underline{\tilde{r}}\|_{\ell^2}^2 = \sigma \|\tilde{r}\|_b^2 .$$

As in the proof of Theorem 3.1

$$\|e_{m+1}\| \leq (c(1 + \rho^2) + c_0 \rho^2) h_k^2 \|r\|$$

$$\leq (C(1 + \rho^2) + C_0 \rho^2)\sigma^{1/2} h_k^2 \|r\|_b .$$

Now

$$\|\tilde{r}\|_b = \| \sum_{i=1}^N \alpha_i \tilde{\lambda}_i (1 - \frac{\tilde{\lambda}_i}{\tilde{\lambda}_N})^m \tilde{\psi}_i \|_b$$

$$\leq \|e_0\|_b \max_{0 \leq \lambda \leq \tilde{\lambda}_N} (\lambda(1 - \frac{\lambda}{\tilde{\lambda}_N})^m)$$

$$\leq \|e_0\|_b \frac{\lambda_N}{m+1}$$

where the first inequality uses orthogonality of the eigenfunctions $\{\tilde{\psi}_i\}$ in the inner product $b(\cdot, \cdot)$. Then since

$$\|e_0\|^2 = \underline{e}_0^T M \underline{e}_0 = (M^{1/2} \underline{e}_0)^T (M^{1/2} \underline{e}_0)$$

(3.3.3)

$$= \|M^{1/2} \underline{e}_0\|_{\ell^2}^2 \geq \frac{1}{\sigma} \|\underline{e}_0\|_{\ell^2}^2 = \frac{1}{\sigma} \|e_0\|_b^2$$

and since the maximum eigenvalue $\tilde{\lambda}_N$ satisfies the bound (2.3.2) the rest of the proof goes through exactly as before. $\square$

This theorem shows that by twice approximately solving the lower dimensional linear system for M_{k-1} and doing a fixed number of simultaneous displacement iterations on M_k one can obtain a second order accurate solution in M_k. Since the convergence shown by Theorem 3.2 is independent of $k \geq 2$, the two related problems in M_{k-1} can be solved by recursively applying the same algorithm to their smaller linear systems. Continuing the recursion one ends up repeatedly having to solve the linear system for M_1. This can be done either directly or by any convenient iterative method.

To estimate the computational cost of this algorithm observe that from (3.2.1) the dimensions of the finite element spaces $\{M_j\}$ must satisfy

$$c_1 \rho^{2j} \leq N_j \leq c_2 \rho^{2j} , \qquad 1 \leq j < \infty ,$$

for c_1, $c_2 > 0$ independent of j. This assumes that h_1, the mesh size of M_1, is fixed and the degree of the highest degree polynomial on any element is bounded uniformly in j. Then, if s_0 is the storage required for the direct or iterative inversion of the linear system for M_1, the total storage must be proportional to

$$s_0 + \sum_{j=2}^{k} N_j \leq s_0 + c_2 \sum_{j=2}^{k} \rho^{2j}$$

$$\leq s_0 + c_2 \frac{\rho^{2k}}{1 - \rho^{-2}}$$

$$\leq s_0 + \frac{c_2}{c_1} (\frac{1}{1 - \rho^{-2}}) N_k$$

for $\rho > 1$. Since s_0 is fixed, assuming the mesh ratio ρ is greater than one, the total storage required depends only linearly on the dimension of the solution space M_k.

To bound the computation time, observe that the time required to do one of the simultaneous displacement iterations (3.3.2) for M_j is proportional to N_j. Let t_0 be the computation time for the approximate solution of the linear system for M_1. Then taking into account that we must recursively apply the algorithm twice to M_{k-1}, four times to M_{k-2}, and so on the computation time T is bounded by

$$(3.3.4) \qquad T \leq 2^k t_0 + cm \sum_{j=2}^{k} 2^{k-j} N_j$$

$$\leq 2^k t_0 + cc_2 m \sum_{j=2}^{k} 2^{k-j} \rho^{2k}$$

$$\leq 2^k t_0 + cc_2 m\rho^{2k} \sum_{j=2}^{k} (\frac{2}{\rho^2})^{k-j} .$$

We have neglected the computational cost of carrying out the mappings of functions on one grid to another in steps one and three of algorithm (3.2.5)-(3.2.8). Despite the fact that the spaces $\{M_j\}$ are nested, the mappings required in steps one and three are not free since in general each space M_j uses a different basis. It is however easy to see that the above bound on the computation time is unaltered by taking into account this additional cost.

Now from (3.3.4) we have the bounds

$$T \leq \begin{cases} O(N_k) & \text{for } \rho > \sqrt{2} \\[2ex] O(N_k \log(N_k)) & \text{for } \rho = \sqrt{2} \\[2ex] O\left(N_k^{\frac{\log 2}{2 \log \rho}}\right) & \text{for } 1 < \rho < \sqrt{2} . \end{cases}$$

The natural choice $\rho = 2$ gives an optimal order algorithm and is convenient in programming.

4. LOCALLY REFINED GRIDS

This chapter, which contains the principal results of this
thesis, presents two multi-level algorithms achieving O(N) complexity
bounds on locally refined grids. One of these attains this optimal
complexity bound under weaker than expected restrictions on the
dimensions of the finite element spaces used by the method. These
optimal order convergence results are a consequence of an approximation
theorem proved in this chapter. Aside from its use in extending multi-
level convergence theory to locally refined grids, this approximation
result is of interest since it relies only on local properties of the
finite element spaces used and is not based on elliptic regularity.
As a consequence it provides an independent verification of the O(N)
convergence of multi-level methods on non-convex domains shown previously
by Bank and Dupont.

4.1. Introduction

There are a variety of reasons why one might wish to use locally
refined grids. They can be used to maintain high order accuracy in the
face of singularities caused by discontinuous coefficients or the corners
of the domain, or to resolve boundary layers. They may simply result
from an adaptive discretization. They can also be used to give good
resolution of the part of the solution of greatest interest without
incurring the computational cost of a fine global grid. This situation
occurs, for example, on unbounded domains. As long as the ratio of the
diameter of the largest element to that of the smallest remains uniformly
bounded on the family of finite element spaces of interest, multi-level
convergence results for quasi-uniform grids, such as those in the last

chapter, remain valid. However, in many cases of interest this ratio of element diameters becomes unbounded on the family of finite element spaces being used. In this case, while results for quasi-uniform spaces, such as those in the last chapter, yield an O(N) work bound for each finite element space, the constant hidden in the O(N) work bound will depend on the ratio of element diameters. Such bounds are clearly unsatisfactory for locally refined finite element spaces.

While nearly all of the convergence results in the literature apply only to the case of quasi-uniform finite difference or finite element grids, multi-level methods are being used increasingly for locally refined grids, Brandt (1977), Bank and Sherman (1978). In fact, one of their principal advantages is that their rate of convergence remains very good on locally refined grids. By contrast other iterative methods, whose rates of convergence depend on the condition number of the stiffness matrix, do poorly on locally refined grids. For example, the number of iterations required with conjugate gradient iteration goes as $\sqrt{K(K)}$, where $K(K)$ is the spectral condition number of the stiffness matrix, Axelsson (1977). This condition number satisfies

$$K(K) \geq c \, h_{min}^{-2}$$

where h_{min} is proportional to the diameter of the smallest element in the grid. Thus, $K(K)$ may grow much faster as a function of the number of unknowns on a locally refined grid. This presents a very severe difficulty for nearly all iterative methods. As will be shown in this chapter, multi-level methods do not suffer from this difficulty.

The algorithms in this chapter differ considerably from that in the last chapter, aside from their application to locally refined grids. In the algorithm of the last chapter, to approximately solve the elliptic

problem on the finest grid, a related problem on the next coarser grid
was solved twice. It was solved once at the beginning to obtain a
starting value for the simultaneous displacement iteration and once at
the end as a final correction. In the algorithms of this chapter and
in other algorithms in the literature, solution of a related problem on
the next coarser grid is used more frequently, typically after every
few smoothing iterations on the finest grid. Thus, the basic operation
in such an algorithm is best thought of as a multi-level iteration
consisting of a simple relaxation iteration periodically accelerated
by the solution of a related problem on the next coarser grid.

Algorithms like those considered in this chapter have been
described for quasi-uniform grids by Bank and Dupont, Nicolaides,
Hackbusch, and others. They are also related to methods for locally
refined grids suggested by Brandt and very closely to the method used in
the adaptive finite element code of Bank and Sherman. An algorithm
similar to that of chapter three could have been used, but probably
would not have been as practical as the algorithms here. One could argue
that in any case it is best to consider different modifications of the
standard approach separately rather than in combination.

To our knowledge there is only one convergence result in the
literature for multi-level methods on locally refined grids. This is
given in Bank and Dupont (1978) where a two-level multi-level iteration
is shown to converge at a rate independent of the mesh size. Since this
result relies only on local properties of the finite element space and
not on global approximation properties, it applies automatically to
locally refined grids. The limitation to this result is that it deals
only with their two-level scheme. With such a scheme, as one decreases
the mesh size, the dimensions of both the finer and coarser finite

element spaces grow. In their algorithm the linear system for the

coarser finite element space must be approximately solved once per

multi-level iteration. Whether this is done directly or by one of

the common iterative methods, the cost of approximately solving this

linear system grows faster than linearly in the dimension of the

linear system. As a consequence, such a two-level scheme can never be of

optimal order. It may, however, be quite attractive as a practical

algorithm since inversion of the lower dimensional linear system for

the coarser space may be much cheaper than the cost of inverting the

linear system for the finer space in which the solution is sought.

The use of direct methods to solve these linear systems for the coarser

space is particularly attractive since after the first multi-level

iteration only back solves are required.

One might expect that a simple generalization of their

convergence proof for the two-level case could be used to show optimal

order convergence of a multi-level method using many levels on locally

refined grids. In some cases it can, for example, for Poisson's equation

on certain finite element grids. In general, however, the error

reduction per iteration, $\gamma \in (0, 1)$, of their two-level scheme, while

independent of the mesh size, depends strongly on the PDE and on certain

properties of the discretization. The inductive argument used to show

the optimal order convergence of multi-level methods using many levels

requires that we be able to choose a sufficiently small value of γ.

For example, for the first algorithm in this chapter, γ must be chosen

in $(0, \frac{1}{3})$. Since there is no easy way of controlling γ in their two-level

scheme, it seems to be quite hard to get general optimal order convergence

results from a direct generalization of their two-level result. This is

unfortunate, since the argument showing the convergence of their

two-level algorithm is far simpler than that given here.

The results here and the technique of their proof are actually

much more closely related to the second kind of algorithm considered in

their paper. This second algorithm is a recursive algorithm using many

grid levels just as the algorithm in the last chapter did. Their

algorithm is shown to converge in $O(N)$ operations on quasi-uniform

finite element spaces. Unlike the results in chapter three, their

results apply also to non-convex domains, on which the elliptic problem

is less than H^2 regular. The analysis of algorithms for locally refined

grids here, which also applies to non-convex domains, is largely

modeled on their analysis of the second algorithm in their paper. The

notations here have also been kept almost the same as theirs.

Analysis of multi-level methods is harder than that of most

iterative methods since it involves both algebraic and approximation

questions. The algebraic and complexity questions for locally refined

grids are slightly harder, since the dimensions of the finite element

spaces used in the multi-level algorithm must be taken into account more

carefully. By contrast, the approximation questions for locally refined

grids seem to be a great deal harder. In the first half of this chapter

we consider the algebraic and complexity questions involved in treating

multi-level methods on locally refined grids. The analysis here

largely follows that in the paper by Bank and Dupont, while the algorithms

themselves are motivated by the adaptive finite element code of Bank

and Sherman. Two algorithms are considered; the first is similar to

that in the Bank and Sherman code. The second is a modification yielding

an improved complexity bound. This modification, which appears to be

new, allows greater freedom in the choice of finite element spaces used and may have significant practical applications.

The second half of this chapter deals with the approximation questions involved in treating locally refined grids. The approximation result proved there is the main result of this thesis. In previous work on multi-level methods, the approximation questions were answered by using the elliptic regularity of the problem to show smoothness of the functions being approximated. Then the approximation error could be estimated using standard finite element techniques including duality arguments and the Bramble-Hilbert lemma. The approximation result here uses a completely different kind of argument making no use of the regularity of the problem. This is why it applies to non-convex domains where the regularity is weaker than H^2. Perhaps of greater interest from a computational point of view is the fact that this approximation result relies only on local properties of the finite element spaces and the PDE operator. In consequence, it would be quite easy to compute rigorous upper bounds on the rate of convergence of the algorithms in this chapter. Up until now there were good heuristic bounds on the rate of convergence of multi-level methods on finite difference grids, Brandt (1977), and in a few cases rigorous bounds. No previous bounds for the rate of convergence of multi-level methods on irregular finite element grids appear to be available.

4.2. Notation

To develop the algorithms considered in this chapter, it is necessary to go into greater detail concerning the finite element spaces

used than was required in the last chapter. Additional notation must also be developed. These are the tasks of this section. To keep the mathematical issues in this chapter clear, our analysis will be restricted to C^0 triangular and rectangular elements. Though the analysis in this chapter seems to be extendable to all or nearly all families of finite elements, the details of the proofs here can be considerably harder for some families of elements. Using the C^0 square and triangular elements considered here permits considerable simplification of the analysis and may also cover the cases of greatest interest, since the majority of finite element computation seems to be done with these elements. Linear C^0 triangular elements, called Turner triangles, are used in the Bank and Sherman adaptive multi-level code, so at least for polygonal domains, the solution space used by their code is covered by the theory here. However, as will be seen, their algorithm violates the assumptions of the theory developed here in a number of other ways.

There are really two reasons why more detailed specification of the finite element spaces used in the multi-level algorithm is needed here than was required in the last chapter. The first is that locally refined spaces are intrinsically more complex than quasi-uniform spaces since they are not characterized by a single mesh parameter. The second is that, unlike the analysis in the last chapter, which was based on standard finite element error estimates, the analysis here is based on an approximation result developed here. The proof of this result depends on detailed analysis of the finite element spaces used. While this result could have been proven more abstractly, the approach here is certainly more straightforward. The reader with a background

in finite element theory should have little trouble in seeing the generalizations to elements with higher order continuity.

The two elements considered here are the standard C^0 triangular and square elements. The C^0 triangular elements of degree k, $k \geq 1$, use as element trial space the space P_k of polynomials in x and y of degree at most k. The C^0 square elements of degree k, $k \geq 1$, are tensor product elements based on the trial space Q_k consisting of polynomials whose terms are of degree k or less in x and y separately. Both types of elements may be taken as nodal finite elements with node placement as in figures 1 and 2.
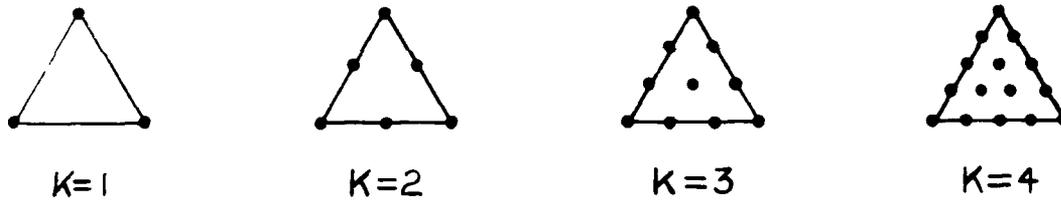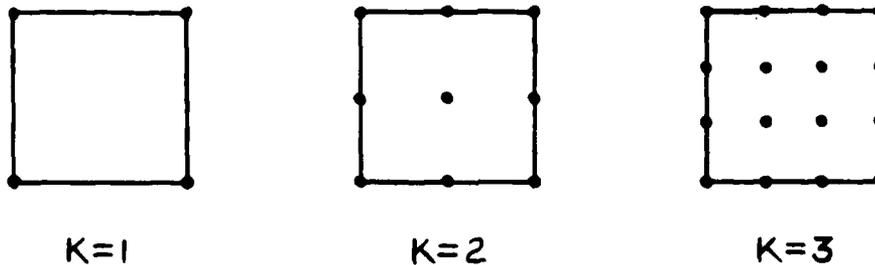

Figure 1. C^0 Lagrangian Triangular Elements

Figure 2. C^0 Tensor Product Elements

The nodal parameters used are just the function values at these points, providing a substantial simplification of the analysis in this chapter over that which would be required for the more complex Hermite elements. Equally important from our point of view, for any $k \geq 1$, either type of element constitutes an affine family, Ciarlet (1979). What this means is that we may fix a reference element in the plane, on which we have a reference trial space L, which will be either P_k or Q_k for triangular or square elements, respectively. For any other element of the same type anywhere in the plane we can find an affine transformation from the reference element to the given element. This affine transformation carries the region occupied by the given element, the nodes, and the element trial space from the reference element to any other element of the same type anywhere in the plane.

Affine transformations are mappings of the form

$$\underline{x}' = A\underline{x} + \underline{b}$$

where $\underline{b}$ is a fixed coordinate vector and A is a nonsingular 2×2 matrix. For triangular elements all affine transformations may be used, carrying the reference element into the entire six parameter family of triangles in the plane. For square elements, A may be taken as a multiple of the identity. That is, there is no need for rotation and distortion in this case.

The point of all this is that affine transformations and the use of reference elements provide an elegant simplification of the theory. The reference element and its element trial space basically contain all properties intrinsic to this type of finite element. On the other hand, the affine transformations contain all information about the geometry, mesh size, and distortion of any particular element in the

finite element grid. Thus, analysis of a finite element space is conveniently factored into two parts, a property we will use to advantage in section 4.5.

Aside from the choice of element, two types of grids need to be considered. These may be conveniently labeled "aligned" and "unaligned" grids. Aligned grids are those which are standard in finite element theory, where each edge of an element is the edge of another, or part of the boundary. Unaligned grids are those where the edge of an element may be only part of the edge of another. See figures 3, 4, and 5 for examples of these two types of grids. In general, triangular grids may always be chosen to be aligned, without sacrificing efficiency, but it is often necessary to choose grids of square or rectangular elements to be unaligned, or much of the benefit of local refinement will be lost.

To be admissible, the finite element spaces must be C^0 and satisfy the essential, that is Dirichlet, boundary conditions. On aligned grids there is no problem obtaining C^0 continuity. One simply requires agreement of the nodal parameter on the edge between adjacent elements. This does not work for unaligned grids. To simplify this problem we restrict the unaligned grids as follows. Assume that every edge of an element is either part of the boundary, the edge of another element, or the union of the edges of two other elements. This last situation is shown in figure 6.

In this last situation we consider the single element to dominate the two elements it adjoins across this edge. For convenience call the single element the "larger" since it almost always is larger than the other two. Then we may say that the "larger" dominates the
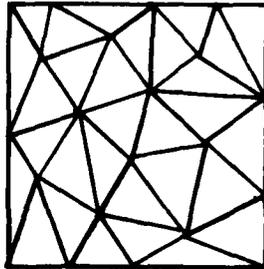
Figure 3. Aligned Triangular Grid
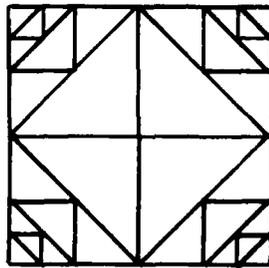
Figure 4. Unaligned Triangular Grid
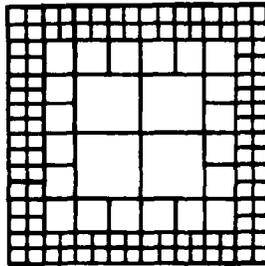
Figure 5. Unaligned Grid with Square Elements
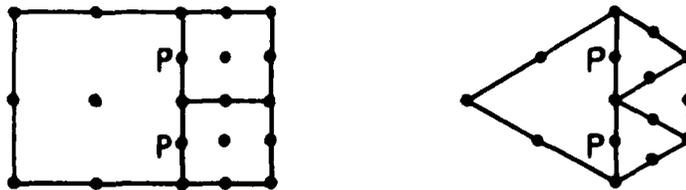
Figure 6. Adjacent, Unaligned Elements

other two in the sense that its nodes on this edge are the real nodes of the finite element space there. The nodes of the "smaller" elements may be called parasitic since their values will be taken from the larger element. Parasitic nodes of the "smaller" elements in figure 6 not coinciding with nodes of the "larger" element are labeled with a "P." It is easy to see that this system preserves the required C^0 continuity and is fairly straightforward to code.

One more item needs to be discussed before we consider construction of the locally refined spaces needed in this chapter. This is the angle condition. It was mentioned in section 2.3 that the mass matrix will usually be well conditioned uniformly in the level of refinement of the grid as long as the element geometry does not degenerate. While this is not always true for locally refined grids, it is still important that the element geometry not degenerate. One way of stating the required condition is to say that the minimum interior angle of any triangle in the grid must be bounded below uniformly in the level of refinement. More formally, if $\{\tau_j\}_{j=1}^{\infty}$ is the family of triangulations being considered, we want

$$(4.2.1) \qquad\qquad \inf_{j\geq 1} \left\{ \min_{T\in\tau_j} \phi_T \right\} > 0$$

where ϕ_T is the smallest interior angle of the triangle T. There are several alternate formulations of this condition in the finite element literature, Oden and Reddy (1976), but this one is adequate for our purposes. Obviously, no such condition is needed on square grids.

We next turn to consideration of the class of finite element grids on which the multi-level algorithms of this chapter can be shown to converge rapidly. This class, which includes both locally refined and

quasi-uniform spaces, is quite large and may be thought of as an indexed

family $\{M_\alpha\}_{\alpha \in A}$ of spaces. It consists of all spaces which can be

generated by the refinement process described below and which satisfy

an additional condition given in section 4.4. An $O(N)$ complexity bound

holds uniformly on the class of spaces in $\{M_\alpha\}_{\alpha \in A}$ satisfying this

condition uniformly. In general, this class of spaces with a uniform

$O(N)$ complexity bound is large enough to show the improved convergence

to singular solutions obtainable by discretization with locally refined

grids. A partial exception to this is the case of point singularities,

since the best discretizations for these problems do not satisfy the

conditions of section 4.4 uniformly. For these problems $O(N)$ bounds

can be shown by the theory here only for suboptimal, but still very good

locally refined grids.

Fixing a coarse space M_1, the refinement process described

below will generate sequences $\{M_{\beta,j}\}_{j=1}^\infty$, $\beta \in B$ of finite element spaces,

where the index β ranges over the uncountable family of choices allowed

in this refinement process. The family of spaces $\{M_\alpha\}_{\alpha \in A}$ is just the

union of all spaces which can be produced by this refinement process,

$$\{M_\alpha\}_{\alpha \in A} = \bigcup_{\beta \in B} \{M_{\beta,j}\}_{j=1}^\infty .$$

The condition mentioned above, which is described more fully in section 4.4,

is that for each fixed $\beta \in B$, the dimensions of the spaces $\{M_{\beta,j}\}_{j=1}^\infty$

in this sequence of refinements grow geometrically. The $O(N)$ complexity

bound given here depends only on the problem, the coarsest finite

element space M_1, and this geometric growth rate.

The multi-level algorithms here actually use only the finite family of spaces $\{M_{\beta,j}\}_{j=1}^{k}$, for fixed $\beta \in B$ and fixed $k \geq 2$, in their computation. Since β is fixed, it is convenient to drop it, although we retain j, and in fact consider the entire sequence of refinements $\{M_j\}_{j=1}^{\infty}$ since that simplifies the statements of some of the theorems here. One word of warning is needed, however. Though the spaces in the sequence $\{M_j\}_{j=1}^{\infty}$ become infinite dimensional, they do not usually become dense in H_E' and thus convergence to the true solution will not ordinarily be obtained in $\{M_j\}_{j=1}^{\infty}$. In order to get a sequence of solutions converging to the true solution, one must go back to consideration of the family $\{M_\alpha\}_{\alpha \in A}$.

Description of the families $\{M_j\}_{j=1}^{\infty}$ of locally refined grids is essentially the same for grids based on square and triangular elements. We consider first the triangular case since it is the case of greatest interest in this chapter. In this case, the family $\{M_j\}_{j=1}^{\infty}$ of finite element spaces will be based on a nested family of triangulations $\{\tau_j\}_{j=1}^{\infty}$. By nested we mean that each triangle T of τ_j, $j \geq 2$, is contained in a triangle of τ_{j-1}. Once we have chosen the triangulations $\{\tau_j\}_{j=1}^{\infty}$, and the particular element to be used, the family of finite element spaces $\{M_j\}_{j=1}^{\infty}$ is completely determined.

The construction of the nested family of triangulations $\{\tau_j\}_{j=1}^{\infty}$ may be done by a process Bank and Sherman call regular subdivision. In regular subdivision of a triangle T, it is divided into four smaller triangles by pairwise connecting the midpoints of the edges together as shown in figure 7. Let τ_1 be a fixed, aligned triangulation of Ω. Now inductively suppose τ_j has been chosen for some $j \geq 1$. Select a subset $\bar{\tau}_j$ of τ_j. We

may take τ_{j+1} as the (unaligned) triangulation of Ω formed by regular subdivision of the triangles in $\bar{\tau}_j$ while leaving the remaining triangles of τ_j unaltered. Proceeding is this way, the entire family of triangulations $\{\tau_j\}_{j=1}^{\infty}$ may be constructed.

Figure 7. Regular Subdivision of a Triangular Element

In our approach the subsets $\bar{\tau}_j \subset \tau_j$, $j \geq 1$, of triangles to be refined cannot be chosen arbitrarily, but most satisfy certain conditions necessary for the operation and analysis of the algorithms here. Let $\Omega_{j+1} \subset \Omega$ be the subdomain convered by triangles in $\bar{\tau}_j$, for $j \geq 1$. For simplicity of notation it is convenient to set $\Omega_1 = \Omega$. The first condition required is

$$(4.2.2) \qquad \Omega_{j+1} \subset \Omega_j \ , \ j \geq 1 \ .$$

This condition means that for $j > 1$, only triangles of τ_j which were formed by regular subdivision of triangles in τ_{j-1} may be subdivided in creating τ_{j+1}. Equivalently, for $j \geq 1$, each triangle of $\bar{\tau}_{j+1}$ is contained in a triangle of $\bar{\tau}_j$.

Now assume that the diameters d_T of the triangles T in the coarsest triangulation τ_1 satisfy

$$(4.2.3) \qquad \frac{1}{\sigma} h_1 \leq d_T \leq \sigma h_1 \ ,$$

for some fixed $\sigma > 1$ and mesh parameter h_1. As before set

(4.2.4) $$h_j = \rho^{1-j} h_1 \ , \ j \geq 1 \ ,$$

where ρ is a mesh ratio, which for the regular subdivision process here is two. Conditions (4.2.2)-(4.2.4) imply that the diameters of triangles in τ_j, $j \geq 1$, must satisfy

(4.2.5a) $$\frac{1}{\sigma} h_j \leq d_T \leq \sigma h_j \ \text{ for } T \subset \bar{\Omega}_j$$

(4.2.5b) $$\frac{1}{\sigma} h_i \leq d_T \leq \omega h_i \ \text{ for } T \subset \bar{\Omega}_i \setminus \Omega_{i+1}$$

$$\text{with } 1 \leq i \leq j - 1 \ .$$

The overbar here denotes closure, necessary since triangles are closed, and our domains $\{\Omega_j\}_{j=1}^{\infty}$ are not.

Intuitively, the triangles in τ_1 are all of about the same size. By (4.2.3) or (4.2.5a) their diameters are all *comparable* to h_1. The triangles in τ_2 come in two sizes: fine ones in $\bar{\Omega}_2$ with diameter comparable to h_2, produced by regular subdivisions, and coarse ones outside Ω_2 with diameters about h_1. In general, τ_j may be thought of as having j sizes of triangles, the finest lying in $\bar{\Omega}_j$. Of course, this intuitive point of view should not be pushed too far. Because of the constant σ in (4.2.5) these sizes may overlap a great deal.

Only one other condition is really essential for triangular grids. That is, that the number of elements adjoining a given element along one edge be at most two. The "two" here is arbitrary, and any other finite bound could be chosen, but in practice two is probably all one wants. The coding is difficult enough as it is. This restriction, that at most two "smaller" elements adjoin one "larger" element across an edge, together with the angle condition (4.2.1) limits the rate of change of the mesh size as one can easily show.

Locally refined grids of square elements may be constructed along the same lines as the triangular grids just considered. One begins with an aligned square grid G_1 on Ω and then generates a family of finer (unaligned) grids $\{G_j\}_{j=1}^{\infty}$, using an analogous regular subdivision process. This process is shown in figure 8. The subdomains $\{\Omega_j\}_{j=1}^{\infty}$ can be defined analogously, and the analogs of (4.2.3)-(4.2.5) will hold, now with the notation "d_e" instead of "d_T", using "e" for "element." Since such square grids are only being considered in this section and will thereafter be dropped for simplicity, there is no need to go into these details more deeply.
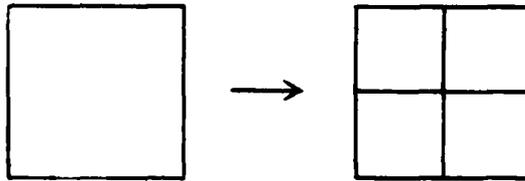
Figure 8. Regular Subdivision of a Square Element

A problem arises on locally refined grids not present on quasi-uniform grids. Unless special care is taken, the condition numbers of the mass matrices for a family of locally refined finite element spaces may not be uniformly bounded. For aligned quasi-uniform grids and the usual types of elements, including those considered here, the uniform boundedness of the condition number of the mass matrices is equivalent to the angle condition (4.2.1). Unaligned quasi-uniform grids are rarely considered, for obvious reasons, but the same thing can be shown for such grids.

The problem with locally refined grids is that the number of basis functions which overlap with a given basis function may be very large, becoming unbounded as the grid is refined. This violates the hypotheses of the result of Fried described in section 2.3, and in fact mass matrices with condition numbers becoming unbounded do occur. A simple finite element grid where this problem occurs is shown in figure 9. Here bilinear elements are used with the standard "hat function" or "pagoda" basis functions. Notice that the conditioning problem in this example occurs despite the fact that the mesh size changes slowly in the sense that the diameters of adjacent elements differ at most by a factor of two. Proper scaling of the basis functions mitigates the problem but does not eliminate it.
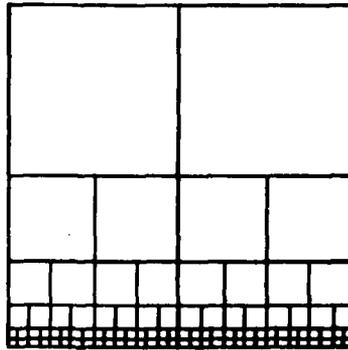
Figure 9. Finite Element Space with Poorly Conditioned Mass Matrix

The only real solution is to arrange things so that each basis function overlaps only a bounded number of elements, with the bound independent of the level of refinement. This can be done either by modifying the basis elements to prevent excessive overlap, or by selecting

only grids where such overlap does not occur. The second approach is probably preferable in most cases since it tends to be simpler. Consider the approach of modifying the basis functions for the simple bilinear elements. The situation is similar for the other square and triangular elements here. Ordinarily, the nodal parameter at the intersection of the four elements shown in figure 10 would be associated with the "hat function" whose support is the union of the four elements. When one or more of these four elements is refined, the basis function associated with its center node can be modified to have support on a smaller region. Up to rotation, there are four cases to consider, as shown in Figure 11. Clearly the case where all four elements are refined need not be dealt with since then the same considerations apply on the resulting smaller elements. The dark lines in figure 11 show the boundaries of the supports of the modified basis functions. The squares marked with asterisks may be subject to further refinement without violating the requirement that the mesh size change slowly. The point is that all squares so marked lie outside the support of these basis functions, so the number of elements these basis functions overlap, and hence the number of other basis functions they overlap, will be bounded by a fixed constant independent of the level of refinement. The only difficulty with this approach is that computations of the element integrals is somewhat more complicated since these basis functions are more complex. This approach was suggested to the author by D. Gannon, who has done considerable work with rectangular tensor product elements on locally refined grids, Gannon (1980).
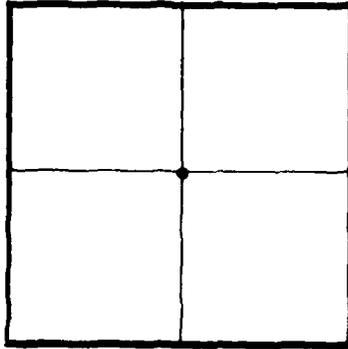
Figure 10. Support of a Basis Function
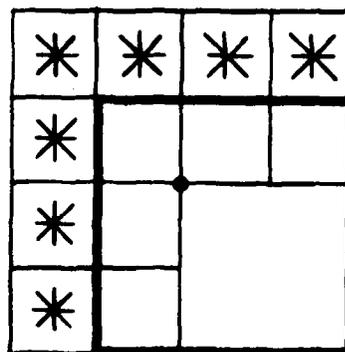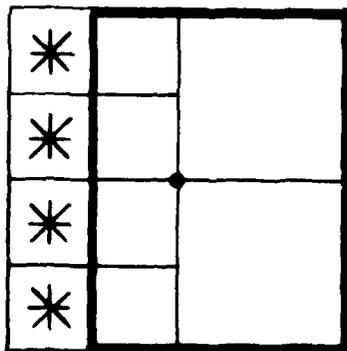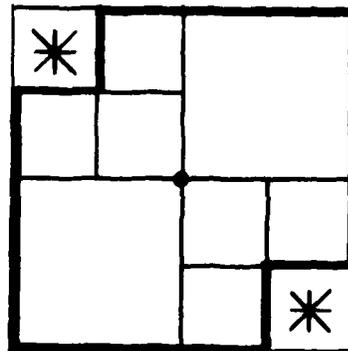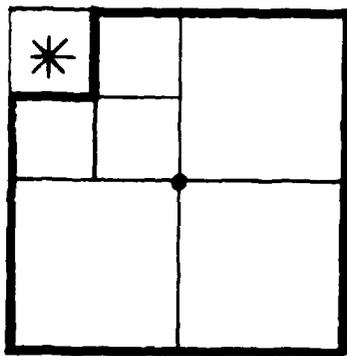
Figure 11. Modified Basis Functions

In the second approach, one retains the simple Lagrangian basis functions, "hat functions," and their higher order analogues, but restricts or modifies the grids so that the problem of one basis function overlapping an increasingly large number of others never arises. With triangular grids modification is not difficult. This is the approach used in the Bank and Sherman code. Beginning with an unaligned triangulation τ_j, $1 \leq j < \infty$, belonging to a nested family of triangulations $\{\tau_j\}_{j=1}^{\infty}$ satisfying the conditions given so far, their algorithm constructs an aligned triangulation $\tilde{\tau}_j$, refining τ_j, on which this condition number problem does not arise. This is done in two steps. In the first step each triangle with the property that there are vertices of other triangles in the middle of more than one of its edges is regularly refined, as in figure 12. Once all such triangles are refined, including those produced in carrying out this process, the only triangles in the grid are those with either one vertex in the middle of an edge or no vertices in the middle of its edges. The former are refined by a different subdivision process, called "green" refinement, shown in figure 13. The resulting aligned triangulation $\tilde{\tau}_j$ is then used to generate the finite element space M_j used by the multi-level algorithm.

Notice that green refinement may decrease the smallest interior angle present on the grid. This would be a problem if successive green refinements were applied. In Bank and Sherman's approach this is avoided by first generating the unaligned triangulations $\{\tau_j\}_{j=1}^{\infty}$ and then forming the corresponding aligned triangulation $\{\tilde{\tau}_j\}_{j=1}^{\infty}$, rather than taking τ_{j+1}, $j \geq 1$, as a refinement of $\tilde{\tau}_j$. This ensures that the angle condition (4.2.1) is satisfied. Even so, the triangulations $\{\tilde{\tau}_j\}_{j=1}^{\infty}$ used by their code are not generally admissible in the theory here, even when
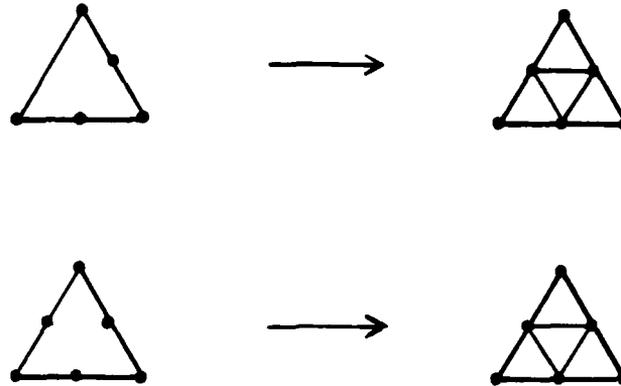
Figure 12.   Regular Refinement of Triangles with Edges Containing Vertices

Figure 13.   Green Refinement of a Triangle

the underlying unaligned triangulations $\{\tau_j\}_{j=1}^{\infty}$ satisfy all of our

assumptions. The problem is that in producing an aligned triangulation

$\tilde{\tau}_j$, $j \geq 1$, a triangle may be green refined while in a finer unaligned

triangulation τ_i, $i > j$, this same triangle may be regularly refined.

If so, the triangulations $\{\tilde{\tau}_j\}_{j=1}^{\infty}$ will not be nested, nor will the resulting

finite element spaces $\{M_j\}_{j=1}^{\infty}$. While this does not seem to be a problem

for their code, it is certainly a problem for the theory here. The cure

for this seems to be to require the unaligned triangulations $\{\tau_j\}_{j=1}^{\infty}$

to contain only triangles with a vertex of another triangle in at most

one of their edges. If so, in going from $\{\tau_j\}_{j=1}^{\infty}$ to the corresponding

aligned triangulations $\{\tilde{\tau}\}_{j=1}^{\infty}$ only green refinement will be done. Since

the triangles of τ_j, $j \geq 1$, green refined in producing $\tilde{\tau}_j$ must lie

outside Ω_j (the portion of τ_j lying in Ω_j is aligned), there is no

danger of subsequently regularly refining them. Further, it is easy

to see that those triangles of $\tilde{\tau}_j$ produced by green refinement must be

green refined in producing $\tilde{\tau}_i$, $i > j$, so the finite element spaces based

on the family of triangulations $\{\tilde{\tau}\}_{j=1}^{\infty}$ will be nested and the theory of

this chapter will go through for them.

The advantage of this approach is that the condition number

of the mass matrix for a finite element space based on an aligned

triangulation will be uniformly bounded, as long as the angle condition

is satisfied. This is so for linear elements since each pyramid basis

function overlaps only those elements meeting at the vertex with which

its nodal parameter is associated. The support is thus a polygon formed

from these elements, as in figure 14. The other kinds of basis functions

required for higher order elements, those associated with nodes on the

edges or interiors of triangles, are not a problem either, since their

supports are smaller than those associated with the vertices.
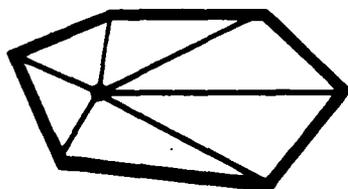
Figure 14. Support of a Basis Function on an Aligned Triangulation

Though the above approach is quite attractive for triangular

grids, since aligned triangulations are convenient in programming, it

does not generalize directly to square elements, since there is no

analog of green refinement for these elements. Instead, we must either

use the modified basis functions described above, or restrict the rate

of change of the mesh size so that the conditioning problem does not

occur when using the standard basis functions. The simplest way to

describe the required restriction on the rate of change of the mesh

size is in terms of the process for constructing the grids $\{G_j\}_{j=1}^{\infty}$ of

square elements we wish to consider. Say that two elements, e_1, $e_2 \in G_j$,

$1 \leq j < \infty$, touch if they have any point in common; that is, if they

share a vertex or part of an edge as in figure 15. Let G_j', $j \geq 1$, be

the set $\{e \in G_j : e \subset \Omega_j\}$. That is, G_j' is the set consisting of the

most refined elements of G_j. Define the interior of G_j', denoted

$\text{int}(G_j')$ as the set of elements of G_j' not touching any elements in

$G_j \setminus G_j'$. Then a sufficient condition for the mass matrices for the locally

refined spaces $\{M_j\}_{j=1}^{\infty}$, based on the grids $\{G_j\}_{j=1}^{\infty}$, to be uniformly

well conditioned is

$$\overline{G}_j \subset \text{int}(G_j') \ , \ j \geq 1 \ .$$

This condition suffices for the mass matrices to be uniformly bounded

since no basis functions centered at nodes outside of G_j' have supports

extending into $\text{int}(G_j')$. It follows that they only overlap elements outside

Ω_j, and so they only overlap a bounded number of elements.
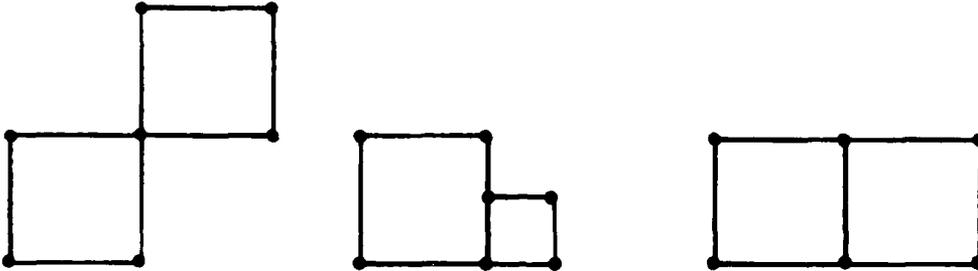
Figure 15. Touching Elements

With the two types of elements considered here and the variety of ways of ensuring the uniform boundedness of the condition numbers of the mass matrices, a number of options are available. However, it would be unwieldy to treat the theory in the rest of this chapter in the same generality, particularly the work in section 4.5, which relies on detailed analysis of the finite element spaces used. For this reason, we restrict the analysis that follows to C^0 spaces on aligned triangular grids. This choice is motivated partly by convenience, and partly by the use of such spaces in the Bank and Sherman code. There would be no difficulty in treating the other types of finite element spaces considered here, however, the interpolation mappings of section 4.5 are slightly more complicated to analyze on unaligned grids.

4.3. Algorithms

In this section we describe the multi-level algorithms to be considered in this chapter and begin the analysis of their convergence. The first of these algorithms is identical to one of those described by Bank and Dupont for quasi-uniform grids and is similar to algorithms described by Nicolaides, Hackbusch, Brandt, Bokhvalov, and others. It is also similar to the algorithm used in the adaptive finite element code of Bank and Sherman though there is a significant difference here which will be discussed. The second algorithm to be considered here appears to be new. It can be used to show the $O(N)$ complexity bound given in the next section under weaker assumptions than otherwise necessary.

Both of the algorithms described here differ substantially from that described in the last chapter. There, to approximately solve the finite element equations on a member M_j, $j \geq 2$, of a family $\{M_j\}_{j=1}^{\infty}$ of finite element spaces, approximate solution of a related problem in M_{j-1} was required twice; once at the beginning to obtain a starting value for smoothing iteration on M_j, once at the end of the iteration as a final correction. The algorithms in this chapter and other multi-level algorithms in the literature use approximate solution of the equations for M_{j-1} more often. After obtaining a starting value from M_{j-1}, the accepted solution is obtained through a sequence of multi-level (outer) iterations. Each of these outer iterations consists of a number of smoothing (inner) iterations on M_j, followed by approximate solution of a related problem in M_{j-1}. Thus, approximate solution of the related problem in M_{j-1} is required an indeterminate number of times depending on how far the outer iteration on M_j is carried.

This kind of algorithm is somewhat more complicated than that of chapter three. However it made sense to analyze the application of this kind of algorithm to locally refined grids rather than the application of algorithms like those in the last chapter for several reasons. The main one was the desire to establish the optimal order convergence of an algorithm as close as possible to those being used in practice. This was more fully accomplished than it would have been if we had examined algorithms more like those in chapter three. It might also be pointed out that relatively little of the complexity of this chapter is caused by the choice of algorithm.

Description of the algorithms here and analysis of their convergence properties will involve the eigenfunctions and eigenvalues. Following the notation in chapter three let N_j be the dimensions of M_j, $1 \leq j < \infty$. Then for the problem (2.1.3) we have eigenfunctions $\{\psi_i^{(j)}\}_{i=1}^{N_j}$ in M_j and eigenvalues $\{\lambda_i^{(j)}\}_{i=1}^{N_j}$ satisfying

$$a(\psi_i^{(j)}, \phi) = \lambda_i^{(j)} (\psi_i^{(j)}, \phi), \quad \phi \in M_j .$$

Since more than one finite element space may be involved in our considerations, we will retain the superscript j. As in chapter three we may replace the L^2 inner product by the bilinear form $b_j(\cdot, \cdot)$ defined there, where we now use the subscript j to keep track of the finite element space in $\{M_j\}_{j=1}^{\infty}$ giving rise to this bilinear form. With this replacement we have as before eigenfunctions $\{\tilde{\psi}_i^{(j)}\}_{i=1}^{j}$ and eigenvalues $\{\tilde{\lambda}_i^{(j)}\}_{i=1}^{j}$ satisfying

$$a(\tilde{\psi}_i^{(j)}, \phi) = \tilde{\lambda}_i^{(j)} b_j(\tilde{\psi}_i^{(j)}, \phi), \quad \phi \in M_j .$$

As usual we assume the eigenvalues are numbered in order of increasing magnitude:

$$0 < \lambda_1^{(j)} \le \lambda_2^{(j)} \le \ldots \le \lambda_{N_j}^{(j)} ,$$

$$0 < \tilde{\lambda}_1^{(j)} \le \tilde{\lambda}_2^{(j)} \le \ldots \le \tilde{\lambda}_{N_j}^{(j)} .$$

For both of the algorithms considered in this chapter the basic multi-level iteration is the same. In both cases each outer iteration consists of a sequence of smoothing inner iterations followed by approximate solution of a related problem in a coarser finite element space. The difference between the two algorithms will lie in the manner of solving these related problems. In order to obtain a solution in a member M_k, $k \ge 2$, of a family $\{M_j\}_{j=1}^{\infty}$ of locally refined spaces, multi-level iteration will need to be applied to all of the spaces $\{M_j\}_{j=2}^{k}$; thus, the iteration will be described for any space M_j in $\{M_j\}_{j=1}^{\infty}$.

Beginning with a trial solution $u_{0,n}^{(j)}$ in M_j the outer iteration produces a sequence of iterates $\{u_{0,n}^{(j)}\}_{n=1}^{\infty}$ converging to the true discrete solution $u^{(j)}$ in M_j. Each of these outer iterations is similar to the algorithm of the last chapter. Beginning with a trial solution $u_{0,n}^{(j)} \in M_j$ an outer iteration produces an improved solution $u_{0,n+1}^{(j)}$ given by

$$u_{0,n+1}^{(j)} = u_{m+1,n}^{(j)} ,$$

where $\{u_{\ell,n}^{(j)}\}_{\ell=1}^{m+1}$ is a sequence of inner iterates. All but the last of these inner iterates are generated by a simultaneous displacement smoothing iteration on M_j. The last uses a correction based on the approximate solution of a related problem in the coarser finite element space M_{j-1}. More precisely, beginning with a trial solution $u_{0,n}^{(j)} \in M_j$, an outer iteration consists of the steps:

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1. For $\ell = 1,\ldots,$ m define $u_{\ell,n} \in M_j$ by

(4.3.1) $b_j(u_{\ell,n} - u_{\ell-1,n}, \phi) =$

$$-\frac{1}{\tilde{\lambda}_{N_j}^{(j)}} (a(u_{\ell-1,n}, \phi) - (f, \phi)) , \phi \in M_j .$$

2. Letting $\bar{v} \in M_{j-1}$ be given by

(4.3.2) $a(\bar{v}, \phi) = a(u_{m,n}, \phi) - (f, \phi) , \phi \in M_{j-1} ,$

let v be an approximation to $\bar{v}$ and set

(4.3.3) $u_{m+1,n} = u_{m,n} - v .$

The quality required in the approximation of $\bar{v}$ by v will be specified in theorems 4.1 and 4.3.

The intuitive meaning of these two steps is that in the first the more oscillatory components of the convergence error are rapidly reduced by the smoothing iteration. In the second step the smoother components of the error, for which the simultaneous displacement iteration is ineffective, are reduced through approximate solution of the related problem (4.3.2). Consequently convergence will be rapid for all components of the error.

To make these considerations precise, let $\theta_j \in (0, 1)$ be an arbitrary parameter. Let $J(\theta_j)$ be the integer such that

$$\lambda_i^{(j)} \leq \theta_j \lambda_{N_j}^{(j)} \text{ for } i \leq J(\theta_j) ,$$

$$\lambda_i^{(j)} > \theta_j \lambda_{N_j}^{(j)} \text{ for } i > J(\theta_j) .$$

Similarly let $\tilde{J}(\theta_j)$ be the integer such that

$$\tilde{\lambda}_i^{(j)} \leq \theta_j \, \tilde{\lambda}_{N_j}^{(j)} \quad \text{for } i \leq \tilde{J}(\theta_j) \; ,$$

$$\tilde{\lambda}_i^{(j)} > \theta_j \, \tilde{\lambda}_{N_j}^{(j)} \quad \text{for } i > \tilde{J}(\theta_j) \; .$$

Then we can define the subspace $S_j \; \mathit{0}_j \; \tilde{S}_j \; \tilde{\mathit{0}}_j$ of M_j by

$$S_j = \text{span } \{\psi_i^{(j)}, \; i \leq J(\theta_j)\} \; ,$$

$$\mathit{0}_j = \text{span } \{\psi_i^{(j)}, \; i > J(\theta_j)\} \; ,$$

$$\tilde{S}_j = \text{span } \{\tilde{\psi}_i^{(j)}, \; i \leq \tilde{J}(\theta_j)\} \; ,$$

$$\tilde{\mathit{0}}_j = \text{span } \{\tilde{\psi}_i^{(j)}, \; i > J(\theta_j)\} \; .$$

The spaces S_j and $\tilde{S}_j$ consist of relatively smooth functions while $\mathit{0}_j$ and $\tilde{\mathit{0}}_j$ contain more oscillatory functions. For any $\theta_j \in (0, 1)$ one has:

$$M_j = S_j \; \oplus \; \mathit{0}_j = \tilde{S}_j \; \oplus \; \tilde{\mathit{0}}_j$$

In fact, $\mathit{0}_j$ is the orthogonal complement of S_j in M_j in both the energy and L^2 inner products, while $\tilde{\mathit{0}}_j$ is the orthogonal complement of $\tilde{S}_j$ in energy and the inner product $b_j \; (\cdot, \cdot)$.

For the time being only the subspaces $\tilde{S}_j$ and $\tilde{\mathit{0}}_j$ will be important in analyzing the convergence of the multi-level iteration (4.3.1)-(4.3.5), since the approximation question involved will be deferred until sections 4.5-4.7. There S_j and $\mathit{0}_j$ will play an important role. To establish the rapid convergence of iteration (4.3.1)-(4.3.3) we will show that error components in the oscillatory space $\tilde{\mathit{0}}_j$ are rapidly reduced in step 1, while components in the smooth space $\tilde{S}_j$ are reduced in step 2.

The multi-level iteration (4.3.1)-(4.3.3) differs from that in the Bank and Sherman adaptive finite element code primarily in the smoothing inner iteration used. Their code uses a symmetric Gauss-Seidel

smoothing iteration rather than the simultaneous displacement iteration

used here. While comparisons for quasi-uniform grids have shown that

the choice of smoothing iteration is not very important, Brandt (1977),

Bank and Sherman (1979), for locally refined grids the choice is significant.

In Fourier terms, the iteration (4.3.1) rapidly annihilates Fourier error

components whose wave length is comparable to h_j, the finest mesh spacing

of M_j. Away from Ω_j, the most refined region, this iteration has little

effect. The reason for this is that the residual on the right hand

side of 4.3.1 is multiplied by the constant factor $-\dfrac{1}{\tilde{\lambda}_{N_j}^{(j)}}$. By contrast

in Jacobi or Gauss-Seidel iteration each component of the residual is

multiplied by the inverse of the corresponding diagonal entry of the

stiffness matrix. In other words, in these latter cases the scaling

factor looks like $-ch_{local}^2$ rather than $-ch_j^2$, where h_{local} is a function

which takes on a different value for each component of the residual

vector, depending on the size of the corresponding elements. As a

result of this difference, the smoothing iteration used in the Bank and

Sherman code is effective everywhere on the grid, rather than just on the

most refined parts. This is important in their code since they a'low

any element of M_{j-1}, $j \geq 2$, to be refined in creating a finer space M_j.

We do not, and the simultaneous displacement iteration considered here

is perfectly adequate in this simpler case.

The choice of smoothing iteration is intimately connected with

the decomposition of M_j into smooth and oscillatory components. The

decomposition here, $M_j = \tilde{S}_j \oplus \tilde{O}_j$, is natural only for simultaneous

displacement iteration. The hard part of the analysis here is the

approximation question involved in demonstrating the effectiveness of

step 2 of the multi-level iteration (4.3.1)-(4.3.3). For the decomposition

$M_j = \tilde{S}_j \oplus \tilde{O}_j$ and the case considered here where only the finest elements

of M_{j-1} are refined in creating M_j, this approximation qu stion can be

answered. The author was unable to handle the more general case where

any element of M_{j-1} may be refined.

Though this approximation result will not be shown until

section 4.6, we now state it since it is needed for the analysis in

this section. Let $\{M_j\}_{j=1}^{\infty}$ be a family of locally refined spaces

satisfying the assumptions of section 4.2.

Let $j \geq 2$ and $\theta_j \in (0, 1)$ be arbitrary, and for any

$\eta \in \tilde{S}_j$ let $\bar{\eta} \in M_{j-1}$ be given by

$$a(\eta - \bar{\eta}, \phi) = 0 , \phi \in M_{j-1} ,$$

that is, $\bar{\eta}$ is the elliptic projection of η onto M_{j-1}. Then

(4.3.4) $$|||\eta - \bar{\eta}||| \leq c_0 \, \theta_j^{1/4} \, |||\eta||| ,$$

where $c_0 > 0$ is independent of j, θ_j, and η. This inequality, which

is the main result of this thesis, shows that eigenfunctions $\tilde{\psi}_1^{(j)}$ in

M_j whose eigenvalues are relatively small can be well approximated in the

coarser finite element space M_{j-1}.

With these preliminaries behind we are ready to consider the

first convergence theorem for the multi-level iteration (4.3.1)-(4.3.3).

This theorem is the same as one found in Bank and Dupont for the case

of quasi-uniform grids and is similar to theorems proved by Nicolaides,

Hackbusch, and others. Our notation largely follows Bank and Dupont

although there are some differences.

Theorem 4.1. There exists $\gamma \in (0, 1)$ and a positive integer m, both independent of $j \geq 2$, such that if the approximate solution v of (4.3.2) satisfies

$$(4.3.5) \qquad \||v - \bar{v}\|| \leq \gamma^2 \||\bar{v}\|| \; ,$$

then

$$(4.3.6) \qquad \||u_{m+1,n} - u^{(j)}\|| \leq \gamma \; \||u_{0,n} - u^{(j)}\|| \; .$$

Except for our use of (4.3.4) rather than a similar inequality for quasi-uniform grids, the proof of this theorem is identical to the proof of the analogous theorem for quasi-uniform grids given by Bank and Dupont.

Proof. Let $\theta_j \in (0, 1)$ be free to be chosen later and consider the decomposition of M_j:

$$M_j = \tilde{S}_j \oplus \tilde{O}_j$$

Let e_ℓ, $0 \leq \ell \leq m + 1$, be the convergence error,

$$e_\ell = u_{\ell,n} - u^{(j)} \; ,$$

since the second subscript on u plays no role in the theorem. Then

$$e_\ell = \eta_\ell + \xi_\ell \; ,$$

for some $\eta_\ell \in \tilde{S}$ and $\xi_\ell \in \tilde{O}$. Expanding e_0 in eigenfunctions,

$$e_0 = \sum_{i=1}^{N} \alpha_i \tilde{\psi}_i^{(j)} \; ,$$

we have for $\ell = 0, 1, \ldots, m,$

$$e_\ell = \sum_{i=1}^{N} \alpha_i (1 - \frac{\tilde{\lambda}_i^{(j)}}{\tilde{\lambda}_N^{(j)}})^\ell \tilde{\psi}_i^{(j)}$$

by (4.3.1). Thus,

$$\||\xi_m\||^2 = \sum_{i=J(\theta)+1}^{N} \alpha_i^2 (1 - \frac{\tilde{\lambda}_i^{(j)}}{\tilde{\lambda}_N^{(j)}})^{2m} \||\tilde{\psi}_i^{(j)}\||^2$$

$$\leq (1 - \theta_j)^{2m} \sum_{i=J(\theta)+1}^{N} \alpha_i^2 \||\tilde{\psi}_i^{(j)}\||^2 ,$$

so

$$\||\xi_m\|| \leq (1 - \theta_j)^m \||\xi_0\|| \leq (1 - \theta_j)^m \||e_0\|| ,$$

$$\||\eta_m\|| \leq \||\eta_0\|| \leq \||e_0\|| .$$

Let $\bar{\xi}$ and $\bar{\eta}$ be the elliptic projections on M_{j-1} of ξ_m and η_m, respectively,

$$a(\bar{\xi} - \xi_m, \phi) = 0 , \quad \phi \in M_{j-1} ,$$

$$a(\bar{\eta} - \eta_m, \phi) = 0 , \quad \phi \in M_{j-1} .$$

Since $\bar{\xi}$ is the projection relative to the energy inner product

$$\||\bar{\xi} - \xi_m\|| \leq \||\xi_m\|| \leq (1 - \theta_j)^m \||e_0\||$$

and by (4.3.4)

$$\||\bar{\eta} - \eta_m\|| \leq c_0 \theta_j^{1/4} \||\eta_m\|| \leq c_0 \theta_j^{1/4} \||e_0\|| .$$

Since $\bar{v} = \bar{\eta} + \bar{\xi}$ and $e_m = \eta_m + \xi_m$,

$$\||e_m - \bar{v}\|| \leq c_0 \theta_j^{1/4} \||e_0\|| + (1 - \theta_j)^m \||e_0\|| .$$

Then since $e_{m+1} = e_m - v$,

$$\||e_{m+1}\|| \leq \||e_m - \bar{v}\|| + \||v - \bar{v}\||$$

$$\leq c_0 \, {}_j^{1/4} \||e_0\|| + (1 - \theta_j)^m \||e_0\|| + \gamma^2 \||\bar{v}\||$$

by the hypothesis of the theorem. Since $\bar{v}$ is the projection of e_m on M_{j-1},

$$|||\bar{v}||| \leq |||e_m||| \leq |||e_0||| \; .$$

Thus,

$$(4.3.7) \qquad |||e_{m+1}||| \leq (c_0\theta_j^{1/4} + (1 - \theta_j)^m + \gamma^2) \; |||e_0||| \; .$$

To complete the proof, let $\gamma \in (0, 1)$ be such that

$$\gamma^2 < \frac{\gamma}{3} \; .$$

Then choose $\theta_j \in (0, 1)$ such that

$$c_0\theta_j < \frac{\gamma}{3}$$

and finally m such that

$$(1 - \theta_j)^m < \frac{\gamma}{3} \; .$$

Since these choices of γ, θ_j and m are independent of j, the proof
is complete. $\square$

Inequality (4.3.5) in this theorem is essentially an induction
hypothesis. Suppose that for fixed $j_0 \geq 2$ using iteration (4.3.1)-(4.3.3)
with $j = j_0$ causes the energy norm of the convergence error to be reduced
by the factor $\gamma \in (0, 1)$ according to this theorem. Also suppose we
have an arbitrary iteration on M_{j_0-1} which reduces the energy norm of the
convergence error on M_{j_0-1} by the same factor $\gamma \in (0, 1)$ at each iteration.
Then this iteration on M_{j_0-1} can be used to approximately solve the
related problem (4.3.2) on M_{j_0-1} arising in using the multi-level
iteration (4.3.1)-(4.3.3) on M_{j_0}. Beginning with the trial solution

$$v_0 \equiv 0$$

in M_{j_0-1} the iteration on M_{j_0-1} produces a sequence of iterates $\{v_\ell\}_{\ell=0}^{\infty}$
satisfying

$$|||v_\ell - \bar{v}||| \leq \gamma^\ell \; |||\bar{v}||| \; .$$

Thus, two of these iterations on M_{j_0-1} would be enough to satisfy the hypothesis (4.3.4) of theorem 4.1.

According to this theorem the constant $\gamma \in (0, 1)$ by which the energy norm of the convergence error is reduced per iteration of (4.3.1)-(4.3.3) is independent of $j \geq 2$. It follows that if $j_0-1 \geq 2$ the iteration on M_{j_0-1} can be taken as the multi-level iteration (4.3.1)-(4.3.3) with $j = j_0-1$. In this case we must solve in turn related problems on M_{j_0-2}. Again this can be done recursively with iteration (4.3.1)-(4.3.3) if $j_0-2 \geq 2$.

To make these ideas clear, we may consider the following informal procedure. Calling this recursive procedure with $j = j_0$ causes n of the multi-level iterations (4.3.1)-(4.3.3) to be carried out on M_{j_0}.

Procedure Outer_1_on_M_j (n)

>   If (j = 1) {solve the equations on M_j directly; then return}
>
>   else {repeat n times{
>
>   >   1.   do m smoothing inner iterations on M_j according to (4.3.1).
>   >
>   >   2.   approximately solve the related problem (4.3.2) on M_{j-1}
>   >   by calling Outer_1_on_M_{j-1} (2); then make the correction (4.3.3).
>
>   }}
>
>   return

The computational cost of this multi-level iteration, which will be analyzed in the next section, depends on the dimensions of the spaces $\{M_j\}_{j=1}^{k}$. In order to have a cost per iteration proportional to the number of unknowns, the cost of the smoothing inner iteration on

M_{j_0} must dominate the cost of recursively solving the related problems on the coarser spaces. This will only be true if the dimensions of the spaces $\{M_j\}_{j=1}^{\infty}$ grow fast enough. Since these dimensions will grow quite slowly for many families $\{M_j\}_{j=1}^{\infty}$ of locally refined grids th. presents a problem.

To ameliorate this difficulty, we consider now a second approach to solving the related problem (4.3.2) in the multi-level iteration (4.3.1)-(4.3.3). This second approach is based on a slightly sharper version of theorem 4.1. First we require a simple algebraic lemma.

Lemma 4.2. Let x,y be constants in (0, 1). If m is given by

$$m = \left\lceil \frac{1-y}{xy} \right\rceil$$

then

$$(1 - x)^m \leq y .$$

Proof. We have

$$\frac{(1-x)^{m+1} - 1}{(1-x) - 1} = 1 + (1-x) + \ldots + (1-x)^m$$

$$\geq (m+1)(1-x)^m ,$$

so

$$(1 - x)^{m+1} - 1 \leq -x(m+1)(1-x)^m .$$

Then

$$1 \geq (1-x)^{m+1} + x(m+1)(1-x)^m = (1 + mx)(1-x)^m ,$$

so

$$(1 - x)^m \leq (1 + mx)^{-1} \leq (1 + \frac{1-y}{y})^{-1} = y . \quad \square$$

Using this lemma, we can show the following theorem, which for our purposes improves on theorem 4.1. All notation is as before.

Theorem 4.3. Let $r \in (0, 1)$ be fixed and let $\gamma \in (0, 1)$ and $j \geq 2$ be arbitrary. Then there exists a positive integer m independent of j such that if the approximate solution v of (4.3.2) satisfies

$$(4.3.8) \qquad |||v - \bar{v}||| \leq r \gamma |||\bar{v}||| \ ,$$

then

$$(4.3.9) \qquad |||u_{m+1} - u^{(j)}||| \leq \gamma |||u_0 - u^{(j)}||| \ .$$

Moreover, m can be chosen to satisfy

$$(4.3.10) \qquad m \leq m_0 \gamma^{-5}$$

where $m_0 > 0$ is a constant independent of j and γ.

Proof. The proof is the same as that of theorem 4.1, except for the choice of the parameters m, θ_j, γ at the end of the proof. Instead of inequality (4.3.7) we have

$$(4.3.11) \qquad |||e_{m+1}||| \leq (c_0 \theta_j^{1/4} + (1 - \theta_j)^m + r\gamma) \ |||e_0||| \ ,$$

because the factor $r\gamma$ in (4.3.8) replaces the factor γ^2 in (4.3.5). Since we wish to show

$$c_0 \theta_j^{1/4} + (1 - \theta_j)^m + r\gamma \leq \gamma \ ,$$

it is enough to choose θ_j and m such that

$$c_0 \theta_j^{1/4} + (1 - \theta_j)^m \leq (1 - r)\gamma \ .$$

For convenience assume $c_0 \geq 1$, without loss of generality, and define y by

$$y = \frac{(1 - r)\gamma}{2c_0} \ .$$

Then we may choose θ_j as

$$\theta_j^{1/4} = y$$

and m as

$$m = \left\lceil \frac{1 - y}{\theta_j y} \right\rceil .$$

Using lemma 4.2,

$$c_0 \theta_j^{1/4} + (1 - \theta_j)^m \leq \frac{(1 - r)\gamma}{2} + \frac{(1 - r)\gamma}{2c_0} \leq (1 - r)\gamma ,$$

establishing the first part of the theorem. For the second part we have

$$m \leq \left\lceil \frac{1}{\theta_j y} \right\rceil = \left\lceil y^{-5} \right\rceil = \left\lceil (\frac{2c_0}{1 - r})^5 \gamma^{-5} \right\rceil$$

$$\leq 1 + (\frac{2c_0}{1 - r})^5 \gamma^{-5}$$

$$\leq (1 + (\frac{2c_0}{1 - r})^5) \gamma^{-5} ,$$

completing the proof. $\square$

Using this theorem, we can now analyze a modified version of the multi-level iteration given. This modified iteration will be the same as (4.3.1)-(4.3.3) except that m, the number of smoothing inner iterations, will now be made to depend on the level, j. In applying this modified iteration to M_{j_0}, $j_0 \geq 3$, the related problem in M_{j_0-1} will be solved by doing only one of the modified outer iterations on M_{j_0-1}. However, to adequately solve this related problem on M_{j_0-1} using only one outer iteration, rather than two as before, the number of smoothing inner iterations must be increased.

An informal statement of this modified outer iteration
follows. Calling this procedure with $j = j_0$ causes n multi-level
iterations to be performed on M_{j_0}. Here $m(\cdot, \cdot)$ is an integer valued
function to be given following the description of this procedure.

Procedure   Outer_2_on_M_j (n, j_0)

   If (j = 1) {solve the equations on M_j directly; then return}

   else {repeat n times{

   1.   do $m(j, j_0)$ smoothing inner iterations on M_j according

        to (4.3.1).

   2.   approximately solve the related problem (4.3.2) on M_{j-1}

        by calling Outer_2_on_M_{j-1} (1, j_0), then make the

        correction (4.3.3).

   }}

   return

It remains only to choose the function $m(\cdot, \cdot)$ required by
this procedure. Let $r \in (0, 1)$ be arbitrary and let $m_0 > 0$ be the
corresponding constant from inequality (4.3.10) of theorem 4.3. Also, let
$\gamma_0 \in (0, 1)$ be arbitrary. Then we may set

$$(4.3.12) \qquad m(j, j_0) = \left\lceil m_0 \, \gamma_0^{-5} \, r^{5(j-j_0)} \right\rceil .$$

Since the argument needed to show that the multi-level iteration in
Outer_2 converges rapidly is somewhat more complex than that for
Outer_1, we state this as a theorem. Let $\{u_{0,n}\}_{n=1}^{\infty}$ be successive
iterates produced by the multi-level iteration in Outer_2. Then we have

<u>Theorem 4.4.</u> Let $j_0 \geq 2$ be arbitrary. Then with $m(\cdot, \cdot)$ given by (4.3.12) the iterates $\{u_{0,n}\}_{n=1}^{\infty}$ satisfy

$$\||u_{0,n+1} - u^{(j_0)}_0\|| \leq \gamma_0 \||u_{0,n} - u^{(j_0)}_0\||,$$

for all $n \geq 1$.

<u>Proof.</u> In order for each outer iteration on M_{j_0}, $j_0 \geq 2$, to reduce the energy norm of the convergence error by the factor γ_0, according to theorem 4.3, it suffices to do m of the inner iterations (4.3.1) on M_{j_0} for some

$$m \leq m_0 \, \gamma_0^{-5},$$

and then approximately solve the related problem (4.3.2) with relative accuracy $r\gamma_0$. Examination of (4.3.11) shows that doing extra inner iterations never hurts so the choice

$$m = m(j_0, j_0) = \left\lceil m_0 \, \gamma_0^{-5} \right\rceil$$

is fine. Thus, for the case where $j_0 = 2$ and the related problem on M_{j_0-1} is solved directly the proof is done. Otherwise, we solve the related problem on M_{j_0-1} with one outer iteration. Choosing $\gamma = r\gamma_0$ in theorem 4.3 it suffices to do m inner iterations on M_{j_0-1} for some

$$m \leq m_0(r\gamma_0)^{-5}$$

and then solve the related problem on M_{j_0-2} with relative accuracy $r^2\gamma_0$. Doing

$$m = m(j_0-1, j_0) = \left\lceil m_0 \, \gamma_0^{-5} \, r^{-5} \right\rceil$$

inner iterations is certainly sufficient, so the proof is done for $j_0 = 3$. Continuing the induction in the obvious way completes the proof. □

The computational cost of the multi-level iterations described in this section and construction of an attractive overall algorithm for obtaining solutions to the elliptic problem (2.1.3) will be considered in the following section.

4.4. Complexity

In this section an estimate of the computational cost of the multi-level iterations described in the last section is given, and these iterations are then used as building blocks for computationally attractive algorithms for solving finite element equations on locally refined grids. Under certain restrictions these algorithms will be shown to be of optimal order, producing a good solution to the finite element equations for M_k, $k \geq 1$, in $O(N_k)$ operations, where N_k is the dimension of the finite element space M_k. This complexity analysis is somewhat harder than the corresponding analysis in chapter three, since the complexity of the multi-level iterations in the last section depends on the dimensions of the finite element spaces $\{M_j\}_{j=1}^k$ used. This complication was absent from the last chapter, since for the family $\{M_j\}_{j=1}^k$ of quasi-uniform spaces there, the dimensions of the spaces were essentially determined by the dimension of the coarsest space M_1 and the mesh ratio ρ.

A second minor difficulty also arises in proving the optimal rate convergence of the algorithms here. A multi-level algorithm is considered to be of optimal order if, when applied to any finite element space $\{M_j\}_{j=1}^{k}$, it produces a good solution $\tilde{u}^{(k)}$ in $O(N_k)$ operations. By "a good solution" means, an error bound applicable to all grids is needed. We now proceed to derive such a bound. so that the error bound derived not be overly

pessimistic. If it were, the results here would be correspondingly weakened. To have shown a multi-level method produces a solution containing a convergence error smaller than the expected truncation error means very little if one expects far more truncation error than actually results.

Only a simple error estimate is required here. Such an estimate can be derived directly from the Bramble-Hilbert lemma. First we need the notion of a Sobolev seminorm. For ℓ a non-negative integer, the seminorm $|\cdot|_{H^\ell}$ is the same as the Sobolev norm $\|\cdot\|_{H^\ell}$ except that it contains only derivatives of order ℓ, not those of lower order. That is, for $u \in H^\ell(\Omega)$,

$$|u|_{H^\ell(\Omega)} = \left(\sum_{|\alpha|=\ell} \|D^\alpha u\|_{L^2}^2 \right)^{1/2} .$$

Just a for the Sobolev norm, these seminorms can be defined for non-integral ℓ in a more complicated way.

Now let M be a finite element space of C^0 Lagrangian elements, based on a triangulation τ of Ω and satisfying conditions (4.2.1)-(4.2.5). Suppose the element trial functions are polynomials of degree $k - 1$. Then M is said to be a finite element space of degree k. For $u \in H_E^\ell(\Omega)$, $1 \le \ell \le k$, the nodal interpolant $u_I \in M$ of u is well defined and we have

Lemma 4.5. For any triangle $T \in \tau$ and for $s = 0, 1$

$$|u - u_I|_{H^s(T)} \le \sigma_s \, d_T^{\ell-s} \, |u|_{H^\ell(T)}$$

where d_T is the diameter of T.

The variable ℓ here need not be an integer. This lemma is a special case of the Bramble-Hilbert lemma, which may be found in, for example, Strong and Fix or Oden and Reddy.

Applying this lemma to the spaces in a family $\{M_j\}_{j=1}^{\infty}$ of locally refined finite element spaces satisfying the assumptions of section 4.2 yields the required error bound. Assume that the differential equation (2.1.3) is $H^{1+\alpha}$ regular for $\alpha \in (0, 1)$, that is, that its solutions lie in $H^{1+\alpha}$. Then the following error estimate holds on M_j, $j \geq 1$.

Theorem 4.6. Let $u^{(j)}$ be the finite element solution of (2.1.3) in M_j. Then

(4.4.1) $$\||u - u^{(j)}\|| \leq c_0 \left(\sum_{T \in \tau_j} d_T^{2\alpha} \, |u|_{H^{1+\alpha}(T)}^2 \right)^{1/2}$$

where $c_0 > 0$ is independent of u, j, and the family of locally refined spaces $\{M_i\}_{i=1}^{\infty}$.

Proof. Let $u_I^{(j)}$ be the interpolant of u in M_j. Then

$$\||u - u^{(j)}\|| = \min_{v \in M_j} \||u - v\|| \leq \||u - u_I^{(j)}\|| .$$

Since the energy and H^1 norms are equivalent,

(4.4.2) $$\||u - u^{(j)}\|| \leq c \, ||u - u_I^{(j)}||_{H^1} .$$

Applying the above lemma on each triangle in τ_j,

$$||u - u_I^{(j)}||_{L^2(\Omega)}^2 \leq \sigma_0^2 \sum_{T \in \tau_j} d_T^{2(1+\alpha)} \, |u|_{H^{1+\alpha}(T)}^2$$

$$\left| u - u_I^{(j)} \right|^2_{H^1(\Omega)} \leq \sigma_1^2 \sum_{T \in \tau_j} d_T^{2\alpha} \left| u \right|^2_{H^{1+\alpha}(T)} \ .$$

Thus,

$$\left\| u - u_I^{(j)} \right\|^2_{H^1(\Omega)} \leq c^2 \sum_{T \in \tau_j} \cdot d_T^{2\alpha} \left| u \right|^2_{H^{1+\alpha}(T)}$$

for $c^2 = \max\{\sigma_0^2, \sigma_1^2\}$, assuming $d_T \leq 1$ for all triangles $T \in \tau_j$. Now combining this with (4.4.2) the result follows. $\square$

Since this bound contains d_T, in effect the local mesh parameter, it is much tighter for locally refined grids than bounds in terms of the maximum element diameter. Moreover, the exponent of d_T here is optimal. For sufficiently smooth u the reverse inequality, bounding the error from below, may be expected to hold although this is difficult to show. Establishing that the multi-level methods of this section produce solutions whose convergence error is comparable to the truncation error expected from this error bound may be regarded as more than adequate for most purposes. In particular, for problems with singular solutions the improved convergence of finite element spaces using the proper locally refined grids may be readily shown using this bound. The multi-level algorithms here will inherit this improved convergence to singular solutions on properly refined grids.

With this error bound established we are ready to turn to the design of multi-level algorithms for solving the elliptic problem (2.1.3) on locally refined grids. Let $\{M_j\}_{j=1}^{\infty}$ be a family of locally refined finite element spaces satisfying the assumptions of section 4.2. The algorithms we wish to consider will produce a good solution $\tilde{u}^{(k)}$ in any number M_k of this family, $k \geq 1$. The first step in these algorithms is

to solve the finite element equations for M_1 directly to obtain an approximate solution $\tilde{u}^{(1)}$ to the problem (2.1.3). After this, solutions $\tilde{u}^{(j)} \in M_j$, $2 \le j \le k$, will be produced, each a good approximate solution of (2.1.3). Each solution $\tilde{u}^{(j)}$, $2 \le j \le k$, is obtained by beginning with the previous solution $\tilde{u}^{(j-1)} \in M_{j-1}$, considering it instead as a function in M_j, and then carrying out a sequence of multi-level iterations to obtain the improved solution $\tilde{u}^{(j)} \in M_j$. This multi-level iteration may be taken as either of the algorithms of the last section.

Two questions arise immediately: what conditions must the above process satisfy in order to obtain an accurate solution $\tilde{u}^{(k)}$, and what determines the computational complexity of this process? With the error bound (4.4.1) already proven, the first question is easier. We wish to obtain a solution $\tilde{u}^{(k)}$ which contains only convergence error comparable to the expected truncation error. That is, we want the convergence error to satisfy

$$\||u^{(k)} - u^{(k)}\|| \le c_0 \left(\sum_{T \in \tau_k} d_T^{2\alpha} |u|^2_{H^{1+\alpha}(T)} \right)^{1/2} .$$

In order to have an algorithm whose solution $\tilde{u}^{(k)}$ satisfies this inequality, it is convenient to ask that the intermediate solutions $\tilde{u}^{(j)}$, $1 \le j \le k - 1$, also satisfy this type of inequaltiy. That is, for $1 \le j \le k$,

$$(4.4.3) \qquad \||\tilde{u}^{(j)} - u^{(j)}\|| \le c_0 \left(\sum_{T \in \tau_j} d_T^{2\alpha} |u|^2_{H^{1+\alpha}(T)} \right)^{1/2} .$$

A simple condition insuring this is given in the following theorem.

Theorem 4.7. Suppose that $\tilde{u}^{(1)}$ satisfies (4.4.3), and for $2 \leq j \leq k$

(4.4.4) $\quad \||\tilde{u}^{(j)} - u^{(j)}\|| \leq (1 + 2^{1+\alpha})^{-1} \||\tilde{u}^{(j-1)} - u^{(j)}\||$.

Then (4.4.3) holds for $1 \leq j \leq k$.

Proof. By assumption

$$\||\tilde{u}^{(1)} - u^{(1)}\|| \leq c_0 \left(\sum_{T \in \tau_1} d_T^{2\alpha} |u|_{H^{1+\alpha}(T)}^2 \right)^{1/2} .$$

Now inductively suppose

$$\||\tilde{u}^{(j)} - u^{(j)}\|| \leq c_0 \left(\sum_{T \in \tau_j} d_T^{2\alpha} |u|_{H^{1+\alpha}(T)}^2 \right)^{1/2}$$

for some j, $1 \leq j \leq k$. Then

$$\||\tilde{u}^{(j)} - u\|| \leq \||\tilde{u}^{(j)} - u^{(j)}\|| + \||u^{(j)} - u\|| ;$$

so

$$\||\tilde{u}^{(j)} - u\|| \leq 2c_0 \left(\sum_{T \in \tau_j} d_T^{2\alpha} |u|_{H^{1+\alpha}(T)}^2 \right)^{1/2}$$

using the error bound (4.4.1). Then, since

$$\||u^{(j+1)} - u\|| \leq c_0 \left(\sum_{T \in \tau_{j+1}} d_T^{2\alpha} |u|_{H^{1+\alpha}(T)}^2 \right)^{1/2} ,$$

we have

(4.4.5) $\quad \||\tilde{u}^{(j)} - u^{(j+1)}\|| \leq c_0 \left(\sum_{T \in \tau_{j+1}} d_T^{2\alpha} |u|_{H^{1+\alpha}(T)}^2 \right)^{1/2}$

$$+ 2c_0 \left(\sum_{T \in \tau_j} d_T^{2\alpha} |u|_{H^{1+\alpha}(T)}^2 \right)^{1/2} .$$

Now let $T \in \tau_j$ and let S_T be the set of subtriangles of T in τ_{j+1}. That is S_T consists either of T or of the two or four triangles formed by refining T. Then we have the following relations.

$$d_T \leq 2d_{T'} \quad , \quad T' \in S_T$$

$$|u|^2_{H^{1+\alpha}(T)} = \sum_{T' \in S_T} |u|^2_{H^{1+\alpha}(T')} \; .$$

Applying these to (4.4.5) yields

$$\||\tilde{u}^{(j)} - u^{(j+1)}\|| \leq c_0 \Big(\sum_{T \in \tau_{j+1}} d_T^{2\alpha} |u|^2_{H^{1+\alpha}(T)} \Big)^{1/2}$$

$$+ 2c_0 \Big(\sum_{T \in \tau_j} \sum_{T' \in S_T} (2d_T)^{2\alpha} |u|^2_{H^{1+\alpha}(T')} \Big)^{1/2}$$

$$= c_0 (1 + 2(2)^{\alpha}) \Big(\sum_{T \in \tau_{j+1}} d_T^{2\alpha} |u|_{H^{1+\alpha}(T')} \Big)^{1/2} \; .$$

Finally, making use of the hypothesis (4.4.4) the induction is complete. □

The dependence of this result on the regularity of the problem will affect the complexity of the multi-level algorithm here in an interesting way. Consider the construction of such algorithms based on this theorem. As described above, after obtaining $\tilde{u}^{(1)}$ by solving the equations for M_1 directly, approximate solutions $u^{(j)} \in M_j$, $2 \leq j \leq k$, are obtained by iteratively improving the previous solution $u^{(j-1)} \in M_{j-1}$. The amount of iterative improvement required is given by (4.4.5). If the problem is $H^{1+\alpha}$ regular, we must reduce the convergence error by the factor $(1 + 2^{1+\alpha})^{-1}$ in going from $\tilde{u}^{(j-1)}$ to $\tilde{u}^{(j)}$. The interesting point here is that more error reduction is required for smoother problems. Since the regularity did not enter the convergence theorems of the last section, obtaining a given amount of reduction of the convergence error is apparently no easier for problems with smooth solutions than for problems with weak regularity. Thus, the computational cost per element of the algorithms here will be greater for problems with smooth solutions.

Of course, there is no real dilemma here. The algorithms here produce a solution satisfying (4.4.5). Since the finite element solution $u^{(k)}$ satisfies (4.4.1), the approximate solution produced by the multi-level algorithm must satisfy

$$|||\tilde{u}^{(k)} - u||| \leq 2c_0 (\sum_{T \in \tau_k} d_T^{2\alpha} |u|^2_{H^{1+\alpha}(T)})^{1/2} ,$$

a much stronger inequality for large α. Thus, even though one must work harder per element for smooth problems, it will still be cheaper to produce a good answer.

Now we are ready to consider the design of multi-level algorithms based on theorem 4.7 and the multi-level iterations of section 4.3. The multi-level iteration used may be that described in either procedure Outer_1 or Outer_2. In either case the dimensions of the finite element space $\{M_j\}_{j=1}^{\infty}$ will be of central importance. If N_j is the dimension of M_j, $j \geq 1$, optimal complexity bounds can be shown under the assumption that the dimensions $\{N_j\}_{j=1}^{\infty}$ grow at least geometrically. That is,

(4.4.6) $$N_{j+1} \geq \delta N_j , \quad j \geq 1 ,$$

for fixed $\delta > 1$. Actually, this can be weakened slightly. It is enough that we have a sequence of numbers $\{x_j\}_{j=1}^{\infty}$ growing at least geometrically,

$$x_{j+1} \geq \delta x_j , \quad j \geq 1 ,$$

for $\delta > 1$, and that N_j and x_j are comparable,

$$\frac{1}{c} x_j \leq N_j \leq c x_j$$

for fixed $c > 0$. For simplicity, assume that (4.4.6) holds. Using either of the multi-level iterations of section 4.3, the storage required to

approximately solve the finite element equations for M_k, $k \geq 1$, can be estimated as follows. Let c_1 be the storage required for direct solutions of the equations for M_1. Then the total storage required is proportional to

$$\sum_{j=2}^{k} N_j + c_1 \leq N_k \sum_{j=2}^{k} \delta^{j-k} + c_1$$

$$\leq N_k \frac{1}{1 - \frac{1}{\delta}} + c_1 \ .$$

Since c_1 is fixed, for any $\delta > 1$ the storage required is proportional to N_k.

Next consider the use of the multi-level iteration described by procedure Outer_1. According to theorem 4.1, when this multi-level iteration is applied to M_j, $j > 1$, the convergence error is reduced by a factor γ at every iteration, where $\gamma < 1$ is independent of j. Actually, the proof of theorem 4.1 shows γ can be chosen arbitrarily, though this is unimportant, since for any $\gamma < 1$ we can find $\nu \geq 1$ such that

$$\gamma^\nu \leq (1 + 2^{1+\alpha})^{-1} \ .$$

Let $\gamma < 1$ and an integer ν satisfying this inequality be fixed. Then according to theorem 4.7, a solution $\tilde{u}^{(k)}$ in M_k, satisfying (4.4.5) can be produced by carrying out ν of the multi-level iterations described by Outer_1 on each space M_j, $2 \leq j \leq k$. For each M_j, $2 \leq j \leq k$, the approximate solution produced in the next coarser space M_{j-1} is used as a starting value for this iteration. To estimate the complexity of this procedure, note that in applying Outer_1 to M_j, $j \geq 2$, $O(N_j)$ operations are required plus the cost of recursively solving the related

problem in M_{j-1}. This related problem is solved in Outer_1 applied to M_j, by recursively applying Outer_1 twice to the equations for M_{j-1}, if $j - 1 > 1$. Continuing the recursion one sees that the total cost in doing ν multi-level iterations of this type on the equations for M_j is proportional to

$$\nu(\sum_{i=2}^{j} 2^{j-i} N_i + 2^{j-1} c_2) \leq \nu N_j (\sum_{i=2}^{j} (\frac{2}{\delta})^{j-i} + (\frac{2}{\delta})^{j-1} c_2) ,$$

where c_2 is the cost of solving the linear system for M_1. Since c_2 is a fixed constant, the computation time $T(M_j)$ to do ν of these multi-level iterations on M_j must satisfy

$$T(M_j) \leq \begin{cases} O(N_j) , & \delta > 2 \\ O(N_j \log (N_j)) , & \delta = 2 \\ O(N_j^{\frac{\log 2}{\log \delta}}) , & 1 < \delta < 2 . \end{cases}$$

Then following the procedure described above where the approximate solution $\tilde{u}^{(k)}$ in M_k is obtained by successively obtaining good solutions in M_j, $1 \leq j \leq k$, the total computation time T to obtain $\tilde{u}^{(k)}$ must satisfy

$$T = \sum_{j=1}^{k} T(M_j) \leq \begin{cases} (N_k) , & \delta > 2 \\ (N_k \log (N_k)) , & \delta = 2 \\ (N_k^{\frac{\log 2}{\log \delta}}) , & 1 < \delta < 2 . \end{cases}$$

For many problems a bound like this is quite adequate. For example, in doing local refinement along a plane singularity or boundery in a three dimensional problem, the dimensions $\{N_j\}_{j=1}^{\infty}$ would at least quadruple at each level. Since the two dimensional results here readily extend to three dimensional problems, this example is important. In other problems, for example, refining along a line sigularity in two

or three dimensions, the dimensions typically only double at each level, yielding the N log (N) bound. Stil' other problems arise for which the growth of these dimensions is even slower. In refining around point singularities, these dimensions will usually not grow geometrically, and the results of this section are of little use. This does not necessarily mean that multi-level methods should be avoided in these cases, only that theoretical results showing their effectiveness have not yet been given.

When the dimensions of the spaces M_j grow geometrically according to (4.4.7), but the rate of growth, δ, is not greater than 2, the complexity results above can be improved by replacing the multi-level iteration given by Outer_1 with that given by Outer_2. In equation (4.3.12), governing the number of inner iterations used on each level in Outer_2, we are free to choose both γ_0 and r. For convenience, γ_0 may be chosen as

$$\gamma_0 = (1 + 2^{1+\alpha})^{-1} .$$

With this choice in applying Outer_2 to the equation for M_{j_0}, $2 \leq j_0 \leq k$, only one iteration of Outer_2 is needed to get $\tilde{u}^{(j_0)}$ satisfying (4.4.4) from $\tilde{u}^{(j_0-1)}$. The other free parameter, r, in (4.3.12) will be chosen as

$$r = \delta^{-\frac{1}{6}} .$$

Then for any $\delta > 1$, $r \in (0, 1)$, so theorem 4.4 shows that applying the multi-level iteration described by Outer_2 to M_{j_0}, $2 \leq j_0 \leq k$, reduces the convergence error by γ_0 per iteration as required. From equation (4.3.12), governing the number of inner iterations on each level, we can

estimate the computation time to produce $\tilde{u}^{(j_0)}$ from $\tilde{u}^{(j_0-1)}$, $j_0 \geq 2$.
This time, $T(M_{j_0})$ is proportional to

$$\sum_{j=2}^{j_0} r^{-5(j_0-j)} N_j + c_2 \leq N_{j_0} \sum_{j=2}^{j_0} (\frac{r}{\delta})^{-5} {}^{j_0-j} + c_2$$

$$\leq N_{j_0} \sum_{j=2}^{j_0} (\delta^{-\frac{1}{6}})^{j_0-j} + c_1$$

$$\leq N_{j_0} (1 - \delta^{-\frac{1}{6}})^{-1} + c_2 .$$

Thus, the amount of computation required to produce $\tilde{u}^{(j_0)}$ from $\tilde{u}^{(j_0-1)}$,
$j_0 \geq 2$, is $\mathcal{O}(N_{j_0})$. Then, since

$$\sum_{j_0=2}^{k} N_{j_0} \leq N_k (1 - \delta^{-1})^{-1}$$

for any $\delta > 1$, the total computation time is $\mathcal{O}(N_k)$.

This is an optimal bound, showing $\mathcal{O}(N)$ complexity under the
weaker assumption that the dimensions $\{N_j\}_{j=1}^{\infty}$ grow geometrically, with
any growth rate $\delta > 1$. Unfortunately, the dependence of this $\mathcal{O}(N)$ bound
on δ is quite severe, though this could be improved by choosing the
various constants differently, and by using a more effective inner
iteration. The hidden constant in the $\mathcal{O}(N)$ bound just given contains
not only the factors $(1 - \delta^{-1/6})^{-1}$ and $(1 - \delta^{-1})^{-1}$ seen above, but
also the constant m_0 of (4.3.12), which depends indirectly on δ. By
the proof of theorem 4.3 it can be seen that m_0 goes as

$$(1 - r)^{-5} = (1 - \delta^{-\frac{1}{6}})^{-5}$$

Thus, including the dependence on δ, the complexity bound is

$$O((1 - \delta^{-\frac{1}{6}})^{-6} (1 - \delta^{-1})^{-1} N) \ .$$

While the factor $(1 - \delta^{-1/6})^{-6} (1 - \delta^{-1})^{-1}$ already exceeds 10^6 for

$\delta = 2$, in practice one would expect the dependence on δ to be far

less severe than this, although the appropriate numerical experiments

have not yet been carried out.

This completes the analysis of the algebraic aspects of the

multi-level algorithms of this chapter. It remains to prove the

approximation result (4.3.4), on which the complexity results just

given are based.

4.5. Interpolation

This section begins the analysis of the approximation properties

of locally refined finite element spaces needed to justify the convergence

and complexity results of the last two sections. For convenience this

analysis is split into two parts. The first of these, given in this

section, deals with properties of the interpolation mapping from M_j to

M_{j-1}, $j \geq 2$, for M_j, M_{j-1} members of the family $\{M_j\}_{j=1}^{\infty}$ of locally

refined spaces. The second part of this analysis, given in the

following section will use the results of this section to obtain the

required approximation theorem.

While the analysis in the next section is completely general,

the analysis here is specific to the C^0 triangular elements and grids

based on aligned triangulations being considered. However, there is no

reason to believe that similar results could not be shown for more general

grids and for most common elements including triangular elements with higher continuity, square or quadrilateral elements, and elements with curved edges. The results here were not proven in greater generality primarily because it did not seem to aid understanding of the multi-level algorithms here and because the notations and arguments required would have been burdensome to both author and reader. Proof of results analogous to those in this section for most reasonable finite element spaces is not difficult.

The work in this section is closely related to the work Bank and Dupont did in analyzing their two-level scheme discussed briefly in section 4.1. Much of the notation here and one of the lemmas, lemma 4.10, are taken directly from their paper. The difference here is mainly in emphasis. While lemma 4.10 plays a fairly small role here, it is central to the convergence proof for the two-level scheme in their paper. Lemma 4.11, which is not from their paper, is really much more central here.

The reason for considering first the interpolation mapping is the usual one: while projection with respect to the energy inner product is global, interpolation gives a local projection more easily analyzed. Yet the two mappings are sufficiently alike that properties of one can be derived from the other. This is a common point of view, but the way in which properties of the elliptic projection are derived here from the interpolation projection is unusual. Instead of bounding the approximation error in the interpolate and using this as a bound on the approximation error in the elliptic projection, we begin by establishing properties of the null space of the interpolation mapping. From these

properties, in the next section we will derive properties of the null

space of the energy projection. Finally these lead to the required

approximation result.

This proof may seem circuitous, but the author was unable to

establish an approximation result such as (4.3.4) in any other way.

The problem encountered with the usual finite element approach is not

hard to see. Since elliptic projection is global, an error estimate,

such as theorem 4.6, for a locally refined space M, must involve the

maximum element diameter h_{max}. However, the eigenvalue bounds involve

h_{min}. Thus, one will be able to prove inequalities such as (4.3.4) but

the constant c occuring will contain some power of the factor $\frac{h_{max}}{h_{min}}$.

This factor becomes unbounded for the locally refined grids being

considered, so such a result would be too weak to prove $O(N)$

complexity bounds like those in the last section.

Since the finite element spaces considered in this chapter

are based on C^0 Lagrangian elements, the interpolation mapping from M_j

to M_{j-1}, $j \geq 2$, with M_j, M_{j-1} members of $\{M_j\}_{j=1}^{\infty}$, is automatically well

defined. For any $u \in M_j$, u is evaluated at the points corresponding to

the nodal parameters of M_{j-1}. Then there is a unique $u_I \in M_{j-1}$ attaining

the same values at these points. One of the problems encountered when

using elements with higher continuity is that this interpolation may not

be well defined. There are, for example, C^0 elements using first

derivatives as nodal parameters and almost all C^1 elements use second

derivatives. In these cases care must be taken that the nodes of M_{j-1}

lie at points where the corresponding values of functions in M_j are

well defined. In those rare cases where nodal interpolation is not well

defined, more complex interpolation projections based on various local averaging techniques must be used.

Now let $\hat{M}_j$, $j \geq 2$, denote the null space of the interpolation from M_j to M_{j-1}. $\hat{M}_j$ is the subspace of M_j consisting of functions vanishing at the nodes of M_{j-1}. Much of this section and part of the next are devoted to establishing properties of these null spaces. For now we note that $\hat{M}_j$ yields a direct sum decomposition of M_j:

$$M_j = \hat{M}_j \oplus M_{j-1} .$$

This is easy to show, for example, by dimensionality arguments.

Besides the direct method of defining the interpolate $u_I \in M_{j-1}$ of a function $u \in M_j$, there is also a more abstract way which we now consider.

The interpolation mapping from M_j to M_{j-1} considered here is completely local in the sense that the value of the interpolate $u_I \in M_{j-1}$ on a triangle $T \in \tau_{j-1}$ depends only on the value of $u \in M_j$ on T. For this reason the interpolation mapping from M_j to M_{j-1} decomposes into separate mappings from $M_j\big|_T$ to $M_{j-1}\big|_T$ for each $T \in \tau_{j-1}$. Rather than considering separately each of the function spaces $M_j\big|_T$ and $M_{j-1}\big|_T$ as T ranges over M_{j-1} we can consider only a few canonical function spaces on T_R and the interpolation mappings between these canonical spaces. These canonical spaces on T_R can then be transformed into the spaces $M_j\big|_T$ and $M_{j-1}\big|_T$ through the affine transformation F_T.

As in section 4.2 let L be the element trial space on the reference triangle T_R. We must consider three different refinements of T_R: regular refinement, green refinement, and the null refinement,

in which T_R is not altered. There are three possible green refinements but we need only one, say the one with the new edge leading to the top vertex.

Regular   Green   Null

Figure 16. Refinements of T_R

Since each of these refinements of T_R is a triangulation, finite element spaces on each of them can be generated in the usual way from the reference element trial space L.

Let K_R be the finite element space on T_R generated in this way from the regular refinement of T_R and let K_G be the finite element space generated by this green refinement. Null refinement just reproduces L. All three of these spaces generated by refinement, K_R, K_G, and L contain L as a subspace. Moreover, there are interpolation mappings I_R, I_G, and I_L from K_R, K_G, and L, respectively to L. The last is of course just the identity on L. Just as for the interpolation mapping from M_j to M_{j-1}, we denote the kernels of I_R and I_G by $\hat{K}_R$ and $\hat{K}_G$, respectively. The kernel of I_L is the trivial function space $\{0\}$.

Now let $T \in \tau_{j-1}$ be arbitrary. In going from τ_{j-1} to τ_j the triangle T is regular, green, or null refined. Consider the same type of refinement of T_R. Then there is an affine transformation $F_T: T_R \to T$ carrying the subtriangles of T_R generated by this refinement to the subtriangles created in refining T. When T is regular or null refined, any of the six affine transformations have this property. When T is green refined two of the six affine transformations carry the green

subtriangles of T_R to those of T. The other four affine transformations correspond to the other possible green refinements of T.

To see the importance of all this, let I_T be the interpolation mapping from $M_j|_T$ to $M_{j-1}|_T$ and let F_T' be the dual of F_T, carrying functions on T to functions on T_R ,

$$F_T': H^1(T) \rightarrow H^1(T_R)$$

by

$$F_T'(u) = u \circ F_T .$$

For each of the three types of refinement of T we get a commutative diagram. For regular refinement we have

$$
\begin{array}{ccc}
L & \xleftarrow{\ F_T'\ } & M_{j-1}|_T \\
\uparrow{\scriptstyle I_R} & & \uparrow{\scriptstyle I_T} \\
K_R & \xleftarrow{\ F_T'\ } & M_j|_T
\end{array} .
$$

For green refinement we have

$$
\begin{array}{ccc}
L & \xleftarrow{\ F_T'\ } & M_{j-1}|_T \\
\uparrow{\scriptstyle I_G} & & \uparrow{\scriptstyle I_T} \\
K_G & \xleftarrow{\ F_T'\ } & M_j|_T
\end{array} .
$$

And finally for null refinement we have

$$L \xleftarrow{\quad F'_T \quad} M_{j-1}\big|_T$$

$$\uparrow I_L \qquad\qquad \uparrow I_T$$

$$L \xleftarrow{\quad F'_T \quad} M_j\big|_T$$

That these diagrams are in fact commutative is a consequence of the fact that F_T carries not only T_R and its subtriangles into T and its subtriangles, but carries nodes of T_R and its subtriangles into corresponding nodes of T and its subtriangles. We omit the straight-forward but messy verification.

The lemmas which follow just elucidate the properties of these three commutative diagrams and the various function spaces involved. The first result here shows that functions in L cannot well approximate nonzero functions in $\hat{K}_R$ or $\hat{K}_G$, the kernels of I_R and I_G. The kernel, $\{0\}$, of I_L contains no nonzero functions so the question does not arise there. The functions in $\hat{K}_R$ and $\hat{K}_G$ vanish at the nodes of T_R while nonzero functions in L cannot vanish at all nodes of T_R. The results follow from this and the finite dimensionality of all spaces involved. For convenience define the interpolation

$$\pi_L: C^0(T_R) \to L .$$

The mappings I_R, I_G, and I_L described above are just restrictions of π_L to the subspaces K_R, K_G, and L of $C^0(T_R)$, respectively.

Lemma 4.8. If $w \in \hat{K}_R$ or $w \in \hat{K}$, for any $v \in L$

(4.5.1a) $$\|w - v\|_{L^2(T_R)} \geq c \|w\|_{L^2(T_R)} ,$$

(4.5.1b) $$|w - v|_{H^1(T_R)} \geq c |w|_{H^1(T_R)} ,$$

for $c > 0$ independent of v and w.

Proof. Let $\hat{K} = \hat{K}_R + \hat{K}_C$. Then $\hat{K}$ is contained in the null space of π_L.

Since π_L is the identity mapping on L,

(4.5.2) $$\hat{K} \cap L = \vec{0} .$$

Now consider the functional

$$f(w, v) = \|w - v\|_{L^2(T_R)}$$

defined on $\hat{K} \times L$. Let C be the compact subset

$$\{(w, v): \|w\|_{L^2(T_R)} = \|v\|_{L^2(T_R)} = 1\}$$

of $\hat{K} \times L$. Since f is continuous it attains its intimum of some $(w_0, v_0) \in C$, but by (4.5.2) we have

$$\|w_0 - v_0\|_{L^2(T_R)} \geq c > 0 ,$$

or

$$\|w_0 - v_0\|_{L^2(T_R)} \geq c \|w_0\|_{L^2(T_R)} .$$

Since $f(w, v)$ is minimal at (w_0, v_0), (4.5.1a) holds on all of C, and then on $\hat{K} \times L$ by linearity.

For (4.5.1b) note that since the functions in $\hat{K}$ are continuous and piecewise C^1, the only functions $u \in \hat{K}$ for which $|u|_{H^1} = 0$ are the constant functions. The only constant function in $\hat{K}$ is the function identically zero, since $\hat{K}$ is contained in the null space of π_L and the function values at the vertices of T_R are nodal parameters for L. Thus,

the seminorm $|\cdot|_{H^1}$ is norm on $\hat{K}$. Because of this, the verification of (4.5.1b) goes through exactly the same as for (4.5.1a). □

For any triangle $T \subseteq \mathbb{R}^2$, each affine transformation $F_T: T_R \to T$ induces a dual mapping F_T' carrying functions on T back to functions on T_R:

$$F_T': H^1(T) \to H^1(T_R)$$

by

$$F_T'(u) = u \circ F_T .$$

One can then ask for the norm of this dual mapping F_T' either as a mapping from $H^1(T)$ to $H^1(T_R)$ or from $L^2(T)$ to $L^2(T_R)$. For the H^1 seminorm the answer is that $|u|_{H^1(T)}$ and $|u \circ F_T|_{T^1(T_R)}$ are essentially the same since the change in area over which $|\cdot|_{H^1}$ is computed and change in the size of the derivatives are opposite effects and cancel perfectly (in two dimensions). Only triangles $T \in \bar{\tau}_{j-1}$ are considered in the following lemma both because they are the only ones of interest here and because the requirements of section 4.2 force all triangles in $\bar{\tau}_{j-1}$ to have diameter comparable to h_{j-1}, simplifying the statement of the lemma.

Lemma 4.9. Let $T \in \bar{\tau}_{j-1}$, $j \geq 2$ and let F be an affine transformation from T_R onto T. If $u \in H^1(T)$, then $u^* = u \circ F \in H^1(T_R)$ and

(4.5.3a) $\qquad \dfrac{1}{c} |u|_{H^1(T)} \leq |u^*|_{H^1(T_R)} \leq c |u|_{H^1(T)} ,$

(4.5.3b) $\dfrac{1}{c}\, \|u\|_{L^2(T)} \leq h_j \, \|u*\|_{L^2(T_R)} \leq c\, \|u\|_{L^2(T)}$,

for fixed $c > 1$ independent of u, j, and T.

<u>Proof</u>. $F = Ax + b$ for some 2×2 matrix A and $b \in R^2$. Then as shown

in Oden and Reddy (1976, eqs. (6.167) and (6.168)), for any $s \geq 0$

(4.5.4a) $|u*|_{H^s(T_R)} \leq \|A\|^s\, |\det A|^{-1/2}\, |u|_{H^s(T)}$,

(4.5.4b) $|u|_{H^s(T)} \leq \|A^{-1}\|^s\, |\det A|^{1/2}\, |u*|_{H^s(T_R)}$,

where $\|A\|$ is the spectral norm. Now let d_T and d_{T_R} be the diameters of

T and T_R, respectively and let ρ_T and ρ_{T_R} be the diameters of the

inscribed circles of T and T_R. Then as given in Oden and Reddy,

lemma 6.2, the following simple bounds on the norms of A and A^{-1} apply

$$\|A\| \leq \frac{d_T}{\rho_{T_R}}$$

$$\|A^{-1}\| \leq \frac{d_{T_R}}{\rho_T} \ .$$

We also have

$$\det A = \frac{\text{area }(T)}{\text{area }(T_R)}$$

The triangle T satisfies the angle condition (4.2.1) uniformly for

$T \in \bar{\tau}_{j-1}$ and $j \geq 2$. Using this condition it can be shown by simple

geometric arguments that

$$\frac{1}{c} \, d_T \leq \rho_T$$

$$\frac{1}{c} \, d_{T_R} \leq \rho_{T_R}$$

$$\frac{1}{c} \, d_T^2 \leq \text{area}(T) \leq c \, d_T^2$$

$$\frac{1}{c} \, d_{T_R}^2 \leq \text{area}(T_R) \leq c \, d_{T_R}^2 \, ,$$

for $c > 1$ independent of $T \in \bar{\tau}_{j-1}$ and $j \geq 2$. Then since $d_{T_R} = 1$ by assumption it follows that

$$\|A\| \leq c \, d_T$$

$$\|A^{-1}\| \leq c \, d_T^{-1}$$

$$|\det A|^{1/2} \leq c \, d_T$$

$$|\det A|^{-1/2} \leq c \, d_T^{-1} \, .$$

Combining these relations with (4.5.4) yields (4.5.3a) and

$$\frac{1}{c} \, \|u\|_{L^2(T)} \leq d_T \, \|u*\|_{L^2(T_R)} \leq c \, \|u\|_{L^2(T)} \, .$$

Since

$$\frac{1}{c} \, h_{j-1} \leq d_T \leq c \, h_{j-1}$$

by (4.2.), and $\rho = 2$, inequality (4.5.3b) follows and the proof is complete. $\square$

Lemma 4.9 above shows how the norm of a function u on a triangle T is affected by transforming it to a function $u* = u \circ F_T$ on T_R. Lemma 4.8 showed that functions in $\hat{k}_R$ or $\hat{k}_C$ were not well approximated by those in L. Combining these two lemmas we should be able to show

that for any triangle T, functions in $\hat{K}_R \circ F_T^{-1}$ or $\hat{K}_G \circ F_T^{-1}$ are not well approximated by those in $L \circ F_T^{-1}$. This is equivalent to the statement that for any $T \in \tau_{j-1}$ functions in $\hat{M}_j|_T$ cannot be well approximated by those in $M_{j-1}|_T$, since $\hat{M}_j|_T$ is either $\{0\}$, $\hat{K}_R \circ F_T^{-1}$, or $\hat{K}_G \circ F_T^{-1}$ for one of the affine transformations $F_T: T_R \to T$. Applying this observation to all triangles $T \in \tau_{j-1}$ yields the following lemma, which shows that functions in M_{j-1} cannot well approximate those in $\hat{M}_j$. This lemma was shown previously by Bank and Dupont.

__Lemma 4.10.__ If $v \in M_{j-1}$ and $w \in \hat{M}_j$ the strengthened Cauchy inequaltiy

(4.5.5) $$a(v, w) \le \gamma \ |||v||| \ |||w|||$$

holds for fixed $\gamma \in (0, 1)$ independent of v, w and $j \ge 2$.

__Proof.__ The proof will be done locally over the triangles of τ_{j-1}. Let $T \in \tau_{j-1}$ be arbitrary. Then there exists an affine transformation $F_T: T_R \to T$. Letting $w \in \hat{M}_j$ be arbitrary we can define $w^* \ w \circ F_T$. Then $w^* \in \hat{K}_R$ or $w^* \in \hat{K}_6$ even when $T \notin \bar{\tau}_{j-1}$ and $w^* \equiv 0$. Also we have $\{v^* = v \circ F_T$ for $v \in M_{j-1}\} \ c \ L$ where the inclusion may be strict because of boundary conditions or interelement continuity conditions. Thus,

$$\inf_{v \in M_{j-1}} |v \circ F_T - w^*|_{H^1(T_R)} \ge \inf_{v^* \in L} |v^* - w^*|_{H^1(T_R)} \ge c \ |w^*|_{H^1(T_R)}$$

$$\inf_{v \in M_{j-1}} ||v \circ F_T - w^*||_{L^2(T_R)} \ge \inf_{v^* \in L} ||v^* - w^*||_{L^2(T_R)} \ge c \ ||w^*||_{L^2(T_R)} \ ,$$

where the inequalities on the right are from lemma 4.8. By lemma 4.9 these inequalities can be converted into

$$\inf_{v \in M_{j-1}} |v - w|_{H^1(T)} \geq c \, |w|_{H^1(T)}$$

$$\inf_{v \in M_{j-1}} \|v - w\|_{L^2(T)} \geq c \, \|w\|_{L^2(T)} \, ,$$

where the constants here are independent of d_T. Combining these relations yields

$$\inf_{v \in M_{j-1}} \|v - w\|_{H^1(T)}^2 \geq \inf_{v \in M_{j-1}} |v - w|_{H^1(T)}^2 + \inf_{v \in M_{j-1}} \|v - w\|_{L^2(T)}^2$$

$$\geq c(|w|_{H^1(T)}^2 + \|w\|_{L^2(T)}^2) = c \, \|w\|_{H^1(T)} \, .$$

Then summing over the triangles of τ_{j-1}

$$\inf_{v \in M_{j-1}} \|v - w\|_{H^1}^2 = \inf_{v \in M_{j-1}} \sum_{T \in \tau_{j-1}} \|v - w\|_{H^1(T)}^2$$

$$\geq \sum_{T \in \tau_{j-1}} \inf_{v \in M_{j-1}} \|v - w\|_{H^1(T)}^2$$

$$\geq c \sum_{T \in \tau_{j-1}} \|w\|_{H^1(T)}^2 = c \, \|w\|_{H^1}^2 \, .$$

So using the equivalence of the H^1 and energy norms

$$\inf_{v \in M_{j-1}} \|\|v - w\|\| \geq c \, \|\|w\|\| \, .$$

This is equivalent to (4.5.5). To see this, assume $\|\|w\|\| = 1$. Then

$$\inf_{\substack{\|\|v\|\|=1 \\ v \in M_{j-1}}} \|\|v - w\|\| \geq \inf_{v \in M_{j-1}} \|\|v - w\|\| \geq c \, \|\|w\|\| = c \, ,$$

so if $\|\|v\|\| = \|\|w\|\| = 1$

$$a(v, w) = 1 - \frac{1}{2} \|\|v - w\|\|^2 \leq 1 - c = (1 - c) \, \|\|v\|\| \, \|\|w\|\| \, .$$

The general case of (4.5.5) follows by iinearity. $\square$

One more lemma is needed, which will be, for our purposes, the most important. It states that functions in $\hat{M}_j$, the kernel of the interpolation mapping from M_j to M_{j-1}, are all quite oscillatory. Intuitively, the relatively smooth functions in M_j agree quite closely with their interpolate in M_{j-1}. Only the most oscillatory functions in M_j can have the zero function as their interpolate. This observation can be made precise quite easily through the use of lemma 4.9. For our purposes a function is "oscillatory" when its H^1 norm or seminorm is much larger than its L^2 norm. That functions in $\hat{M}_j$ are oscillatory in this sense is shown by the following lemma.

Lemma 4.11. If $w \in \hat{M}_j$, $j \geq 2$ then

$$(4.5.6) \qquad \frac{1}{c} \|w\|_{L^2} \leq h_j \, |w|_{H^1} \leq c \, \|w\|_{L^2} \, ,$$

with $c > 1$ independent of w and j.

Proof. It is enough to show this locally on the triangles of $\bar{\tau}_{j-1}$. To see this, suppose that for all $w \in \hat{M}_j$ and $T \in \bar{\tau}_{j-1}$

$$(4.5.7) \qquad \frac{1}{c} \|w\|_{L^2(T)} \leq h_j \, |w|_{H^1(T)} \leq c \, \|w\|_{L^2(T)} \, ,$$

with $c > 1$ independent of w, T, and j. Then for any $w \in \hat{M}_j$

$$\frac{1}{c^2} \sum_{T \in \bar{\tau}_{j-1}} \|w\|^2_{L^2(T)} \leq h_j^2 \sum_{T \in \bar{\tau}_{j-1}} |w|_{H^1(T)} \leq c^2 \sum_{T \in \bar{\tau}_{j-1}} \|w\|^2_{L^2(T)} \, ,$$

which is the same as (4.5.6).

To show (4.5.7), let $w \in \hat{M}_j$ and let T be an arbitrary triangle in $\bar{\tau}_{j-1}$. Then there is an affine transformation $F: T_R \to T$ such that $w^* = w \circ F$ is in $\hat{K}_R$ or $\hat{K}_G$. Since $|\cdot|_{H^1}$ is a norm on $\hat{K}_R$ and on $\hat{K}_G$ and

$u \in \hat{K}_R$ or $u \in \hat{K}_6$

$$\frac{1}{c} \|u\|_{L^2(T_R)} \leq |u|_{H^1(T_R)} \leq c \|u\|_{L^2(T_R)} \ ,$$

with $c > 1$ independent of u. Then

$$\frac{1}{c} \|w^*\|_{L^2(T_R)} \leq |w^*|_{H^1(T_R)} \leq c \|w^*\|_{L^2(T_R)} \ ,$$

so using lemma 4.9

$$\frac{1}{c} \|w\|_{L^2(T)} \leq h_j \ |w|_{H^1(T)} \leq c \|w\|_{L^2(T)} \ ,$$

completing the proof. $\square$

4.6. Approximation

In this section the approximation inequality (4.3.4) needed to justify the convergence and complexity results of sections 4.3 and 4.4 will be proven. This inequality generalizes a result of Bank and Dupont for quasi-uniform grids. Assume the elliptic equation is $H^{1+\alpha}$ regular for $\alpha \in (0, 1)$. Then their result is that if $\{M_j\}_{j=1}^{\infty}$ is a family of quasi-uniform spaces and S_j and O_j, $j \geq 2$, are defined as before, the following inequality holds for $\eta \in S_j$,

$$\||\eta - \bar{\eta}\|| \leq c \ \theta_j^{\alpha/2} \ \||\eta\|| \ ,$$

where $\bar{\eta}$ is the elliptic projection of η on M_{j-1}. A domain with a slit, the worst case ordinarily arising, yields $\alpha = \frac{1}{2}$ and their result becomes

$$\||\eta - \bar{\eta}\|| \leq c \, \theta_j^{1/4} \, \||\eta\|| \, ,$$

the same as the result here.

The result here generalizes theirs in the sense that it applies to locally refined grids as well as to quasi-uniform grids, although the exponent of θ_j is usually poorer in our inequality. This is of secondary importance for multi-level convergence since $\mathcal{O}(N)$ complexity can be shown for any positive exponent of θ_j. Only the hidden constant in the $\mathcal{O}(N)$ bound is affected by the exponent.

Enough has been said about this approximation result that no further introduction seems necessary. The technique of proof is quite interesting and begins with a sequence of strengthened Cauchy inequalities that follow more or less directly from the results of the last section. Each of these inequalities is, in effect, a lower bound on the dihedral angle between two subspaces.

Lemma 4.12. Let $w \in \hat{M}_j$, $\eta \in S_j$, $s \in S_{j-1}$ and $o \in O_{j-1}$. Then the following strengthened Cauchy inequalities hold:

(4.6.1a)
$$a(\eta, \, w) \leq c \, \theta_j^{1/2} \, \||\eta\|| \, \||w\|| \, ,$$

(4.6.1b)
$$a(s, \, w) \leq c \, \theta_{j-1}^{1/2} \, \||s\|| \, \||w\|| \, ,$$

(4.6.1c)
$$a(\eta, \, o) \leq c \, \theta_j^{1/2} \, \theta_{j-1}^{-1/2} \, \||\eta\|| \, \||o\|| \, .$$

Proof. For (4.6.1a) let η be expanded as

$$\eta = \sum_{n=1}^{J(\theta_j)} \alpha_i \, \psi_i^{(j)} \, .$$

Then the corresponding finite element data is

$$f = \sum_{i=1}^{J(\theta_j)} \alpha_i \, \lambda_i^{(j)} \, \psi_i^{(j)} \; ,$$

with

(4.6.2) $\qquad a(\eta, \phi) = (f, \phi) \; , \; \phi \in M_j \; .$

Then

$$\| f \|^2 = \sum_{i=1}^{J(\theta_j)} \alpha_i^2 \, (\lambda_i^{(j)})^2 \, \| \psi_i^{(j)} \|^2$$

$$\leq \theta_j \, \lambda_{N_j}^{(j)} \sum_{i=1}^{J(\theta_j)} \alpha_i^2 \, \lambda_i^{(j)} \, \| \psi_i^{(j)} \|^2$$

$$= \theta_j \, \lambda_{N_j}^{(j)} \sum_{i=1}^{J(\theta_j)} \alpha_i^2 \, \vert\!\vert\!\vert \psi_i^{(j)} \vert\!\vert\!\vert^2$$

$$= \theta_j \, \lambda_{N_j}^{(j)} \, \vert\!\vert\!\vert \eta \vert\!\vert\!\vert^2 \; .$$

Thus

(4.6.3) $\qquad \| f \| \leq (\theta_j \, \lambda_{N_j}^{(j)})^{1/2} \, \vert\!\vert\!\vert \eta \vert\!\vert\!\vert \; .$

Setting $\phi = w$ in (4.6.2)

(4.6.4) $\quad a(\eta, w) = (f, w) \leq \| f \| \, \| w \| \leq (\theta_j \, \lambda_{N_j}^{(j)})^{1/2} \, \vert\!\vert\!\vert \eta \vert\!\vert\!\vert \, \| w \|$

so using lemma 4.11,

(4.6.5) $\qquad a(\eta, w) \leq c \, h_j (\theta_j \, \lambda_{N_j}^{(j)})^{1/2} \, \vert\!\vert\!\vert \eta \vert\!\vert\!\vert \; \vert\!\vert\!\vert w \vert\!\vert\!\vert$

$$\leq c \, \theta_j^{1/2} \, \vert\!\vert\!\vert \eta \vert\!\vert\!\vert \; \vert\!\vert\!\vert w \vert\!\vert\!\vert \; ,$$

using the eigenvalue bound (2.3.2). The proof of (4.6.1b) is the same

except θ_{j-1} and the eigenfunctions and eigenvalues $\{\psi_i^{(j-1)}\}_{i=1}^{N_{j-1}}$ and $\{\lambda_i^{(j-1)}\}_{i=1}^{N_{j-1}}$ occur throughout. Thus, instead of (4.6.5) we get

$$a(s, w) \leq c \, h_j \, (\theta_{j-1} \, \lambda_{N_{j-1}}^{(j-1)})^{1/2} \, |||s||| \, |||w|||$$

$$\leq c \, \rho^{-1} \, \theta_{j-1}^{1/2} \, |||s||| \, |||w|||$$

$$\leq c \, \theta_{j-1}^{1/2} \, |||s||| \, |||w||| \,,$$

since by assumption $\rho = 2$. For (4.6.1c) we have as in (4.6.4)

$$a(\eta, o) \leq (\theta_j \, \lambda_{N_j}^j)^{1/2} \, |||\eta||| \, ||o|| \,.$$

Expanding o in eigenfunctions

$$o = \sum_{i=J(\theta_{j-1})}^{N_{j-1}} \alpha_i \, \psi_i^{(j-1)} \,.$$

Thus,

$$|||o|||^2 = \sum_{i=J(\theta_{j-1})+1}^{N_{j-1}} \alpha_i^2 \, |||\psi_i^{(j-1)}|||^2$$

$$= \sum_{i=J(\theta_{j-1})+1}^{N_{j-1}} \alpha_i^2 \, \lambda_i^{(j-1)} \, ||\psi_i^{(j-1)}||^2$$

$$\geq \theta_{j-1} \, \lambda_{N_{j-1}}^{(j-1)} \sum_{i=J(\theta_{j-1})+1}^{N_{j-1}} \alpha_i^2 \, ||\psi_i^{(j-1)}||^2$$

$$= \theta_{j-1} \, \lambda_{N_{j-1}}^{(j-1)} \, ||o||^2 \,,$$

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$$a(\eta, o) \le (\theta_j \, \lambda_{N_j}^{(j)})^{1/2} \, (\theta_{j-1} \, \lambda_{N_{j-1}}^{(j-1)})^{-1/2} \, |||\eta||| \; |||o||| \; .$$

By the eigenvalue bound (2.3.2)

$$\lambda_{N_j}^{(j)} \le c \, h_j^{-2} \; .$$

The inequality in the eigenvalue bound (2.3.2) also goes the other way. One way to see this is to expand w in eigenfunctions in lemma 4.11:

$$w = \sum_{i=1}^{N_j} \alpha_i \, \psi_i$$

Then the left side of (4.5.6) is

$$\frac{1}{c} \, \| \sum_{i=1}^{N_j} \alpha_i \, \psi_i \| \le h_j \, | \sum_{i=1}^{N_j} \alpha_i \, \psi_i |_{H^1} \le c \, h_j \, ||| \sum_{i=1}^{N_j} \alpha_i \, \psi_i |||$$

where the last inequality is from the equivalence of the H^1 and energy norms. It follows easily from this that

$$\lambda_{N_{j-1}}^{(j-1)} \ge c \, h_{j-1}^{-2} \; ,$$

so we get

$$a(\eta, o) \le c \, \rho \, \theta_j^{1/2} \, \theta_{j-1}^{-1/2} \, |||\eta||| \; |||o|||$$

$$\le c \, \theta_j^{1/2} \, \theta_{j-1}^{-1/2} \, |||\eta||| \; |||o||| \; ,$$

since the mesh ratio, ρ, is fixed. This completes the proof. $\square$

Each of equations (4.6.1) of this lemma states that the energy inner product of a relatively smooth function with a relatively oscillatory function is small compared to the product of their norms. In other words, the angle between these functions, as measured by the energy inner

product, is large. In the appropriate limit as θ_j on θ_{j-1} goes to zero each of equations (4.6.1) expresses an orthogonality relation.

Our aim is to show that the function in S_j can be well approximated in the coarser finite element space M_{j-1} when θ_j is small. Let $M_{j-1}^{\perp}$ be the orthogonal complement of M_{j-1} in M_j with respect to the energy inner product. We approach the question of approximation of functions of S_j in M_{j-1} by the back door. Instead of showing directly that M_{j-1} contains a function near any function in S_j, we show that S_j and $M_{j-1}^{\perp}$ are nearly orthogonal. Since S_j consists of relatively smooth functions, by analogy with the lemma just proven we should try to show $M_{j-1}^{\perp}$ consists of oscillatory functions. This is the content of the following lemma.

<u>Lemma 4.13.</u> If $u \in M_{j-1}^{\perp}$, then

$$u = s + o + w ,$$

for $s \in S_{j-1}$, $o \in O_{j-1}$, and $w \in \hat{M}_j$ and

(4.6.7a)
$$|||s||| \leq c \, \theta_{j-1}^{1/2} \, |||u||| ,$$

(4.6.7b)
$$|||o||| \leq c \, |||u||| ,$$

(4.6.7c)
$$|||w||| \leq c \, |||u||| ,$$

with $c > 0$ independent of u and j.

<u>Proof.</u> Since $M_j = M_{j-1} \oplus \hat{M}_j$, we can set

$$u = v + w ,$$

for some $v \in M_{j-1}$, $w \in \hat{M}_j$. Then using the elementary inequality

$$xy \leq \frac{1}{2} (x^2 + y^2) , \quad x , y \in \mathbb{R} ,$$

and lemma 4.5.3,

$$|||u|||^2 = |||v + w|||^2 = |||v|||^2 + |||w|||^2 + 2a(v, w)$$

$$\geq |||v|||^2 + |||w|||^2 - 2\gamma \, |||v||| \, |||w|||$$

$$\geq (1 - \gamma)(|||v|||^2 + |||w|||^2) \; .$$

Thus,

(4.6.8a) $$|||v||| \leq c \, |||u||| \; ,$$

(4.6.8b) $$|||w||| \leq c \, |||u||| \; ,$$

where $c > 0$ depends only on the constant $\gamma \in (0, 1)$ from lemma 4.10, giving (4.6.7c). Now since $u = v + w \in M_{j-1}^{\perp}$,

$$a(v + w, \phi) = 0 \; , \; \phi \in M_{j-1} \; ,$$

so v is the elliptic projection of $-w$ on M_{j-1}. Writing

$$v = s + o \; ,$$

for $s \in S_{j-1}$, $o \in O_{j-1}$,

$$a(s + w, \phi) = 0 \; , \; \phi \in S_{j-1} \; ,$$

$$a(o + w, \phi) = 0 \; , \; \phi \in O_{j-1} \; ,$$

since S_{j-1} and O_{j-1} are energy orthogonal. That is, s and o are the elliptic projections of $-w$ on S_{j-1}, O_{j-1}, respectively. Then

$$|||s|||^2 = -a(s, w) \leq c \, \theta_{j-1}^{1/2} \, |||s||| \, |||w||| \; ,$$

by lemma 4.12, (4.6.1b). Thus, using (4.6.8b) we have

$$|||s||| \leq c \, \theta_{j-1}^{1/2} \, |||w||| \leq c \, \theta_{j-1}^{1/2} \, |||u|||$$

and

$$|||o||| \leq |||w||| \leq c \, |||u||| \; ,$$

completing the proof. $\square$

The lemma just proved shows that $M_{j-1}^{\perp}$, the orthogonal complement of M_{j-1} in M_j, is contained in

$$S_{j-1} \oplus O_{j-1} \oplus M_j$$

and for any $u \in M_{j-1}^{\perp}$, the component of u in S_{j-1} is relatively small when θ_{j-1} is small. In other words, $M_{j-1}^{\perp}$ consists of relatively oscillatory functions. Since S_j consists of smooth functions when θ_j is small, one might expect that the inner product of functions in S_j with functions in $M_{j-1}^{\perp}$ would be small when θ_j was small. This is the content of the next lemma.

<u>Lemma 4.14.</u> If $\eta \in S_j$ and $u \in M_{j-1}^{\perp}$, then

$$a(\eta, u) \le c \, \theta_j^{1/4} \, |||\eta||| \, |||u||| \,.$$

<u>Proof.</u> Since θ_j is the only member of the family $\{\theta_i\}_{i=1}^{\infty}$ of parameters occuring either implicitly or explicitly in the statement of this lemma, we are free to choose the remaining parameters $\{\theta_i\}_{i \ne j}$ arbitrarily. In particular, $\theta_{j-1} \in (0, 1)$ may be considered a free parameter until its value is chosen at the end of the proof. Let η and u satisfy the hypotheses. According to lemma 4.13, u can be decomposed as

$$u = s + o + w \,,$$

with $s \in S_{j-1}$, $o \in O_{j-1}$ and $w \in \hat{M}_j$. Then

$$a(\eta, u) = a(\eta, s) + a(\eta, o) + a(\eta, w) \,.$$

Bounding each of these terms separately

$$a(\eta, s) \le |||\eta||| \, |||s||| \le c \, \theta_{j-1}^{1/2} \, |||\eta||| \, |||u|||$$

by lemma 4.13, (4.6.7a). By lemma 4.12, (4.6.1c),

$$a(\eta, o) \le c \, \theta_j^{1/2} \, \theta_{j-1}^{-1/2} \, |||\eta||| \, |||o||| \le c \, \theta_{j-1}^{1/2} \, |||\eta||| \, |||u||| \; ,$$

where the last inequality is from lemma 4.13. Similarly by lemma 4.12, (4.6.1d),

$$a(\eta, w) \le c \, \theta_j^{1/2} \, |||\eta||| \, |||w||| \le c \, \theta_j^{1/2} \, |||\eta||| \, |||u||| \; .$$

Combining these inequalities

$$a(\eta, u) \le c(\theta_{j-1}^{1/2} + \theta_j^{1/2} \, \theta_{j-1}^{-1/2} + \theta_j^{1/2}) \, |||\eta||| \, |||u||| \; .$$

Choosing $\theta_{j-1} = \theta_j^{1/2}$ completes the proof. $\square$

Our first approximation result is a simple consequence of this lemma.

<u>Theorem 4.15</u>. Let $\eta \in S_j$ and let $\bar{\eta} \in M_{j-1}$ be the elliptic projection of η on M_{j-1},

$$a(\eta - \bar{\eta}, \phi) = 0 \; , \quad \phi \in M_{j-1} \; .$$

Then

$$|||\eta - \bar{\eta}||| \le c \, \theta_j^{1/4} \, |||\eta||| \; .$$

<u>Proof</u>. We have

$$|||\eta - \bar{\eta}|||^2 = a(\eta, \eta - \bar{\eta}) \le c \, \theta_j^{1/4} \, |||\eta||| \, |||\eta - \bar{\eta}||| \; ,$$

by lemma 4.14, since $\eta - \bar{\eta} \in M_{j-1}^{\perp}$. Dividing by $||\eta - \bar{\eta}||$ the result follows. $\square$

An important feature of the result just proven is that the constant c occurring here depends entirely on local properties of the

finite element spaces M_{j-1} and M_j. In fact, c depends only on the
constants in the eigenvalue bounds and on those of the last section
on interpolation. As a consequence, it is readily computable.
Moreover, the approximation property just shown does not depend
significantly on either the domain or the boundary conditions. This
is in contrast to the result of Bank and Dupont where both the
exponent of θ_j and the constant occurring depend on the regularity
of the PDE and hence on the domain.

The approximation result just proven is almost the result
we needed in section 4.3. Its defect is that it is expressed in
terms of the subspace S_j and indirectly the eigenfunctions $\{\psi_i\}_{n=1}^{N_j}$,
rather than in terms of $\tilde{S}_j$ and indirectly $\{\tilde{\psi}_i\}_{i=1}^{N_j}$. Since it would
be difficult to analyze the simultaneous displacement iteration of
section 4.3 in terms of the eigenfunctions $\{\psi_i\}_{i=1}^{N_j}$ we must prove
the analog (4.3.4) of theorem 4.15 which has S_j replaced by $\tilde{S}_j$. Exactly
the same thing occurred in chapter three. While it may be possible
to get the required analog directly from theorem 4.15, it is easier to
repeat the steps in its proof with slight modifications. Actually,
this could have been done in the first place, but it seemed more
natural this way. First we show the analog of lemma 4.12. Only S_j
need be replaced by $\tilde{S}_j$. There is no necessity to replace S_{j-1} and
0_{j-1} since these subspaces play only an intermediate role.

Lemma 4.16. Let $w \in \hat{M}_j$, $\eta \in \tilde{S}_j$, $s \in S_{j-1}$ and $o \in 0_{j-1}$. Then the
following strengthened Cauchy inequalities hold:

(4.6.9a) $\qquad a(\eta, w) \le c\, \theta_j^{1/2}\, |||\eta|||\ |||w|||\ ,$

(4.6.9b) $\qquad a(s, w) \le c\, \theta_{j-1}^{1/2}\, |||s|||\ |||w|||\ ,$

(4.6.9c) $\qquad a(\eta, o) \le c\, \theta_j^{1/2}\, \theta_{j-1}^{-1/2}\, |||\eta|||\ |||o|||\ .$

Proof. Equation (4.6.8b) is the same as (4.6.1b) so is already established. For (4.6.8a) expand η in the eigenfunctions $\{\tilde{\psi}_i^{(j)}\}$,

$$\eta = \sum_{i=1}^{\tilde{J}(\theta_j)} \alpha_i\, \tilde{\psi}_i^{(j)}\ ,$$

and let f be given by

$$f = \sum_{i=1}^{\tilde{J}(\theta_j)} \alpha_i\, \tilde{\lambda}_i^{(j)}\, \tilde{\psi}_i^{(j)}\ .$$

Then

(4.6.10) $\qquad a(\eta, \phi) = b(f, \phi)\ ,\ \phi \in M_j\ .$

Now instead of (4.6.3) we get by a similar argument

$$\|f\|_b \le (\theta_j\, \tilde{\lambda}_{N_j}^{(j)})^{1/2}\, |||\eta|||\ ,$$

so with $\phi = w$ in (4.6.10),

(4.6.11) $\qquad a(\eta, w) = b(f, w) \le (\theta_j\, \tilde{\lambda}_N^{(j)})^{1/2}\, |||\eta|||\ \|w\|_b\ .$

Then as in (3.3.3), since the eigenvalues of the mass matrix lie on an interval $[\frac{1}{\sigma}, \sigma]$ for $\sigma > 1$ independent of j,

(4.6.12) $\qquad \|w\|_b \le \sigma^{1/2}\, \|w\| \le c\, \sigma^{1/2}\, h_j\, |||w|||\ ,$

where we used lemma 4.11. Combining (4.6.11) and (4.6.12) and using the fact that σ is fixed,

$$a(\eta, w) \leq c \, h_j (\theta_j \, \tilde{\lambda}_{N_j}^{(j)})^{1/2} \, |||\eta||| \, |||w|||$$

$$\leq c \, \theta_j^{1/2} \, |||\eta||| \, |||w|||$$

by the eigenvalue bound (2.3.2). For (4.6.9c) we have as in (4.6.11)

$$a(\eta, o) \leq (\theta_j \, \tilde{\lambda}_{N_j}^{(j)})^{1/2} \, |||\eta||| \, ||o||_b \leq \sigma^{1/2} \, (\theta_j \, \tilde{\lambda}_{N_j}^{(j)})^{1/2} \, |||\eta||| \, ||o|| .$$

Then as in (4.6.6)

$$||o|| \leq (\theta_{j-1} \, \lambda_{N_{j-1}}^{(j-1)})^{-1/2} \, |||o||| ,$$

so

$$a(\eta, o) \leq \sigma^{1/2} (\theta_i \, \tilde{\lambda}_{N_j}^{(j)})^{1/2} \, (\theta_{j-1} \, \lambda_{N_{j-1}}^{(j)})^{-1/2} \, |||\eta||| \, |||o|||$$

$$\leq c(\theta_j \, h_j^{-2})^{1/2} \, (\theta_{j-1} \, h_{j-1}^{-2})^{-1/2} \, |||\eta||| \, |||o||| ,$$

using the eigenvalue bounds as before. Then since the mesh ratio is fixed,

$$a(\eta, o) \leq c \, \theta_j^{1/2} \, \theta_{j-1}^{-1/2} \, |||\eta||| \, |||o|||$$

as required. □

Since we are retaining S_{j-1} and O_{j-1} and only replacing S_j by $\tilde{S}_j$, there is no need to alter lemma 4.13. Then following the same proof as for lemma 4.14, but using the strengthened Cauchy inequalities of lemma 4.16 rather than lemma 4.12, we get the analog of lemma 4.14.

<u>Lemma 4.17.</u> If $\eta \in \tilde{S}$ and $u \in M_{j-1}^{\perp}$ then

$$a(\eta, u) \le c \, \theta_j^{1/4} \, |||\eta||| \, |||u||| \, .$$

The required approximation theorem follows easily from this lemma just as the analogous theorem 4.15 followed from lemma 4.14.

<u>Theorem 4.18.</u> Let $\eta \in \tilde{S}_j$ and let $\bar{\eta} \in M_{j-1}$ be the elliptic projection of η on M_{j-1},

$$a(\eta - \bar{\eta}, \phi) = 0 \, , \quad \phi \in M_{j-1} \, .$$

Then

(4.6.13) $$|||\eta - \bar{\eta}||| \le c \, \theta_j^{1/4} \, |||\eta||| \, .$$

With this approximation result, the convergence and complexity results of sections 4.3 and 4.4 are now completely justified. As for theorem 4.15, the constant c occurring here depends only on local properties of the finite element spaces. For particular spaces one could easily compute this constant and determine a rigorous upper bound on the rate of convergence of the multi-level iterations of section 4.3. This generalizes the heuristic local Fourier mode analysis to irregular finite element grids, and is, moreover, completely rigorous. The hidden constants in the $O(N)$ complexity bounds of section 4.4 are not so easily computed since they depend on the generally unknown constant in the error estimate (4.4.1). However, for most purposes it is enough to have a bound on the spectral radius of the multi-level iteration, which this analysis gives since this is what determines an iteration's practicality.

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