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COHERENT SYSTEM REPAIR MODELS.

BY

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## CHAPTER I

### INTRODUCTION

#### 1. Summary

Consider a system which is composed of some number of subunits or components. The structure of the system is the relationship between the functioning of various components and that of the whole system. Structure can be specified completely in terms of the minimum cut sets of a system. These are sets of components for which:

- (i) the system can't work if they are removed or failed,
- (ii) if any component in the min cut set is restored, then the system will work.

Example 1.1 (parallel system of 2 components): The system works  $\Leftrightarrow$  at least one component is working; min cut set =  $\{1, 2\}$ .

Example 1.2 (series system of 3 components): The system works  $\Leftrightarrow$  all components work; min cut sets =  $\{1\}, \{2\}, \{3\}$ .  $\{2, 3\}$  is not a min cut set because even though the system is failed when 2 and 3 are failed, restoring component 2 or 3 does not restore the system.

Intuitively speaking, a system is coherent<sup>\*</sup> if:

- (i) Suppose it is down when a certain subset of components is failed; it will not start working if even more components fail.

\*See Ref. [2], Chapter 1, for precise mathematical definition.

- (ii) If it is working when a certain subset of components is working, it will not fail if even more components are restored to working order.

A Coherent System Repair Model models the maintenance over an infinite continuous time horizon of a stochastically deteriorating finite set of components which form a coherent system. There are costs for repair and system failure and the lifetime of each component is a random variable with known expected value. The objective is to, assuming decisions can be made at the instant of a component failure, minimize the long run expected cost per unit time or the total expected cost, whichever is desired. Decisions can be made to do nothing or to repair some subset of the failed components; when the system is down, the do-nothing option is eliminated except in cases of noninstantaneous repair. A more detailed discussion of the parameters which can be varied in these models will appear later in the chapter.

The purpose of this thesis is to formulate and describe different types of coherent system repair models and then gather as much and as general information as possible on the nature and form of optimal policies involved. In the case where system components have exponential (not necessarily identical) lifetime distributions, the model can be formulated as a continuous time, infinite horizon Markov decision chain with no discounting and a finite state and decision space. States, which are nontime-dependent, depend on the states of the individual components: working, failed, or under

repair. The bulk of information will come for models with exponential components, as Markov chain theory is a very powerful tool. If the components are nonexponential (and non-Erlang) the process as defined above will be non-Markov. In that case, it is no longer sufficient to have nontime-dependent states. The problem is much more difficult and will be beyond the scope of this thesis.

After initial general discussion, three types of models will be looked at in detail, all requiring exponential or Erlang components. The Basic Model (Chapters II & III) allows exponential components to be working or failed (on/off). Repair time is assumed instantaneous with unlimited service facilities. The Degradation Model (Chapter IV) has the same component lifetime and repair assumptions as the Basic Model except that here each component can be in any one of a finite number of observable degradation states, with the first state being "new" and the last state being "failed". Following, the Noninstantaneous Repair Model (Chapter V) takes the Basic Model and allows for exponential repair times (noninstantaneous).

Results obtainable for each of the three models, using Veinott's solution technique for Markov decision chains [see Section 1.4] (i.e., finding the long run expected cost/time for each policy and then among those policies for which this is minimized, find the total expected cost, etc.) are included in the model Chapters II through V. In each case, the specific structure of the underlying Markov chain allows us to get conditions which eliminate certain types of policies from being optimal. For some cases, notably

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those for which components are identical, a complete general optimal solution is obtained.

Chapter VI looks at the theory and algorithms available for getting computational results, and implements a linear programming method. This provides a look at optimal solutions in cases where policy enumeration by hand is too messy or none of the theorems in Chapters II-V apply. Some test problem results are presented.

In Chapter VII, results from the previous five chapters are compiled and compared. Basic policy forms and restrictions which interrelate the models are presented, and conclusions about how varying model parameters affect the optimal solution are drawn. Then some of the possible applications of these models or variations on them are discussed. Extensions and topics for future research conclude the chapter and thesis.

## 2. A Survey of Maintenance Model Literature

A maintenance model models the control and surveillance of a stochastically deteriorating system. For two decades there has been a large and continuing interest in such models and the number of relevant papers in the literature reflects it. Figure 1.1 below gives a classification of maintenance models according to the type of maintenance problem modeled:

FIGURE 1.1 Ref.[29]

Classification of Maintenance Models

<u>Discrete Time</u>	<u>Continuous Time</u>
1. Complete Information	1. Control Theory
2. Incomplete Information	2. Age Dependent Replacement Models
3. Maintenance and Inventory	3. Shock Models
	* 4. Interacting Repair Activities Models
	5. Incomplete Information

\*Further broken down in Figure 1.2.

Discrete time models select actions at discrete points in time. These models, as appear in the literature, utilize information regarding the degree of deterioration of the unit or units in order to select the best repair or maintenance action at certain discrete points in time. In some cases, an inspection must be made to ascertain the state of the system before repair decisions are made, while in others it is assumed the current state of the system is always known. For incomplete information models, actions must be taken under uncertainties about costs, underlying failure laws, or observations of state. Inventory models involve decisions concerning periodic restocking of inventories of spare parts. Note that in all these models, the system is treated as a unit in terms of formulation,

the individual components being ignored. Most formulations for discrete time models are based on Markov decision theory or inventory theory, thus, linear and dynamic programming are primary solution techniques. See [21], [26], [33] for details and specific models.

Continuous time models do not restrict maintenance or inspection activity to a particular set of discrete points in time.

Control theory models permit maintenance activity to occur as a continuous stream. The decision maker must optimize over functions  $m(\cdot)$  where  $m(t)$  is the maintenance expenditure rate at time  $t$ .

Age Dependent Replacement Models allow maintenance only at certain discrete points in time, e.g., replacing an item when it reaches a certain age. This problem is just the continuous time analog of the discrete time one with deterioration stages. Different models vary cost assumptions, types of repairs allowed and numbers of spares around. Again, the system is treated as a unit.

Shock Models regard the unit as subject to exterior shocks, each of which damages (causes wear) in such a way that the damage accumulation up to a particular time defines the unit's probability of failure of that time. This assumption differs from the standard assumption that the time-to-failure random variable of a unit is intrinsic to that unit.

The Coherent System Repair Model falls into the category of interacting repair activities models, the only ones where maintenance policies exploit interactions among the units of a system. Figure 1.2 presents ways in which a maintenance policy achieves this:

FIGURE 1.2

Types of Maintenance Policies Which Exploit System Structure

- |  |  |   |
|--|--|---|
| * 1. Opportunistic<br>(economies of scale) |  | → how many? (cheaper<br>to repair more at once) |
| 2. Cannibalization                         | ] units of same<br>type utilized<br>at different<br>locations in<br>system | ← no new items in system                        |
| 3. Multi-stage Repair                      |  | ← new items can enter<br>system                 |
| 4. Variable Rate<br>Repair                 | repair capacity limited and under<br>decision maker's control              |   |
| * 5. Which ones?                           | (if system has nonidentical components<br>with several items failed)       |   |
| * 6. Repair or not?                        | (depending on how much of the system<br>is still operative)                |   |

\* type of decisions allowed in a coherent system repair model.

Research on problems involving cannibalization, multi-stage repair, variable rate repair, and opportunistic policies has been going on for some 10-15 years, (see [33], Reference list). "Repair or not" type policies for deteriorating units were among the first to be studied, the best example being a "control limit policy" which says "repair or replace the unit when it reaches a certain level of deterioration".

However, until just recently, no work has been done on optimal repair order, i.e., choosing which item to repair if several components are down, or on models which can treat the type of system as a variable.

It was a desire to get some results in these areas that motivated the Coherent System Repair Model formulation. It models a deteriorating system of components as do other interacting repair activities models but, unlike the others, the state of the components explicitly determines the state of the system. Varying the component configurations for which a "penalty" is incurred for system failure allows introduction of the system type as a variable.

Two recent works on optimal repair order have been by Derman, Lieberman, and Ross (1978) [15] and D. R. Smith (1978) [29]. Derman, Lieberman, and Ross consider the same type problem as a coherent system repair model with exponential components, using an N-server queuing system to show that the policy which always repairs the failed component whose failure rate is smallest stochastically maximizes the number of working components. However:

- (a) decisions to not repair even though the server is free are not allowed
- (b) decisions to repair more than one unit at a time are not allowed, (single repairman)
- (c) different objective function - no cost structure  
(maximize number of working components)

Smith considers a series system of  $n$  independent components as an irreducible continuous time Markov chain and gets the same result as Derman, et al. However:

- (a) he has no cost structure, his objective is minimizing the long run fraction of time the system is up

- (b) he does allow for "do nothing" options but only a single server
- (c) he solves using a different technique
- (d) results apply to a series system only.

In a coherent system repair model, time is not a variable - the level of deterioration of the system is measured by which and how many of the components are working or are in a certain level of degradation. In general, with complicated systems having different components, one would expect some very complicated optimal policies which would be impossible to guess but well worth finding. When the components are identical, things simplify considerably for k-of-n systems, as the number of components up is now the only determining factor in system deterioration and failure.

Although, in the most general cases, one cannot hope for exact solutions, except by computer, one can and does look for restrictions on the very large initial decision space (can repair any subset of failed components at the instant of any component failure). In the course of investigation of various models, two such restriction types will appear:

- (1) never repair more than a certain number of components at a time in certain states
- (2) never repair when more than a certain number of components are working.

Both these results, besides being useful from a system operator's

viewpoint, are useful in lowering the computation time needed to find an optimal policy in cases where it cannot be done by hand.

Initially, maintenance/inspection models possessed an elegance and a simplicity which led to easily implementable results and exact optimal policies. More recent results have become increasingly complex, requiring a large computer to implement policies. Coherent system repair model results are some of each - in general cases it is necessary to use the computer or state theorems which eliminate certain policy types but in many specific cases, exact "optimal" policies can and have been worked out, especially in cases where components are identical. The formulation is simple and makes for easy usage of the model under a great variety of hypotheses.

The following section gives a description of coherent system repair models, their basic features and parameters.

### 3. Description of Model

The formulation of a coherent system repair model is quite general. Figure 1.3 summarizes the features of such a model. As is easily seen, there are a lot of unspecified parameters which have to be defined before using the model. These include: the objective, type of system, distribution of component lifetimes, costs, and several others described in Figure 1.4.

FIGURE 1.3

Features of a Coherent System Repair Model

- (1) Models the maintenance of a stochastically deteriorating system of  $n$  independent components
- (2) System states (and, thus, decisions) are functions of component states but not functions of time (due mainly to exponential components)
- (3) Cost structure for repair or system failure
- (4) Formulation as a continuous time Markov decision chain, infinite horizon, no discounting (finite state and action space)
- (5) Large decision space: at each state, can decide
  - a. repair or not (must have some repair action going if system is down)
  - b. if so, which subset of failed components to repair
- (6) Decision times are limited to times of a component failure and are made immediately following such
- (7) Some resulting policy limitations:
  - a. don't repair unless a certain number of components have failed
  - b. don't repair more than a certain number of components at one time
- (8) Can get exact "optimal" policies in some cases, especially when components are identical
- (9) Objective: minimize  $V_{-1}$  (long run expected cost per unit time) or  $V_0$  (total expected cost)  
Start with all components up
- (10) Solution techniques: use structure of underlying Markov chain to get conditions on  $V_{-1}$ ,  $V_0$ .

FIGURE 1.4

Model Parameters

<u>parameter</u>	<u>possible values</u>
1. Objective:	<ul style="list-style-type: none"> <li>- <math>\min V_{-1}</math>, the long run expected cost per unit time</li> <li>- <math>\min V_0</math>, the total expected cost</li> </ul>
2. System Type:	<ul style="list-style-type: none"> <li>- general coherent</li> <li>- k-of-n <math>\rightarrow</math> k = 1 parallel</li> <li style="padding-left: 1.5em;"><math>\rightarrow</math> k = n series</li> </ul>
3. Component Lifetime Distributions:	<ul style="list-style-type: none"> <li>- exponential - identical components</li> <li style="padding-left: 1.5em;">- nonidentical components</li> <li>- gamma, integer shape parameter</li> </ul>
4. States:	<ul style="list-style-type: none"> <li>- components on/off (which)</li> <li>- components in degradation states (which)</li> <li>- components on/off or in service (which)</li> <li>- <u>number</u> of components in various states (k-of-n system, identical components only)</li> </ul>
5. Repair:	<ul style="list-style-type: none"> <li>- instantaneous</li> <li>- exponential (s servers)</li> <li>- Erlang, integer shape parameter</li> </ul>
6. Costs: in all models $K_i$ to fix unit i p per system breakdown	<p style="margin-left: 1em;">plus</p> <ul style="list-style-type: none"> <li>- fixed charge L, per repair decision</li> <li>- no fixed charge L</li> <li>- labor cost <math>l</math>/server/unit time (noninstantaneous repair only)</li> </ul>

The two objective functions used are among those proposed by Veinott [30, 31] in his method for solving continuous time Markov decision chains with infinite time horizon and no discounting. See Section 1.4, [31] or [11] for a brief presentation of the theory behind it.

Changing parameters which affect the number and type of states in the Markov chain cause such overwhelming changes in the model that each is studied separately as a "model type". Such parameters include the component lifetime distributions, types of component states existing, and the repair option. The following three model types were selected, a chapter being devoted to each.

(I) Basic Model	comp. life = exponential states = on/off repair = instantaneous	[Chapters II, III]
(II) Degradation Model	comp. life = exponential states = degradation levels repair = instantaneous	[Chapter IV]
(III) Noninstantaneous Repair Model	comp. life = exponential states = on/off, in service repair = exponential/ Erlang	[Chapter V]

Changing system type or costs affects only the cost structure and such variations are studied within each model type.

Obviously, other combinations of parameters could be used to get other, more complicated, model types. However, these three are the simplest, give the most results, and give a good feeling for

the types of optimal policies being dealt with. Given any other combination with exponential or Erlang repair and component lifetimes, an optimal solution can be sought using computer algorithms like those in Chapter VI if the number of states is not too large.

#### 4. Discussion of Optimality Criteria

In this section, some optimality criteria are defined. Which ones will be used in this thesis and why, is then discussed.

Suppose we are given a continuous time, infinite horizon Markov renewal programming model with finite state and action space and no discounting. It is sufficient to consider only nonrandomized policies, (see [11]). The case of exponential transition times simplifies to the Markov decision chain which will be used to represent a coherent system repair model. In this case the decisions or the holding time distributions in a state depend only on the current state, not on the next state as for the general Markov renewal programming model. Also, only stationary policies need be considered (see [8], Example 4).

The problem being infinite horizon, the objective is to minimize the expectation of undiscounted cost incurred over all time, i.e., minimize  $\lim_{t \rightarrow \infty} V_i^Y(t)$  over  $\gamma$  where  $V_i^Y(t)$  is the expectation of undiscounted cost incurred during interval  $[0, t]$  under policy  $\gamma$  if the system starts in state  $i = \{1, 2, \dots, n\}$  = all components working.

Unfortunately,  $\lim_{t \rightarrow \infty} V_1^Y(t) = \infty$  in most cases and, thus, is not a good quantity to compare policies with. To render things finite if  $V(t)$  grows slower than exponentially, a positive "interest" or "discount" rate "s" can be introduced into the model.

Let 
$$v_1^Y(s) = \int_{0^-}^{\infty} e^{-st} dV_1^Y(t). \quad (\text{taking Laplace-Stieltjes transforms})$$

Definition:  $\delta$  is s-optimal if  $v^\delta(s) \geq v^Y(s) \forall Y \in \Delta$ ,

$\Delta$  = policy space.

Definition: (Blackwell)  $\delta$  is optimal if it is s-optimal  $\forall$  sufficiently small s. ( $\Delta_{\text{opt}}$  = set of optimal policies, possibly empty).

As the behavior of  $v(s)$  as  $s \rightarrow 0^+$  is intimately related to that of  $V(t)$  as  $t \rightarrow \infty$ , we use Blackwell's criterion as the definition of optimality.

Given a definition of optimality, it is now necessary to show how to compute on optimal policy given a specific Markov decision chain (or renewal program). The next theorem will demonstrate an approach towards doing so.

A Markov renewal program is defined by the:

- (i) states 1, 2, ..., N
- (ii) decisions  $D_i$  = set of possible decisions in state i
- (iii) transition probabilities  $Q_{ij}^k(x)$ ,

$$Q_{ij}^k(x) = P\{s_{n+1} = j, t_{n+1} \leq t + x | s_n = i, t_n = t, \delta(i) = k\}$$

- (iv) cost structure  $R_i^k(x)$  = expected cost incurred during time interval  $[0, \min(x, t)]$ ,  $s_0 = i$ ,  $\delta(i) = k$ .

This is assumed time invariant.

In the case of exponential transition probabilities, i.e., the Markov decision process,

$$Q_{ij}^k(x) = p_{ij}^k \begin{bmatrix} -\lambda_i^k x \\ 1 - e^{-\lambda_i^k x} \end{bmatrix}; \quad R_i^k(x) = r_i^k \begin{bmatrix} -\lambda_i^k x \\ 1 - e^{-\lambda_i^k x} \end{bmatrix}.$$

Definition: Normalized moments

$$Q_n = \int_{0-}^{\infty} t^n dQ(t)/n!$$

$N \times N$  matrix

$$R_n = \int_{0-}^{\infty} t^n dR(t)/n!$$

$N \times 1$  vector

Note:  $Q_0$  = transition matrix of embedded Markov chain.

Note: Well known Markov chain theory says as  $n \rightarrow \infty$ , (if only one ergodic class present)  $Q_0^n \rightarrow P^*$ , the stationary transition probability matrix which can be computed using the equations

$$P^*(I-Q_0) = 0 \quad \text{and} \quad P^* \cdot \underline{1} = \underline{1}$$

Theorem: Suppose  $Q_{n+2}$  is finite and  $R_{n+1}$  is defined and finite. Then (eliminating  $\delta, i$  from notation to simplify)

$v(s) = s^{-1}V_{-1} + V_0 + sV_1 + \dots + s^n V_n + o(s^n)$ . Moreover,  $\forall i = -1, 0, \dots, n$ , the vector  $V_i$  is the unique solution of equations:

$$(I-Q_0)V_i = b_i \quad \text{and} \quad P^*Q_1V_i = P^*c_i$$

where

$$c_{-1} = R_0, \quad b_{-1} = 0, \quad b_i = c_{i-1} - Q_1V_{i-1}$$

and

$$\forall i \geq 0, \quad c_i = (-1)^{i+1}R_{i+1} + \sum_{j=2}^{i+2} (-1)^j Q_j V_{i+1-j}.$$

Proof: see [11].

This suggests that when  $s$  is sufficiently small, the decision maker can go about finding an optimal policy by first selecting policies  $\delta$  so as to minimize  $V_{-1}^\delta$ , break ties by minimizing  $V_0^\delta$ , etc., until a unique policy is arrived at, if one exists. In the exponential case, it does (see [31]). In addition, there it can be shown that  $\Delta_{N-1} = \Delta_\infty$  where  $\Delta_{-2} = \Delta$ ,  $\Delta_k = \{\delta \in \Delta_{k-1} \mid V_k^\delta \geq V_k^\delta \forall \delta \in \Delta_{k-1}\}$ , and  $\Delta_\infty = \lim_{k \rightarrow \infty} \Delta_k$ .

It can also be shown that  $\Delta_{\infty} = \Delta_{opt}$  under most cases (see [11]) in Markov renewal programming.

Definition:  $V_{-1}$  optimal policies are termed gain optimal and are those for which the long run average cost per unit time is minimized.

$$V_{-1} = \frac{P^* R_0}{P^* Q_1} .$$

Definition:  $V_0$  optimal policies are called bias optimal and are those for which the total expected cost is minimized. In particular  $V_0$  solves  $(I-Q_0)V_0 = R_0 - Q_1 V_{-1}$  and  $P^* Q_1 V_0 = P^* (-R_1 + Q_2 V_{-1})$ .

In solving specific coherent system repair models in successive chapters,  $V_{-1}$  and  $V_0$  will be used as optimality criteria. With other past maintenance models which use a Markov decision chain structure with costs, the objective was almost always to minimize  $V_{-1}$ , the average cost per unit time. An example will be given that demonstrates the need for going to  $V_0$  if it is desired to find the optimal decisions for the transient states.

Example 1.3 (parallel system,  $n = 2$  components, no fixed charge, states = which components up = {12, 1, 2, 0}):

decisions:  $\Delta_{12} = \{A\}$      $\Delta_1 = \{A, R_2\}$      $\Delta_2 = \{A, R_1\}$   
 $\Delta_0 = \{R_1, R_2, R_{12}\} .$

where "A" denotes doing nothing and  $R_1$  denotes repairing  $i^{th}$  unit.

cost:

$$r_{12}^A = 0$$

$$r_1^A = r_2^A = 0$$

$$r_0^{R_1} = p + K_1$$

$$r_1^{R_2} = K_2; r_2^{R_1} = K_1$$

$$r_0^{R_2} = p + K_2$$

$$r_0^{R_{12}} = p + K_1 + K_2 .$$

It turns out that if  $p$  is small enough,  $R_1$  in 0 is  $V_{-1}$  optimal if component 1 has a smaller  $\frac{\text{expected cost}}{\text{expected life}}$  ratio than component 2. The  $V_{-1}$  optimal policy does not specify what to do in states 1 or 2 in this case.

Basically, given a policy  $\delta$ , an underlying Markov chain structure is formed which contains some ergodic states and some transient ones. Finding a  $V_{-1}$  optimal policy locates that ergodic chain of states which minimizes long run average cost/unit time over all other policies which have a different ergodic structure. However, except for a few special cases, there will be many  $V_{-1}$  optimal policies i.e., when one can change any decision in the transient states without forming other ergodic states,  $V_{-1}$  will be unchanged. Thus, to find out the optimal thing to do in transient states before reaching the ergodic chain, from which one never leaves, it is necessary to look at  $V_0$ .

The next question is why then, given the above argument, not keep going with  $V_1, V_2, \dots$  etc., until a unique policy is found?

Besides being computationally messy, it is unnecessary. Any ties with  $V_0$  can be broken by changing one of the costs or mean component lifetimes by some  $\epsilon$ , however small. Any data measured to use in such a model is only going to be precise to a certain decimal point so one can always change the insignificant portion around to break ties or just leave them and call them equivalent policies.

The fact that in a coherent system repair model, the states and costs are nontime-dependent with no discounting, enables us to specify an "optimal" decision in every state by only considering  $V_{-1}$ ,  $V_0$ . The  $V_0$  criterion places equal weight on costs at different times, which suffices to determine an optimal policy given model assumptions to minimize total expected cost, i.e., there is no advantage given to a policy which has the same total costs as another but less of the cost is likely to be incurred earlier, thus, allowing for more cash-on-hand for a while, even though the total spent will end up the same (type of policies  $V_1$  distinguishes). For further economic interpretations of  $V_{-1}$ ,  $V_0$ ,  $V_1$ , see Veinott [31].

CHAPTER II  
THE BASIC MODEL

1. Description of Model

The Basic Model is the simplest of the coherent system repair models, but by no means uninteresting. In it, assumptions are as follows regarding fixing model parameters:

States: components are on or off, either

- (a) depending on which components up if system  $\neq$  k-of-n or different components
- (b) depending on number of components up if system = k-of-n and same components.

Repair: instantaneous, unlimited service.

Component Lifetimes: exponential, nonidentical components order the components so that number 1 has smallest mean lifetime (or largest parameter), i.e.,

$$L_i = \text{lifetime of number } i \sim 1 - e^{-\lambda_i t} = p\{L_i \leq t\}$$

where  $\lambda_1 > \lambda_2 > \dots > \lambda_n$ .

Variables within the Basic Model structure are the objective ( $V_{-1}$  or  $V_0$ ), the type of system, and the existence or not of a fixed charge as well as costs to repair a component, penalty cost for system failure, and the mean lifetimes of the various components. Section 2 gives results for the case of no fixed charge,  $V_{-1}$  optimal. In Section 3, what happens for positive fixed charge is looked at.

It should be noted that the type of system can be anything but series. Due to the instantaneous repair, the requirement that repair be undertaken on some component when the system fails and the fact that the decision space is to repair some subset of the failed components, there are no decisions to be made in the series case. The only possible states are {all components up} or {i down}, some  $1 \leq i \leq n$ . In the former, no repair is done since all components are up and in the latter, component  $i$  is repaired automatically. One could drop the requirement of repair when the system breaks down, thus, allowing more states and decisions but, besides unnecessarily complicating the problem, it would be unrealistic to assume that a decision maker would want to do nothing if the whole system he is in charge of breaks down, (if the system is coherent).

The fixed charge is denoted  $L$ , and is  $\geq 0$ . It costs  $K_i > 0$  to repair component  $i$  (an instantaneous lump sum since instantaneous repair) and  $p > 0$  is charged each time the system fails.

Component lifetimes being exponential, the model can be formulated as a continuous time Markov decision chain with infinite time horizon as mentioned in Chapter I. To define the model completely, we need:

- (1) State space (states depend on which components up)
- (2) Decision space (decisions available in each state)
- (3) Transition structure (probabilities of changing states)
- (4) Cost structure (cost of decisions in each state)
- (5) Objective function

The following notation is now introduced and will be used throughout.

Figure 2.1

Coherent System Repair Model - Notation

- $n$  = number of components
- $s$  = a state,  $s \subseteq \{1, \dots, n\} = \{\text{all components working}\}$
- $\delta$  = a policy involving decisions in each state  $s$  (to repair or not and, if so, which ones and how many)
- $A$  = decision to do nothing (can be made in any state in which system is not failed)
- $R_{\Omega_s}$  = decision to repair the set  $\Omega_s$  of components (can be made in any state  $s$  where all elements of  $\Omega_s$  are failed)
- $Q_0^\delta$  = matrix of transition probabilities between various states (depends on policy  $\delta$  and lifetime distributions of components)
- $F_i(t)$  = lifetime distribution of component  $i$   
 $= 1 - e^{-\lambda_i t}$  in Basic Model
- $\mu_i$  = expected lifetime of component  $i = 1/\lambda_i$

- $Q_1^\delta = v^\delta$  = vector of expected times to transition  
 (holding times) given various states  $s$  and  
 policy  $\delta$
- $P_\delta^*$  = vector of stationary transition probabilities  
 for policy  $\delta = \langle P_{\cdot s} \rangle_{s \in \mathcal{S}}$ . Get by solving  
 $P_\delta^*(I - Q_0^\delta) = \underline{0}$
- $\mathcal{S}$  = set of states,  $s$  ( $2^n$  of them - some may be  
 inaccessible given starting state and policy)
- $p$  = penalty cost incurred when system fails
- $K_i$  = cost to repair component  $i$
- $L$  = fixed cost for repair (same no matter how many  
 items ( $> 0$ ) are repaired at once)
- $R_0^\delta$  = vector of costs given various states and  
 policy  $\delta$
- $V_{-1}^\delta$  = long run expected cost per unit time of policy  
 $\delta$  (scalar)
- $V_{-1}$  = min over  $\delta$  of  $V_{-1}^\delta$  = optimal long run average  
 expected cost/time
- $V_0^\delta$  = expected total cost for policy  $\delta$ . This is a  
 vector depending on  $s$
- $V_0$  =  $\min_\delta V_0^\delta$ . It turns out that the policy which  
 minimizes  $V_{0,s}^\delta$  for one  $s$  does it for all  $s$   
 (see Veinott [27]).

We are now ready to specify the Markov chain completely for  
 the Basic Model:

State Space:

States = which components working, i.e.,  $s = i_1 i_2 \cdots i_k$   
 if components  $i_1, i_2, \dots, i_k \subseteq \{1, \dots, n\}$  are working.

If system = k-of-n and components are identical,  
 then states = number of components working, i.e.,  
 $s = i$  if  $i$  components left working.

Decision Space:

Let  $\Omega = \{1, \dots, n\}$ . Then  $\Omega - s =$  set of components  
 which are failed in  $s$ . Then the possible decisions  
 in state  $s$  are:  $R_{\Omega_s}$ , where  $\Omega_s \subseteq \Omega - s$  ( $\Omega_s$  is  
some subset of the failed components given state  $s$ ).

If  $\Omega_s = \phi$ , then  $R_{\Omega_s} = A$  (do nothing). If system is  
 down in  $s$ , then  $\Omega_s \neq \phi$  (must do something)

Transition Structure:

The transition matrix,  $Q_0$ : (assume  $\delta : R_{\Omega_s}$  in  $s$ )

$$(Q_0)_{s, s \cup \Omega_s \sim i}^{\delta} = \left\{ \begin{array}{l} \frac{\lambda_i}{\sum_{j \in s \cup \Omega_s} \lambda_j}, \quad i \in s \cup \Omega_s \\ 0, \quad \text{otherwise} \end{array} \right\}$$

$$Q_{1,s}^{\delta} = \frac{1}{\sum_{j \in s \cup \Omega_s} \lambda_j}$$

Cost Structure:

$$R_{0,s}^{\delta} = \left\{ \begin{array}{ll} L(1-I_{\Omega_s}^{\phi}) + \sum_{i \in \Omega_s} K_i, & \text{if system up in } s \\ L + p + \sum_{i \in \Omega_s} K_i, & \text{if system down in } s \end{array} \right\}$$

where

$$I_A^B = \left\{ \begin{array}{ll} 1, & \text{if } A = B \\ 0, & \text{otherwise} \end{array} \right\}.$$

Objective Function:

$$V_{-1}^{\delta} = \frac{P_{\delta}^* R_0^{\delta}}{P_{\delta}^* Q_1^{\delta}} \text{ if } \delta \text{ forms a single irreducible ergodic chain.}$$

$$V_0^{\delta} \text{ solves (i) } (I - Q_0^{\delta}) V_0^{\delta} = R_0^{\delta} - Q_1^{\delta} V_{-1}^{\delta}$$

$$(ii) P_{\delta}^* Q_1^{\delta} V_0^{\delta} = P^* (-R_1^{\delta} + Q_2^{\delta} V_{-1}^{\delta}),$$

$Q_2$  being the normalized variance of the holding time distributions in various states.

A couple of examples are given below:

Example 2.1 ( $n = 2$ , nonidentical components, parallel system, no fixed charge):

<u>states</u> (s):	12	1	2	0
<u>decisions</u> (k):	A	A, R <sub>2</sub>	A, R <sub>1</sub>	R <sub>1</sub> , R <sub>2</sub> , R <sub>12</sub>
<u>cost</u> (R <sub>0,s</sub> <sup>k</sup> ):	0	0, K <sub>2</sub>	0, K <sub>1</sub>	K <sub>1</sub> + p, K <sub>2</sub> + p, K <sub>1</sub> + K <sub>2</sub> + p

transitions (Q<sub>0</sub>): Q<sub>0,12</sub><sup>A</sup> = 12th row of Q<sub>0</sub>

$$\begin{aligned}
 & \begin{matrix} & 12 & 1 & 2 & 0 \\ & = & \left[ \begin{array}{cccc} 0 & \frac{\lambda_2}{\lambda_1 + \lambda_2} & \frac{\lambda_1}{\lambda_1 + \lambda_2} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \\ Q_{0,1}^A = Q_{0,2}^A = & \begin{matrix} R_1 \\ R_2 \\ Q_{0,0} \end{matrix} \\ & = \left[ \begin{array}{cccc} 0 & \frac{\lambda_2}{\lambda_1 + \lambda_2} & \frac{\lambda_1}{\lambda_1 + \lambda_2} & 0 \end{array} \right] \\ Q_{0,1}^{R_2} = Q_{0,2}^{R_1} = & \begin{matrix} R_{12} \\ Q_{0,0} \end{matrix} \end{aligned}
 \end{aligned}$$

Example 2.2 (n = 4, identical components, 2 of 4 system, fixed charge L.

<u>states</u> (s):	4	3	2	1
<u>dimensions</u> (k):	0	A, R <sub>1</sub>	A, R <sub>1</sub> , R <sub>2</sub>	R <sub>1</sub> , R <sub>2</sub> , R <sub>3</sub>
<u>cost</u> (R <sub>0,s</sub> <sup>k</sup> ):	0	0, K+L	0, K+L, 2K+L	K+L+p, 2K+L+p, 3K+L+p

where R<sub>i</sub> denotes repairing i units simultaneously.

transitions ( $Q_0$ ):

$$\begin{array}{l}
 Q_{0,4}^A = Q_{0,3}^{R_1} = Q_{0,2}^{R_2} = Q_{0,1}^{R_3} = \begin{bmatrix} & 4 & 3 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
 Q_{0,3}^A = Q_{0,2}^{R_1} = Q_{0,1}^{R_2} = \begin{bmatrix} & & & & \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
 Q_{0,2}^A = Q_{0,1}^{R_1} = \begin{bmatrix} & & & & \\ & & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{array}$$

These two examples will be referred to in succeeding sections as solution techniques are developed.

## 2. No Fixed Charge, $V_{-1}$ Results

Initially, assume we have a general coherent system (identical or nonidentical components). The following theorem is the basis for many of the results in this section. It expresses, given  $\delta, V_{-1}^\delta$  as a convex combination of certain quantities which in the case of no fixed charge, turn out to be  $V_{-1}^{\gamma_i}$  for other possible policies, say  $\{\gamma_i\}$ . The minimal  $V_{-1}$  must, thus, be  $V_{-1}^{\gamma_i}$  for some  $i$ , and since  $\min_i V_{-1}^{\gamma_i}$  turns out to be easy to find, the optimal  $V_{-1}$  and corresponding  $V_{-1}$  optimal policy can be obtained for any coherent system.

Theorem 2.1: Suppose we have the Basic Model with no fixed charge, and a general coherent system. Fix a policy  $\delta$  such that:

(i)  $R_{\Omega_s}$  in state  $s$

(ii) The underlying Markov chain defined by  $\delta$  is irreducible.

Then 
$$V_{-1}^\delta = \sum_{s \in \mathcal{S}} \alpha_s V_s \quad \text{where} \quad \alpha_s = \frac{P \cdot s Q_{1,s}}{P^* Q_1}$$

and

$$V_s = \sum_{i \in (s \cup \Omega_s - 1) \text{ is "up" for system}} \frac{K_i}{\mu_i} + \sum_{i \in (s \cup \Omega_s - 1) \text{ is "down" for system}} \frac{K_i + p}{\mu_i}$$

Note: There is a result in Denardo [8] which holds for all Markov decision chains which says  $V_{-1}$  is a convex combination of  $\frac{R_{0,s}^\delta}{Q_{1,s}^\delta}$  for all  $s$ . Theorem 2.1 expresses  $V_{-1}$  as a convex combination of different quantities. While it isn't true (or applicable) for a general Markov decision chain, it is much more useful in obtaining results for the repair models under discussion.

Note: The assumption (ii) in Theorem 2.1 is not really a restriction as far as finding a  $V_{-1}$  optimal policy is concerned. If  $\delta$  defines a process with several ergodic chains,  $C$ , each with its own  $V_{-1}^C$ , the policy  $\delta^*$ , which is irreducible and has single

erodic chain  $V_{-1} = \min_C V_{-1}^C$  is better than  $\delta$ . Thus, if  $\delta$  has a chance at being  $V_{-1}$  optimal, it must define an irreducible Markov chain.

Proof: Since  $\delta$  is fixed, we drop it in notation for brevity.

$$V_{-1} = \frac{P^* R_0}{P^* Q_1}.$$

Thus, it is sufficient to prove that

$$P^* R_0 = \sum_s P \cdot s Q_{1,s} V_s = \text{RHS}.$$

This will be done by proving that the corresponding coefficients of  $K_i$ ,  $1 \leq i \leq n$  and of  $p$  are equal for  $P^* R_0$  and RHS.

Let  $\mathcal{P}_i = \{s \in \mathcal{P} \mid \text{component } i \text{ is repaired in state } s\}$

$\mathcal{P}_p = \{s \in \mathcal{P} \mid \text{system is failed in } s\}$ .

$$P^* R_0 = \sum_{s \in \mathcal{P}} P \cdot s R_{0,s} = \sum_{i=1}^n K_i \left( \sum_{s \in \mathcal{P}_i} P \cdot s \right) + p \left( \sum_{s \in \mathcal{P}_p} P \cdot s \right)$$

$$\text{RHS} = \sum_s P \cdot s Q_{1,s} \begin{bmatrix} \sum_{i \in s \cup \Omega_s} \lambda_i K_i + \sum_{i \in s \cup \Omega_s} \lambda_i (K_i + p) \\ i: s \cup \Omega_s^{-i} & i: s \cup \Omega_s^{-i} \\ \text{is up} & \text{is failed} \end{bmatrix}$$

where

$$Q_{1,s} = \frac{1}{\sum_{i \in \Omega_s} \lambda_i} .$$

First, the coefficients of  $p$ :

$$\text{coefficient } p \text{ } P^* R_0 = \sum_{s \in \mathcal{S}_p} P \cdot s = \sum_{s \in \mathcal{S}_p} \sum_{\substack{(i,t): \\ t \cup \Omega_t - i \\ = s}} \frac{\lambda_i}{\sum_{j \in t \cup \Omega_t} \lambda_j} P \cdot t,$$

using the  $s^{\text{th}}$  equation of  $P^*(I-Q_0) = 0$

$$= \sum_t P \cdot t \left[ \frac{\sum_{\substack{j \in t \cup \Omega_t \\ j: t \cup \Omega_t - j \in \mathcal{S}_p}} \lambda_j}{\sum_{j \in t \cup \Omega_t} \lambda_j} \right]$$

$$= \sum_t \sum_{\substack{i \in t \cup \Omega_t \\ i: t \cup \Omega_t - i \\ \in \mathcal{S}_p}} \lambda_i Q_{1,t} P \cdot t$$

= coefficient  $p$  of RHS.

Now, the coefficients of  $K_i$ , and  $i$ :

$$K_i \text{ coefficient } P^* R_0: \sum_{s \in \mathcal{J}_i} P \cdot s$$

$$K_i \text{ coefficient RHS: } \sum_{s: i \in s \cup \Omega_s} \lambda_i P \cdot s Q_{1,s} = \sum_{s: i \in s \cup \Omega_s} \frac{\lambda_i}{\sum_{j \in s \cup \Omega_s} \lambda_j} P \cdot s,$$

since RHS without p-terms

$$\begin{aligned} &= \sum_s P \cdot s Q_{1,s} \sum_{i \in s \cup \Omega_s} \frac{K_i}{\mu_i} \\ &= \sum_s \sum_{i \in s \cup \Omega_s} P \cdot s \lambda_i Q_{1,s} K_i \\ &= \sum_{i=1}^n K_i \sum_{s: i \in s \cup \Omega_s} \lambda_i P \cdot s Q_{1,s} \end{aligned}$$

$$\text{so coefficient } K_i \text{ of RHS} = \sum_{s: i \in s \cup \Omega_s} \lambda_i P \cdot s Q_{1,s}$$

$$\begin{aligned} &= \sum_{s: i \in s} \frac{\lambda_i}{\sum_{j \in s} \lambda_j} \left( \sum_{\substack{t: t \cup \Omega_t \\ =s}} P \cdot t \right) \\ &= \sum_{\ell=1}^n \sum_{\substack{|s|=\ell \\ i \in s}} \frac{\lambda_i}{\sum_{j \in s} \lambda_j} \left( \sum_{\substack{t: t \cup \Omega_t \\ =s}} P \cdot t \right) \end{aligned}$$

where  $|s|$  = number of elements in  $s$ .

From the appropriate columns of  $P^*(I-Q_0) = 0$ , we get:

$$i^{\text{th}} \text{ column: } P_{\cdot i} = \sum_{l_1 \neq i} \frac{\lambda_{l_1}}{\lambda_i + \lambda_{l_1}} \left( \sum_{\substack{t: t \cup \Omega_t \\ = i l_1}} P_{\cdot t} \right)$$

$i l_1 \dots l_k$  <sup>th</sup> column:

$$P_{\cdot i l_1 \dots l_k} = \sum_{l_{k+1} \notin \{i, l_1, \dots, l_k\}} \frac{\lambda_{l_{k+1}}}{\lambda_i + \sum_{j=1}^{k+1} \lambda_{l_j}} \left( \sum_{\substack{t: t \cup \Omega_t \\ = i l_1 \dots l_{k+1}}} P_{\cdot t} \right);$$

$$k \leq n - 1.$$

thus,

$$P_{\cdot i} + \sum_{\substack{s: |s|=2 \\ i \in s}} \frac{\lambda_i}{\sum_{j \in s} \lambda_j} \left( \sum_{\substack{t: t \cup \Omega_t \\ = s}} P_{\cdot t} \right) = \sum_{\substack{s: |s|=2 \\ i \in s}} \sum_{\substack{t: t \cup \Omega_t \\ = s}} P_{\cdot t}$$

and

$$\sum_{\substack{l_1, \dots, l_k \\ l_k \neq i}} P_{\cdot i l_1 \dots l_k} + \sum_{\substack{s: |s|=k+1 \\ i \in s}} \frac{\lambda_i}{\sum_{j \in s} \lambda_j} \left( \sum_{\substack{t: t \cup \Omega_t \\ = s}} P_{\cdot t} \right) = \sum_{\substack{s: |s|=k+1 \\ i \in s}} \sum_{\substack{t: t \cup \Omega_t \\ = s}} P_{\cdot t}; \quad 1 \leq k \leq n - 1.$$

Thus, coefficient  $K_i$  of RHS

$$\begin{aligned}
&= \sum_{\ell=1}^n \sum_{\substack{s: |s|=\ell \\ i \in s}} \sum_{\substack{t: t \cup \Omega_t = s \\ i \in t}} P_{\cdot t} - P_{\cdot i} - \sum_{k=j}^{n-1} \sum_{\ell_1, \dots, \ell_k \neq i} P_{\cdot i \ell_1 \dots \ell_k} \\
&= \sum_{\ell=1}^n \sum_{\substack{s: |s|=\ell \\ i \in s}} \left( \sum_{\substack{t: t \cup \Omega_t = s \\ i \in t}} P_{\cdot t} + \sum_{\substack{t: t \cup \Omega_t = s \\ i \notin t}} P_{\cdot t} \right) - P_{\cdot i} \\
&\quad - \sum_{k=1}^{n-1} \sum_{\ell_1, \dots, \ell_k \neq i} P_{\cdot i \ell_1 \dots \ell_k} \\
&= \sum_{s \in \mathcal{P}_i} P_{\cdot s} = \text{coefficient } K_i \text{ in } P^* R_0. \quad \square
\end{aligned}$$

The following lemma associates a feasible policy with each  $V_s$ :

**Lemma 2.2:**  $V_s$ , as defined in Theorem 2.1, is equal to  $V_{-1}^\gamma$ , where  $\gamma$  is a policy which has  $\{s \cup \Omega_s \sim i\}_{i \in s \cup \Omega_s}$  as its chain of ergodic states, i.e.,  $\gamma$  is the policy:  $R_i$  in  $s \cup \Omega_s \sim i$  [keep the set of components  $s \cup \Omega_s$  working].

**Proof:** To get  $V_{-1}^\delta = V_s$ , some  $s$  in Theorem 2.1, need  $\delta : P_{\cdot t}^\delta = 0$  unless  $t \cup \Omega_t = s \cup \Omega_s$ . This occurs when  $\delta = \gamma$ .  $\square$

**Corollary 2.3:** A  $V_{-1}$  optimal policy for the Basic Model with no fixed charge will never involve repair of more than one unit at a time in an ergodic state.

**Proof:** Uses Theorem 2.1 and Lemma 2.2 directly. Any policy  $\delta$ , which repairs more than one unit at a time in an ergodic state has

$V_{-1}^\delta$  = convex combination of  $V_s$ ,  $s \in \mathcal{S}$ . Thus,  $V_{-1}^\delta \geq V_s$ , where  $V_s = V_{-1}^\gamma$ , some  $\gamma$  for which repair is never done on more than one unit simultaneously.  $\square$

The following theorem which follows directly from Theorem 2.1, Lemma 2.2, and Corollary 2.3, gives an exact procedure for finding the  $V_{-1}$  optimal policies for the Basic Model.

Theorem 2.4: Given the Basic Model with no fixed charge, a  $V_{-1}$  optimal policy  $\delta$  is one for which

$$V_{-1}^\delta = \min_{\substack{s \in \mathcal{S} \\ \text{system up in } s}} \hat{V}_s$$

where

$$\hat{V}_s = \sum_{\substack{i \in s \\ i: s-i \notin \mathcal{S}_p}} \frac{K_i}{\mu_i} + \sum_{i \in s} \frac{K_i + p}{\mu_i}.$$

Proof: From Theorem 2.1, Lemma 2.2, and Corollary 2.3.

Thus, given any coherent system with state space  $\mathcal{S}$  under the Basic Model, to find a  $V_{-1}$  optimal policy, compute  $\hat{V}_s \forall s \in \mathcal{S}$  and find the one(s) which are minimum, say  $V_s^*$ . By Lemma 2.2,  $V_s^*$  defines a  $V_{-1}$  optimal policy  $\gamma$  which keeps the set of  $s$  components working by repairing any component of  $s$  as soon as it fails.

Notice that there will be many  $V_{-1}$  optimal policies since the above procedure only defines what happens in the ergodic states, i.e., those for which the process once having entered, will stay in forever. If keeping the components of state "s" working is  $V_{-1}$  optimal, then all other states besides  $\{s-i\}_{i \in s}$  must be transient or unable to be reached from the initial state of all components working (or would have multiple ergodic chains, in which case the policy wasn't  $V_{-1}$  optimal). To determine the optimal policy for those states, we need  $V_0$  results, looked at in the next chapter.

The no fixed charge  $L$  assumption is essential. Although if  $L > 0$ , a version of Theorem 2.1 holds, not every  $V_s$  corresponds to a feasible optimal policy in that case, limiting its usefulness.

Some examples of specific systems conclude the section.

Example 2.3 (parallel system, different components (n))

$$\mathcal{P} = \{0\}$$

$$V_s = \sum_{\substack{i \in s \\ s-i \neq \{0\}}} \frac{K_i}{\mu_i} + \sum_{s-i=\{0\}} \frac{K_i+p}{\mu_i} .$$

If  $|s| > 1$ , then  $s \sim i \neq \{0\} \forall i \in s$

$|s| = 1$  ( $s = \{i\}$ , some  $i$ ), then  $s \sim i = \{0\}$ .

Possible min  $\hat{V}_s^s$  (optimal  $V_{-1}^s$ ) are:

	$\hat{V}_s$	<u>ergodic chain</u>	<u>policy</u>
$s = \{i\}$	$\frac{K_i + p}{\mu_i}$	$\{0\}$	$R_i$ in 0 (keep unit i working)
$s = \{i, j\}$	$\frac{K_i}{\mu_i} + \frac{K_j}{\mu_j}$	$\{i, j\}$	$R_i$ in j $R_j$ in i (keep i & j working) .

If  $s : |s| > 2$ ,  $\hat{V}_s \geq \frac{K_i}{\mu_i} + \frac{K_j}{\mu_j}$ , some  $i$  &  $j$ . A  $V_{-1}$  optimal policy is one which has

$$V_{-1} = \min \left\{ \underbrace{\min_j \frac{K_j + p}{\mu_j}}_{\text{keep } j \text{ working}} ; \underbrace{\min_{i,j} \frac{K_i}{\mu_i} + \frac{K_j}{\mu_j}}_{\text{keep } i \text{ \& } j \text{ working}} \right\}$$

If  $n = 2$ , then we have Example 2.1. Below are some examples of policy behavior:

Suppose  $n = 3$

		<u>states</u>						
		$s :  s  > 1$	1	2	3	0		
Let	$\delta_1:$	A	$R_2$	$R_1$	A	$R_2$		
	$\delta_2:$	A	$R_2$	$R_1$	$R_1$	( $R_2$ )		
	$\delta_3:$	12 $R_3$	13 $R_2$	23 A	$R_2$	$R_1$	$R_1$	--

→  $\delta_1$  is not  $V_{-1}$  optimal, since it has two ergodic chains {12} and {0} both accessible from the initial state.

→  $\delta_2$  is possibly  $V_{-1}$  optimal, as even though it has {12} and {0} as absorbing chains, {12} is the unique one accessible from the starting state.

→  $\delta_3$  is also possibly  $V_{-1}$  optimal, and has the same  $V_{-1}$  as  $\delta_2$ .  $V_0$  is needed to distinguish between  $\delta_2$  and  $\delta_3$ .

Example 2.4 (k-of-n system (system works  $\Leftrightarrow$  at least  $k \leq n$  units functioning)):

$k = 1$  is parallel

$k = n$  is series

$$\mathcal{L}_p = \{s \mid |s| \leq k - 1\}.$$

Note that  $s : |s| < k - 1$  are inaccessible due to the instantaneous repair and requirement of repair in a failed state.

$$\hat{V}_s = \sum_{i \in S} \frac{K_i}{\mu_i}, \quad \text{if } |s| > k$$

$$\sum_{i \in S \cup \Omega_s} \frac{K_i + p}{\mu_i}, \quad \text{if } |s| = k$$

not defined for  $|s| < k$  (system down).

Possible optimal  $V_{-1}^s$ :

$$V_{-1} = \sum_{j=1}^k \frac{K_{i_j} + p}{\mu_{i_j}}, \text{ some } \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$$

has ergodic chain:  $\{i_1, \dots, i_k\} \sim i, i \in \{i_1, \dots, i_k\}$

policy: keep  $\{i_1, \dots, i_k\}$  working

or

$$V_{-1} = \sum_{j=1}^{k+1} \frac{K_{i_j}}{\mu_{i_j}}, \text{ some } \{i_1, \dots, i_{k+1}\} \subseteq \{1, \dots, n\}$$

has ergodic chain:  $\{i_1, \dots, i_{k+1}\} \sim i, i \in \{i_1, \dots, i_{k+1}\}$

policy: keep  $\{i_1, \dots, i_{k+1}\}$  working.

True for  $1 \leq k \leq n - 1$  ( $k \neq n$  (series) by assumptions on Basic Model).

Thus,  $\delta$  optimal is either

- (a) keep the cheapest set of  $k$  components working if  $p$ ,  
the penalty cost is low enough, or

- (b) keep the cheapest set of  $k + 1$  components working  
if  $p$ , the penalty cost is high enough.

Example 2.5 (system up  $\Leftrightarrow$  component  $i$  is up):

$$\mathcal{S}_p = \{s \mid i \notin s\}$$

$$\hat{V}_s \text{ defined for } s : i \in s$$

$$\hat{V}_s = \frac{K_i + p}{\mu_i} + \sum_{j \in s - i} \frac{K_j}{\mu_j}$$

$$\min_s \hat{V}_s = \frac{K_i + p}{\mu_i} \text{ at } s = \{i\}.$$

So, a  $V_{-1}$  optimal policy has absorbing state  $\{0\}$  and keeps component  $i$  only working.

Example 2.6 ( $n = 3$  system defined by the min cut sets  $\{1\}$ ,  $\{13\}$ ):

$$\mathcal{S}_p = \{23, 2, 3, 1, 0\}$$

$$\hat{V}_s \text{ defined for } s \in \{123, 12, 13\}$$

$$\hat{V}_{12} = \frac{K_1 + p}{\mu_1} + \frac{K_2 + p}{\mu_2}$$

$$\hat{V}_{13} = \frac{K_1 + p}{\mu_1} + \frac{K_3 + p}{\mu_3}$$

$$\hat{V}_{123} = \frac{K_1+p}{\mu_1} + \frac{K_2}{\mu_2} + \frac{K_3}{\mu_3}, \text{ any of which could be min.}$$

So,  $\delta$  optimal is to keep {12}, {13}, or {123} working, depending on parameters.

### 3. Fixed Charge Results

In this, the final section of Chapter II, a brief look will be taken at Basic Model results for cases when there is a fixed charge  $L > 0$  per repair decision. A larger fixed charge is going to induce more multiple repair decisions and theorems in Sections 1-2 involving conditions under which one never repairs more than a single unit at a time will be no longer true in many cases. However, a number of the other results from Section 2 either apply directly or can be modified for use here. The case of a k-of-n system with identical components is looked at in detail as it yields a precise optimal solution using  $V_{-1}$  only which can be compared to the  $L = 0$  case.

The basic result from Section 2 upon which all the  $V_{-1}$  results for the  $L = 0$  case were based is Theorem 2.1, which expressed  $V_{-1}^\delta$  for any policy  $\delta$  as a convex combination of  $V_{-1}^{\gamma_i}$ , where  $\gamma_i$  were policies which never repaired more than a single component at once in ergodic states. An extension of Theorem 2.1 to the  $L > 0$  case is direct. Unfortunately, the quantities which  $V_{-1}^\delta$  is a convex combination of, are not all  $V_{-1}$ 's of certain other policies, rendering the result less useful but still worthwhile stating.

Theorem 2.5: (extension of Theorem 2.1) Suppose we have the Basic Model with fixed charge  $L$ , and a general coherent system.

Fix a policy  $\delta$  such that:

- (i)  $R_{\Omega_s}$  in states where  $\Omega_s \subseteq \{1, \dots, n\} \sim s$
- (ii) the underlying Markov chain defined by  $\delta$  is irreducible.

Then

$$V_{-1}^\delta = \sum_{s \in \mathcal{S}} \alpha_s V_s \quad \text{where} \quad \alpha_s = \frac{P \cdot s Q_{1,s}}{P^* Q_1}$$

and

$$\begin{aligned}
 V_s &= \sum_{\substack{i \in s \cup \Omega_s \\ i: s \cup \Omega_s - i \\ \text{is up}}} \frac{K_i + L}{\mu_i} + \sum_{\substack{i \in s \cup \Omega_s \\ i: s \cup \Omega_s - i \\ \text{is down}}} \frac{K_i + p + L}{\mu_i} \quad \text{if } \Omega_s \neq \phi \\
 &= \sum_{\substack{i \in s \\ i: s \cup \Omega_s - i \\ \text{is up}}} \frac{K_i}{\mu_i} + \sum_{\substack{i \in s \\ i: s \cup \Omega_s - i \\ \text{is down}}} \frac{K_i + p}{\mu_i} \quad \text{if } \Omega_s = \phi .
 \end{aligned}$$

Proof: As in the proof of Theorem 2.1, it suffices to show  $P^* R_0 = \sum_s F \cdot s Q_{1,s} V_s$  (RHS) by proving the corresponding coefficients of  $K_i$ ,  $1 \leq i \leq n$ , of  $p$ , and of  $L$  are equal. Those of  $K_i$  and  $p$ , unchanged from  $L = 0$  case are still equal so only need to check the coefficients of  $L$ .

$$P^* R_0 = \sum_{s \in \mathcal{P}} P \cdot s R_{0,s} = \sum_{i=1}^n K_i \left( \sum_{s \in \mathcal{P}_i} P \cdot s \right) + p \left( \sum_{s \in \mathcal{P}_p} P \cdot s \right)$$

$$(\mathcal{P}_p, \mathcal{P}_1, \mathcal{P} \text{ defined on page 30}) + L \left( \sum_{s \in \mathcal{P}^U} P \cdot s \right)$$

where  $\mathcal{P}^U = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \dots \cup \mathcal{P}_n =$  set of states in which some repair is performed.

$$\begin{aligned} \text{Coefficient } L \text{ in (RHS)} &= \sum_{s \in \mathcal{P}^U} P \cdot s Q_{1,s} \cdot \sum_{i \in s \cup \Omega_s} 1/\mu_i \\ &= \sum_{s \in \mathcal{P}^U} P \cdot s \cdot \frac{1}{\sum_{i \in s \cup \Omega_s} \lambda_i} \cdot \sum_{i \in s \cup \Omega_s} \lambda_i \\ &= \sum_{s \in \mathcal{P}^U} P \cdot s \\ &= \text{coefficient } L \text{ in } P^* R_0. \quad \square \end{aligned}$$

In the  $L = 0$  case, one then has Lemma 2.2 which states that  $V_s$ , as defined in Theorem 2.1 is equal to  $V_{-1}^\gamma$ , where  $\gamma$  is a policy which keeps the set of components  $\{s \cup \Omega_s\}$  working. (Whenever one fails, repair it and, thus, never repair more than one unit at a time.) This then eliminates from being optimal, all policies which allow repair of  $>$  one unit at a time in an ergodic state. If  $L > 0$ , this is no longer true; as the following example illustrates:

Example 2.7 [refer to Example 2.3] ( $n = 3$  parallel system):

Fix  $\delta$

$$\left. \begin{array}{l} \text{If } s = \{1\}, R_2 \text{ in } s \\ \text{or } s = \{0\}, R_{12} \text{ in } s \end{array} \right\} : V_s = \lambda_1(K_1+L) + \lambda_2(K_2+L)$$

But, if  $s = \{12\}$ , A in  $s$ , then  $V_s = \lambda_1 K_1 + \lambda_2 K_2$ . Then policy  $\gamma$ , of keeping components 1 and 2 working, has  $V_{-1}^\gamma = \lambda_1(K_1+L) + \lambda_2(K_2+L)$ . So, if we pick  $\delta$  so that "A" in state 12, then  $V_{12}$  cannot be expressed as  $V_{-1}^\gamma$ , where  $\gamma$  keeps components number 1 and number 2 working.

Given the failures of Lemma 2.2 and Theorem 2.1 in limiting possible  $V_{-1}$  optimal policies in the fixed charge case, it is now clear that the finding of a general optimal policy would be extremely time consuming, if not impossible without the aid of a computer (see Chapter VI for suggestions). Thus, in the interests of getting a feel for how a fixed charge affects things, we close out the section by looking at two special cases where results are obtainable. These are:

- (1) identical components, k-of-n system
- (2) no penalty cost ( $p = 0$ ).

In case (1), we can get a general optimal policy.

Definition: Let  $f_k(i) = (i+k)[1/k + \dots + 1/(i+k-1)] - i$ ,  $1 \leq k \leq n$ ,  $1 < i < n - k + 1$ ,  $f_k(0) \triangleq 0$ ,  $f_k(n - k + 1) \triangleq \infty$ .

Notice that

$$(a) f_k(i) + i, + k$$

$$(b) f_k(i) - f_k(i-1) + i, + k.$$

Theorem 2.6: (optimal policy for  $L > 0$ , k-of-n identical components).

Suppose we have the Basic Model with

- (1) a k-of-n system
- (2) identical components ( $\lambda_i \equiv \lambda$ ,  $K_i \equiv K$ ), and
- (3) a fixed charge  $L > 0$  per repairman visit.

Then the following are true:

- (i) Never optimal to repair until k left working.
- (ii) Among policies which repair when there are k left (label  $R_\ell$  = repair  $\ell$  units at once),  $R_{\ell-1}$  is optimal  $\Leftrightarrow f_{k+1}(\ell-1) < L/K < f_{k+1}(\ell)$ , where  $1 \leq \ell \leq n - k$  and  $f_{k+1}(n-k) = \infty$ .
- (iii) Among policies which repair only when the system fails (k - 1 left),  $R_\ell$  is optimal  $\Leftrightarrow f_k(\ell-1) < \frac{L+p}{K} < f_k(\ell)$ ,  $1 \leq \ell \leq n - k + 1$ .
- (iv) The optimal policy is the better of that in (ii), (iii).

Proof:

- (i) is direct from Theorem 3.3 in the next chapter. As we have a k-of-n system with identical components, the

state and decision spaces are greatly simplified with states =  $n, n - 1, \dots, k, k - 1$ , the number of components working, and decisions limited to the number of components desired to repair in a state. By (i), the optimal decision is "A" in states  $n, n - 1, \dots, k + 1$ . In state  $k$ , possible decisions are:  $A, R_1, R_2, \dots, R_{n-k}$ , in state  $k - 1$ , they are:  $R_1, R_2, \dots, R_{n-k}, R_{n-k+1}$ . No state  $i < k - 1$  can be reached by the necessity of repair upon system failure assumption. The possible policies then reduce to the following  $2(n-k) + 1$  possibilities:

1 policy			2		
$\delta_k$	$\delta_{k-1}$	$\mathcal{E} = \{\text{ergodic states}\}$	$\delta_k$	$\delta_{k-1}$	$\mathcal{E} = \{\text{ergodic states}\}$
A	$R_1$	$\{k-1\}$	$R_1$	never reach	$\{k\}$
A	$R_2$	$\{k-1, k\}$	$R_2$	k-1 so	$\{k, k+1\}$
A	$R_3$	$\{k-1, k, k+1\}$	$R_3$	irrelevant	$\{k, k+1, k+2\}$
.	.	.	.		.
.	.	.	.		.
A	$R_{n-k+1}$	$\{k-1, k, \dots, n-1\}$	$R_{n-k}$		$\{k, k+1, \dots, n-1\}$

Those policies in column 1 are those referred to in (iii) and those in column 2 are treated in (ii). It remains now to compute  $V_{-1}^s$  for various policies and compare them. Below is a table of such values. Comparisons of column 1 values lead to result (iii) and of column 2 values lead to result (ii); (iv) is obvious.  $\square$

**Figure 2.2**

**Table of  $V_{-1}^\delta$  for  $\delta$  in Theorem 2.6 Model**

Column 1 (iii)		Column 2 (ii)	
$\delta$	$V_{-1}$	$\delta$	$V_{-1}$
$AR_1$	$\frac{k + L + p}{\mu/k}$	$R_1 -$	$\frac{K + L}{\nu/k+1}$
$AR_2$	$\frac{2K + L + p}{\mu(1/k + 1/k+1)}$	$R_2 -$	$\frac{2K + L}{\mu(1/k+1 + 1/k+2)}$
$AR_3$	$\frac{3K + L + p}{\mu(1/k + 1/k+1 + 1/k+2)}$	$R_3 -$	$\frac{3K + L}{\mu(1/k+1 + 1/k+2 + 1/k+3)}$
$\vdots$		$\vdots$	
$AR_{n-k+1}$	$\frac{(n-k+1)K + L + p}{\mu(1/k + \dots + 1/n)}$	$R_{n-k} -$	$\frac{(n-k)K + L}{\mu(1/k+1 + \dots + 1/n)}$

It is worth noting that  $V_{-1}$  alone here gives us our desired "optimal" policy - no  $V_0$  needed.

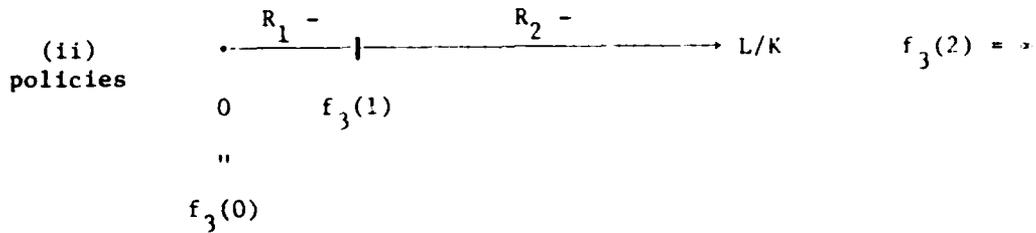
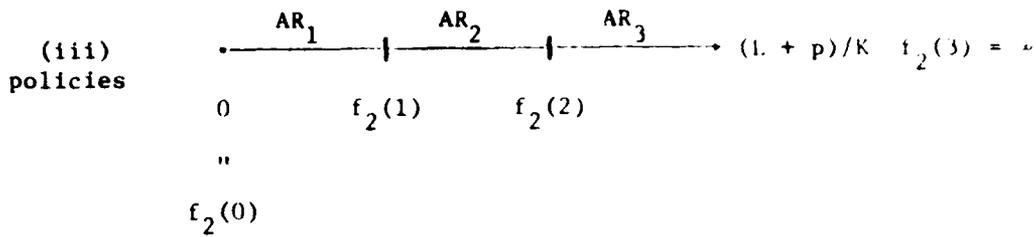
Example 2.8 (consider Example 2.2 from Section 2.1):

$n = 4, k = 2$  costs: with fixed charge  $L > 0$   
 component repair  $K/\text{component}$

states: 4, 3, 2, 1      $\delta_4^{\text{opt}} = A, \delta_3^{\text{opt}} = A$

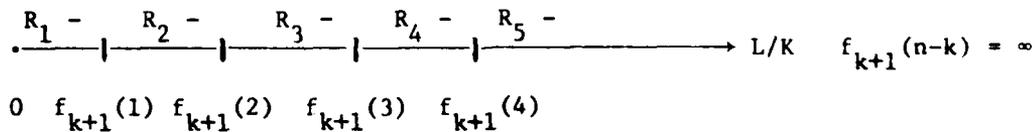
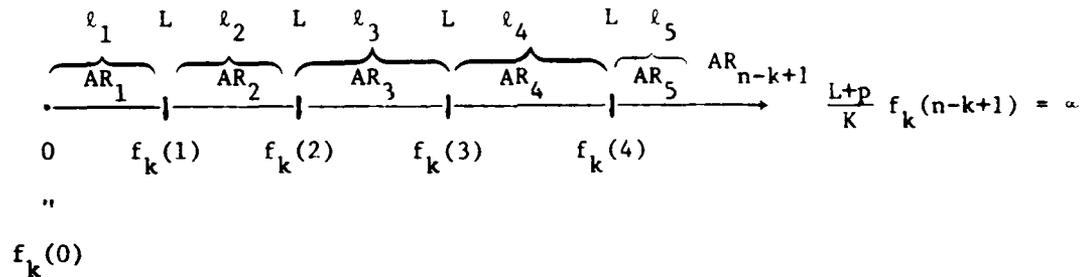
mean lifetimes  
 of components:  $\mu = 1/\lambda$

policies:  $AR_1, AR_2, AR_3, R_1 -, R_2 - .$



Compare optimal among (ii) and (iii) to get optimal.

In general,



The  $L/K$  ratio =  $\frac{\text{fixed charge}}{\text{component repair charge}}$  is the key here. A large  $L/K$  favors repair of several components simultaneously while a small ratio favors single repairs. Obviously, a large  $p$  favors an " $R_1 -$ " policy while a small  $p$  favors " $AR_1$ " types.

Now, fix all parameters except the type "k-of-n" system to see how k and n affect the optimal policy.  $(L+p)/K$  and  $L/K$  are now fixed. Increasing k lowers  $f_k(i)$  for any fixed i. This fact enables us to state:

Lemma 2.7: Given the Basic Model with identical components and a k-of-n system with a fixed number of components, n. Let  $1 \leq k_1 < k_2 < n$ . Then the number of components repaired in the optimal policy for the  $k_1$ -of-n system is less than or equal to the number repaired in that for the  $k_2$ -of-n system.

Increasing n, the number of components while leaving k fixed leaves the previously existing  $f_k(i)$  unchanged but does add more possible values of i and thus further subdivides the interval between the previously last  $f_k(i)$  and  $\infty$ . This leads to:

Lemma 2.8: Given the Basic Model with identical components and a k-of-n system with fixed k, let  $k < n_1 < n_2$ . Then:

- (i) If the number of components repaired in the optimal policy for the k-of- $n_1$  system is less than all possible, then the same policy holds for k-of- $n_2$ .
- (ii) If the number repaired in the optimal policy for the k-of- $n_1$  system is all possible (i.e.,  $n_1-k$  in state k or  $n_1-k+1$  in state k-1), then the number repaired optimally for k-of- $n_2$  is greater than or equal to that for  $n_1$ .

Another interesting question that is easily answered is:  
 What happens to the optimal policy asymptotically as  $k, n \rightarrow \infty$   
 with  $n - k$  held constant?

Lemma 2.9: Given the Basic Model with identical components  
 and a  $k$ -of- $n$  system, let  $n, k \rightarrow \infty$  with  $n - k$  held fixed.  
 Then the optimal policy is either  $AR_{n-k+1}$  or  $R_{n-k}$ .

Proof: Notice that  $\lim_{k \rightarrow \infty} f_k(i) = 0$  for fixed  $i$ . Since  
 $\frac{L+p}{K}, \frac{L}{K} > 0, \frac{L}{K} > f_{k+1}(n-k-1)$  and  $\frac{L+p}{K} > f_k(n-k)$ , the  
 optimal policy will be either to allow system failure or to get  
 down to  $k$  left and repair all failed components.  $\square$

Notice that if  $L = 0$  (no fixed charge), the optimal policy  
 is either  $AR_1$  or  $R_1$  - since in that case, one never repairs more  
 than one at a time.

Now consider the  $p = 0$  case briefly (components no longer  
 need be identical) using a  $k$ -of- $n$  system. Using arguments from  
 Theorem 3.3 (next chapter), it is clear that repair need never be  
 done until system failure ( $k-1$  left) since there is no penalty.  
 Since  $\delta_s = A$  if  $|s| \geq k$ , only  $\delta_s, |s| = k - 1$  need be determined,  
 which is possible using only  $V_{-1}$  optimality. For the case of  
 identical components, the optimal policy in (iii), Theorem 2.6 gives  
 the overall optimal policy since those in (ii) will never be optimal  
 given  $p = 0$ .

See Chapter VII (Conclusions) for a summary of results  
 presented in this and other model-result chapters.

## CHAPTER III

### THE BASIC MODEL, $V_0$ - RESULTS

#### 1. No Fixed Charge, General Results

In this chapter we will try to pinpoint optimal decisions in transient states. The problem is much too complicated to solve for a general coherent system except by computer, so let's restrict ourselves to a  $k$ -of- $n$  system. In this case, it will be shown that one never repairs when more than  $k$  units are working and that, with no fixed charge, one never repairs up to more than  $k + 1$  units working from any state if  $p$  is small enough. As  $V_{-1}$  determines what to do when in  $s : |s| = k - 1$  and one does nothing until  $k$  - left, the only unknowns are what to do in transient states where  $k$  - units are working. If  $k = 1$  (parallel system), the problem can be solved exactly in many cases and a general  $V_0$  optimal solution obtained. This requires considerable work and is done in the next section (2). The other case for which an exact optimal policy is obtainable is that of identical components. Here, the state space of the model simplifies, causing every possible optimal policy to have a unique  $V_{-1}$ , in which case  $V_{-1}$  is sufficient in computing an optimal policy.

To prove the aforementioned results, we need the following lemmas:

Lemma 3.1: Suppose we have a  $k$ -of- $n$  system and the Basic Model. If it is never optimal to repair more than one unit at a time in states  $s : |s| = k' > k$ , then it is optimal to do nothing in states  $s : |s| = k'$ .

Proof: First, from Chapter II, all states  $s, |s| \geq k + 1$  are transient in a  $V_{-1}$  optimal policy so any realization of the process starting in  $s : |s| = k'$  using an optimal policy must eventually reach a state  $s : |s| = k' - 1$ . This will happen at the first instant decision "A" is chosen.

The basic argument is: suppose you start in state  $s$  and follow policy  $\delta$ , under which  $R_2$  in  $s$ , until a state  $t : |t| = k' - 1$  is hit. Then the policy  $\gamma$ , which first lets a component fail from  $s$  and then does everything  $\delta$  does will be better than  $\delta$  (with respect to  $V_0$ ). Since  $\gamma$  never repairs until  $k' - 1$  units are left, the result is true. To be precise: let the starting state  $s = \{i_1, \dots, i_k\}$ . It is desired to compare policies with respect to  $V_0$ , where  $V_0$  is a vector whose length is the total number of states and which satisfies:

$$(1) \quad (I - Q_0)V_0 = R_0 - Q_1V_{-1}$$

$$(2) \quad P^*Q_1V_0 = P^*(-R_1 + Q_2V_{-1}) .$$

Let  $V_{0,\delta}^{s'}$  = total expected cost given start in  $s'$  and follow policy  $\delta$ . It suffices to compare  $V_{0,\delta}^{s'}$  and  $V_{0,\gamma}^{s'}$  for any one  $s'$  for if a

policy minimizes  $V_0^s$  for one  $s'$ , it does so for all  $s'$  (see Veinott [31]). Let  $s' = s$  where  $s : |s| = k'$  is our starting state. In comparing  $V_0^s$ , it is assumed that an optimal  $V_{-1}$  policy and a corresponding set of ergodic states have been found. We are concerned now with transient states only, as ergodic states and decisions there have been determined by  $V_{-1}$ . By equation (1), if  $R_j$  in state  $t$ ,  $t = \{i_1, \dots, i_q\}$

$$(3) \quad V_{0,R_j}^t = \sum_{\ell \in t} \frac{\lambda_\ell}{\sum_{p \in t} \lambda_p} V_0^{t \cup j - \ell} + K_j - \frac{V_{-1}}{\sum_{p \in t} \lambda_p}$$

or, combining  $V_0^t$  terms

$$(4) \quad V_{0,R_j}^t = \sum_{\ell \in t} \frac{\lambda_\ell}{\sum_{p \in t} \lambda_p} V_0^{t - \ell \cup j} + \frac{\sum_{p \in t \cup j} \lambda_p}{\sum_{p \in t} \lambda_p} \cdot K_j - \frac{V_{-1}}{\sum_{p \in t} \lambda_p}$$

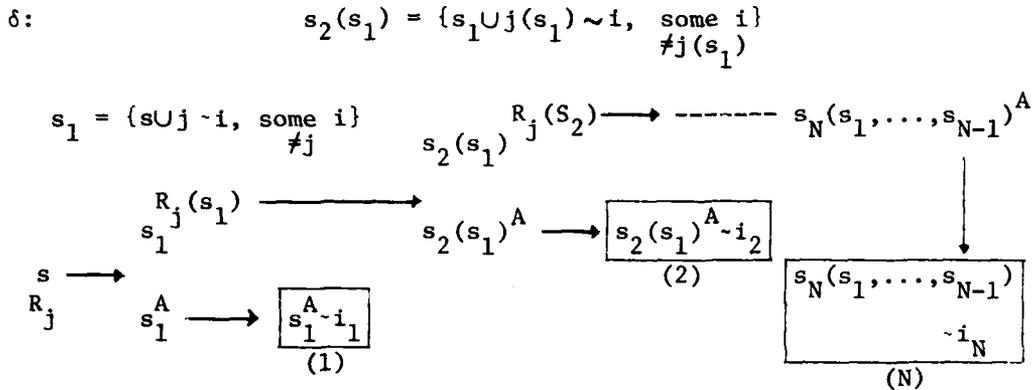
where  $V_{0,\delta(t)}^t =$  total expected cost given start in  $t$ , use  $\delta(t)$  in  $t$  and follow any policy thereafter.

If "A" in state  $t$ ,

$$(5) \quad V_{0,A}^t = \sum_{\ell \in t} \frac{\lambda_\ell}{\sum_{p \in t} \lambda_p} V_0^{t - \ell} - \frac{V_{-1}}{\sum_{p \in t} \lambda_p}.$$

Now, pick any policy  $\delta$ , which has  $R_j$  in the initial state  $s$  and specifies what to do each possible time another state  $s'$ ,  $|s'| = k'$  is entered.

As previously mentioned, there are a finite number of such possibilities.  $\delta$  can be represented by the following sample path diagram:



Where:

- $N$  = maximum possible number transitions before going to a state with  $|s| = k' - 1$  (i.e., making the decision "A"). Note  $N < \infty$ .
- "Boxed in" ( $\square$ ) states (1), ..., (N) represent "termination", i.e., entry into a state with only  $k' - 1$  units working after 1, 2, ..., or  $N$  transitions.
- $s_j(s_1, \dots, s_{j-1})$  represents the state entered after the  $j^{\text{th}}$  transition (if termination has not yet occurred).

Clearly, this is a function of  $s_1, \dots, s_{j-1}$  where

$$s_1 \in s_1^R, \dots, s_{j-1} \in s_{j-1}^R \text{ (or would have had termination).}$$

→  $j(s_\ell)$  represents the item specified by  $\delta$  to be repaired in state  $s_\ell$ . If  $j(s_\ell) = A$ , then termination occurs at the  $\ell^{\text{th}}$  transition.

→  $s_\ell(s_1, \dots, s_{\ell-1}) = s_{\ell-1}(s_1, \dots, s_{\ell-2}) \cup j(s_{\ell-1}) \sim i, i \in s_{\ell-1}$  by definition of the process involved.

→  $s_j^A = \{s_j \mid \delta_{s_j} = A\}$ .

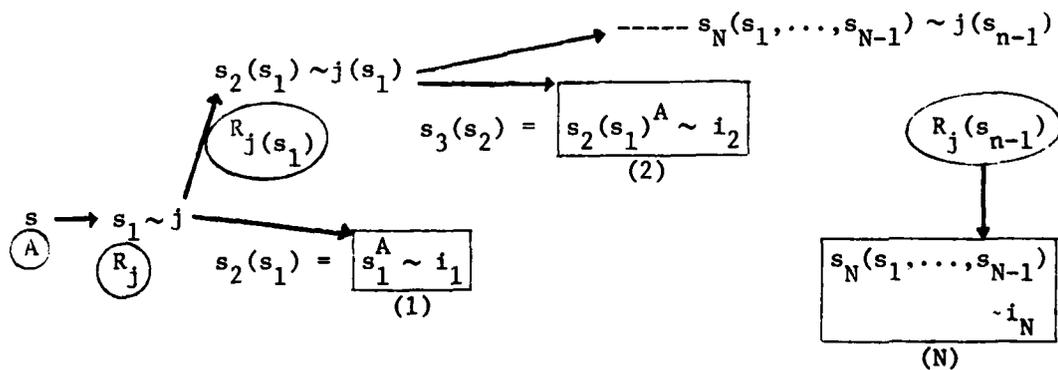
→  $s_j^{R_k} = \{s_j \mid \delta_{s_j} = R_k\}$ .  $s_N^{R_k} = \phi$  by definition of  $N$ .

→  $s_j^R = \{s_j \mid \delta_{s_j} \neq A\}$ .

It will be shown that the policy  $\gamma$ , represented by the sample path diagram below, has  $V_{0,\gamma}^S \leq V_{0,\delta}^S$ . Note that the two diagrams have identical probabilistic structure and termination states (1) ... (N).

$\gamma$ :  $s_\ell \in s_\ell^R, 1 \leq \ell \leq N-1$ .

$s_2(s_1) = \{s_1 \cup j(s_1) \sim i, \text{ some } i, s_1 \in s_1^R\}$



By (4),

$$(6) \quad v_{0,\delta}^s = \sum_{\ell \in s} \frac{\lambda_\ell}{\sum_{p \in s} \lambda_p} v_0^{s \cup j - \ell} + \frac{\sum_{p \in s \cup j} \lambda_p}{\sum_{p \in s} \lambda_p} \cdot K_j - \frac{v-1}{\sum_{p \in s} \lambda_p}.$$

By (5),

$$(7) \quad v_{0,\gamma}^s = \sum_{\ell \in s} \frac{\lambda_\ell}{\sum_{p \in s} \lambda_p} v_0^{s-\ell} - \frac{v-1}{\sum_{p \in s} \lambda_p}.$$

If  $N = 1$ , i.e.,  $\delta_{s_1} = \delta_{s \cup j - \ell} = A \forall \ell$ , then, using (4) and

(5), plug  $v_0^{s \cup j - \ell}$  into (6)  $\forall \ell$  and  $v_0^{s-\ell}$  into (7)  $\forall \ell$ . (6),

(7) become equal except for the coefficients of  $K_j$ . Thus,

$$(6)-(7) = v_{0,\delta}^s - v_{0,\gamma}^s = \left[ \frac{\sum_{p \in s \cup j} \lambda_p}{\sum_{p \in s} \lambda_p} - 1 \right] K_j$$

$$= \frac{\lambda_j K_j}{\sum_{p \in s} \lambda_p} > 0$$

so,  $\gamma$  is better than  $\delta$  if  $N = 1$ . We now proceed by induction on  $N$ , the object being to prove

$$(8) \quad v_{0,\delta}^s - v_{0,\gamma}^s = \sum_{w|s} p_w \sum_{j=0}^{\ell(w)-1} \frac{\lambda_{d(s_j)}^{K_{d(s_j)}}}{\sum_{p \in s_j} \lambda_p}$$

where  $w|s$  is any sample path starting from  $s$ ,  $l(w) =$  length of sample path  $w$ , i.e.,  $w = \langle s, s_1, s_2, \dots, s_{l(w)} \rangle$ ,  $d(s_j) =$  decision at state  $s_j$  of sample path  $w$ . Since (8)  $> 0$  obviously, the desired result would then be proved. (8) has been shown true for  $N = 1$ . Suppose true for  $N - 1$ ,  $\delta$  is given. Let state  $t_\ell = s \cup j \sim \ell$ ,  $\ell \neq j$ . Suppose

$$\delta : R_{j(\ell)} \quad \text{in } t_\ell, \ell \in r$$

$$A \quad \text{in } t_\ell, \ell \in a, r \cup a = s$$

$r, a$  are defined to satisfy the above.

Let  $\delta_\ell$  be the policy which starts in  $t_\ell$ ,  $\ell \in r$  with " $R_{j(\ell)}$ " and follows  $\delta$  thereafter. Let  $\gamma_\ell$  be the policy which starts in  $t_\ell$ ,  $\ell \in r$  with " $A$ " and follows  $\gamma$  thereafter.  $\delta_\ell$  has a maximum number of transitions to a state  $t : |t| = k' - 1 = N - 1$  so the hypothesis holds for  $\delta_\ell$ ,  $\ell \in r$ .

By (6) and (5)

$$v_{0,\delta}^s = \left( 1 + \frac{\lambda_j}{\sum_{p \in s} \lambda_p} \right) K_j - \frac{v_{-1}}{\sum_{p \in s} \lambda_p} + \sum_{\ell \in r} \frac{\lambda_\ell}{\sum_{p \in s} \lambda_p} v_0^{t_\ell} + \sum_{\ell \in a} \frac{\lambda_\ell}{\sum_{p \in s} \lambda_p} \left( \sum_{k \in t_\ell} \frac{\lambda_k}{\sum_{p \in t_\ell} \lambda_p} v_0^{t_\ell - k} - \frac{v_{-1}}{\sum_{p \in t_\ell} \lambda_p} \right)$$

By (7) and (3)

$$V_{0,\gamma}^s = K_j - \frac{V_{-1}}{\sum_{p \in s} \lambda_p} + \sum_{\ell \in r \cup a} \frac{\lambda_\ell}{\sum_{p \in s} \lambda_p} \left( \sum_{k \in t_\ell} \frac{\lambda_k}{\sum_{p \in t_\ell} \lambda_p} V_0^{t_\ell - k} - \frac{V_{-1}}{\sum_{p \in t_\ell} \lambda_p} \right).$$

So,  $V_{0,\delta}^s - V_{0,\gamma}^s$

$$= \frac{\lambda_j K_j}{\sum_{p \in s} \lambda_p} + \sum_{\ell \in r} \frac{\lambda_\ell}{\sum_{p \in s} \lambda_p} \left[ V_0^{t_\ell} - \left( \sum_{k \in t_\ell} \frac{\lambda_k}{\sum_{p \in t_\ell} \lambda_p} V_0^{t_\ell - k} - \frac{V_{-1}}{\sum_{p \in t_\ell} \lambda_p} \right) \right]$$

$$= \frac{\lambda_j K_j}{\sum_{p \in s} \lambda_p} + \sum_{\ell \in r} \frac{\lambda_\ell}{\sum_{p \in s} \lambda_p} \left[ V_{0,\delta_\ell}^{t_\ell} - V_{0,\gamma_\ell}^{t_\ell} \right] \text{ by (5) and definitions of } \delta_\ell, \gamma_\ell.$$

$$= \frac{\lambda_j K_j}{\sum_{p \in s} \lambda_j} + \sum_{\ell \in r} \frac{\lambda_\ell}{\sum_{p \in s} \lambda_p} \left[ \sum_{w|t_\ell} P_w^{\ell(w)-1} \frac{\lambda_{d(s_j)} K_{d(s_j)}}{\sum_{p \in s_j} \lambda_p} \right] \text{ by induction hypothesis}$$

$$= \sum_{\ell \in r} \frac{\lambda_\ell}{\sum_{p \in s} \lambda_p} \left[ \frac{\lambda_j K_j}{\sum_{p \in s} \lambda_p} + \sum_{w|t_\ell} P_w^{\ell(w)-1} \frac{\lambda_{d(s_j)} K_{d(s_j)}}{\sum_{p \in s_j} \lambda_p} \right]$$

$$= \sum_{w|s} P_w^{\ell(w)-1} \frac{\lambda_{d(s_j)} K_{d(s_j)}}{\sum_{p \in s_j} \lambda_p}, \text{ which is what was to be proved. } \square$$

**Lemma 3.2:** Suppose we have the Basic Model and a  $k$ -of- $n$  system. Let  $k' > k$ . Then, if it is optimal to do nothing in all states  $s : |s| > k'$ , it is never optimal to repair more than one unit at a time in states  $s : |s| = k'$ .

Proof: The basic approach will be to use the  $V_0$  equations to show that if one starts in state  $s$ ,  $|s| = k'$  and repairs some subset  $\Omega_s = \{j_1, \dots, j_m\}$ ,  $m > 1$  of the failed components and then follows any other policy upon again getting back to a state  $t : |t| = k'$  (we have assumed "do nothing" optimal for  $|t| > k'$ ), that this policy is worse than at least one of the following:

- (i)  $R_{\Omega_s - j_\ell}$  in  $s$ ,  $1 \leq \ell \leq m$ , then anything
- (ii)  $A$  in  $s$  and  $R_{\Omega_s}$  at the next transition.

This shows the desired result since

- if  $m = 2$ , any policy repairing two at once in state  $s$  is dominated either by a policy which repairs one at a time or does nothing in  $s \Rightarrow m = 2$  never optimal
- (induction argument) suppose  $R_{\Omega_s}$ ,  $|\Omega_s| = m - 1$  is never optimal in any  $s$ . But any  $R_{\Omega_s}$ ,  $|\Omega_s| = m$  is dominated either by a policy which does nothing in  $s$  or by a policy which repairs  $m - 1$  units in  $s \Rightarrow R_{\Omega_s}$ ,  $|\Omega_s| = m$  is never optimal.

Now for the specifics:

Let  $s = (i_1, i_2, \dots, i_{k'})$ ;  $|s| = k' > k$  ( $k$  of  $n$  system)

$\Omega_s = (j_1, j_2, \dots, j_m)$ ;  $|\Omega_s| = m > 1$ .

Now

$$V_{0, R_{\Omega_s}}^s = V_{0, A}^{s \cup \Omega_s} + \sum_{p=1}^m K_{j_p} \quad (\text{since instantaneous repair})$$

$$(9) \quad = \sum_{p=1}^{k'} \frac{\lambda_{i_p}}{\sum_{q=1}^{k'} \lambda_{i_q} + \sum_{q=1}^m \lambda_{j_q}} V_{0, A}^{s \cup \Omega_s - i_p} + \sum_{p=1}^m \frac{\lambda_{j_p}}{\sum_{i_q} \lambda_{i_q} + \sum_{j_q} \lambda_{j_q}} V_{0, A}^{s \cup \Omega_s - j_p}$$

$$+ \sum_{p=1}^m K_{j_p} - \frac{V-1}{\sum_{i_q} \lambda_{i_q} + \sum_{j_q} \lambda_{j_q}} \quad \text{by (5) with } t = s \cup \Omega_s .$$

Let

$$(10) \quad z_s(j) \triangleq 1$$

$$z_s(j_1, j_2) \triangleq 1 + \frac{\lambda_{j_1}}{\lambda_{j_2} + \sum_{i_q} \lambda_{i_q}} + \frac{\lambda_{j_2}}{\lambda_{j_1} + \sum_{i_q} \lambda_{i_q}}$$

$$\vdots$$

$$z_s(\Omega_s) \triangleq 1 + \sum_{p=1}^m \frac{\lambda_{j_p}}{\sum_{\substack{q=1 \\ q \neq p}}^m \lambda_{j_q} + \sum_{i_q} \lambda_{i_q}} z_s(\Omega_s \sim p) .$$

Note that by (5) and instantaneous repair,

$$V_{0, A \rightarrow R_{\Omega_s}}^s = \sum_{p=1}^{k'} \frac{\lambda_{i_p}}{\sum_{i_q} \lambda_{i_q}} V_{0, A}^{s \cup \Omega_s - i_p} + \sum_{p=1}^m K_{j_p} - \frac{V-1}{\sum_{i_q} \lambda_{i_q}}$$

where  $A \rightarrow R_{\Omega_s}$  denotes doing nothing in  $s$  but  $R_{\Omega_s}$  at the first transition after  $s$ .

If in (9), one repeatedly applies (5) to those  $V_0^t$  terms left on the RHS for which  $t \supset s$  until there are none such left, then brings all coefficients of  $V_{0,R_{\Omega_s}}^s$  over to the left hand side, and multiplies the whole equation by

$$\frac{\sum \lambda_{i_q} + \sum \lambda_{j_q}}{\sum \lambda_{i_q}},$$

the result is: (using (4) to combine terms into  $V_{0,R_{\Omega_s \sim j_p}}^s$ )

$$V_{0,R_{\Omega_s}}^s \cdot z_s(\Omega_s) = \sum_{p=1}^k \frac{\lambda_{i_p}}{\sum \lambda_{i_q}} V_{0,R_{\Omega_s \sim i_p}}^{s \cup \Omega_s} + \left( 1 + \frac{\sum_{q=1}^m \lambda_{j_q}}{\sum \lambda_{i_q}} \right) \sum_{p=1}^m K_{j_p}$$

$$- \frac{V_{-1}}{\sum \lambda_{i_q}} + \sum_{p=1}^m \frac{\lambda_{j_p}}{\sum \lambda_{i_q} + \sum_{q \neq p} \lambda_{j_q}} V_{0,R_{\Omega_s \sim j_p}}^s \cdot z_s(\Omega_s \sim j_p)$$

$$- \frac{\sum_p \sum_q \lambda_{i_p} K_{i_q}}{\sum \lambda_{i_q}}.$$

⇒

$$(10') \quad v_{0, R_{\Omega_s}}^s \cdot z_s(\Omega_s) = v_{0, A \rightarrow R_{\Omega_s}}^s + \sum_{p=1}^m \frac{\lambda_{j_p}}{\sum_{q \neq p} \lambda_{i_q} + \sum_{q \neq p} \lambda_{j_q}} v_{0, R_{\Omega_s - j_p}}^s$$

$$\cdot z_s(\Omega_s \sim j_p) + \frac{\sum_{p=1}^m \lambda_{j_p} K_{j_p}}{k' \sum_{q=1}^m \lambda_{i_q}}$$

⇒

$$\left( v_{0, R_{\Omega_s}}^s - v_{0, A \rightarrow R_{\Omega_s}}^s \right) + \sum_{p=1}^m \frac{\lambda_{j_p}}{\sum_{q \neq p} \lambda_{i_q} + \sum_{q \neq p} \lambda_{j_q}} z_s(\Omega_s \sim j_p) \left( -v_{0, R_{\Omega_s}}^s - v_{0, R_{\Omega_s - j_p}}^s \right)$$

$$+ \frac{\sum_{p=1}^m \lambda_{j_p} K_{j_p}}{k' \sum_{q=1}^m \lambda_{i_q}}$$

by the recursive  
definition  
(10) of  $z_s(\cdot)$

> 0 .

Thus, at least one of the following must be true:

- 0.  $v_{0, R_{\Omega_s}}^s - v_{0, A \rightarrow R_{\Omega_s}}^s > 0$
- 1.  $v_{0, R_{\Omega_s}}^s - v_{0, R_{\Omega_s - j_1}}^s > 0$
- 
- 
- 
- m.  $v_{0, R_{\Omega_s}}^s - v_{0, R_{\Omega_s - j_m}}^s > 0$

If 0., then  $V_{0, R_{\Omega_s}}^s > V_{0, A+R_{\Omega_s}}^s \Rightarrow$  not optimal  
 j., then  $V_{0, R_{\Omega_s}}^s > V_{0, R_{\Omega_s} - j}^s \Rightarrow$  not optimal.  $\square$

Note: In the previous Lemma 3.2, the assumption  $k' = |s| > k$ , the type of system, is crucial. If  $k' = k$ , then the policy  $A \rightarrow R_{\Omega_s}$  (let a component fail, then repair same set of components at next transition) cannot be used without incurring the penalty cost for system failure. In this case,  $V_{0, A+R_{\Omega_s}}^s$  will have a "p" term in it, causing the proof of Lemma 3.2 to break down. We can now state the following theorem:

Theorem 3.3: If we have a k-of-n system and the Basic Model, then it is never optimal to repair until you get down to k items left functioning.

Proof: Consider  $s : |s| = k', k + 1 \leq k' \leq n$ . If  $k' = n - 1$ , we know it is never optimal to repair  $> 1$  unit at a time since only one unit is down.

Then, Lemma 3.1  $\Rightarrow$  "A" optimal for  $s : |s| = n - 1$   
 (also for  $k' = n$  since none down)  
 Lemma 3.2  $\Rightarrow$  never  $R > 1$  for  $k' = n - 2$   
 Lemma 3.1  $\Rightarrow$  "A" optimal for  $s : |s| = n - 2$ .

Continue alternating applications of Lemmas 3.1 and 3.2 until  $k' = k + 1$  is reached and then we are done, i.e., have shown "A" optimal for  $s : |s| = k' > k$  or do nothing until  $k$  - left.  $\square$

Using the results of Chapter II and Theorem 3.3, we now have a pretty good idea of the optimal policy for a  $k$ -of- $n$  system. Since one does nothing in states  $s : |s| > k$  and will never reach any states  $s : |s| < k - 1$ , only states  $s : |s| = k - 1, k$  need be considered. From Example 2.5, possible ergodic chains are  $C_s = \{s \sim i; i \in s\}; s : |s| = k + 1$  or  $k$ . For states in  $C_s$ , the optimal decisions have been previously described,  $R_i$  in  $s \sim i$ . In other states, decisions are restricted so as to form no other ergodic chains. Note that for non-ergodic states of sizes  $k - 1, k$ , decisions have not been restricted to one unit of repair at a time.

It would be nice to be able to also say that in the remaining transient states, it is optimal to never repair more than one unit at a time. For  $k = 1$  (the parallel case), this is almost true by the next Theorem 3.4.

Theorem 3.4: Given the Basic Model and a parallel system of components, (a) it is never optimal to repair more than one unit at a time in a state  $s : |s| = 1$ , (b) it is also never optimal to repair more than two at a time in the failed state 0.

Proof: We do (b) first.

Let {components repaired in state 0} =  $\Omega_0$ ,  $|\Omega_0| = m$ .

$$= \{j_1, j_2, \dots, j_m\}.$$

Want to show  $m \geq 3$  is never optimal. Take  $m \geq 3$ .

$$\begin{aligned} (11) \quad v_{0, R_{\Omega_0}}^0 &= \sum_{p=1}^m K_{j_p} + p + v_{0, A}^{\Omega_0} \\ &= p + \sum_{p=1}^m K_{j_p} + \sum_{p=1}^m \frac{\lambda_{j_p}}{\sum_{q=1}^m \lambda_{j_q}} v_{0, A}^{\Omega_0 \sim j_p} - \frac{v_{-1}}{\sum_{q=1}^m \lambda_{j_q}} \end{aligned}$$

(since  $|\Omega_0 \sim j_p| \geq 2$ , "A" is optimal there by Theorem 3.3)

$$(12) \quad v_{0, R_{\Omega_0 \sim j_p}}^0 = \sum_{\substack{q=1 \\ q \neq p}}^m K_{j_q} + p + v_{0, A}^{\Omega_0 \sim j_p}$$

$$\text{or} \quad v_{0, A}^{\Omega_0 \sim j_p} = v_{0, R_{\Omega_0 \sim j_p}}^0 - p - \sum_{\substack{q=1 \\ q \neq p}}^m K_{j_q}.$$

Substituting (12) into (11) gives

$$v_{0, R_{\Omega_0}}^0 = \sum_{p=1}^m \frac{\lambda_{j_p}}{\sum_{q=1}^m \lambda_{j_q}} v_{0, R_{\Omega_0 \sim j_p}}^0$$

$$\begin{aligned}
& + \frac{1}{\sum_{q=1}^m \lambda_{j_q}} \left[ \sum_{q=1}^m \lambda_{j_q} K_{j_q} - V_{-1} \right] \\
& \geq \sum_{p=1}^m \frac{\lambda_{j_p}}{\sum_{q=1}^m \lambda_{j_q}} V_{0, R_{\Omega_0}^{-j_p}}
\end{aligned}$$

since

$$V_{-1} \leq \min_{j,k} \lambda_j K_j + \lambda_k K_k \leq \sum_{q=1}^m \lambda_{j_q} K_{j_q}$$

$\Rightarrow$

$V_{0, R_{\Omega_0}}^0 \geq V_{0, R_{\Omega_0}^{-j_p}}^0$ , at least one  $p$ . Setting  $m = 3$  here

proves  $m = 3$  is never optimal. Given  $m - 1$  is never optimal,  $m$  is never optimal by the above argument so (b) is true by induction.

Now to prove (a). Suppose we are in state  $\{i\} = s$ .

Let  $\Omega_i =$  set of components repaired in  $i$   
 $= \{j_1, j_2, \dots, j_m\}$ .

The object is to show that an  $m > 1$  decision is never optimal. The method will be to show  $V_{0, R_{\Omega_i}}^i$  to be  $\geq$  a certain convex combination

of  $v_{0,R_{j_q}}^i$   $q = 1, \dots, m$ . This is sufficient to prove

$v_{0,R_{\Omega_i}}^i \geq v_{0,R_{j_q}}^i$ , some  $i \leq q \leq m$  which proves the desired result

that repair of more than one unit at a time in a state  $\{i\}$  is never

optimal. It now remains to show  $v_{0,R_{\Omega_i}}^i \geq$  convex combination of

$v_{0,R_{j_q}}^i$  :

$$\text{Let } \eta_{j_q} = \frac{\lambda_i + \sum_{p=1}^m \lambda_{j_p}}{\lambda_i} \cdot P\{\Omega_s \cup i \rightarrow j_q \mid \text{no repairs}\}$$

= Probability that if one starts with components

$\Omega_s \cup i$  working and does no repairs, that

component  $j_q$  is the last one left.

Notice that  $\sum_{q=1}^m \eta_{j_q} = z_i(\Omega_i)$  as defined in (10). Define  $\bar{\eta} = \sum_{q=1}^m \eta_{j_q}$ .

Our desired convex combination is

$$\begin{aligned} (13) \quad & \sum_{q=1}^m \frac{\eta_{j_q}}{\bar{\eta}} v_{0,R_{j_q}}^i \\ &= \sum_{q=1}^m \frac{\eta_{j_q}}{\bar{\eta}} \left( v_0^{j_q} + \frac{\lambda_i + \lambda_{j_q}}{\lambda_i} \cdot K_{j_q} - \frac{v-1}{\lambda_i} \right) \quad \text{using} \\ &= \sum_{q=1}^m \frac{\eta_{j_q}}{\bar{\eta}} v_0^{j_q} + \frac{1}{\bar{\eta}} \sum_{q=1}^m \frac{\eta_{j_q} (\lambda_i + \lambda_{j_q})}{\lambda_i} \cdot K_{j_q} - \frac{v-1}{\lambda_i} \quad (4) \end{aligned}$$



$$b_i(j_p j_q, \Omega_i) = \frac{\prod_{r=1}^m \lambda_{j_r}}{\lambda_{j_p} + \lambda_{j_q}} \left[ \sum_{\substack{r_3=1 \\ \neq p,q}}^{m+1} \frac{1}{\lambda_{j_p} + \lambda_{j_q} + \lambda_{j_{r_3}}} \sum_{\substack{r_4=1 \\ \neq p,q,r_3}}^{m+1} \frac{1}{\lambda_{j_p} + \lambda_{j_q} + \sum_{t=3}^4 \lambda_{j_{r_t}}} \right. \\ \left. \sum_{\substack{r_5=1 \\ \neq p,q,r_3,r_4}}^{m+1} \frac{1}{\lambda_{j_p} + \lambda_{j_q} + \sum_{t=3}^5 \lambda_{j_{r_t}}} \dots \sum_{\substack{r_m=1 \\ \neq p,q,r_3,\dots,r_{m-1}}}^{m+1} \frac{1}{\lambda_{j_p} + \lambda_{j_q} + \sum_{t=3}^m \lambda_{j_{r_t}}} \right].$$

If  $\Lambda_i \subseteq \Omega_i$ , order  $j_1, \dots, j_m$  so that  $\{j_1, \dots, j_k\} = \Lambda_i, k \leq m$ .

Define

$$b_i(\Lambda_i, \Omega_i) = \quad (\text{set } j_{m+1} = i)$$

$$= \frac{\prod_{r=k+1}^m \lambda_{j_r}}{\sum_{r=1}^k \lambda_{j_r}} \left[ \sum_{r_{k+1}=k+1}^{m+1} \frac{1}{\sum_{r=1}^k \lambda_{j_r} + \lambda_{j_{r_{k+1}}}} \sum_{\substack{r_{k+2}=k+1 \\ \neq r_{k+1}}}^{m+1} \frac{1}{\sum_{r=1}^k \lambda_{j_r} + \sum_{t=k+1}^{k+2} \lambda_{j_{r_t}}} \right. \\ \left. \sum_{\substack{r_{k+3}=k+1 \\ \neq r_{k+1}, r_{k+2}}}^{m+1} \frac{1}{\sum_{r=1}^k \lambda_{j_r} + \sum_{t=k+1}^{k+3} \lambda_{j_{r_t}}} \dots \sum_{\substack{r_m=k+1 \\ \neq r_{k+1}, \dots, r_{m-1}}}^{m+1} \frac{1}{\sum_{r=1}^k \lambda_{j_r} + \sum_{t=k+1}^m \lambda_{j_{r_t}}} \right].$$

Note

$$\begin{aligned}
 y(\Omega_i \cup 1) &= z_i(\Omega_i) + \lambda_i \sum_{\substack{s \in \Omega_i \\ |s| \geq 2}} b_i(s, \Omega_i) \\
 &= \bar{n} + \lambda_i \sum_{\substack{s \in \Omega_i \\ |s| \geq 2}} b_i(s, \Omega_i)
 \end{aligned}$$

so,

$$\begin{aligned}
 (14) \quad v_{0, R_{\Omega_i}}^i &= \sum_{p=1}^m \frac{\eta_{j_p}}{\bar{n}} v_0^{j_p} + \left( \sum_{p=1}^m K_{j_p} \right) \frac{\lambda_i + \sum_{p=1}^m \lambda_{j_p}}{\bar{n} \cdot \lambda_i} \\
 &\quad - \frac{v_{-1}}{\lambda_i} - \frac{v_{-1}}{\bar{n}} \sum_{\substack{s \in \Omega_i \\ |s| \geq 2}} b_i(s, \Omega_i) .
 \end{aligned}$$

want to show (14)-(13)  $\geq 0$ .

$$\begin{aligned}
 (14)-(13) &= \frac{1}{\lambda_i \bar{n}} \left[ \sum_{p=1}^m K_{j_p} \left( \lambda_i + \sum_{q=1}^m \lambda_{j_q} - \eta_{j_p} (\lambda_i + \lambda_{j_p}) \right) \right. \\
 &\quad \left. - v_{-1} \cdot \lambda_i \sum_{\substack{s \in \Omega_i \\ |s| \geq 2}} b_i(s, \Omega_i) \right] .
 \end{aligned}$$

A lot of algebraic manipulation leads to:

$$(15) \quad K_{j_p} \left( \lambda_i + \sum_{q=1}^m \lambda_{j_q} - \eta_{j_p} (\lambda_i + \lambda_{j_p}) \right)$$

$$= \lambda_{j_p} K_{j_p} (\lambda_i + \lambda_{j_p}) \sum_{\substack{j_p \in S \\ |s| \geq 2}} b_i(s, \Omega_i) .$$

Thus (14)-(13)

$$= \frac{1}{\lambda_i^n} \left[ \sum_{p=1}^m \lambda_{j_p} K_{j_p} (\lambda_i + \lambda_{j_p}) \sum_{\substack{j_p \in S \\ |s| \geq 2}} b_i(s, \Omega_i) - v_{-1} \cdot \lambda_i \sum_{\substack{j_p \in S \\ |s| \geq 2}} b_i(s, \Omega_i) \right]$$

$$= \frac{1}{\lambda_i^n} \sum_{p=1}^m \lambda_{j_p}^2 K_{j_p} \sum_{\substack{j_p \in S \\ |s| \geq 2}} b_i(s, \Omega_i) \geq 0 \quad \text{by definition}$$

$$+ \frac{1}{n} \left[ \sum_{p=1}^m \lambda_{j_p} K_{j_p} \sum_{\substack{j_p \in S \\ |s| \geq 2}} b_i(s, \Omega_i) - v_{-1} \cdot \sum_{\substack{j_p \in S \\ |s| \geq 2}} b_i(s, \Omega_i) \right]$$

$$\geq \frac{1}{n} \left[ \sum_{p=1}^m \lambda_{j_p} K_{j_p} \sum_{\substack{j_p \in S \\ |s| \geq 2}} b_i(s, \Omega_i) - v_{-1} \cdot \sum_{\substack{j_p \in S \\ |s| \geq 2}} b_i(s, \Omega_i) \right]$$

$$= \frac{1}{n} \left[ \sum_{\substack{j_p \in S \\ |s| \geq 2}} b_i(s, \Omega_i) \left\{ \sum_{\substack{k \in S \\ |s| \geq 2}} \lambda_k K_k - v_{-1} \right\} \right]$$

$$\geq 0 \quad \text{since} \quad \sum_{\substack{k \in S \\ |S| \geq 2}} \lambda_k K_k \geq V_{-1} . \quad \square$$

It would be false to say that one never repairs more than one at a time in state zero.

<u>Example:</u>	<u>n = 3</u>	<u>parallel model</u>	
	$\lambda_1 = 3$	$\lambda_2 = 2$	$\lambda_3 = .5$
	$K_1 = 1$	$K_2 = 2$	$K_3 = 20$
		$p = 2$	

$$V_{-1} = \min \left\{ \min_{i=1,2,3} \lambda_i (K_i + p), \min_{(i,j) = 12,13,23} \lambda_i K_i + \lambda_j K_j \right\}$$

$$= \min\{9, 8, 11, 7, 13, 14\} = 7 .$$

So,  $\mathcal{E} = \{1, 2\}$  and  $\delta_1^{\text{opt}} = R_2$

$$\delta_2^{\text{opt}} = R_1$$

Theorem 3.3  $\Rightarrow \delta_{12}^{\text{opt}} = \delta_{13}^{\text{opt}} = \delta_{23}^{\text{opt}} = A$

Theorem 3.4  $\Rightarrow \delta_3^{\text{opt}} \in \{R_1, R_2, A\}$   
 +  
 need single ergodic chain  $\delta_0^{\text{opt}} \in \{R_{12}, R_{13}, R_{23}\} .$

- From state 3, you can
- (i)  $R_1$
  - (ii)  $R_2$
  - (iii)  $A \rightarrow R_{12}, R_{13}$  or  $R_{23}$

$$V_{0,R_1}^3 = -5 \qquad V_{0,R_2}^3 = -3 \qquad V_{0,A \rightarrow R_{12}}^3 = -7\frac{5}{6}$$

(i), (ii) are not optimal, giving an example of a problem for which it is optimal to repair two items in state zero. Notice the negative  $V_0$ 's. Although a  $V_0$  optimal policy minimizes the total expected cost, the  $V_0$ 's themselves are not the total expected cost given a starting state for that would have to be  $\geq 0$  since there are only costs in this model, no benefits.

The remaining question is: Does Theorem 3.4 hold for  $k$ -of- $n$  systems when  $k > 1$ ? In general, the problem is still open. There

is no "nice" convex combination of  $V_{0,R_j}^{i_1 i_2 \dots i_k}$ 's which is

smaller than  $V_{0,R_s}^{i_1 i_2 \dots i_k}$  given  $m \geq 2$  for  $k > 1$  as in

Theorem 3.4. We can say, that if  $p$  is small enough, part (a) of Theorem 3.4 holds for  $k > 1$  using the method of proof of Lemma 3.2.

Lemma 3.5: Suppose we have the Basic Model and a  $k$ -of- $n$  system. Then, if  $p$ , the penalty cost, is less than

$$\frac{\sum_{p=1}^k \lambda_{i_p} K_{i_p}}{\lambda_{i_1} + \dots + \lambda_{i_k}} \qquad \text{where} \qquad \lambda_{i_1} K_{i_1} < \lambda_{i_2} K_{i_2} < \dots < \lambda_{i_n} K_{i_n}$$

it is never optimal to repair more than one unit at a time in a state  $s : |s| = k$ .

Proof: Pick  $s = (i_1, i_2, \dots, i_k)$ ,  $\Omega_s = (j_1, j_2, \dots, j_m)$  .  
 $m > 1$

The proof of Lemma 3.2 holds exactly up to Equation (10') where a "- p" term must be added to the RHS due to system failure when following the policy  $A \rightarrow \Omega_s$ . The RHS is now only

$$> 0 \Leftrightarrow p < \frac{\sum_{j=1}^m \lambda_j K_j}{\sum_{q=1}^{k'} \lambda_{i_q}}$$

which is true given the assumption on  $p$ . □

In the next section, we look at the parallel case in detail; coming up with an exact optimal policy in some cases, by defining optimal decisions in states  $0, 1, \dots, n$  which are transient.

One final note with respect to fixed charges. Recall Theorem 3.3 which states that for a  $k$ -of- $n$  system, it is never optimal to do anything until there are  $k$  or less components working. The proof of this theorem (and thus, those of Lemmas 3.1 and 3.2) does not require  $L = 0$  and, in fact, goes through with only slight modifications to include an "L". Theorem 3.3 is restated below now including specific reference to the fixed charge case:

Theorem 3.3 (restated): If we have a  $k$ -of- $n$  system and the Basic Model with some fixed charge  $L \geq 0$  for repair, then it is never optimal to repair until you get down to  $k$  items left functioning.

Proof: Same as for Lemmas 3.1, 3.2, Theorem 3.3, only with an "L" added.

## 2. The Parallel Case

In Chapter II, a  $V_{-1}$  optimal policy was found for the Basic Model and any general coherent system. Unfortunately, a  $V_{-1}$  optimal policy is only unique up to what goes on in the ergodic states which, at least for the Basic Model, comprise very few of the total possible number of states. Thus, to break ties among  $V_{-1}$  optimal policies, and find the optimal decision in transient states,  $V_0$  was looked at in the previous section for  $k$ -of- $n$  systems. Using it, the possible decision space was restricted significantly, but the exact optimal policy given any possible values for  $p$ ,  $K_1$ 's, and  $\lambda_1$ 's was too complicated computationally to be found. Given specific parameter values, optimal policies can be found using linear programming or policy improvement algorithms on the computer as in Chapter VI.

There is one case, that of the parallel system ( $k = 1$ ), where a general optimal policy form can be found for certain parameter values. Such results being the most desirable, this

whole section will be devoted to the further restrictions obtainable on optimal policies in the parallel case using the Basic Model. General optimal policies will be obtained where possible.

Consider a parallel system with  $n$  independent components:  $1, 2, \dots, n$ , with exponential parameters  $\lambda_1 > \lambda_2 > \dots > \lambda_n$ , i.e., order them by increasing mean lifetime. Let  $i_1, i_2, \dots, i_n$  and  $j_1, j_2, \dots, j_n$  represent orderings of the components by increasing repair costs  $K_{i_1} < K_{i_2} < \dots < K_{i_n}$  and by  $\lambda_{j_1} K_{j_1} < \lambda_{j_2} K_{j_2} < \dots < \lambda_{j_n} K_{j_n}$  for repair cost/mean lifetime ratios respectively.

By the results of Chapter II, we know that

$$V_{-1} = \min \left\{ \begin{array}{l} \min_{i=1, \dots, n} \lambda_i (K_i + p), \quad \min_{\substack{(i,j) \in \\ 1, \dots, n \\ i+j}} (\lambda_i K_i + \lambda_j K_j) \end{array} \right\}$$

$$= \min \left\{ \begin{array}{l} \min_{i=1, \dots, n} \lambda_i (K_i + p); \quad \lambda_{j_1} K_{j_1} + \lambda_{j_2} K_{j_2} \end{array} \right\}.$$

There are two types of optimal policy structures:

(I)  $V_{-1} = \lambda_i (K_i + p)$ , some  $i$ . Then  $\mathcal{E} = \{\text{ergodic states}\} = \{0\}$   $\delta_0^{\text{opt}} = R_i$ . One then has to determine  $\delta_0^{\text{opt}}$  for  $i = 1, \dots, n$ . We know that  $\delta_i^{\text{opt}} = R_{p(i)}$  (where  $R_0 = A$ ) since it is never optimal to repair more than one unit at a time in  $i$  (Theorem 3.4). Thus, if start in  $i$  and  $R_{p(i)}$ , the next new state entered will be  $p(i)$ .

Define a policy  $\delta(i)$  by:

$$(1) \quad \begin{aligned} \delta(i) &= R_{p(i)} \\ \delta(p(i)) &= R_{p^2(i)} \\ &\vdots \\ \delta(p^{\ell}(i)) &= R_{p^{\ell+1}(i)} \\ &\vdots \\ &\vdots \end{aligned}$$

where  $p^{k+1}(i) = 0$  ( $\delta(p^k(i)) = A$ ), some  $k < n$  and  $p^q(i) \neq p^r(i)$  if  $r \neq q$  since  $\mathcal{E} = \{0\}$  must be the single irreducible ergodic chain.

$$(II) \quad V_{-1} = \lambda_{j_1} K_{j_1} + \lambda_{j_2} K_{j_2}. \quad \text{Then}$$

$$\mathcal{E} = \{j_1, j_2\} \quad \delta_{j_1}^{\text{opt}} = R_{j_2} \quad \delta_{j_2}^{\text{opt}} = R_{j_1}.$$

Must determine  $\delta_i^{\text{opt}}$ ,  $i \neq j_1, j_2$  and  $\delta_0^{\text{opt}}$ .

$$\delta_i^{\text{opt}} = R_{p(i)} \quad \text{as before where } R_0 = A \quad \text{and } 0 < p(i) \leq n.$$

$$\delta_0^{\text{opt}} = R_{p_1 p_2} \quad \text{since repair of a single unit would produce two}$$

ergodic chains and repair of three or more units is impossible by

Theorem 3.4 (b).

First, let's examine policies of type (I). Suppose

$V_{-1} = \lambda_{\ell} (K_{\ell} + p)$ . A policy  $\delta$  can be characterized by partitioning

the set  $\{1, \dots, n\}$  of components into  $m$  subsets where  $m =$  number of  $i$  such that  $R_i$  in any state  $j$ . Call these components  $a_1, a_2, \dots, a_m$ , ordered by  $a_1 < a_2 < \dots < a_m$ . Let  $c_i = \{1 \leq j \leq n \mid p^k(a_i) = j, \text{ some } k\}$ . Within each set  $c_i$ , we can find an ordering  $a_i, p(a_i), p^2(a_i), \dots, p^k(a_i)$  some  $k$ ,  $p^{k+1}(a_i) = 0$ . Order the elements of  $c_i$  in such a fashion  $v_i$ .  $c_1 \cup \dots \cup c_m = \{1, \dots, n\}$  but  $c_i \cap c_j \neq \emptyset$  necessarily if  $i \neq j$ . Now, let  $\delta = c_1 c_2 \dots c_m$  and suppose we are in state  $i$  (the first state hit for which one component is left after doing nothing since all components were up). Let  $p$  be defined as in (1).

$V_0^\delta$  solves

$$(2) \quad (I - Q_0^\delta) V_0^\delta = R_0^\delta - Q_1^\delta V_1^\delta$$

and

$$; V_{-1}^\delta = (K_\ell + p) \lambda_\ell .$$

$$(3) \quad P_\delta^* Q_1^\delta V_0^\delta = P_\delta^* Q_2^\delta V_{-1}^\delta$$

Using (2) for  $s \in \mathcal{E}$  with the single equation (3) gives the solution (unique) for  $V_0^s$ ,  $s \in \mathcal{E}$ . This is the same for all possible  $\delta$ 's since the  $\delta$ 's being considered are already  $V_{-1}$  optimal. Thus, equations (2) for  $s \notin \mathcal{E}$  (transient states) are sufficient to differentiate between  $\delta$  and some other  $V_{-1}$  optimal policy. By Veinott [31], minimizing  $V_0^s$  for one  $s$  does it for all  $s$ . Choose  $s = \{i\}$ . As the  $s \in \{j\}^{\text{th}}$  equation of (2) will never depend on

$t : |t| > 1$ , only the rows  $1, 2, \dots, n$  of  $I - Q_0^\delta$  are needed for computing  $V_0^i$ , any  $i$ .

$I - Q_0^\delta$ :		Columns					
		$ s  > 1$	$i$	$j$	$0$	$R_0$	$Q_1 \cdot 1$
row	decision						
$i$	A	$Q$	1	0	-1	0	$1/\lambda_i$
$i$	$R_j$	$Q$	$\frac{\lambda_i}{\lambda_i + \lambda_j}$	$\frac{-\lambda_i}{\lambda_i + \lambda_j}$	0	$K_j$	$1/(\lambda_i + \lambda_j)$

Equation (3)  $\Rightarrow V_0^0 = K_\ell + p$ .

Let

$$z_\delta = \begin{bmatrix} I - Q_0^\delta \end{bmatrix} \quad \begin{array}{l} \text{rows } s = 1, 2, \dots, n \\ \text{columns } s = 1, 2, \dots, n \end{array}$$

$z_\delta$  is invertible (elements given previously).

Equation (2)  $\Rightarrow$  for  $i = 1, \dots, n$ ,

$$(4) \quad V_0^\delta = z_\delta^{-1} (R_0^\delta - Q_1^\delta V_{-1}^\delta).$$

Let  $z_{\delta,i}^{-1} = i^{\text{th}}$  row of  $z_\delta^{-1}$  then

$$(4i) \quad V_{0,\delta}^i = z_{\delta,i}^{-1} \cdot (R_{0,i}^\delta - Q_{1,i}^\delta V_{-1}^\delta) .$$

If  $\delta(i) = A$ , then

$$z_{\delta,i}^{-1} = \langle 0 \cdots 0 \overset{i}{1} 0 \cdots 0 \rangle, R_{0,i}^\delta - Q_{1,i}^\delta V_{-1}^\delta = -\frac{V_{-1}}{\lambda_i}$$

or

$$V_0^i = -\frac{V_{-1}}{\lambda_i}.$$

If  $i \in c_j$ , some  $j$  with

$$\begin{aligned} \delta(i) &= R_{p(i)} \\ \delta(p(i)) &= R_{p^2(i)} \\ &\vdots \\ \delta(p^{k-1}(i)) &= R_j \\ \delta(p^k(i) = j) &= A \quad \text{as in (1),} \end{aligned}$$

then  $z_{\delta,i}^{-1}$

$$= \langle 0 \cdots 0 \frac{\overset{i}{\lambda_i + \lambda} p(i)}{\lambda_i} \frac{\overset{p(i)}{\lambda p(i) + \lambda} p^2(i)}{\lambda p(i)} \frac{\overset{p^2(i)}{\lambda p^2(i) + \lambda} p^3(i)}{\lambda p^2(i)} \cdots \frac{\overset{p^{k-1}(i)}{\lambda p^{k-1}(i) + \lambda} p^k(i)=j}{\lambda p^{k-1}(i)} 1 \ 0 \cdots 0 \rangle$$

$(R_0^\delta - Q_1^\delta V_1^\delta)^c$

$$= \left\langle K_{p(i)} - \frac{\overset{i}{V_{-1}}}{\lambda_i + \lambda p(i)}, K_{p^2(i)} - \frac{\overset{p(i)}{V_{-1}}}{\lambda p(i) + \lambda p^2(i)} \cdots K_j - \frac{\overset{p^{k-1}(i)}{V_{-1}}}{\lambda p^{k-1}(i) + \lambda_j}, \frac{\overset{p^k(i)=j}{-V_{-1}}}{\lambda_j} \right\rangle$$

so,

$$(5) \quad v_{0,\delta}^i = \sum_{n=1}^k \left[ \left( \frac{\lambda p^{n-1}(i) + \lambda p^n(i)}{\lambda p^{n-1}(i) \lambda p^n(i)} \right) \lambda p^n(i) K_{p^n(i)} - \frac{v_{-1}}{\lambda p^n(i)} \right] - \frac{v_{-1}}{\lambda_i}$$

or

$$v_{0,\delta}^i = \sum_{n=1}^k \left[ \left( \mu_{p^{n-1}(i)} + \mu_{p^n(i)} \right) \cdot \frac{K_{p^n(i)}}{\mu_{p^n(i)}} - v_{-1} \cdot \mu_{p^n(i)} \right] - \mu_i v_{-1}$$

where  $\mu_i = \frac{1}{\lambda_i}$  = mean lifetime of component  $i$ .

We can now compare various policies, starting from state  $i$ :

Definition: Let  $i \rightarrow p(i) \rightarrow p^2(i) \rightarrow \dots \rightarrow p^k(i) = j \rightarrow 0$   
denote the policy which has  $\delta(p^{\ell-1}(i)) = R_{p^\ell(i)}$ .

Lemma 3.6: For the Basic Model, parallel case with no fixed charge, if  $\lambda_i > \lambda_j$  and  $K_i > K_j$  for some  $i, j$ , then unit  $i$  will never be repaired in an optimal policy.

Proof: First, we will show that one never repairs number  $i$  in the ergodic states and then in the transient states. To use in an ergodic state, we need

$$\mathcal{E} = \{0\}, R_i \text{ in } 0 \quad (V_{-1} = \lambda_i(K_i + p))$$

or

$$\mathcal{E} = \{ik\}, R_i \text{ in } k \quad (V_{-1} = \lambda_i K_i + \lambda_k K_k) .$$

$$R_k \text{ in } i$$

Note  $\lambda_i > \lambda_j, K_i > K_j \Rightarrow \lambda_i K_i > \lambda_j K_j$ . But  $\lambda_i K_i > \lambda_j K_j$  and  $\lambda_i > \lambda_j$  (so  $\lambda_i p > \lambda_j \cdot p$  for  $p > 0$ )

$\Rightarrow \lambda_i(K_i + p) > \lambda_j(K_j + p)$  so  $\mathcal{E} = \{0\}_{R_i \text{ in } 0}$  loses

to  $\mathcal{E} = \{0\}_{R_j \text{ in } 0}$  and is never  $V_{-1}$  optimal.

Similarly, since  $\lambda_i K_i > \lambda_j K_j, \lambda_i K_i + \lambda_k K_k > \lambda_j K_j + \lambda_k K_k$

$\Rightarrow \mathcal{E} = \{jk\}$  beats  $\mathcal{E} = \{ik\}$

$\Rightarrow$  never repair component  $i$  in an ergodic state.

For the transient states  $\{\ell\}$ , let

$$\left. \begin{array}{l} \delta : R_i \text{ in } \ell, \quad \text{then} \\ \gamma : R_j \text{ in } \ell, \quad \text{then} \end{array} \right\} \begin{array}{l} \text{follow same policy} \\ a_1 \rightarrow a_2 \rightarrow \dots \rightarrow a_k \rightarrow 0 . \end{array}$$

Using (5), we get if  $a_1 = 0$   $\left. \begin{array}{l} \text{then} \\ \delta = \ell \rightarrow i \rightarrow 0 \\ \gamma = \ell \rightarrow j \rightarrow 0 \end{array} \right\}$

$$\delta < \gamma \Leftrightarrow \left[ (\mu_\ell + \mu_i) \frac{K_i}{\mu_i} - v_{-1} \cdot \mu_i \right] < \left[ (\mu_\ell + \mu_j) \frac{K_j}{\mu_j} - v_{-1} \cdot \mu_j \right]$$

which never occurs since  $\frac{K_i}{\mu_i} > \frac{K_j}{\mu_j}$  and  $\mu_i < \mu_j$  and  $K_i > K_j$

so,  $l \rightarrow i \rightarrow 0$  never optimal.

Now suppose  $a_1 \neq 0, a_2 = 0$ . Again using (5),  $\delta < \gamma \Leftrightarrow v_0^\delta < v_0^\delta$

$$\Leftrightarrow \left[ (\mu_l + \mu_i) \frac{K_i}{\mu_i} - v_{-1} \cdot \mu_i \right] + \left[ (\mu_i + \mu_{a_1}) \frac{K_{a_1}}{\mu_{a_1}} - v_{-1} \cdot \mu_{a_1} \right]$$

$$< \left[ (\mu_l + \mu_j) \frac{K_j}{\mu_j} - v_{-1} \cdot \mu_j \right] + \left[ (\mu_i + \mu_{a_1}) \frac{K_{a_1}}{\mu_{a_1}} - v_{-1} \cdot \mu_{a_1} \right]$$

$$(6) \Leftrightarrow \underbrace{\mu_l \left( \frac{K_i}{\mu_i} - \frac{K_j}{\mu_j} \right)}_{> 0} + \underbrace{K_i - K_j}_{> 0} + \underbrace{(\mu_j - \mu_i)}_{> 0} \left( v_{-1} - \frac{K_{a_1}}{\mu_{a_1}} \right) < 0 .$$

Notice that, if  $v_{-1} \geq \frac{K_{a_1}}{\mu_{a_1}}$ , then this is impossible and  $\delta$  is

never optimal.

Consider policy  $\sigma$ :  $l \rightarrow 0$  (A in  $l$ )  $\delta < \sigma \Leftrightarrow v_0^\delta < v_0^\sigma$

$$\Leftrightarrow \left[ (\mu_l + \mu_i) \frac{K_i}{\mu_i} - v_{-1} \mu_i \right] + \left[ (\mu_i + \mu_{a_1}) \frac{K_{a_1}}{\mu_{a_1}} - v_{-1} \cdot \mu_{a_1} \right] < 0$$

$$\Leftrightarrow (\mu_l + \mu_i) \frac{K_i}{\mu_i} + (\mu_i + \mu_{a_1}) \left[ \frac{K_{a_1}}{\mu_{a_1}} - v_{-1} \right] < 0$$

$\phi$ , if  $V_{-1} \leq \frac{K_{a_1}}{\mu_{a_1}}$  and  $\delta$  ( $l \rightarrow i \rightarrow a_1 \rightarrow 0$ ) is never optimal.

Now, suppose we have  $\delta = l \rightarrow i \rightarrow a_1 \rightarrow \dots \rightarrow a_k \rightarrow 0$ ,  $k \geq 2$ ,  $\sigma$  as before, and  $\gamma = l \rightarrow j \rightarrow a_1 \rightarrow \dots \rightarrow a_k \rightarrow 0$ ,  $\delta < \gamma \Leftrightarrow (6)$  holds so,

as with  $k = 1$ ,  $V_{-1} \geq \frac{K_{a_1}}{\mu_{a_1}} \Rightarrow \delta > \gamma$ .

$\delta < \sigma \Leftrightarrow V_0^\delta < V_0^\sigma$ , (using (5)),

$$\Leftrightarrow (\mu_l + \mu_i) \frac{K_i}{\mu_i} - V_{-1} \cdot \mu_i + (\mu_i + \mu_{a_1}) \frac{K_{a_1}}{\mu_{a_1}} - V_{-1} \cdot \mu_{a_1} + \sum_{j=2}^k \left[ (\mu_{a_{j-1}} + \mu_{a_j}) \frac{K_{a_j}}{\mu_{a_j}} - V_{-1} \cdot \mu_{a_j} \right] < 0$$

$$\Leftrightarrow (\mu_l + \mu_i) \frac{K_i}{\mu_i} + \sum_{j=2}^k \mu_{a_{j-1}} \cdot \frac{K_{a_j}}{\mu_{a_j}} + (\mu_i + \mu_{a_1}) \left( \frac{K_{a_1}}{\mu_{a_1}} - V_{-1} \right) + \sum_{j=2}^k \mu_{a_j} \left( \frac{K_{a_j}}{\mu_{a_j}} - V_{-1} \right) < 0,$$

$\phi$ , if  $V_{-1} \leq \frac{K_{a_j}}{\mu_{a_j}}$ ;  $1 \leq j \leq k$ .

But, from Lemma 3.7 which immediately follows, for  $\delta$  to be a possible optimal solution, we need

$$\frac{K_{a_1}}{\mu_{a_1}} < \frac{K_{a_2}}{\mu_{a_2}} < \dots < \frac{K_{a_k}}{\mu_{a_k}} .$$

$$\text{Thus, } V_{-1} \leq \frac{K_{a_1}}{\mu_{a_1}} \Rightarrow V_{-1} \leq \frac{K_{a_j}}{\mu_{a_j}}, \quad 1 \leq j \leq k \Rightarrow \delta > \sigma .$$

Conclusion:

$$V_{-1} > \frac{K_{a_1}}{\mu_{a_1}} \Rightarrow \delta > \gamma$$

$$V_{-1} \leq \frac{K_{a_1}}{\mu_{a_1}} \Rightarrow \delta > \sigma$$

so,  $\delta$  is never optimal. So, never repair  $i$  in any state.  $\square$

Henceforth, in addition to  $\lambda_1 > \dots > \lambda_n$ , I will also assume  $K_1 < K_2 < \dots < K_n$ , which avoids inclusion of "irrelevant" components, i.e., those which are known to never be worth repairing from the start.

The next two lemmas are useful in further reducing the possible optimal policies:

Lemma 3.7: Suppose we have the Basic Model, parallel case with no fixed charge and  $\lambda_1 > \lambda_2 > \dots > \lambda_n$ . Suppose we start in state  $i$ . Then  $i \rightarrow a$  beats  $i \rightarrow b \rightarrow a$  ( $i \rightarrow a$  means  $R_a$  in  $i$ )

if either  $\frac{K_b}{\mu_b} > \frac{K_a}{\mu_a}$  or  $b > i$  and  $b \neq j_1$ .

Proof: Let  $\delta : i \rightarrow a$      $\gamma : i \rightarrow b \rightarrow a$ .

$$V_0^\gamma < V_0^\delta \Leftrightarrow \left[ (\mu_i + \mu_b) \frac{K_b}{\mu_b} - V_{-1} \cdot \mu_b \right] + \left[ (\mu_b + \mu_a) \frac{K_a}{\mu_a} - V_{-1} \mu_a \right]$$

$$< (\mu_i + \mu_a) \frac{K_a}{\mu_a} - V_{-1} \cdot \mu_a$$

$$\Leftrightarrow (\mu_i + \mu_b) \frac{K_b}{\mu_b} + (\mu_b - \mu_i) \frac{K_a}{\mu_a} < \mu_b V_{-1}$$

$$(7) \quad \Leftrightarrow \left(1 + \frac{\mu_i}{\mu_b}\right) \frac{K_b}{\mu_b} + \left(1 - \frac{\mu_i}{\mu_b}\right) \cdot \frac{K_a}{\mu_a} < V_{-1} .$$

Suppose  $\frac{K_b}{\mu_b} > \frac{K_a}{\mu_a}$ . (7) is true

$$\Leftrightarrow \underbrace{\frac{K_b}{\mu_b} + \frac{K_a}{\mu_a}}_{\frac{K_{j_1}}{\mu_{j_1}} + \frac{K_{j_2}}{\mu_{j_2}}} + \frac{\mu_i}{\mu_b} \underbrace{\left[ \frac{K_b}{\mu_b} - \frac{K_a}{\mu_a} \right]}_{> 0} < V_{-1}$$

$$\phi, \text{ since } V_{-1} \leq \frac{K_{j_1}}{\mu_{j_1}} + \frac{K_{j_2}}{\mu_{j_2}} \quad \text{so, } \frac{K_b}{\mu_b} > \frac{K_a}{\mu_a} \Rightarrow \delta < \gamma .$$

Suppose  $\frac{K_b}{\mu_b} < \frac{K_a}{\mu_a}$ . (7) is true

$$\Leftrightarrow \underbrace{\frac{K_b}{\mu_b} + \frac{K_b}{\mu_b}}_{\substack{K_{j_1} & K_{j_2} \\ \mu_{j_1} & \mu_{j_2} \\ \text{if } b \neq j_1}} + \underbrace{\left(1 - \frac{\mu_i}{\mu_b}\right)}_{> 0, \text{ if } b > i} \underbrace{\left(\frac{K_a}{\mu_a} - \frac{K_b}{\mu_b}\right)}_{> 0} < v_{-1}$$

$\phi$ , if  $b > i$  or  $b \neq j_1$  so,  $b > i \Rightarrow \delta < \gamma$   
&  $b \neq j_1$  □

Note: Lemma 3.7 works for

$$\delta : c_1 \rightarrow \dots \rightarrow c_q \rightarrow i \rightarrow a \rightarrow a_1 \rightarrow \dots \rightarrow a_p \rightarrow 0$$

$$\gamma : c_1 \rightarrow \dots \rightarrow c_q \rightarrow i \rightarrow b \rightarrow a \rightarrow a_1 \rightarrow \dots \rightarrow a_p \rightarrow 0$$

since  $\gamma < \delta \Leftrightarrow v_0^{\gamma'} < v_0^{\delta'}$  where  $\gamma' : i \rightarrow b \rightarrow a$ ,  $\delta' : i \rightarrow a$ .

Lemma 3.8: Suppose we have the Basic Model, parallel case with no fixed charge. Suppose we start in state  $i$ . Components are ordered  $\mu_1 < \mu_2 < \dots < \mu_n$ .

$$(\lambda_1 > \lambda_2 > \dots > \lambda_n)$$

Then:

(i)  $i \rightarrow 0$  beats  $i \rightarrow p(i) \rightarrow 0$  if  $p(i) < i$  and  $p(i) \neq j_1$ .

$$(\text{Recall: } j_1 = \underset{j}{\operatorname{argmin}} \lambda_j K_j)$$

and

(ii)  $i \rightarrow 0$  beats  $i \rightarrow p(i) \rightarrow \dots \rightarrow p^k(i) \rightarrow 0$ ,  $k \geq 2$  if  $p^k(i) < i$ .

Proof:

(i) Let  $\delta : i \rightarrow 0$ ,  $\gamma : i \rightarrow a \rightarrow 0$ ,  $v_0^\delta < v_0^\gamma$

$$\Leftrightarrow v_{-1} < \left(1 + \frac{\mu_i}{\mu_a}\right) \frac{\mu_a^K}{\mu_a}, \text{ always true if } a < i \text{ and } a \neq j_1.$$

(ii) Let  $\delta : i \rightarrow 0$ ,  $\gamma : i \rightarrow p(i) \rightarrow \dots \rightarrow p^k(i) \rightarrow 0$ ,  $K \geq 2$

$$v_0^\delta < v_0^\delta$$

$$\Leftrightarrow 0 < \sum_{j=1}^k \left[ \left( \mu_{p^{j-1}(i)} + \mu_{p^j(i)} \right) \frac{\mu_{p^j(i)}^K}{\mu_{p^j(i)}} - \mu_{p^j(i)} v_{-1} \right]$$

$$\Leftrightarrow v_{-1} \sum_{j=1}^k \mu_{p^j(i)} < \sum_{j=1}^k \left( \mu_{p^{j-1}(i)} + \mu_{p^j(i)} \right) \frac{\mu_{p^j(i)}^K}{\mu_{p^j(i)}}, \text{ where } p^0(i) = i$$

$$\Leftrightarrow v_{-1} \left( \sum_{j=1}^{k-1} \mu_{p^j(i)} + \mu_{p^k(i)} \right) < \sum_{j=1}^{k-1} \mu_{p^j(i)} \left( \frac{\mu_{p^j(i)}^K}{\mu_{p^j(i)}} + \frac{\mu_{p^{j+1}(i)}^K}{\mu_{p^{j+1}(i)}} \right) + \mu_i \frac{\mu_{p(i)}^K}{\mu_{p(i)}} + \mu_{p^k(i)} \cdot \frac{\mu_{p^k(i)}^K}{\mu_{p^k(i)}}$$

$$\Leftrightarrow v_{-1} \left( 1 + \frac{\mu_{p^k(i)}}{\sum_{j=1}^{k-1} \mu_{p^j(i)}} \right) < \sum_{j=1}^{k-1} \frac{\mu_{p^j(i)}}{\sum_{j=1}^{k-1} \mu_{p^j(i)}} \left[ \frac{\mu_{p^j(i)}^K}{\mu_{p^j(i)}} + \frac{\mu_{p^{j+1}(i)}^K}{\mu_{p^{j+1}(i)}} \right]$$

$$+ \frac{\mu_i}{\sum_{j=1}^{k-1} \mu_{p^j(i)}} \cdot \frac{K_{p(i)}}{\mu_{p(i)}} + \frac{\mu_{p^k(i)}}{\sum_{j=1}^{k-1} \mu_{p^j(i)}} \cdot \frac{K_{p^k(i)}}{\mu_{p^k(i)}} .$$

True, if

$$\mu_{p^k(i)} \cdot v_{-1} < \mu_i \cdot \frac{K_{p(i)}}{\mu_{p(i)}} + \frac{\mu_{p^k(i)}}{\mu_{p^k(i)}} \cdot \frac{K_{p^k(i)}}{\mu_{p^k(i)}}$$

since

$$v_{-1} \leq \text{convex combination of } \frac{K_i}{\mu_i} + \frac{K_j}{\mu_j} \text{ 's} .$$

So, true if

$$v_{-1} < \frac{\mu_i}{\mu_{p^k(i)}} \cdot \frac{K_{p(i)}}{\mu_{p(i)}} + \frac{K_{p^k(i)}}{\mu_{p^k(i)}} ,$$

which is so as  $i > p^k(i)$  and

$$v_{-1} < \frac{K_{p(i)}}{\mu_{p(i)}} + \frac{K_{p^k(i)}}{\mu_{p^k(i)}} . \quad \square$$

Using the results of the previous two lemmas, we can now state the basic theorem describing optimal policies in the parallel case. Although it does not specify the optimal policy in general, it brings the number of possibilities down to only a few, which

could be differentiated by hand using (5). In some special cases, more can be said and optimal policies described exactly. Such is the case when (i) the cost/lifetime ratios are ordered the same as the mean lifetimes,  $\left(\frac{K_1}{\mu_1} < \dots < \frac{K_n}{\mu_n}\right)$  or (ii) the value "l" in  $V_{-1} = \frac{K_l + p}{\mu_l}$  is small.

Theorem 3.9: Given the Basic Model, a parallel system and no fixed charge, let  $\delta = c_1 c_2 \dots c_m$  be an optimal policy as described on page 77. The elements of  $c_j = \langle i, p(i), p^2(i), \dots, p^k(i) = a_j \rangle$  have the following three restrictions:

- (1)  $p^j(i) < p^k(i) \forall 0 \leq j \leq k - 1$
- (2)  $i > p(i) > \dots > p^{k-1}(i)$
- (3)  $\frac{K_{p(i)}}{\mu_{p(i)}} < \frac{K_{p^2(i)}}{\mu_{p^2(i)}} < \dots < \frac{K_{p^k(i)}}{\mu_{p^k(i)}}$ .

Proof: Directly from Lemmas 3.7 and 3.8. □

Note: Conditions (1) and (2) state that, given state  $i$  is the first one reached with  $|s| = 1$ , the last element repaired before doing nothing must have longer expected life than  $i$  and all the ones in between must have successively shorter ones up to the last one.

Theorem 3.9 simplifies considerably in the case that the component cost/mean lifetime ratios are ordered in the same way as the mean lifetimes (i.e.,  $\frac{K_1}{\mu_1} < \frac{K_2}{\mu_2} < \dots < \frac{K_n}{\mu_n}$ ) condition (3) becomes  $p(i) < p^2(i) < \dots < p^k(i)$ .

Corollary 3.10: Given the hypotheses of Theorem 3.9 plus  $\frac{K_1}{\mu_1} < \frac{K_2}{\mu_2} < \dots < \frac{K_n}{\mu_n}$ , a set  $c_j$  will have no more than 3 elements, i.e.,  $k \leq 2$ .

Proof: Suppose  $k > 2$ . Conditions (2) and (3) in Theorem 3.9 lead immediately to a contradiction.  $\square$

Thus, in this case, given initial  $|s| = 1$  state  $i$ , three things can happen before doing nothing and entering the ergodic state zero:

- (i)  $i \rightarrow 0$  (do nothing)
- (ii)  $i \rightarrow p(i) \rightarrow 0$ ,  $p(i) > i$  (repair a component  $> i$ , then do nothing)
- (iii)  $i \rightarrow p(i) \rightarrow p^2(i) \rightarrow 0$ ,  $p(i) < i < p^2(i)$ .

Which of these three and what exact  $p(i)$ ,  $p^2(i)$  give optimality can be easily determined by comparing  $v_{0,\delta}^i$ 's using equation (5).

Another type of optimal policy restriction occurs when the value " $\ell$ " in  $V_{-1} = \frac{K_\ell + p}{\mu_\ell}$  is small.

Lemma 3.11: Suppose we have the hypotheses of Theorem 2.13

with  $V_{-1} = \frac{K_\ell + p}{\mu_\ell}$ . Then

(i) If  $i \geq \ell \neq j_1$ ,  $i \rightarrow 0$  beats  $i \rightarrow j \rightarrow 0$

(ii) If  $\ell = 1$ ,  $i \rightarrow 0$  beats  $i \rightarrow j \rightarrow 0$  unless  $j_1 = 1$  in which case  $i \rightarrow 1 \rightarrow 0$  might beat  $i \rightarrow 0$ .

Proof:

$$i \rightarrow j \rightarrow 0 \text{ beats } i \rightarrow 0 \Leftrightarrow V_{-1} > \left(1 + \frac{\mu_i}{\mu_j}\right) \frac{K_j}{\mu_j}.$$

Suppose this is true. Then

$$\frac{K_j + p}{\mu_j} > V_{-1} > \left(1 + \frac{\mu_i}{\mu_j}\right) \frac{K_j}{\mu_j} \quad \text{by definition of } V_{-1}.$$

$$\Rightarrow p > \mu_i \cdot \frac{K_j}{\mu_j} \geq \mu_\ell \cdot \frac{K_j}{\mu_j} \quad \text{if } i \geq \ell \text{ or } \frac{p}{\mu_\ell} > \frac{K_j}{\mu_j}.$$

But, also

$$v_{-1} < \frac{K_{j_1}}{\mu_{j_1}} + \frac{K_{j_2}}{\mu_{j_2}}$$

$$\Leftrightarrow \frac{p}{\mu_\ell} < \frac{K_{j_1}}{\mu_{j_1}} + \frac{K_{j_2}}{\mu_{j_2}} - \frac{K_\ell}{\mu_\ell}.$$

Thus,

$$i \geq \ell \Rightarrow \underbrace{\frac{K_j}{\mu_j}}_{\geq \frac{K_{j_1}}{\mu_{j_1}}} < \frac{p}{\mu_\ell} < \frac{K_{j_1}}{\mu_{j_1}} + \underbrace{\frac{K_{j_2}}{\mu_{j_2}} - \frac{K_\ell}{\mu_\ell}}_{\leq 0, \text{ if } \ell \neq j_1}.$$

So,  $i \geq \ell \neq j_1 \Rightarrow$  CONTRADICTION Q.E.D. (i).

If  $\ell = 1$ , then  $i \geq \ell$  automatically.

If  $j_1 \neq 1$ , then CONTRADICTION, as before.

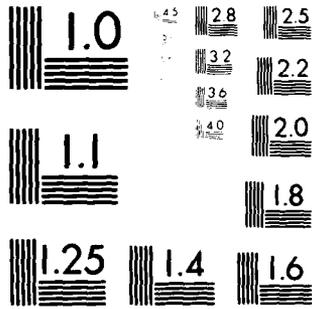
If  $j_1 = 1 = \ell$  and  $j \neq 1$ , get CONTRADICTION. Q.E.D. (ii).  $\square$

Using Lemmas 3.7, 3.8 and 3.11, some special "small  $\ell$ " cases have particularly "nice" optimal policy forms.

Case I:  $v_{-1} = \frac{K_1 + p}{\mu_1} \quad (\ell = 1)$

A.  $j_1 \neq 1$ . Then Lemma 3.11  $\Rightarrow i \rightarrow 0$  beats  $i \rightarrow j \rightarrow 0 \forall i, j$ .





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$$\Rightarrow i \rightarrow p(i) \rightarrow \dots \rightarrow p^k(i) \rightarrow 0$$

$$\circlearrowleft i \rightarrow 0$$

i.e., do nothing in all states  $i$  is optimal.

B.  $j_1 \neq 1$ . Lemma 3.11  $\Rightarrow i \rightarrow 0$  beats  $i \rightarrow j \rightarrow 0$  if  $j \neq 1$ .

So,

$$i \rightarrow p(i) \rightarrow \dots \rightarrow p^k(i) \rightarrow 0$$

$$> i \rightarrow p(i) \rightarrow \dots \rightarrow p^q(i) \rightarrow 1 \rightarrow 0 .$$

Lemma 3.7  $\Rightarrow i \rightarrow 1$  beats  $i \rightarrow p(i) \rightarrow 1$ , since

$$\frac{K_1}{\mu_1} < \frac{K_{p(i)}}{\mu_{p(i)}} \quad (j_1 = 1), \quad \text{so}$$

$$> \circlearrowleft i \rightarrow 1 \rightarrow 0 \quad \text{or} \quad \circlearrowleft i \rightarrow 0 .$$

Thus, given state  $i$ , the optimal action is to either do nothing or to repair component number 1. If  $i = 1$ , then do nothing is optimal.

$$i \rightarrow 0 < i \rightarrow 1 \rightarrow 0 \Leftrightarrow v_{-1} < \left(1 + \frac{\mu_i}{\mu_1}\right) \frac{K_1}{\mu_1} \Rightarrow .$$

If

$$\left(1 + \frac{\mu_i}{\mu_1}\right) \frac{K_1}{\mu_1} < V_{-1} < \left(1 + \frac{\mu_{i+1}}{\mu_1}\right) \frac{K_1}{\mu_1}, \quad \text{then policy}$$

1 2 3 ... i i+1 ... n

A R<sub>1</sub> R<sub>1</sub> ... R<sub>1</sub> A ... A is optimal where  $\mu_{n+1} = \infty$ .

Case II:  $V_{-1} = \frac{K_2+p}{\mu_2} \quad (l = 2)$

A.  $j_1 \neq 2$ . Lemma 3.11  $\Rightarrow i \rightarrow 0$  ( $<$ ) beats  $i \rightarrow j \rightarrow 0 \forall i \geq 2, j$ .

so,

$$i \rightarrow p(i) \rightarrow \dots \rightarrow p^k(i) \rightarrow 0$$

$$> i \rightarrow p(i) \rightarrow \dots \rightarrow p^q(i) \rightarrow 1 \rightarrow 0 \quad (\text{if } i \neq 1)$$

$$> i \rightarrow 0 \quad \text{by Theorem 2.13} \quad (\text{if } i \neq 1).$$

If  $i = 1$ , then true just by Lemma 3.11.

So, do nothing in all states i is optimal.

B.  $j_1 = 2$ .  $i \rightarrow 0$  beats  $i \rightarrow j \rightarrow 0$  if  $j \neq j_1 = 2, i \geq 2$ .

So,

$$i \rightarrow p(i) \rightarrow \dots \rightarrow p^k(i) \rightarrow 0$$

$$> i \rightarrow p(i) \rightarrow \dots \rightarrow 1 \rightarrow 0 \quad \text{or} \quad i \rightarrow p(i) \rightarrow \dots \rightarrow 2 \rightarrow 0$$

$$> i \rightarrow 0 \quad \text{by Theorem 3.13}$$

$$> i \rightarrow 2 \rightarrow 0 \quad \text{by Theorem 3.9} \\ \text{since } j_1 = 2$$

$$> i \rightarrow 0 \quad \text{if } i > 2 .$$

So, the optimal policy is:

If in  $i \geq 2$ , do nothing

If in  $i = 1$ , do nothing or repair item two, depending on how large  $V_{-1}$  is.

As  $l$  gets larger, Lemma 3.11 eliminates fewer policies, improving little from Theorem 3.9. Even in the case  $l = 3$ , it is no longer true that  $j_1 \neq l$  means do nothing in all states  $i$  is optimal.

Also, notice that Lemma 3.11 and its applications do not require

any particular ordering on the  $\frac{K_j + p}{\mu_j}$ 's, only knowledge of the minimum

$\frac{K_1 + p}{\mu_1}$ . This concludes investigation into  $V_0$  optimal policies when

$$V_{-1} = \frac{K_l + p}{\mu_l}, \text{ some } l.$$

Now suppose

$$V_{-1} = \frac{K_{j_1}}{\mu_{j_1}} + \frac{K_{j_2}}{\mu_{j_2}}, \quad \text{where}$$

$$j_1 = \operatorname{argmin}_j \frac{K_j}{\mu_j}; \quad j_2 = \operatorname{argmin}_{j \neq j_1} \frac{K_j}{\mu_j}.$$

Suppose you start in state  $i$ . A policy is then of the form

$$(1) \quad i \rightarrow p(i) \rightarrow \dots \rightarrow p^k(i) \rightarrow j_1 \text{ or } j_2$$

or

$$(2) \quad i \rightarrow p(i) \rightarrow \overbrace{\dots \rightarrow 0} \rightarrow p^{q+1}(i) \rightarrow \dots \rightarrow p^k(i) \rightarrow j_1 \text{ or } j_2$$

or

$$(3) \quad i \rightarrow p(i) \rightarrow \dots \rightarrow 0 \begin{cases} \rightarrow p_1^{q+1}(i) \rightarrow \dots \rightarrow p_1^k(i) \rightarrow j_1 \text{ or } j_2 \\ \rightarrow p_2^{q+1}(i) \rightarrow \dots \rightarrow p_2^l(i) \rightarrow j_1 \text{ or } j_2. \end{cases}$$

One can define  $y_\delta = \begin{bmatrix} I - Q_0^\delta \end{bmatrix}$  rows  $\{0, 1, \dots, n\} \sim j_1, j_2$   
 cols.  $\{0, 1, \dots, n\} \sim j_1, j_2$

similar to what was done in the  $V_{-1} = \frac{K_\ell + p}{\mu_\ell}$  case. Then

$$V_{0,\delta}^i = y_{\delta,i}^{-1} \cdot (R_{0,i}^\delta - Q_{1,i}^\delta V_{-1}^\delta), \quad \text{where } \delta \text{ is a policy of type (1),}$$

2) or (3). The computational procedure is the same as before only the types of policies are more varied. To get similar restrictions on optimal policies here would take up twice as much space, adding little of importance. Thus, we leave this case as  $R_{j_1}$  in  $j_2$ ,  $R_{j_2}$  in  $j_1$ ;  $R_{p_1 p_2}$  in 0 (some  $p_1 p_2$ ) and  $R_{p(i)}$  in  $i$  ( $p(i)$  could be zero), where  $p_1, p_2, p(i) \forall i$  are to be determined by comparing  $V_0^i$ 's for various policies.

We conclude the section with a couple of other special cases of interest. Lemma 3.12 gives conditions under which

$$V_{-1} = \frac{K_{j_1} + p}{\mu_{j_1}} \quad \text{or} \quad \frac{K_{j_1}}{\mu_{j_1}} + \frac{K_{j_2}}{\mu_{j_2}} .$$

Lemma 3.12: Given the hypotheses of Theorem 3.9, if

$$\max_{i \neq j_1} \frac{\lambda_{j_1} K_{j_1} + \lambda_{j_2} K_{j_2} - \lambda_i K_i}{\lambda_i} < \min_{i > j_1} \frac{\lambda_i K_i - \lambda_{j_1} K_{j_1}}{\lambda_{j_1} - \lambda_i} ,$$

then

$$V_{-1} = \frac{K_{j_1} + p}{\mu_{j_1}} \quad \text{or} \quad \frac{K_{j_1}}{\mu_{j_1}} + \frac{K_{j_2}}{\mu_{j_2}}$$

no matter what  $p$  is.

Proof: 
$$p < \min_{i > j_1} \frac{\lambda_i K_i - \lambda_{j_1} K_{j_1}}{\lambda_{j_1} - \lambda_i} \rightarrow$$

$$\lambda_{j_1} (K_{j_1} + p) < \lambda_i (K_i + p) \quad \forall i \neq j_1 .$$

$$p > \max_{i \neq j_1} \frac{\lambda_{j_1} K_{j_1} + \lambda_{j_2} K_{j_2} - \lambda_i K_i}{\lambda_i} \rightarrow$$

$$\lambda_{j_1} K_{j_1} + \lambda_{j_2} K_{j_2} < \lambda_i (K_i + p) \quad \forall i \neq j_1 .$$

Thus, if

$$\max_{i \neq j_1} \frac{\lambda_{j_1} K_{j_1} + \lambda_{j_2} K_{j_2} - \lambda_i K_i}{\lambda_i} < \min_{i > j_1} \frac{\lambda_i K_i - \lambda_{j_1} K_{j_1}}{\lambda_{j_1} - \lambda_i} ,$$

$$\lambda_i (K_i + p), i \neq j_1 > \lambda_{j_1} (K_{j_1} + p) \quad \underline{\text{or}} \quad \lambda_{j_1} K_{j_1} + \lambda_{j_2} K_{j_2}$$

$\Rightarrow V_{-1}$  is never  $\lambda_i (K_i + p), i \neq j_1$ . □

Lemma 3.12 gives a condition under which the  $V_{-1}$  optimal policy is to keep either the  $j_1^{\text{st}}$  component (one with  $\min \lambda_i K_i$ ) working or the  $j_1^{\text{st}}$  and  $j_2^{\text{nd}}$  (two  $\min \lambda_i K_i$ 's) no matter what the penalty cost is. Some examples for which this is true are:

(i)  $\lambda_i \equiv \lambda$  (identical component lifetime distributions)

as

$$\min_{i > j_1} \frac{\lambda_i K_i - \lambda_{j_1} K_{j_1}}{\lambda_{j_1} - \lambda_i} = \infty$$

$$\max_{i \neq j_1} \frac{\lambda_{j_1} K_{j_1} + \lambda_{j_2} K_{j_2} - \lambda_i K_i}{\lambda_i} = K_{j_1} + K_{j_2} - K_i < \infty$$

(ii) Suppose

$$(8) \quad \lambda_i K_i - \lambda_{j_1} K_{j_1} \geq (\lambda_i - \lambda_{j_1}) K_{j_2} \quad \forall i \neq j_1.$$

Then

$$\lambda_{j_1} K_{j_1} - \lambda_i K_i \leq (\lambda_i - \lambda_{j_1}) K_{j_2} \quad \forall i \neq j_1$$

so

$$\max_{i \neq j_1} \frac{\lambda_{j_1} K_{j_1} + \lambda_{j_2} K_{j_2} - \lambda_i K_i}{\lambda_i} \leq \max_{i \neq j_1} \frac{(\lambda_i + \lambda_{j_2} - \lambda_{j_1}) K_{j_2}}{\lambda_i}$$

$$= K_{j_2} \cdot \max_{i \neq j_1} \left( 1 - \frac{\lambda_1 - \lambda_{j_2}}{\lambda_i} \right) < K_{j_2} \quad \text{as } \lambda_1 > \lambda_j \quad j \neq 1$$

$$\leq \frac{\lambda_i K_i - \lambda_{j_1} K_{j_1}}{\lambda_1 - \lambda_i} \quad \forall i \neq j_1 \quad \text{by assumption (8).}$$

So,

$$\leq \min_{i \neq j_1} \frac{\lambda_i K_i - \lambda_{j_1} K_{j_1}}{\lambda_1 - \lambda_i} \leq \min_{i \neq j_1} \frac{\lambda_i K_i - \lambda_{j_1} K_{j_1}}{\lambda_{j_1} - \lambda_i} \quad \text{since } j_1 \geq 1, \lambda_{j_1} \leq \lambda_1,$$

the hypothesis of Lemma 3.12. The key to having (8) true is to have a sufficiently large spread of the  $\lambda_i K_i$ 's compared to that of the  $\lambda_i$ 's.

In summary, if the differences in expected cost/lifetimes of the components is large enough compared to the differences in expected lifetimes, the  $V_{-1}$  optimal policies are simpler and more "intuitive". If, in addition, the cost/expected lifetimes are ordered in the same way as the mean lifetimes, or  $V_{-1} = \frac{K_\ell + p}{\mu_\ell}$  has  $\ell$  small, the  $V_0$  optimal policies simplify.

One final interesting case is that in which  $c = \lambda_1 K_1 = \dots = \lambda_n K_n$ . Then  $\lambda_1 (K_1 + p) > \dots > \lambda_n (K_n + p) \forall p > 0$ , i.e., if  $\mathcal{E} = \{0\}$ , then repair of the unit with longest expected lifetime is optimal. Also, note that if  $p = 0$ , the optimal policy is always to let the system fail and then repair that component with smallest expected cost/unit time. For the case above in which these are the same, we can

also say that the  $V_0$  optimal policy is to do nothing in all states  $1, \dots, n$  before hitting 0 since  $V_{-1} = c$ . (Of course, then it doesn't matter which component gets repaired.)

For some examples exhibiting  $V_0$  behavior for the Basic Model, see Example 6.2 (Chapter VI) as well as the solution to Example 2.1 presented in Chapter II, which concludes this section and chapter.

Example 2.1 (from Chapter II, Section 1):

Basic Model,  $n = 2$ , parallel,  $L = 0$ .

Case I  $V_{-1} = \lambda_1 K_1 + \lambda_2 K_2$ :

No transient states accessible from  $\{1, 2\}$  for which decisions must be made.

optimal policy	1	2	0	
	$R_2$	$R_1$	-	$\mathcal{E} = \{1, 2\}$ .

---

Case II  $V_{-1} = \lambda_1(K_1 + p)$ .

If  $\lambda_1 K_1 > \lambda_2 K_2$  ( $j_1 = 2$ ), then by Case I.A. (Application of Lemma 3.11),

optimal policy	1	2	0	
	A	A	$R_1$	$\mathcal{E} = \{0\}$ .

---

If  $\lambda_1 K_1 < \lambda_2 K_2$  ( $j_1 = 1$ ), then by Case I.B. (Lemma 3.11 Applications) the optimal policy is:

$$\left. \begin{array}{l} \begin{array}{ccc} 1 & 2 & 0 \\ A & A & R_1 \end{array}, \quad \text{if } v_{-1} < \left(1 + \frac{\mu_2}{\mu_1}\right) \frac{K_1}{\mu_1} \\ \hline \begin{array}{ccc} 1 & 2 & 0 \\ A & R_1 & R_1 \end{array}, \quad \text{if } v_{-1} > \left(1 + \frac{\mu_2}{\mu_1}\right) \frac{K_1}{\mu_1} \end{array} \right\} \mathcal{E} = \{0\} .$$

Case III  $v_{-1} = \lambda_2(K_2 + p)$ :

If  $\lambda_1 K_1 < \lambda_2 K_2$  ( $j_1 = 1$ ), then by Case II.A. of Lemma 3.11

(Applications),

$$\begin{array}{ccc} \text{optimal policy} & 1 & 2 & 0 \\ & A & A & R_2 \end{array} \quad \mathcal{E} = \{0\} .$$

If  $\lambda_1 K_1 > \lambda_2 K_2$  ( $j_1 = 2$ ), then by Case II.B. of Lemma 3.11 applications, the optimal policy is:

$$\begin{array}{l} \begin{array}{ccc} 1 & 2 & 0 \\ A & A & R_2 \end{array}, \quad \text{if } v_{-1} < \left(1 + \frac{\mu_1}{\mu_2}\right) \frac{K_2}{\mu_2} \\ \hline \begin{array}{ccc} 1 & 2 & 0 \\ R_2 & A & R_2 \end{array}, \quad \text{if } v_{-1} > \left(1 + \frac{\mu_1}{\mu_2}\right) \frac{K_2}{\mu_2} . \end{array}$$

CHAPTER IV  
THE DEGRADATION MODEL

1. Description of Model

In the Basic Model it was assumed that system components are either working or failed (on/off) with exponential lifetime distributions. However, for many systems, the components may be observed in some finite number of states of increasing degradation before failure. The more degraded the component at the time of repair, the greater the repair cost. The Degradation Model takes this factor into account, having the following Coherent System Repair Model parameters:

States: each component can be in one of " $\ell$ " states of degradation or new.

$$0^{\text{th}} = \text{new}$$

$$\ell^{\text{th}} = \text{failed}$$

$$1, \dots, \ell - 1 = \text{degraded (not failed)}$$

Repair: instantaneous, unlimited service - brings degraded component to "new" condition

Component Lifetimes: Let  $L_i^k$  = lifetime of  $i^{\text{th}}$  component given it is in the  $k^{\text{th}}$  degradation state,  $0 \leq k < \ell$ .

$$\text{The } L_i^k \sim 1 - e^{-\lambda_i^k t} = p\{L_i^k \leq t\}.$$

Notice that if  $l = 1$  (1<sup>st</sup> degradation state = failed), then the Degradation Model reduces to the Basic Model.

Of course, in the process of increasing the number of states in the model, a price must be paid in the difficulty of obtaining results. Only the general series case (series systems are viable now unless  $l = 1$ ) and the identical component parallel cases are treated, and for  $V_{-1}$  optimality only. Luckily, in the series case,  $V_{-1}$  optimality is sufficient to give the general optimal solution and in the parallel case it gives the key information desired. In utilizing this model, it would certainly be advantageous to keep the number of degradation states as low as possible, while still capturing the essence of the system being modeled, as the total number of states is of the order  $(l + 1)^n$  where

$l$  = number of degradation states

$n$  = number of components.

Section 2 treats the general series case while Section 3 looks at the parallel case in identical components with no fixed charge. Section 4 presents some possible changes in state and/or decision space in this model to provide extensions of the Basic Model to Erlang component lifetimes as well as provide insight into other possible non-constant failure rates for components.

The number of parameters needed to specify the model is increased  $k$ -fold from the basic case and are as follows:

- (1) fixed charge,  $L \geq 0$ ,
- (2) penalty cost for system failure,  $p > 0$ ,
- (3)  $\mu_i^k = 1/\lambda_i^k$  = expected length of time  $i^{\text{th}}$  component spends in the  $k^{\text{th}}$  degradation state (exponential).

These are defined for  $k = 0, 1, \dots, \ell-1$  and  $i = 1, \dots, n$ .  
 $k = \ell$  is not defined since component failed there.

- (4)  $K_i^k$  = cost to repair  $i^{\text{th}}$  component when in the  $k^{\text{th}}$  degradation state.

These are defined for  $i = 1, \dots, n$  and  $k = 1, \dots, \ell$   
( $k = 0$  is new and no need to repair there).

$0 < K_i^1 < K_i^2 < \dots < K_i^\ell$  is assumed to allow for increased costs to repair in higher ("more") degraded states,  
 $\forall i = 1, \dots, n$ .

A state in this model is denoted by a vector,  $\underline{s}$ , of length  $n$ ;  $\underline{s} = \langle s_1, s_2, \dots, s_n \rangle$  where  $s_i = k$  if component  $i$  is in the  $k^{\text{th}}$  state of degradation. A change of state occurs when one of the components enters a higher state of degradation and at such an instant, a decision is made to repair some subset of the degraded components or to do nothing, (some repair need only be done if the system is down). Repair is always assumed to bring a component back to a "new" condition. Each state as defined now has exponential holding time so the Degradation Model is also a continuous time Markov decision chain with infinite time horizon.

As before, specifying

- (1) state space
- (2) decision space
- (3) transition structure
- (4) cost structure
- (5) objective function

defines the model completely.

The Basic Model notation in Figure 2.1 applies here except that the component related cost and mean lifetime parameters now depend on degradation state (superscript) as well as the component (subscript). Also let  $\ell$  = number of degradation states. Obviously, these could vary by component but for notational simplicity, I assume they are the same for all components.

The Markov chain specified by the Degradation Model is as follows:

State space: States = which components in what degradation states  $\underline{s} = \langle s_1, \dots, s_n \rangle$ . If component  $i$  is in the  $s_i^{\text{th}}$  degradation state,  $0 \leq s_i \leq \ell$ . If system is  $k$ -of- $n$  and components are identical, then states are the number of components in various degradation states, i.e.,  $\underline{s} = \langle s_0, s_1, \dots, s_\ell \rangle$ , where  $s_i$  components are in the  $i^{\text{th}}$  degradation state,  $s_0 + s_1 + \dots + s_\ell = n$ .

Decision Space: Let  $\Omega = \{1, \dots, n\}$ . Possible decisions

in state  $\underline{s} = \langle s_1, \dots, s_n \rangle$  are  $R_{\Omega_{\underline{s}}}$  where

$\Omega_{\underline{s}} \subseteq \Omega - \{i : s_i = 0\}$  = set of non-new components in  $\underline{s}$ .

-  $\Omega_{\underline{s}} = \phi \Leftrightarrow R_{\Omega_{\underline{s}}} = A$  (do nothing)

- if system is down in  $\underline{s}$ , then  $R_{\Omega_{\underline{s}}} \neq \phi$ .

Transition Structure: The transition matrix,  $Q_0$  : (assuming

$\delta : R_{\Omega_{\underline{s}}}$  in  $\underline{s}$ ). Fix  $\underline{s} = \langle s_1, s_2, \dots, s_n \rangle$ .

Definition: State  $\underline{t}^i = \langle t_1, \dots, t_{i-1}, t_i+1, t_{i+1}, \dots, t_n \rangle$

if  $\underline{t} = \langle t_0, t_1, \dots, t_n \rangle$  ( $\underline{t}^i$  is  $\underline{t}$  with  $i^{\text{th}}$  component in a higher degradation state).  $\underline{t}^i$  is only defined for  $i | t_i < \ell$

(non-failed components). Also, let state  $\underline{t} \cup \Omega_{\underline{t}} = \langle t_1^1, \dots, t_n^1 \rangle$

where

$$t_i^1 = \begin{cases} 0, & \text{if } i \in \Omega_{\underline{t}} \\ t_i, & \text{if } i \notin \Omega_{\underline{t}} \end{cases} \quad \text{Note } \underline{t} \cup \Omega_{\underline{t}} \text{ is the condition of the system after instantaneous repairs } \Omega_{\underline{t}} \text{ in state } \underline{t}.$$

$$(Q_0)_{s, (s \cup \Omega_s)^i}^\delta = \left\{ \begin{array}{ll} \frac{\lambda_1 (s \cup \Omega_s)_i}{\sum_{j=1}^n \lambda_j (s \cup \Omega_s)_j}, & i | (s \cup \Omega_s)_i \neq \emptyset \\ 0, & \text{otherwise} \end{array} \right\}$$

$$Q_{1, \underline{s}}^\delta = \frac{1}{\sum_{j=1}^n \lambda_j (s \cup \Omega_s)_j}$$

Cost Structure:

$$R_{0, \underline{s}}^\delta = \left\{ \begin{array}{ll} L(1 - I_{\Omega_s}^\phi) + \sum_{i \in \Omega_s} K_i^{s_i}, & \text{if system up in } \underline{s} \\ L + p + \sum_{i \in \Omega_s} K_i^{s_i}, & \text{if system down in } \underline{s} \end{array} \right\}$$

$$\text{where } I_{\Omega_s}^\phi = \left\{ \begin{array}{ll} 1, & \text{if } \Omega_s = \phi \\ 0, & \text{otherwise} \end{array} \right\}$$

Objective Function:

$$V_{-1}^\delta = \frac{P_{\delta 0}^* R_0^\delta}{P_{\delta 1}^* Q_1^\delta} \quad (\text{any possible optimal } \delta \text{ will form a Markov chain with a single irreducible set of ergodic states}).$$

Below are examples of the series and parallel cases:

Example 4.1:  $n = 2, l = 2$ , series system,  $L = 0$

Then, possible component states are:

0 = new

1 = 1<sup>st</sup> degraded state

2 = 2<sup>nd</sup> degraded state (failed).

	system up				system down			
<u>States:</u>	00	01	10	11	02	20	12	21
<u>Decisions:</u>	--	A, R <sub>2</sub>	A, R <sub>1</sub>	A, R <sub>1</sub> , R <sub>2</sub> , R <sub>12</sub>	R <sub>2</sub>	R <sub>1</sub>	R <sub>2</sub>	R <sub>1</sub>
<u>Cost:</u>	0	0, K <sub>2</sub> <sup>1</sup>	0, K <sub>1</sub> <sup>1</sup>	0, K <sub>1</sub> <sup>1</sup> , K <sub>2</sub> <sup>1</sup> K <sub>1</sub> <sup>1</sup> +K <sub>2</sub> <sup>1</sup>	K <sub>2</sub> <sup>2</sup>	K <sub>1</sub> <sup>2</sup>	K <sub>2</sub> <sup>2</sup>	K <sub>1</sub> <sup>2</sup>

Note: Since series model, there is no need for  $p$  because system failure  $\Leftrightarrow$  component failure  $\Rightarrow$  can include  $p$  in  $K_1^2$ 's.

Sample of  $Q_0$ : Let  $\delta$  : A in 01, R<sub>1</sub> in 10, R<sub>1</sub> in 11,  
then,

$$Q_0^\delta = \begin{matrix} & 00 & 01 & 10 & 11 & 02 & 20 & 12 & 21 \\ \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \\ 02 \\ 20 \\ 12 \\ 21 \end{matrix} & \begin{bmatrix} 1 & \frac{-\lambda_2^0}{\lambda_1^0 + \lambda_2^0} & \frac{-\lambda_1^0}{\lambda_1^0 + \lambda_2^0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-\lambda_1^0}{\lambda_1^0 + \lambda_2^0} & \frac{-\lambda_2^1}{\lambda_1^0 + \lambda_2^1} & 0 & 0 & 0 \\ 0 & \frac{-\lambda_2^0}{\lambda_1^0 + \lambda_2^0} & \frac{\lambda_2^0}{\lambda_1^0 + \lambda_2^0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda_2^1}{\lambda_1^0 + \lambda_2^1} & \frac{-\lambda_2^1}{\lambda_1^0 + \lambda_2^1} & 0 & 0 & 0 \\ 0 & \frac{-\lambda_2^0}{\lambda_1^0 + \lambda_2^0} & \frac{-\lambda_1^0}{\lambda_1^0 + \lambda_2^0} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{-\lambda_2^0}{\lambda_1^0 + \lambda_2^0} & \frac{-\lambda_1^0}{\lambda_1^0 + \lambda_2^0} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-\lambda_2^0}{\lambda_1^1 + \lambda_2^0} & 0 & \frac{-\lambda_1^1}{\lambda_1^1 + \lambda_2^0} & 1 & 0 \\ 0 & 0 & 0 & \frac{-\lambda_1^0}{\lambda_1^0 + \lambda_2^1} & \frac{-\lambda_2^1}{\lambda_1^0 + \lambda_2^1} & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Example 4.2:  $n = 2$ ,  $l = 2$ ,  $L = 0$ , parallel system with identical components.

	system up					system down
<u>States:</u>	200	110	101	020	011	002
<u>Decisions:</u>	-	$A, R_1^{(1)}$	$A, R_1^{(2)}$	$A, R_1^{(1)}, R_2^{(1)}$	$A, R_1^{(1)}, R_1^{(2)}, R_1^{(1)+(2)}$	$R_1^{(2)}, R_2^{(2)}$
<u>Cost:</u>	0	$0, K^1$	$0, K^2$	$0, K^1, 2K^1$	$0, K^1, K^2, K^1+K^2$	$K^2+p, 2K^2+p$

where  $R_i^{(j)}$  denotes repair of  $i$  identical units which are in the  $j^{\text{th}}$  degradation state.

Note the  $p$  is now necessary to distinguish between component and system failures.

## 2. The Series Case

Consider now the case of a series system, i.e., one which fails as soon as any one of its components does. As long as no component has reached the  $\ell^{\text{th}}$  degradation state (has failed), decisions can be made to repair any subset of the degraded components. If the system fails due to a failure of component  $i$ , then the required decision is to immediately repair component  $i$  (no matter what condition the others are in).

The following theorem gives the optimal policy:

Theorem 4.1: We have the Degradation Model with a series system:

$$\text{Let } j_i = \operatorname{argmin}_{1 \leq j \leq \ell} \frac{K_i^j}{\sum_{p=1}^j \mu_i^p} .$$

Then the optimal policy is one which does nothing to component  $i$  until it reaches the  $j_i^{\text{th}}$  state of degradation and then repairs it.

Proof: Directly from Lemmas 4.2, 4.3 which follow:

Lemma 4.2: Suppose we have the series Degradation Model.

Fix a policy  $\delta$ . Then

$$V_{-1}^\delta = \sum_{i=1}^n \sum_{j=1}^{\ell} \alpha_i^j \cdot \frac{K_i^j}{\sum_{p=0}^{j-1} \mu_i^p} ,$$

where  $\sum_{j=1}^{\ell} \alpha_i^j = 1$ ,  $\forall i = 1, \dots, n$  and  $\alpha_i^j \geq 0$ .

Proof: Fix  $\delta$ .

Let  $P^*$  = stationary transition probability vector given  $\delta$   
 $= \langle P_{\cdot s} \rangle$  for possible states  $s$ .

Then  $P^*(I - Q_0) = 0$ .

(Since  $\delta$  is fixed, we drop it to simplify notation).

$$v_{-1} = \frac{P^* R_0}{P^* Q_1} = \frac{1}{P^* Q_1} \sum_{j=1}^{\ell} \sum_{i=1}^n K_i^j \sum_{\substack{s: R_i \text{ in } s \\ \text{and } s_i = j}} P \cdot s \quad (s = (s_1, \dots, s_n))$$

recall  $s_i$  = degradation state of  $i$  in  $s$ .

$$= \sum_{j=1}^{\ell} \sum_{i=1}^n \alpha_i^j \cdot \frac{K_i^j}{\sum_{p=0}^{j-1} \mu_i^p}, \quad \text{where}$$

$$\alpha_i^j = \frac{\left( \sum_{p=0}^{j-1} \mu_i^p \right) \sum_{\substack{s: R_i \text{ in } s \\ s_i = j}} P \cdot s}{P^* Q_1}.$$

It remains to show  $\sum_{j=1}^{\ell} \alpha_i^j = 1$ . Without loss of generality, set  $i = 1$ .

To show  $\sum_{j=1}^{\ell} \alpha_1^j = 1$ , it is sufficient to show

$$P^* Q_1 = \sum_{j=1}^{\ell} \left( \sum_{p=0}^{j-1} \mu_1^p \right) \sum_{\substack{s: R_1 \text{ in } s \\ s_1 = j}} P \cdot s,$$

i.e.,

$$(1) \quad P^* Q_1 - \sum_{j=1}^{\ell} \left( \sum_{p=0}^{j-1} \frac{1}{\lambda_1^p} \right) \sum_{\substack{s: R_1 \text{ in } s \\ s_1 = j}} P \cdot s = 0.$$

Some notation is now introduced for use in this proof:

Definition: Fix  $\delta$ . Let  $\mathcal{W}_s^\delta = \sum_{t|t \cup \Omega_t = s} P_{\cdot t}^\delta$  be the sum

of stationary probabilities of all states  $t$  which are sent to  $s$  instantaneously under  $\delta$ .

- Notice that  $\mathcal{W}_s^\delta$  is only defined for  $s$  such that system is working since if the system is failed in  $s$ ,  $\nexists t|t \cup \Omega_t = s$  by model definition.

Definition: Let  $s^{i\uparrow} (s^{i\downarrow})$  be the state for which all components are in the same degradation state as in  $s$  except for the  $i^{\text{th}}$ , which is in one lower (higher) state of degradation. If  $s_i = 0$ , then  $s^{i\uparrow}$  is not defined.

From the definition of  $\mathcal{W}$ ,

$$(2) \quad P_{\cdot s} = \sum_{\substack{i=1 \\ i:s_i \neq 0}}^n \frac{\lambda_i^{s_i-1}}{\lambda_i^{s_i-1} + \sum_{\substack{j=1 \\ j \neq i}}^n \lambda_j^{s_j}} \mathcal{W}_{s^{i\uparrow}}.$$

Thus,

$$(3) \quad \mathcal{W}_s = \sum_{t:t \cup \Omega_t = s} P_{\cdot t} = \sum_{t:t \cup \Omega_t = s} \sum_{\substack{i=1 \\ i|t_i \neq 0}}^n \frac{\lambda_i^{t_i-1}}{\lambda_i^{t_i-1} + \sum_{\substack{j=1 \\ j \neq i}}^n \lambda_j^{t_j}} \mathcal{W}_{t^{i\uparrow}}$$

Let  $\underline{\mathcal{W}} = \langle \mathcal{W}_s \rangle, s | \text{system up.}$

Let  $\underline{R} = ((R_{st}))_{s,t}$ , where

$$R_{st} = \left\{ \begin{array}{ll} \frac{\lambda_i^{t_i}}{\sum_{j=1}^n \lambda_j^{t_j}}, & \text{if } t^{i+} \cup \Omega_{t^{i+}} = s \\ & \left( \text{i.e., if } i^{\text{th}} \text{ component of } t \right. \\ & \left. \text{degrades, go to } s \text{ immediately} \right) \\ & \text{under } \delta \\ 0, & \text{otherwise} \end{array} \right\}$$

Restating (2) in matrix form gives:

$$(I - R) \mathcal{W} = 0 .$$

If  $R_{s\cdot}$  =  $s^{\text{th}}$  row of  $R$  and

$\underline{1}_s$  = vector of zeros except 1 in the  $s^{\text{th}}$  spot, then

$$(4) \quad (\underline{1}_s - R_{s\cdot}) \mathcal{W} = 0, \quad \forall \text{ states } s \text{ system is up.}$$

This is just a restatement of (3) in vector form.

To prove (1), one must get everything in terms of  $\mathcal{W}_s^s$ , rearrange terms and using certain sums of equations (4), show it to be zero.

Now,

$$P^* Q_1 = \sum_t P_{\cdot t} Q_{1,t} = \sum_s \mathcal{W}_t \cdot \frac{1}{\sum_{j=1}^n \lambda_j^{t_j}} .$$

Thus

$$\begin{aligned}
(1) &= P^* Q_1 - \sum_{j=1}^{\ell} \left( \sum_{p=0}^{j-1} \frac{1}{\lambda_1^p} \right) \sum_{\substack{t: R_1 \text{ in } t \\ \#1 \text{ in } j}} P \cdot t \\
&= \sum_{q=0}^{\ell-2} \frac{1}{\lambda_1^q} \left[ \frac{\lambda_1^q}{\sum_{j=1}^n \lambda_j^{s_j}} \mathcal{W}(q, \underline{s}) - \sum_{k=q+1}^{\ell-2} \sum_{\substack{R_1 \text{ in } \\ (k, \underline{s})}} P \cdot (k, \underline{s}) \right]
\end{aligned}$$

where  $\underline{s}$  is a configuration of components  $2, \dots, n = \langle s_2, \dots, s_n \rangle$   
and state  $(q, \underline{s}) = \langle q, s_2, \dots, s_n \rangle$  by (2),

$$P \cdot (k, \underline{s}) = \frac{\lambda_1^{k-1}}{\lambda_1^{k-1} + \sum_{j=2}^n \lambda_j^{s_j}} \mathcal{W}(k-1, \underline{s}) + \sum_{\substack{i=2 \\ i | s_i \neq 0}}^n \frac{\lambda_i^{s_i-1}}{\lambda_i^{s_i-1} + \lambda_1^k + \sum_{\substack{j=2 \\ j \neq i}}^n \lambda_j^{s_j}} \mathcal{W}(k, s^{i+}) .$$

Substitution of this into the previous expression for (1)  $\Rightarrow$

$$(1) = \sum_{q=0}^{\ell-2} \frac{1}{\lambda_1^q} \left[ \sum_{(q, \underline{s})} \left( \frac{1}{\lambda_1^q} - R_{(q, \underline{s})} \right) \mathcal{W} \right]$$

$$= 0 \quad \text{by} \quad (4) . \quad \square$$

Note: The proof is valid for any decisions in working states, but assumes that in states where an item is failed, repair of only the failed item will take place, not an unreasonable assumption.

Lemma 4.3 now identifies the policies which correspond to extreme points  $(\alpha_i^{j^*} = 1, \alpha_i^j = 0, j \neq j^*)$  in the convex set of  $V_{-1}^s$  for possible policies in the Degradation Series Model.

Lemma 4.3: Suppose we have the Series Degradation Model and let  $\alpha_i^j$  be defined as in Lemma 4.2. Then a policy  $\delta$  has

$$\left\{ \begin{array}{l} \alpha_i^{j_i} = 1 \\ \alpha_i^j = 0, \quad j \neq j_i \end{array} \right\}$$

$\Leftrightarrow \delta$  repairs the  $i^{\text{th}}$  component whenever it reaches the  $j_i^{\text{th}}$  state of degradation.

Proof:

$\Leftarrow$  Suppose  $\delta$  repairs the  $i^{\text{th}}$  component whenever it reaches the  $j_i^{\text{th}}$  state of degradation, leaves it alone otherwise.

$$V_{-1} = \sum_{j=1}^{\ell} \sum_{i=1}^n \alpha_i^j \frac{K_i^j}{\sum_{p=0}^{j-1} \mu_i^p} \quad \text{where} \quad \alpha_i^j = \frac{\left( \sum_{p=0}^{\ell-1} \mu_1^p \right)}{p^* Q_1} \sum_{\substack{s: R_1 \text{ in } s \\ s_i = j}} P_{\cdot s} \cdot$$

Since  $\nexists s | R_1 \text{ in } s \text{ if } i \text{ is not in } j_i \text{ and } P_{\cdot s} = 0$   
 $\forall s | s_i > j_i, \alpha_i^j = 0 \text{ for } j \neq j_i \text{ and since}$

$$\sum_{j=1}^{\ell} \alpha_i^j = 1$$

by lemma 4.2,  $\alpha_i^{j_i} = 1$ . □

( $\Rightarrow$ ) Pick  $i$ . Suppose  $\alpha_i^{j_i} = 1$ ,  $\alpha_i^j = 0$ ,  $j \neq j_i$ .  $\alpha_i^j = 0$   
 $\Rightarrow P_{\cdot s} = 0 \forall s \rightarrow R_i$  in  $s$  and  $i$  in  $j \neq j_i$  and  $s$  is in the  
 ergodic chain entered by starting in the state with all components  
 working.

$\Rightarrow$  No such  $s$ , i.e.,  $\delta$  leaves component  $i$  alone unless  
 in state of degradation  $j_i$ . □

These two lemmas immediately prove Theorem 4.1:

Proof: (Theorem 4.1)

Let  $V_{-1}^{\text{opt}} = \min_{\delta} V_{-1}^{\delta}$ . Fix  $\delta$ .

$$\text{Lemma 4.2} \Rightarrow V_{-1}^{\delta} = \sum_{i=1}^n \sum_{j=1}^{\ell} \alpha_i^j \cdot \frac{K_i^j}{\sum_{p=0}^{\ell-1} \mu_i^p} \geq \sum_{i=1}^n C_i,$$

$$C_i = \min_{i \leq j \leq \ell} \frac{K_i^j}{\sum_{p=0}^{\ell-1} \mu_i^p} \quad \text{by convexity.}$$

But  $\sum_{i=1}^n C_i = V_{-1}^{\sigma}$  where  $\sigma$  is the hypothesized optimal policy. □

Note: No assumptions on orderings of the  $\lambda_i^j$ 's with respect to components  $i$  or  $j$  are needed. Also, the  $K_i^1 < \dots < K_i^l \forall i$  is not needed but is added since if  $K_i^j > K_i^{j+m}$ , some  $m \geq 1$ , then servicing  $i$  in degradation state  $j$  is never optimal.

Note: Theorem 2.7 holds trivially here, i.e., series =  $n$  of  $n$  system and never repair until  $n - 1$  items are left working.

To clarify some of the procedures and proofs in this section, consider Example 4.1 of Section 4.1.

Example 4.1 (continued):

series system

$n = 2$  (2 components)       $l = 2$  (0, 1, 2 = degradation states)

states: 

system up			
00	01	10	11

down			
02	20	12	21

automatic decisions

Let  $\delta$  - A     $R_1$     $R_1$                        $R_2$          $R_1$          $R_2$          $R_1$   
(as before)

Using the notation of the proof of Lemma 4.2,

$$\mathcal{W}^\delta = \langle \mathcal{W}_{00}, \mathcal{W}_{01}, \mathcal{W}_{10}, \mathcal{W}_{11} \rangle$$

$$R^\delta = \begin{bmatrix} \frac{\lambda_1^0}{\lambda_1^0 + \lambda_2^0} & \frac{\lambda_2^1}{\lambda_1^0 \lambda_2^1} & \frac{\lambda_1^1}{\lambda_1^1 \lambda_2^0} & 0 \\ \frac{\lambda_2^0}{\lambda_1^0 + \lambda_2^0} & \frac{\lambda_1^0}{\lambda_1^0 + \lambda_2^1} & \frac{\lambda_2^0}{\lambda_1^1 + \lambda_2^0} & \frac{\lambda_1^1}{\lambda_1^1 + \lambda_2^1} \\ \frac{\lambda_1^0}{\lambda_1^0 + \lambda_2^0} & 0 & 0 & \frac{\lambda_2^1}{\lambda_1^1 + \lambda_2^1} \\ 0 & \frac{\lambda_1^0}{\lambda_1^0 + \lambda_2^1} & \frac{\lambda_2^0}{\lambda_1^1 + \lambda_2^0} & 0 \end{bmatrix}$$

$$P^* Q_1 = \frac{\mathcal{W}_{00}}{\lambda_1^0 + \lambda_2^0} + \frac{\mathcal{W}_{10}}{\lambda_1^1 + \lambda_2^0} + \frac{\mathcal{W}_{01}}{\lambda_1^0 + \lambda_2^1} + \frac{\mathcal{W}_{11}}{\lambda_1^1 + \lambda_2^1}$$

$$\sum_{j=1}^2 \left( \sum_{p=0}^{j-1} \frac{1}{\lambda_1^p} \right) \sum_{s: R_1} \text{in } s \quad P_{\cdot s} = \frac{1}{\lambda_1^0} [P_{\cdot 10} + P_{\cdot 11}] + \left[ \frac{1}{\lambda_1^0} + \frac{1}{\lambda_1^1} \right] [P_{\cdot 20} + P_{\cdot 21}]$$

$s_1 = j$

$$= \frac{1}{\lambda_1^0} \left[ \frac{\lambda_1^0}{\lambda_1^0 + \lambda_2^0} \mathcal{N}_{00} + \frac{\lambda_2^0}{\lambda_1^1 + \lambda_2^0} \mathcal{N}_{10} + \frac{\lambda_1^0}{\lambda_1^0 + \lambda_2^1} \mathcal{N}_{01} \right]$$

$$+ \left[ \frac{1}{\lambda_1^0} + \frac{1}{\lambda_1^1} \right] \left[ \frac{\lambda_1^1}{\lambda_1^1 + \lambda_2^0} \mathcal{N}_{10} + \frac{\lambda_1^1}{\lambda_1^1 + \lambda_2^1} \mathcal{N}_{11} \right]$$

so,

$$P^* Q_1 - \sum_{j=1}^2 \left( \sum_{p=0}^{j-1} \frac{1}{\lambda_1^p} \right) \sum_{\substack{s: R_1 \\ s_1=j}} \text{in } s \text{ } P \cdot s = \mathcal{N}_{10} \left\{ \frac{\lambda_1^1}{\lambda_1^1 + \lambda_2^0} + \frac{\lambda_2^0}{\lambda_1^1 + \lambda_2^0} \right\} + \frac{\lambda_1^1}{\lambda_1^1 + \lambda_2^1} \mathcal{N}_{11}$$

$$= (1_{(00)} - R_{00}) \mathcal{N}_{10} + (1_{(01)} - R_{01}) \mathcal{N}_{11}$$

$$= 0.$$

Set of policies which could possibly be optimal:

$V_1^\delta$	Component		Correspond. Decisions, $\delta$			$\mathcal{E} =$ {ergodic states}
	#1	#2	01	10	11	
$\frac{K_1^1}{\mu_1} + \frac{K_2^1}{\mu_2}$	repair in degradation state 1	repair in state 1	$R_2$	$R_1$	--	{10,01}
$\frac{K_1^1}{\mu_1} + \frac{K_2^2}{\mu_2 + \mu_1}$	repair in state 1	repair in state 2 (failed)	A	$R_1$	$R_1$	{10,01,11,02}
$\frac{K_1^2}{\mu_1 + \mu_1} + \frac{K_2^1}{\mu_2}$	repair in state 2 (failed)	repair in state 1	$R_2$	A	$R_2$	{10,01,11,20}
$\frac{K_1^2}{\mu_1 + \mu_1} + \frac{K_2^2}{\mu_2 + \mu_2}$	repair in state 2 (failed)	repair in state 2 (failed)	A	A	A	{02,10,01,11,20,12,21}

Suppose  $C_1 = \frac{K_1^1}{\mu_1}$ ,  $C_2 = \frac{K_2^2}{\mu_2 + \mu_1}$ . Then the optimal policy would be

01    10    11  
A     $R_1$      $R_1$     by Theorem 4.1.

Note:

(1) Given the general form of an optimal policy in this case, it is clear that one never repairs more than one item at a time in

any state since only one item at a time can change degradation state and repair is done immediately and instantaneously upon entry into the "key" state for repair.

(2) If the components are identical, then the optimal policy is to repair whenever any component reaches a certain degradation level

$$d = \operatorname{argmin}_{1 \leq j \leq l} \frac{K^j}{\sum_{p=0}^{j-1} \mu^p}; \quad K_i^j \equiv K^j \quad \forall i$$

$$\mu_i^j \equiv \mu^j \quad \forall j.$$

(3) Theorem 4.1 is so intuitive, there ought to be an easier way to prove it, i.e., without having to use the stationary probability equations  $p^*(I - Q_0) = 0$ , using only independence, exponentiality, and series structure. If such a proof could be developed, one should then easily be able to show Theorem 4.1 is true for the case of different numbers of degradation states per component. I conjecture that the result also holds for "blocks" of components in series, [reference Chapter VI for Basic Model case].

### 3. The Parallel Case, Identical Components

This section treats the Degradation Model for the case of a parallel system. Identical components are assumed to simplify the system to the point where general results could be obtained for  $V_{-1}$  using theorems similar to those encountered in solving the

Basic Model. These state that, since we have a parallel system and instantaneous repair,

- (i) one will never repair more than a single unit at a time in an ergodic state in a  $V_{-1}$  optimal policy
- (ii) one will never repair until the system gets down to a single unit working.

Showing these two results would limit the number of possible  $V_{-1}$  optimal policies to a manageable size. Precise  $V_{-1}$  results will be presented in some examples for  $l = 2$  and  $l = 3$ , ( $l =$  number of degradation states) and compared to Basic Model results.

In the Basic Model we were able to state Theorem 2.1 which expressed  $V_{-1}^\delta$  for any possible  $\delta$  as a convex combination of  $V_{-1}^{\gamma_i}$ , where  $\gamma_i$ 's are policies which involve repair of at most one unit at a time in any state. For the parallel case with different components these  $V_{-1}^{\gamma_i}$  were such that

$$V_{-1}^{\gamma_i} \in \left\{ \frac{K_i + p}{\mu_i}, \text{ some } i; \quad \sum_{j \in s} \frac{K_j}{\mu_j}, \text{ some } s : 2 \leq |s| \leq n \right\},$$

or, for identical components,

$$V_{-1}^{\gamma_i} \in \left\{ \frac{K+p}{\mu}, \frac{2K}{\mu}, \frac{3K}{\mu}, \dots, \frac{nK}{\mu} \right\}.$$

Given no fixed charge, the  $V_{-1}$  optimal policy was extremely simple: do nothing until one component is left working and then, if the penalty cost is large enough, repair one unit ( $\delta_1$ ) (continuing this policy on forever) or if  $p$  is small enough, let the system fail and then repair ( $\delta_0$ ) one unit ad infinitum.

Under  $\delta_1$ ,  $\mathcal{E} = \text{set of ergodic states} = \{1\}$ ,

under  $\delta_0$ ,  $\mathcal{E} = \text{set of ergodic states} = \{0\}$ ,

where state  $i$  indicates the number of working components.

In the series case of the Degradation Model, we were able to eliminate most policies  $\delta$  from being  $V_{-1}$  optimal by again

expressing  $V_{-1}^\delta$  for any  $\delta$  as  $\sum_{i=1}^n C_i^\delta$ , where the  $C_i^\delta$  are convex combinations of  $\frac{K_i^j}{\sum_{p=0}^{j-1} \mu_i^p}$ . An optimal policy was found by computing

$$C_i^{\text{opt}} = \min_{1 \leq j \leq \ell} \frac{K_i^j}{\sum_{p=0}^{j-1} \mu_i^p} \text{ for } 1 \leq i \leq n, \text{ the optimal policy being to}$$

repair component  $i$  whenever it reaches the  $j_i^{*th} = \operatorname{argmin}_{1 \leq j \leq \ell} \frac{K_i^j}{\sum_{p=0}^{j-1} \mu_i^p}$

state of degradation. In the identical component case, this corresponds to repairing any component that reaches the  $j^{*th}$

degradation state,  $j^* = \operatorname{argmin}_{1 \leq j \leq \ell} \frac{K^j}{\sum_{p=0}^{j-1} \mu^p}$ .

Unfortunately, in the parallel case as is exhibited in the example below, there are no such "nice" convex combinations of simpler policies that  $V_{-1}^{\delta}$  can be written as for any  $\delta$ . The series result depended heavily on the fact that when the system (and, thus, any component) failed, repair of the failed component had to be done immediately, a fact obviously not true for a parallel system. In this case, even if  $p = 0$ ,  $V_{-1}^{\delta} \neq \delta$  is not a convex combination of  $\frac{\kappa^j}{\sum_{p=0}^{j-1} \mu^p}$ , or a sum of such combinations. However, the relationship

between them still appears to be the principal factor in determining at what degradation state to repair a component.

Since the model is parallel, repair is instantaneous and there is no fixed charge, one intuitively expects that as with the Basic Model that repair will never be undertaken on more than one unit at a time and repair will never be done until there is only one unit left working. These results, though true for the Basic Model identical component case are much more difficult to prove given failure of the Basic Model theorems or likenesses thereof and the increased complexity of model structure in the Degradation case. I conjecture these two results are true even if the components are different but leave it as a topic for future research (see Chapter VII).

Example 4.2:  $n = 2, l = 2, L = 0$ , parallel system with identical components.

	system up					system down
<u>States:</u>	200	110	020	101	011	002
<u>Decisions:</u>	-	$A, R_1^{(1)}$	$A, R_1^{(1)}, R_2^{(1)}$	$A, R_1^{(2)}$	$A, R_1^{(1)}, R_1^{(2)}, R_1^{(1)\&(2)}$	$R_1^{(2)}, R_2^{(2)}$

The optimal policy will be found by enumeration of all the 17 possible different  $V_{-1}$ 's. Table 4.1 lists the possibilities, the set of ergodic states induced, and the  $V_{-1}$ 's.

Let

$K^i$  = cost to repair a component in  $i^{\text{th}}$  degradation state

$\mu_i$  = expected length of time spent in the  $i^{\text{th}}$  degradation state by a component.

TABLE 4.1 - ENUMERATION OF POLICIES FOR EXAMPLE 4.2

Policy No.	$\delta$					$\mathcal{E}$ = (ergodic states)	$v_{-1}$
	110	020	101	011	022		
* 1	-	-	-	$R_1^{(1)}$	-	{011}	$\frac{K^1}{\nu_0}$
* 2	-	-	-	A	$R_1^{(2)}$	{011,002}	$\frac{K^2+p}{\nu_0+\nu_1}$
* 3	-	A	A	$R_1^{(2)}$	-	{020,101,011}	$\frac{K^2}{\nu_0 + \frac{\nu_1}{2(\nu_0+\nu_1)}}$
(1) 4	-	$R_1^{(1)}$	A	$R_1^{(2)}$	-	{020,101,011}	$\frac{\nu_1 K^1 + \nu_0 K^2}{\nu_0(\nu_0+\nu_1)}$
(23) 5	A	A	A	A	$R_2^{(2)}$	{110,020,101,011,002}	$\frac{2(\nu_0+\nu_1)K^2+p(\nu_0+\nu_1)}{\frac{3}{2}\nu_0^2 + \frac{3}{2}\nu_0\nu_1 + \frac{3}{2}\nu_1^2}$
(1) 6	A	$R_2^{(1)}$	A	$R_1^{(2)}$	-	{110,020,101,011}	$\frac{2\nu_1 K^1 + \nu_0 K^2}{\nu_0(\nu_0 + \frac{3}{2}\nu_1)}$
(3) 7	A	A	$R_1^{(2)}$	$R_1^{(2)}$	-	{110,020,101,011}	$\frac{2K^2}{\nu_0+\nu_1}$
(1) 8	A	$R_1^{(1)}$	$R_1^{(2)}$	-	-	{110,020,101}	$\frac{\nu_0 K^2 + \nu_1 K^1}{\nu_0(\frac{\nu_0}{2} + \nu_1)}$
(13) 9	A	A	A	$R^{(1)(2)}$	-	{110,020,101,011}	$\frac{(\nu_0+\nu_1)(K^1+K^2)}{\frac{3}{2}\nu_0^2 + \frac{3}{2}\nu_0\nu_1 + \frac{3}{2}\nu_1^2}$
(1) 10	A	$R_2^{(1)}$	$R_1^{(2)}$	-	-	{110,020,101}	$\frac{\nu_0 K^2 + 2\nu_1 K^1}{\nu_0(\frac{\nu_0}{2} + \frac{3}{2}\nu_1)}$
(123) 11	A	$R_2^{(1)}$	A	A	$R_2^{(2)}$	{110,020,101,011,002}	$\frac{2(\nu_1 K^1 + \nu_0 K^2) + \nu_0 p}{\nu_0(\frac{3}{2}\nu_0^2 + \frac{3}{2}\nu_1^2)}$
(1) 12	$R_1^{(1)}$	-	-	-	-	{110}	$\frac{2K^1}{\nu_0}$
(13) 13	A	A	$R_1^{(2)}$	$R^{(1)(2)}$	-	{110,020,101,011}	$\frac{\nu_1 K^1 + (\nu_0+\nu_1)K^2}{\frac{\nu_0}{2} + \frac{3}{2}\nu_0\nu_1 + \frac{\nu_1}{2}}$
(1) 14	A	$R_1^{(1)}$	A	$R^{(1)(2)}$	-	{110,020,101,011}	$\frac{(\nu_0+\nu_1)K^1 + \nu_0 K^2}{\nu_0(\frac{3}{2}\nu_0 + \nu_1)}$
(1) 15	A	$R_2^{(1)}$	A	$R^{(1)(2)}$	-	{110,020,101,011}	$\frac{(2\nu_1+\nu_0)K^1 + \nu_0 K^2}{\nu_0(\frac{3}{2}\nu_0 + \frac{3}{2}\nu_1)}$
(12) 16	A	$R_1^{(1)}$	A	A	$R_2^{(2)}$	{110,020,101,011,002}	$\frac{\nu_1 K^1 + 2\nu_0 K^2 + \nu_0 p}{\nu_0(\frac{3}{2}\nu_0 + 2\nu_1)}$
(23) 17	A	A	$R_1^{(2)}$	A	$R_2^{(2)}$	{110,020,101,011,002}	$\frac{(\nu_0+2\nu_1)K^2 + \nu_1 p}{\frac{\nu_0}{2} + \frac{3}{2}\nu_0\nu_1 + \frac{3}{2}\nu_1^2}$

\* = possible optimal policy.  
 (i) = losses to policy i.  
 (ijk) = losses to i, j, or k.

The following facts/patterns appear from Table 4.1:

- (1) The optimal policies are either:
  - a. Do nothing until one left working in first degradation state, then repair it.
  - b. Do nothing until one left working in first degradation state and then repair the failed unit.
  - c. Do nothing until system fails, then repair a component.
- (2) Never repair until one component left working and never repair more than one unit at a time.
- (3)  $\mu_0$  divides denominator of  $V_{-1}^\delta \Rightarrow$  do not need policy 3 to dominate  $\delta$ .
- (4) No "p"  $\Rightarrow$  do not need 2 to dominate  $\delta$ .
- (5) No "k<sup>1</sup>" i.e., never repair in first degradation state  $\Rightarrow$  do not need policy 1.
- (6) Although the optimal policies are simple and intuitive, the  $V_{-1}^\delta$ 's for other  $\delta$  are not convex combinations of these policies. They are, however,  $\geq$  some convex combinations of these policies.

Given the results of Example 4.2 and others tested, three facts appear to be true in the identical component case and probably in general for a parallel system. These are:

- (1) Never repair more than one unit at a time in an ergodic state.

- (2) Never repair until there is one unit or less working.
- (3) The  $V_{-1}$  optimal policy for a parallel degradation model with  $n$  identical components and  $\ell$  states of degradation is the same as that for a 2-component model with  $\ell$  states of degradation.

Proofs of these facts using Degradation Model stationary probability equations to eliminate certain policies as convex combinations of others as in the Basic Model case are nearly impossible, due to the greatly increased complexity of their structure. They are left as a topic for future research (Chapter VII).

#### 4. Extension of Basic Model to Include Erlang Component Lifetimes

The purpose of this section is to describe how the Degradation Model can be used to extend the Basic Model to the case where component lifetimes are no longer exponential, but Erlang. An example will be solved of such a model where components are non-exponential. Its solution will demonstrate that changing component lifetime distributions does change results given for exponential - in particular it is no longer true that, for no fixed charge, one never repairs more than one unit simultaneously.

Suppose now that we have the Basic Model with  $L = 0$  except that  $L_i$ , the random variable representing the lifetime of the  $i^{\text{th}}$  component, has an Erlang distribution, i.e.,

$$\begin{aligned}
p\{L_i \leq t\} &= \int_0^t \frac{\lambda^n z^{n-1}}{(n-1)!} e^{-\lambda z} dz = G_{\lambda, n}(t) \\
&= 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}, \quad n \geq 1, \text{ an integer, } \lambda > 0.
\end{aligned}$$

Definition:  $G_{\lambda, n}(t)$  is a gamma distribution with parameters  $\lambda, n$ , where  $\lambda, n > 0$ . If  $n$  is an integer  $\geq 1$ , then  $G_{\lambda, n}(t)$  is called an Erlang distribution. If  $n = 1$ , then  $G_{\lambda, 1}(t)$  is exponential.

If  $n \leq 1$ ,  $G_{\lambda, n}(t)$  is a DFR (decreasing failure rate) distribution and if  $n \geq 1$ , it is IFR (increasing failure rate) where:

Definition: Suppose  $F(t)$  is a probability distribution function with density  $f(t)$ . Then the failure rate at time  $t$  at a unit whose lifetime distribution is  $F(t)$  is  $r(t) = f(t)/1-F(t)$ .

Intuitively, this is the rate of change of probability of unit failure at time  $t$ . (DFR)/IFR indicates a (lesser)/greater chance of failure with age. Exponential distributions have a constant failure rate, i.e., the chance of failure is independent of how long the component has been working.

Up to now, the Basic Model has assumed components with constant failure rate. Components having  $G_{\lambda, n}$  distribution allow modeling of systems where components might have IFR (for  $n \geq 1$ ) distributions. Figure 4.2 indicates the failure rate curves for  $G_{\lambda, 1}$ ;  $G_{\lambda, 2}$  and  $G_{\lambda, 4}$  for  $\lambda = 1$ .

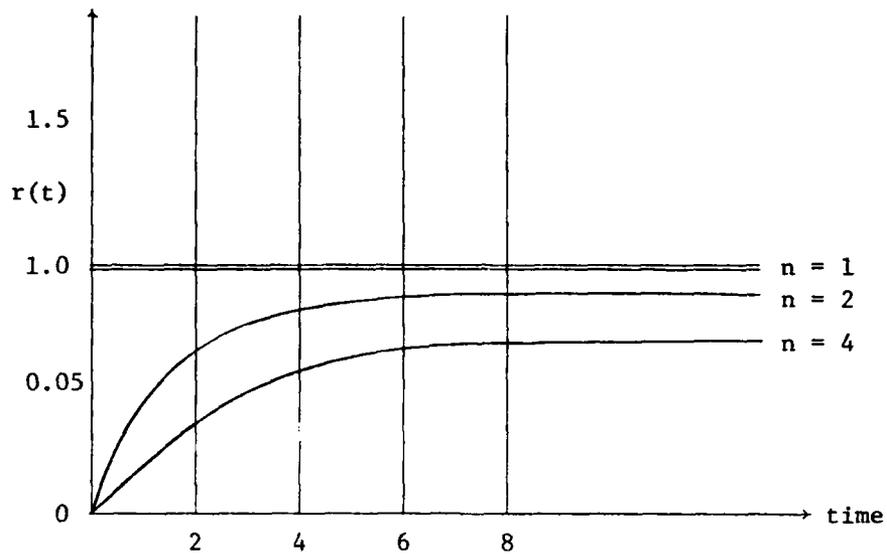


FIGURE 4.2 - FAILURE RATE CURVES FOR GAMMA DISTRIBUTION FOR  $\lambda = 1$ .

Notice that as  $n$  increases, the probability density for  $G_{\lambda,n}(t)$  becomes more peaked, i.e., the variance of the lifetime from its expected value gets smaller as  $n$  gets larger. Since  $G_{\lambda,n}(t)$  has a mean (expected) value of  $1/n\lambda$ , the distributions  $G_{\lambda/n,n}(t)$  and  $G_{\lambda,1}(t) = 1 - e^{-\lambda t}$  will have the same mean,  $1/\lambda$  but  $G_{\lambda/n,n}(t)$  will have a much lower variance (more confident of expected value).

Recall from elementary probability distribution theory that if a random variable  $X \sim G_{\lambda,n}(t)$  [ $p(X \leq t) = G_{\lambda,n}(t)$ ], then  $X$  can be written as a sum of  $n$  exponential random variables, each with

mean  $1/\lambda$ . Thus,  $X = \sum_{i=1}^n X_i$  where  $X_i \sim 1 - e^{-\lambda t}$ . It is this idea

which allows the use of the Degradation Model in extending the Basic Model to the Erlang case by allowing components to pass through  $n$  "fictitious" states of degradation before failure. Of course, unlike the Degradation Model in which these intermediate states can be observed and actions taken in them, decisions here are still limited to the instant of component failures since the intermediate states do not really exist. Thus, if we want to extend the Basic Model to the case where the  $n$  independent components have lifetimes  $G_{\lambda_i, k_i}(t) = p\{L_i \leq t\}$ , then a Degradation Model for  $n$  components can be used with the following modifications:

- (1) Allow  $k_i$  degradation states for component  $i$ , ( $k_i^{\text{th}} = \text{failure}$ ) each with mean holding time  $1/\lambda_i$ .
- (2) Restrict repair decisions to times when a component fails (enters state  $k_i$ ), otherwise do nothing (use same decision in states with same configuration of failed components).
- (3) Component  $i$  has mean life  $1/k_i \lambda_i$  here. To get desired mean life, can adjust  $\lambda_i$ . ( $\lambda_i = \lambda/k_i$  gives mean  $1/\lambda$ ).

Given the limitations on decisions, the theorems for Degradation Model solutions will not apply here but the model as stated is a Markov decision chain and can be solved as such either by hand or on the computer.

To illustrate, consider the following simple example:

Example 4.3. Basic Model,  $n = 2$ ,  $L = 0$ , parallel system.

Let

$$L_1 \sim G(\mu_1/2, 2) \quad \text{mean} = 1/\mu_1$$

$$L_2 \sim G(\mu_2, 1) = 1 - e^{-\mu_2 t} \quad \text{mean} = 1/\mu_2$$

be  $M_1$ , the extended Basic Model.

Let

$$L_1 \sim 1 - e^{-\mu_1 t} \quad \text{mean} = 1/\mu_1$$

$$L_2 \sim 1 - e^{-\mu_2 t} \quad \text{mean} = 1/\mu_2$$

be  $M_2$ , the standard Basic Model.

$M_2$ has states:	12	1	2	0	(Basic Model Notation)
decisions:	--	$A, R_2$	$A, R_1$	$R_1, R_2, R_{12}$	
$M_1$ has states:	$\overbrace{00 \ 10 \ 01 \ 11}^{\text{both up (12)}}$	$\overbrace{02 \ 12}^{\#1 \text{ up (1)}}$	$\overbrace{20 \ 21}^{\#2 \text{ up (2)}}$	$\overbrace{22}^{\text{failed (0)}}$	
decisions:	--	$\left. \begin{array}{l} A, R_2 \\ \text{same decn.} \\ \text{applied to} \\ \text{both} \\ 02, 12 \end{array} \right\}$	$\left. \begin{array}{l} A, R_1 \\ \text{same decn.} \\ \text{applied to} \\ \text{both} \\ 20, 21 \end{array} \right\}$	$R_1, R_2, R_{12}$	

The set of possible policies is the same for both  $M_1$  and  $M_2$ .

Below, in Table 4.3 is a list of possible policies and corresponding

$V_{-1}$ 's under  $M_1$  and  $M_2$ .

TABLE 4.3 - TABLE OF  $V_{-1}$ 's FOR  $M_1$  AND  $M_2$  OF EXAMPLE 4.3.

Policy			$\mathcal{E}_{M_1}$	$\mathcal{E}_{M_2}$	$\frac{M_1}{V_{-1}}$	$\frac{M_2}{V_{-1}}$
1	2	0				
-	-	$R_1$	{0}	{22,12}	$\frac{K_1+p}{\mu_1}$	$\frac{K_1+p}{\mu_1}$
-	-	$R_2$	{0}	{22,21}	$\frac{K_2+p}{\mu_2}$	$\frac{K_2+p}{\mu_2}$
$R_2$	$R_1$	-	{1,2}	{10,01,11,02 20,12, 21}	$\frac{K_1}{\mu_1} + \frac{K_2}{\mu_2}$	$\frac{K_1}{\mu_1} + \frac{K_2}{\mu_2}$
A	A	$R_{12}$	{0,1,2}	{22,10,01,11, 02,12,20,21}	$\frac{(\mu_1+\mu_2)(K_1+K_2+p)}{\mu_1^2+\mu_1\mu_2+\mu_2^2+\frac{2\mu_1^2(\mu_2+\frac{\mu_1}{2})}{\mu_1+\mu_2}}$	$\frac{(\mu_1+\mu_2)(K_1+K_2+p)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}$
A	$R_1$	$R_{12}$	{0,1,2}	{22,10,01,11, 02,12,20,21}		
etc., (different for $M_1, M_2$ )						
$R_2$	A	$R_{12}$	{0,1,2}	{22,10,01,11, 02,12,20,21}		

$M_1$  = revised model       $M_2$  = standard model

The important thing to notice from Table 4.3 is that, for policies where repair is only done on one component at a time in a given state,  $V_{-1}$  is the same for  $M_1$  and  $M_2$ . However, for policy  $AAR_{12}$ , which lets the system fail and repairs both components,  $V_{-1}^{AAR_{12}}$  under  $M_1$  is lower than that under  $M_2$ . The lower variance on the component lifetime of number 1, now with an increasing failure rate, makes the multiple repair policies more attractive relative to single repair ones which are always optimal for exponential cases. Thus, one would expect that it is no longer true that single repair policies are always optimal and indeed it is so:

Let  $K_1 = K_2 = \mu_1 = \mu_2 = 10$      $p = 2$  in Example 4.3.

$$\text{Table 4.3} \Rightarrow \frac{K_1+p}{\mu_1} = 1.2 = \frac{K_2+p}{\mu_2}$$

$$\frac{K_1}{\mu_1} + \frac{K_2}{\mu_2} = 1 + 1 = 2$$

$$V_{-1}^{AAR_{12}} = \frac{(20)(22)}{300+150} = \frac{440}{450} < 1$$

so, a policy with  $R_{12}$   
in 0 must  
be optimal.

This concludes the results for the Degradation Model. A summary along with further conclusions and comparisons to other models is found in Chapter VII (Conclusions).

## CHAPTER V

### THE NON-INSTANTANEOUS REPAIR MODEL

#### 1. Description of Model

Up to now, in both the Basic Model and the Degradation Model, attention has been focused on the components. Such factors as what states they can be observed in, what type of system they make up, their lifetime distributions and mean lifetimes have been looked at. The repair assumption throughout has been that it is done instantaneously and can be done as often as desired (unlimited service). The purpose of the Non-instantaneous Repair Model is to treat cases where repair is non-instantaneous (exponential service in most cases) and the number of servers may be finite. Comparisons to the Basic Model (instantaneous case) can then be made.

Given the purpose of investigating repair assumption effects, component assumptions were chosen to be as simple as possible, i.e., Basic Model assumptions. The Degradation Model could be modified to incorporate non-instantaneous repair in a similar fashion but the number of states would be large, results would be hard to come by except on the computer, and attention would be diverted from the repair aspects.

The Non-instantaneous Repair Model has the following Coherent System Repair Model parameters:

States: each component can be in one of three states:  
working, failed and under repair, or failed  
and not under repair.

Repair: non-instantaneous (exponential or Erlang), when completed it brings a component to "new" condition. The number of servers is  $1 \leq s \leq \infty$  and each has identical service distributions,  $R_s \sim 1 - e^{-(1/\sigma)t}$ , mean  $\sigma$ .

Component Lifetimes: exponential,  $L_i \sim 1 - e^{-\lambda_i t}$ .

Note that  $\begin{bmatrix} s = \infty \\ \sigma = 0 \end{bmatrix}$  gives the instantaneous repair (Basic Model) case.

Again, given the increased complexity of the model, results are much more difficult to obtain. As with the Degradation Model,  $V_{-1}$  results only are looked at. Section 2 considers the case of identical components (k-of-n system), no fixed charge, exponential repair and a single server. In Section 3, results of Section 2 are compared to multiple server results. The final section demonstrates other possible formulations of the model to include Erlang service, non-identical servers, or component degradation states.

Unlike the Degradation Model's k-fold increase in the number of parameters over the Basic Model's, the Non-instantaneous Repair Model requires only the addition of three new parameters to completely specify the model (in the exponential repair case).

These are:

- (1) mean repair time  $\sigma \geq 0$
- (2) labor cost  $\ell$  | server | unit time
- (3) number of servers,  $s \geq 1$

in addition to the Basic Model carryovers:

- (4) fixed charge  $L \geq 0$
- (5) penalty cost for system failure  $p > 0$
- (6) mean lifetime of component  $i$ ,  $\mu_i = \frac{1}{\lambda_i}$   $i = 1, \dots, n$
- (7) number of components,  $n$
- (8) cost to repair  $i^{\text{th}}$  component,  $K_i$ ,  $i = 1, \dots, n$
- (9) type of system (specification of states for which penalty is incurred) - series case impossible as with Basic Model.

The objective is to minimize  $V_{-1}$ , the long run expected cost per unit time. It is interesting to note that if  $K_i \equiv 0 = \ell$  and  $p = \sigma$ , the objective becomes minimizing the fraction of time the system is failed as is used by Smith [29] in his Optimal Repair of a Series System model.

A state in this model can be denoted by a vector  $\underline{s}$  of 2 parts;  $\underline{s} = \langle s_1, s_2 \rangle$  where

$s_1$  = vector of which components are working

$s_2$  = vector of which are in service.

$|s_2| \leq s = \text{number of servers. } s_1 \cup s_2 \subseteq \{1, \dots, n\}$ . Changes of state occur when either one of the working components fails or repair on one of the components in service is completed. At such an instant, decisions can be made to repair some subset of the components which

are failed but not in service if the servers are not all busy. Once repair is started on a component, it must be completed. In the special case of a k-of-n system with identical components, discussed in Sections 2 and 3, the vector  $\underline{s}$  has only two components,  $\underline{s} = \langle i, j \rangle$  where  $i$  = number of working components;  $j$  = number of components in service,  $i + j \leq n$ ,  $j \leq s$ .

The total number of states in an n-component Non-instantaneous Repair Model is at most  $3^n$  in the case of  $s \geq n$ . In cases of small  $s$ , the number can often be considerably lower than that, although still greater than the number of states in the corresponding Basic Model. Thus, although results here are just as difficult (if not more so) to come by as for the Degradation Model, on the computer they will be easier due to the lesser state space enlargement.

Repair times being exponential, each state as defined previously has exponential holding time so this model qualifies as a continuous time Markov decision chain with infinite planning (time) horizon. Notation from the Basic Model in Figure 2.1 all applies here with the addition of repair parameters  $s, \ell, \sigma$  defined previously. The Markov chain specified by the Non-instantaneous Repair Model with given parameters and exponential repair is as follows:

State Space:

states = which components are working, failed, or in service

$$\underline{s} = \langle \underline{s}_1, \underline{s}_2 \rangle \text{ where } \underline{s}_1, \cup \underline{s}_2 \subseteq \{1, \dots, n\}$$

$\underline{s}_1$  = which components are working

$\underline{s}_2$  = which components are in service ( $|\underline{s}_2| \leq s$ )

If system = k-of-n and components are identical, then

$$\underline{s} = \langle i, j \rangle, \quad i = \text{number of working components}$$

$$j = \text{number in service} \quad j \leq s$$

$$i + j \leq n.$$

Decision Space:

Let  $\Omega = \{1, \dots, n\}$ . Possible decisions in state

$$\underline{s} = \langle \underline{s}_1, \underline{s}_2 \rangle \text{ are } R_{\Omega_s}$$

where

$$(i) \quad \Omega_s \subseteq \Omega \sim \underline{s}_1 \cup \underline{s}_2 \quad (\text{repair failed components not in service already})$$

and

$$(ii) \quad |\Omega_s \cup \underline{s}_2| \leq s \quad (\text{number in service} \leq \text{number of servers}).$$

Restrictions:

-  $\Omega_s \neq \phi$  if system down in  $\underline{s}$  and no components under repair.

-  $\Omega_s = \phi$  ( $R_{\Omega_s} = A =$  "do nothing") if  $\underline{s}_2 = s$ .

Transition Structure:

The transition matrix,  $Q_0$ : (assuming  $\delta : R_{\Omega_s}$  in  $s$ ).

Fix  $s = \langle s_1, s_2 \rangle$

$$(Q_0)_{s, \langle s_1 - i, s_2 \cup \Omega_s \rangle}^{\delta} = \frac{\lambda_1}{\sum_{j \in s_1} \lambda_j + (1/\sigma) |s_2 \cup \Omega_s|}, \quad i \in s_1$$

$$(Q_0)_{s, \langle s_1, s_2 \cup \Omega_s - i \rangle}^{\delta} = \frac{1/\sigma}{\sum_{j \in s_1} \lambda_j + (1/\sigma) |s_2 \cup \Omega_s|}, \quad i \in s_2 \cup \Omega_s$$

$$(Q_0)_{s, t}^{\delta} = 0, \quad \text{if } t \neq \langle s_1 - i, s_2 \cup \Omega_s \rangle, \text{ some } i \in s_1$$

or  $\neq \langle s_1, s_2 \cup \Omega_s - i \rangle,$

some  $i \in s_2 \cup \Omega_s$

$$Q_{1, s}^{\delta} = \frac{1}{\sum_{j \in s_1} \lambda_j + (1/\sigma) |s_2 \cup \Omega_s|}.$$

Cost Structure:

$$R_{0, s}^{\delta} = \left\{ \begin{array}{l} L(1 - I_{\Omega_s}^{\phi}) + \sum_{i \in \Omega_s} K_i + |s_2 \cup \Omega_s| \cdot \ell \cdot Q_{1, s}^{\delta}, \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{if system up in } s \\ \\ L + p + \sum_{i \in \Omega_s} K_i + |s_2 \cup \Omega_s| \cdot \ell \cdot Q_{1, s}^{\delta}, \end{array} \right.$$

if system down in  $s$  and  $s$  is a min cut set of the system

where  $I_{\Omega_s}^{\phi} = \begin{cases} 1, & \Omega_s = \phi \\ 0, & \text{o.w.} \end{cases}$

Objective Function:

$$V_{-1}^{\delta} = \frac{P_{\delta}^* R_0^{\delta}}{P_{\delta}^* Q_1^{\delta}} \left( \begin{array}{l} \text{any possible optimal } \delta \text{ will form} \\ \text{a Markov chain with a single} \\ \text{irreducible set of ergodic states} \end{array} \right).$$

The following examples are in the cases of identical components with "s" servers:

Example 5.1:  $n = 2, s = 2, \text{ parallel}, L = 0$

states:	2,0	1,0	1,1	0,0	0,1
decisions:	A	A, R <sub>1</sub>	A	R <sub>1</sub> , R <sub>2</sub>	R <sub>1</sub> , A
cost:	0	$0, K+l \cdot \frac{1}{\lambda+\sigma-1}$	0	$K+l\sigma+p,$ $2K+l\sigma+p$	$K+l\sigma+p,$ $l\sigma+p$

R<sub>i</sub> = initiate repair of i components.

Notes:

- state 0,2 is not listed because, although it is theoretically feasible, it will never be reached given a starting state with all components up and the given decision possibilities.

- in state 0,1, even though the system is failed, the decision to "do nothing" is allowed since repair has already been initiated on some other component and there is no way to get the system operative instantaneously under this model. In state 0,0, however, some form of repair is required.
- The penalty cost is incurred once each time the system fails. This is due to the fact that, once the parallel system fails, all components are failed so the next state change must be due to a service completion, sending the system to a working state.

Example 5.2:  $n = 3, s = 1, k = 2$  (2 of 3 system),  $L = 0$

	system up			system down		
states:	3,0	2,0	(2,1)	1,0	1,1	0,1 <sup>1</sup>
decisions:	A	A, R <sub>1</sub>	A	R <sub>1</sub> (since system failed)	A	A
cost:	0	$0, K+l \cdot \frac{1}{2\lambda+\sigma-1}$	-	$K+p+l \cdot \frac{1}{\lambda+\sigma-1}$	$p+l \cdot \frac{1}{\lambda+\sigma-1}$	$l\sigma$

(1, j) denotes a state which will never be entered.

Notes:

- unlike the Basic Model for a k-of-n system with  $k > 1$ , states for which there are less than  $k - 1$  components

working can be reached due to the non-instantaneous repair.

- The penalty cost is now incurred each time the system enters a state in which  $k - 1$  units are working. This occurs either at a system failure or at the instant of a service completion still leaving the system down.
- There are actually only two possible policies here:
  - a) do nothing when 2 components are left, or
  - b) repair when 2 components are left.

under a):  $\mathcal{E} = \{2,0 \quad 1,0 \quad 0,1\}$

$$\begin{array}{c}
 \\
 2,0 \\
 1,0 \\
 0,1
 \end{array}
 \begin{bmatrix}
 & 2,0 & 1,0 & 0,1 \\
 \left[ \begin{array}{ccc}
 0 & 1 & 0 \\
 \frac{1/\sigma}{\lambda+1/\sigma} & 0 & \frac{\lambda}{\lambda+1/\sigma} \\
 0 & 1 & 0
 \end{array} \right]
 \end{bmatrix}$$

These examples will appear later in the chapter.

In the parallel case, the instances of penalty cost assessment are clear - whenever the system fails. For the  $k$ -of- $n$  model, the fact that states with less than  $k - 1$  components working can be reached means that this penalty could be assessed several times while the system is still down, how often depending upon the decision the modeler makes on which states to apply  $p$  in. If it is desired to actually have  $p$  assessed only once per system failure, this can be accomplished by dividing the states

with  $k - 1$  units into two categories: those due to system failure and those not. I assume that the  $p$  gets incurred in such states no matter what the cause of entry to stick to the model framework introduced previously.

## 2. Exponential Service, Single Server Results

In this section, we look at the Non-Instantaneous Repair Model in which the components are identical,  $L = 0$ , and service times are exponential for the case of a single server ( $s = 1$ ). A policy is found which is  $V_{-1}$  optimal for first the parallel case and then the  $k$ -of- $n$  case for  $k < n$ . Changing behavior of this policy for variable  $k$  and/or  $n$  is looked at.

Some notation is now needed:

Definition: Let the policy  $R^{(j)}$  (for the aforementioned single server model) denote the policy which repairs whenever the number of working components is  $\leq j$  and the server is idle.

Let

$$z_j \triangleq \mu_j + j\sigma\mu^{j-1} + j(j-1)\sigma^2\mu^{j-2} + \dots$$

$$\dots + j(j-1)\dots(3)\sigma^{j-2}\mu^2 + (j!)\sigma^{j-1}(\sigma+\mu), \quad j = 1, 2, \dots$$

$$z_0 \triangleq 1.$$

Let  $f_k$  be defined so that:  $f_1 = 1$ ,

$$f_k = \frac{(k-1)! \sigma^{k-1}}{\mu^{k-2} (\mu + (k-1)\sigma)} ; k \geq 2 .$$

The theorem giving the general  $V_{-1}$  optimal solution can now be stated:

Lemma 5.1: Suppose we have the Non-instantaneous Repair Model with a single server. Assume

- (i) identical components, k-of-n system
- (ii)  $L = 0$  (no fixed charge)
- (iii) exponential service times.

Then:

(A) the  $V_{-1}$  optimal policy is among  $R^{(j)}$ ,  $k - 1 \leq j \leq n - 1$ , policy  $R^{(j)}$  being defined previously,

(B)  $R^{(j)} < R^{(j-1)} \Leftrightarrow$

$$p \cdot \frac{(j-1)!}{j+1} \mu \sigma^{j-1} > f_k (K + \rho \sigma) \left[ \frac{z_j}{j} - \frac{\mu z_{j-1}}{j+1} \right] .$$



TABLE 5.1:  $V_{-1}^{R(j)}$  for  $k$ -of- $n$  System

System type	$k=1$ (parallel)	$k=2$	$k=3$	...	$(k=1)$	...	$k=n-1$
$R(0)$	$\frac{K+to}{u+o}$	NP	NP		NP		NP
$R(1)$	$\frac{(u+o)(K+to)+op}{\frac{u^2}{2} + o(u+o)}$	$\frac{(u+o)(K+to)+c(u)p}{\frac{u^2}{2} + o(u+o)}$			NP		NP
$R(2)$	$\frac{z_2(K+to)+2c^2p}{z_3/3}$	$\frac{z_2(K+to)+2c(c+u)p}{z_3/3}$	$\frac{z_2(K+to)+\frac{2c^2p}{3}}{z_3/3}$		NP		NP
$R(3)$	$\frac{z_3(K+to)+6o^3p}{z_4/4}$	$\frac{z_3(K+to)+6o^2(u+o)p}{z_4/4}$	$\frac{z_3(K+to)+\frac{6o^3p}{3}}{z_4/4}$		NP		NP
$R(j)$ $(j \geq 1-1)$	$\frac{z_j(K+to)+(j!)c^{j-1}p}{z_{j+1}/j+1}$	$\frac{z_j(K+to)+(j!)c^{j-1}(c+u)p}{z_{j+1}/j+1}$	$\frac{z_j(K+to)+(j!)c^{j-1}p}{z_{j+1}/j+1}$		$\frac{z_j(K+to)+(j!)c^{j-1}p}{z_{j+1}/j+1}$		NP
$R(n-1)$	$\frac{z_{n-1}(K+to)+(n-1)!c^{n-1}p}{z_n/n}$	$\frac{z_{n-1}(K+to)+(n-1)!c^{n-2}(c+u)p}{z_n/n}$	$\frac{z_{n-1}(K+to)+(n-1)!c^{n-1}p}{z_n/n}$		$\frac{z_{n-1}(K+to)+(n-1)!c^{n-1}p}{z_n/n}$		$\frac{z_{n-1}(K+to)+(n-1)!c^{n-1}p}{z_n/n}$

NP = decision not possible (if  $j < i - 1$ )

If  $k$  is fixed and  $j \geq k$ , then  $R^{(j)} < R^{(j-1)} \Leftrightarrow$

$$\frac{z_j^{(K+\ell\sigma)+(j!)\frac{\sigma^j \cdot p}{f_k}}}{\frac{z_{j+1}}{j+1}} < \frac{z_{j-1}^{(K+\ell\sigma)+(j-1)!\frac{\sigma^{j-1} \cdot p}{f_k}}}{\frac{z_j}{j}}$$

$$\Leftrightarrow (K + \ell\sigma) \left[ \frac{z_j}{j} - \frac{z_{j-1} z_{j+1}}{j+1} \right] < \frac{p\sigma^{j-1}(j-1)!}{f_k} \left[ \frac{z_{j+1}}{j+1} - \frac{j\sigma \cdot z_j}{j} \right].$$

But, since

$$\frac{z_{j+1}}{j+1} = \frac{\mu^{j+1}}{j+1} + \sigma z_j,$$

the above is true  $\Leftrightarrow$

$$(K + \ell\sigma)\mu^j \left[ \frac{z_j}{j} - \frac{\mu}{j+1} z_{j-1} \right] < \frac{p \cdot \sigma^{j-1}(j-1)! \cdot \mu^{j+1}}{f_k \cdot (j+1)}$$

$$\Leftrightarrow p \cdot \frac{(j-1)!}{j+1} \cdot \mu\sigma^{j-1} > f_k (K + \ell\sigma) \left[ \frac{z_j}{j} - \frac{\mu}{j+1} z_{j-1} \right],$$

our desired result B. □

Collecting all the non- $j$ -dependent terms on the left hand side gives  $R^{(j)} < R^{(j-1)} \Leftrightarrow$

$$(1) \quad \frac{\rho\mu}{f_k \cdot (K+l\sigma)} > \frac{j+1}{\sigma^{j-1} (j-1)!} \left[ \frac{z_j}{j} - \frac{\mu}{j+1} z_{j-1} \right] \triangleq u(j) .$$

Two useful properties of  $u(j)$  are now proved.

Lemma 5.2: Let

$$u(j) = \frac{j+1}{\sigma^{j-1} (j-1)!} \left[ \frac{z_j}{j} - \frac{\mu}{j+1} z_{j-1} \right]$$

where  $z_j$  has been previously defined both recursively and in general form. Then

$$(1) \quad u(j) \geq 0 \quad \forall j \geq 1$$

$$(2) \quad u(j) - u(j-1) \geq 0 \quad \forall j \geq 1 .$$

Proof:  $u(j) \geq 0$

$$\Leftrightarrow \frac{z_j}{j} - \frac{\mu}{j+1} z_{j-1} \geq 0. \quad z_0 = 1, \quad z_1 = \mu + \sigma,$$

thus,

$$\frac{z_1}{1} - \frac{\mu}{2} z_0 = \mu + \sigma - \frac{\mu}{2} = \frac{\mu}{2} + \sigma > 0$$

so,  $u(1) \geq 0$  . Now, suppose  $u(j-1) \geq 0$ .

Notice that for any  $i$ ,

$$\begin{aligned} u(i) \geq 0 &\Leftrightarrow \frac{z_i}{i} - \frac{\mu}{i+1} z_{i-1} \geq 0 \\ &\Leftrightarrow \frac{\mu}{i} + \sigma z_{i-1} - \frac{\mu}{i+1} z_{i-1} \geq 0 \\ &\Leftrightarrow \frac{\mu}{i} + z_{i-1} \left( \sigma - \frac{\mu}{i+1} \right) \geq 0 \end{aligned}$$

thus,

$$\frac{\mu^{j-1}}{j-1} + z_{j-2} \left( \sigma - \frac{\mu}{j} \right) \geq 0 .$$

It is now sufficient to show

$$\frac{\mu^j}{j} + z_{j-1} \left( \sigma - \frac{\mu}{j+1} \right) \geq 0$$

to prove (1).

$$\begin{aligned} \frac{\mu^j}{j} + z_{j-1} \left( \sigma - \frac{\mu}{j+1} \right) &= \frac{\mu^j}{j} - \frac{\mu}{j+1} \left( \frac{\mu^{j-1}}{j-1} + \sigma z_{j-2} \right) + \sigma \left( \frac{\mu^{j-1}}{j-1} + \sigma z_{j-2} \right) \\ &= \mu^j \left[ \frac{1}{j} - \frac{1}{(j+1)(j-1)} \right] + \sigma \left[ \frac{\mu^{j-1}}{j-1} + z_{j-2} \left( \sigma - \frac{\mu}{j+1} \right) \right] \\ &\geq \underbrace{\mu^j \left[ \frac{1}{j} - \frac{1}{(j+1)(j-1)} \right]}_{\geq 0} + \sigma \underbrace{\left[ \frac{\mu^{j-1}}{j-1} + z_{j-2} \left( \sigma - \frac{\mu}{j+1} \right) \right]}_{\geq 0 \text{ by induction}} \geq 0 \quad \square \end{aligned}$$

Now, let  $\Delta(j) = u(j) - u(j - 1)$ ; defined for  $j \geq 2$ .

$$\Delta(j) = u(j) - u(j - 1)$$

$$= \frac{j+1}{\sigma^{j-1}(j-1)!} \left[ \frac{z_j}{j} - \frac{\mu}{j+1} z_{j-1} \right] - \frac{j}{\sigma^{j-2}(j-2)!} \left[ \frac{z_{j-1}}{j-1} - \frac{\mu}{j} z_{j-2} \right]$$

$$= \frac{1}{(j-1)! \sigma^{j-1}} \left[ \frac{j+1}{j} z_j - \mu z_{j-1} - \sigma(j-1)(j) \left\{ \frac{z_{j-1}}{j-1} - \frac{\mu}{j} z_{j-2} \right\} \right]$$

$$= \frac{1}{(j-1)! \sigma^{j-1}} \left[ (j+1) \frac{z_j}{j} - \mu z_{j-1} - \sigma \cdot j \cdot z_{j-1} + \sigma(j-1) \mu z_{j-2} \right]$$

since  $\frac{z_j}{j} = \frac{\mu^j}{j} + \sigma z_{j-1}$ ,  $j \geq 1$ ;

$$= \frac{1}{(j-1)! \sigma^{j-1}} \left[ (j+1) \frac{\mu^j}{j} + z_{j-1}(\sigma - \mu) + \sigma(j-1) \mu z_{j-2} \right]$$

$$= \frac{1}{(j-1)! \sigma^{j-1}} \left[ \frac{\mu^j}{j!} + \sigma \mu^{j-1} + \sigma^2(j-1) z_{j-2} \right] \geq 0 \quad \square$$

Lemma 5.1 along with Lemma 5.2 allow us to state the following optimal policy form:

**Theorem 5.3:** Suppose we have the Non-instantaneous Repair Model with a single server and the other hypotheses of Lemma 5.1.

Let

$$u(j) = \frac{j+1}{\sigma^{j-1}(j-1)!} \left[ \frac{z_j}{j} - \frac{\mu}{j+1} z_{j-1} \right],$$

for  $1 \leq j \leq n-1$  and  $u(0) = 0, u(n) = \infty$ . Suppose the system is  $k$ -of- $n$ . Then the  $V_{-1}$  optimal policy is:

$$R^{(j)}, \quad \text{if } u(j) < \frac{\rho\mu}{f_k \cdot (K+l\sigma)} < u(j+1), \quad j \geq k$$

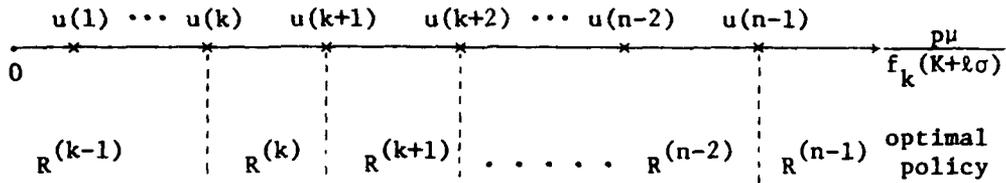
$$R^{(k-1)}, \quad \text{if } 0 < \frac{\rho\mu}{f_k \cdot (K+l\sigma)} < u(k).$$

Proof: Know from Lemma 4.1 that  $R^{(j)} < R^{(j-1)} \Leftrightarrow$

$$\frac{\rho\mu}{f_k \cdot (K+l\sigma)} > \frac{j+1}{\sigma^{j-1}(j-1)!} \left[ \frac{z_j}{j} - \frac{\mu}{j+1} z_{j-1} \right] = u(j).$$

Lemma 5.2  $\Rightarrow 0 \leq u(1) < u(2) < \dots < u(n-1)$  which implies the given optimal solution.  $\square$

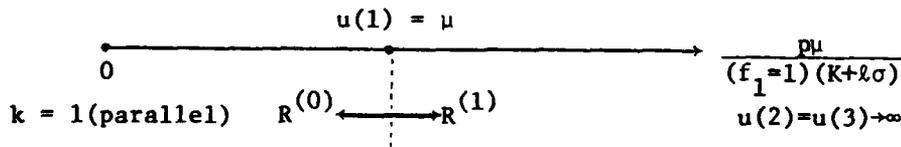
Schematically, given a k-of-n system, s = 1, identical components



Thus, given a fixed k, factors which favor more/(less) repair are:

- (a) (low)/high penalty cost, p for system failure
- (b) (high)/low component repair cost, K, and/or  
(high)/low expected labor cost per server per repair job completed,  $l\sigma$ .

This is because the "u" values are only functions of  $\mu$  and  $\sigma$ , not of system and repair cost parameters. As  $\sigma \rightarrow 0$  (close to instantaneous repair), the u-values all approach infinity except  $u(1) = \mu$ . This means that for small enough  $\sigma$ , one will never repair until there are only k left in a k-of-n system, a familiar Basic Model result.



In the parallel model, the  $\sigma = 0$  result simplifies precisely to the Basic Model result since the optimal policy there needed only a single server. For

$$\left[ \begin{array}{l} k > 1 \\ \sigma \approx 0 \end{array} \right], f_k \approx \frac{(k-1)!}{\mu^{k-1}} \cdot \sigma^{k-1},$$

which must be incorporated into the  $u$ 's. This done,  $\lim_{\sigma \rightarrow 0} f_k \cdot u(j) = \infty$  for  $j \neq k$  and is  $< \infty$  for  $j = k$ ; therefore, the limiting  $\sigma = 0$  policy is the same as for parallel. The next example illustrates:

Example 5.2:  $n = 3, s = 1, K = 2, L = 0$

Possible policies are  $R^{(2)}$  and  $R^{(1)}$ , i.e., if in a state with two components working and an idle server, either repair or don't repair.

$$\begin{array}{c} R^{(1)} \text{ optimal} \qquad R^{(2)} \text{ optimal} \\ \xrightarrow{\hspace{10em}} \frac{p\mu}{(\sigma/\sigma+\mu)(K+l\sigma)} \\ 0 \qquad u(2) = \frac{1}{\sigma} \left[ \frac{\mu^2}{2} + 2\mu\sigma + 3\sigma^2 \right] \end{array}$$

$$\begin{array}{c} R^{(1)} \text{ optimal} \qquad R^{(2)} \text{ optimal} \\ \xrightarrow{\hspace{10em}} \frac{p\mu}{K+l\sigma} \qquad \text{as } \sigma \rightarrow 0 \\ 0 \qquad \lim_{\sigma \rightarrow 0} f_1 \cdot u(2) = \frac{\mu}{2} \qquad \text{(close to} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{instantaneous} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{repair)} \end{array}$$

$$\begin{array}{c} R^{(1)} \text{ optimal} \\ \xrightarrow{\hspace{10em}} \frac{p\mu}{(\sigma/\sigma+\mu)(K+l\sigma)} \\ 0 \qquad u(2) \rightarrow \infty \qquad \text{as } \sigma \rightarrow \infty \\ \text{"takes so long to repair that} \qquad f_k \rightarrow 1 \qquad \text{(large repair} \\ \text{might as well let system fail first"} \qquad \qquad \text{times)} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{+ unlikely case} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{in practice.} \end{array}$$

In the remainder of this section, the problem of how the aforementioned  $V_{-1}$  optimal policy is affected by the type "k" of a k-of-n system is treated. Theorem 5.3 states that, in a k-of-n system, the  $V_{-1}$  optimal policy is:

$$(2) \quad R^{(j)}, \quad \text{if } u(j) < \frac{\rho\mu}{f_k(K+l\sigma)} < u(j+1), \quad j \geq k$$

and

$$(3) \quad R^{(k-1)}, \quad \text{if } 0 < \frac{\rho\mu}{f_k(K+l\sigma)} < u(k) .$$

The only k-dependent quantity is  $f_k$ , except for the value of j at which (2) is cut off and (3) holds. A study of  $f_k$  behavior will yield any system-related policy changes.

Lemma 5.4: Let  $f_1 = 1$  and

$$f_k = \frac{(k-1)! \sigma^{k-1}}{\mu^{k-2} (\mu + (k-1)\sigma)}, \quad k \geq 2 .$$

Let

$$k^* = \frac{1 + \sqrt{1 + 4 \frac{\mu^2}{\sigma^2}}}{2} .$$

Then

$$f_k > f_{k+1} \quad (+) \quad \text{for } k < k^*$$

and

$$f_k < f_{k+1} \quad (+) \quad \text{for } k > k^* .$$



on this line. A smaller  $f_k$  raises  $\frac{p\mu}{f_k(K+l\sigma)}$  and, thus, will cause earlier repair (when fewer components are failed) if it is changed enough. Similarly, a larger  $f_k$  means a wait until more components have failed before commencing repair. Lemma 5.5 states this precisely.

Lemma 5.5: Suppose we have a Non-instantaneous Repair Model with a single server and identical components and the other assumptions of Lemma 5.1. Let  $p$ ,  $\sigma$ ,  $\mu$ ,  $K$ , and  $l$  be given. Let  $M_k$  represent the model for a  $k$ -of- $n$  system,  $1 \leq k \leq n$ .  $k^*$  is

$$\frac{1 + \sqrt{1 + 4 \frac{\mu^2}{\sigma^2}}}{2} .$$

Suppose policy  $R^{(i)}$  was optimal for  $M_k$ ,  $i \geq k$ . Then,

if  $k < k^*$ ,  $R^{(j)}$  is optimal for  $M_{k+1}$ , some  $j \geq i$

if  $k > k^*$ ,  $R^{(j)}$  is optimal for  $M_{k+1}$ , some  $j \leq i$ .

Proof: Obvious from Lemma 4.4 and previous remarks. A final example illustrates:

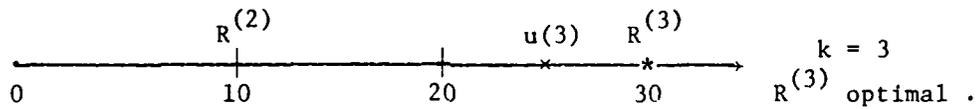
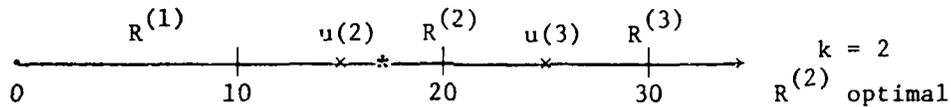
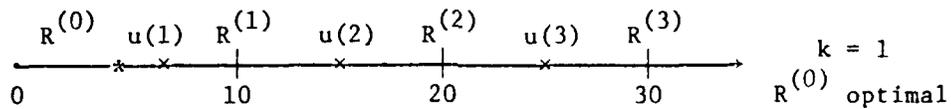
Example 5.3:  $n = 4$ ,  $p$ ,  $K$ ,  $l$ ,  $\mu$ ,  $\sigma$  fixed.  $L = 0$ ,  $k$ -of- $n$  system where  $k = 1, 2$ , or  $3$ .

Let  $p = 2$ ,  $K = 1$ ,  $\sigma = 1$ ,  $\mu = 3$ ,  $\ell = 1/2$ ,

$$u(1) = 2\sigma + \mu = 5 \quad f_1 = 1 \quad \frac{p\mu}{f_1(K+\ell\sigma)} = 4$$

$$u(2) = 13\frac{1}{2} \quad f_2 = \frac{1}{4} \quad \frac{p\mu}{f_2(K+\ell\sigma)} = 16$$

$$u(3) = \frac{97}{4} = 24\frac{1}{4} \quad f_3 = \frac{2}{15} \quad \frac{p\mu}{f_3(K+\ell\sigma)} = 30$$



In this example of a system for which the mean repair time is small compared to the mean component lifetimes, as  $k$  increases, the number of failed components at which repair is started is increased. Note that, if  $f_k$  had been increasing in  $k$ , as would have been true for a small  $\mu/\sigma$  ratio, then increasing  $k$  would have meant waiting longer before initiating repair up to the point where the system fails in which case further increase of  $k$  forces

a lesser wait due to the necessity of repair upon system failure.

So, generally,

$\mu/\sigma$  large  $\Rightarrow$  short repair times  $\Rightarrow$  repair sooner as  $k \uparrow$   
 compared to components'  
 lifetimes

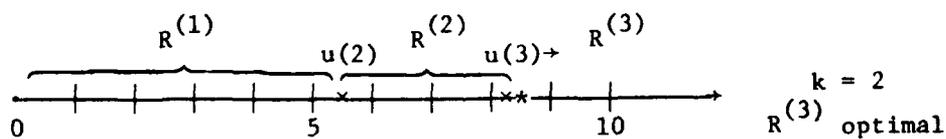
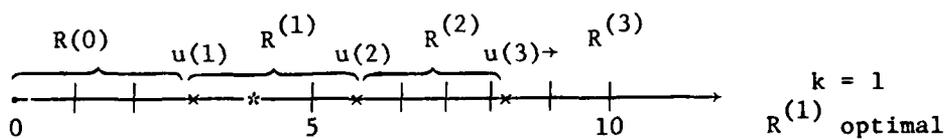
$\mu/\sigma$  small  $\Rightarrow$  long repair times  $\Rightarrow$  repair less as  $k \uparrow$ .  
 compared to component's  
 lifetimes

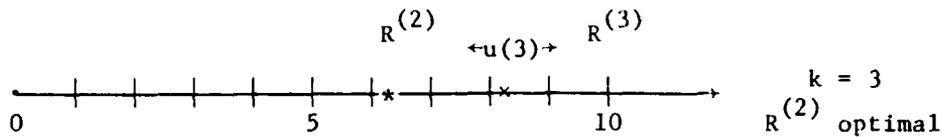
Example 5.4:  $\mu = 1$   $\sigma = 1$   $p = 6.1$   $K = 1$   $\ell = 1/2$

$$u(1) = 3 \qquad f_1 = 1 \qquad \frac{p\mu}{f_1(K+\ell\sigma)} = 4.2$$

$$u(2) = 5\frac{1}{2} \qquad f_2 = \frac{1}{2} \qquad \frac{p\mu}{f_2(K+\ell\sigma)} = 8.4$$

$$u(3) = 8\frac{1}{6} \qquad f_3 = \frac{2}{3} \qquad \frac{p\mu}{f_3(K+\ell\sigma)} = 6.3$$





### 3. Exponential Service, Multiple Servers

In Section 5.2, a single server version of the Non-instantaneous repair Model for an identical component, no fixed charge,  $k$ -of- $n$  system was treated. The  $V_{-1}$  optimal policy was found to be among  $R^{(j)}$ ,  $k - 1 \leq j \leq n - 1$  where  $R^{(j)}$  signifies a decision to repair whenever the server is free and the number of working components is  $j$  or less. The purpose of this section is to investigate what happens for the same model but with multiple servers. For the instantaneous repair (Basic Model) case, it was never optimal to repair more than a single unit at a time in an ergodic state so at least if one was only interested in  $V_{-1}$ , the optimal policy for multiple servers would be the same as if there was only one. The bulk of this section is spent trying to see whether that is still the case for non-instantaneous repair either in general or in some cases.

First, consider Example 5.1.

Example 5.1:  $n = 2$ ,  $s = 2$ , parallel,  $L = 0$

states:      2,0      1,0      1,1      0,0      0,1

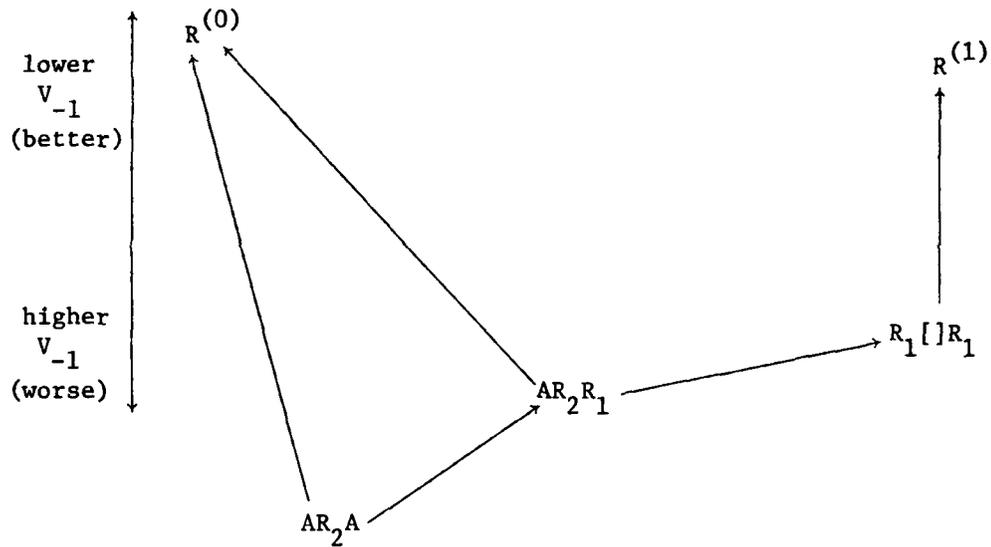
decisions:    A      A,  $R_1$       A       $R_1, R_2$        $R_1, A$

Possible policies,  $\delta$ , giving different ergodic structures are:

$\delta$	$\mathcal{E} = \{\text{Ergodic States}\}$	Decisions	$\delta$ V-1
$R^{(0)}$	{1,0 0,0}	1,0 0,0 A $R_1$	$\frac{K+l\sigma+p}{\mu+\sigma}$
$R^{(1)}$	{2,0 1,0 0,1}	2,0 1,0 0,1 A $R_1$ A	$\frac{(\mu+\sigma)(K+l\sigma)+\sigma p}{\frac{\mu}{2} + \sigma(\sigma+\mu)}$
$AR_2R_1$	{2,0 1,0 1,1 0,0 0,1}	2,0 1,0 1,1 0,0 0,1 A A A $R_2$ $R_1$	$\frac{(2\mu+\sigma)(K+l\sigma)+(\mu+\sigma)p}{\frac{3}{2}\mu + \frac{3}{2}\mu\sigma + \frac{1}{2}\sigma^2}$
$AR_2A$	{2,0 1,0 1,1 0,0 0,1}	2,0 1,0 1,1 0,0 0,1 A A A $R_2$ A	$\frac{2(\mu+\sigma)(K+l\sigma)+(\mu+2\sigma)p}{\frac{3}{2}\mu + \frac{5}{2}\mu\sigma + \frac{3}{2}\sigma^2}$
$R_1[ R_1$	{2,0 1,0 1,1 0,1}	2,0 1,0 1,1 0,1 A $R_1$ A $R_1$	$\frac{(\mu+\sigma)(K+l\sigma)+\sigma p}{\frac{1}{2}\mu^2 + \mu\sigma + \frac{1}{2}\sigma^2}$

The only policies which require only a single server are  $R^{(0)}$  and  $R^{(1)}$ . The relationship between the five policies is summarized below:

Figure 5.2 Policies in Example 5.1



A vertical arrow indicates total domination by the policy located higher in the diagram. Two upward "v" arrows indicate that the policy below is dominated by one of the policies at the ends of the "v".

In this case, it is indeed true that a policy which requires a single server only is always optimal.

Now, suppose we have the same model with two servers but now with 3 components:

Example 5.5:  $n = 3, s = 2, \text{parallel}, L = 0$

states:  $3,0$   $*2,0$   $2,1$   $*1,0$   $*1,1$   $*0,0$   $*0,1$   $0,2$

decisions:  $A$   $A, R_1$   $A$   $A, R_1, R_2$   $A, R_1$   $R_1, R_2$   $A, R_1$   $A$

\* = decision to be made.

There are now quite a number of possible policies, but only three which give ergodic structures which utilize only a single server:  $R^{(0)}$ ,  $R^{(1)}$  and  $R^{(2)}$  as defined in Section 5.2.

Consider the feasible policy  $\delta = R_1 - R_1$  — which gives  $\mathcal{E} = \{3,0 \ 2,0 \ 2,1 \ 1,1 \ 0,2\}$ . Clearly this requires two servers. Computations give

$$V_{-1}^{\delta} = \frac{(\mu + \sigma)^2 (K + l\sigma) + \sigma^2 p}{\frac{\mu^3}{3} + \sigma\mu^2 + \sigma^2\mu + \frac{\sigma^3}{2}}$$

Also,

$$V_{-1}^{R^{(2)}} = \frac{(\mu^2 + 2\sigma\mu + 2\sigma^2)(K + l\sigma) + 2\sigma^2 p}{\frac{\mu^3}{3} + \sigma\mu^2 + 2\sigma^2\mu + 2\sigma^3}$$

$$V_{-1}^{R^{(1)}} = \frac{(\mu + \sigma)(K + l\sigma) + \sigma p}{\frac{\mu^2}{2} + \mu\sigma + \sigma^2}$$

$$V_{-1}^{R^{(0)}} = \frac{K + l\sigma + p}{\mu + \sigma} .$$

Suppose  $K + \lambda\sigma = 1$ ;  $p = 30$ ,  $\sigma = 1$ ,  $\mu = 1$ . Then,

$$V_{-1}^{R(0)} = \frac{31}{2} = 15.5$$

$$V_{-1}^{R(1)} = \frac{32}{2.5} = 12.8$$

$$V_{-1}^{R(2)} = \frac{65}{5\frac{1}{3}} \approx 12.19$$

$$V_{-1}^{\delta} = \frac{34}{2\frac{5}{6}} = 12 .$$

Notice that policy  $\delta$  is better than any of the possible single server options. Thus, for this case, some policy (may or may not be  $\delta$ ) is optimal which does require the services of both servers. Under instantaneous repair, policy  $\delta$  would not have been optimal since it involves repair before getting down to one component left. Thus, under non-instantaneous repair, it is no longer true that no fixed charge  $\Rightarrow$  never repair more than one unit at a time.

The section is concluded with an obvious result but worth nothing:

Lemma 5.6: Suppose policy  $\delta$  is  $V_{-1}$  optimal for a Non-instantaneous Repair Model with  $s$ -servers ( $N_s$ ) but it involves the use of at most only  $s' < s$  servers. Then policy  $\delta$  is optimal for the Non-instantaneous Repair Model with only  $s'$  servers ( $N_{s'}$ ).

Proof: If  $\delta$  is optimal for  $N_s$ , then

$$\delta = \operatorname{argmin}_{\gamma \in \Delta(N_s)} V_{-1}^{\gamma},$$

$\Delta(N_s)$  being the policy space for model  $N_s$ . But  $\Delta(N_s) \supseteq \Delta(N_{s'})$  so given  $\sigma \in \Delta(N_{s'})$ ,

$$V_{-1}^{\sigma} \geq \min_{\gamma \in \Delta(N_{s'})} V_{-1}^{\gamma} \geq \min_{\gamma \in \Delta(N_s)} V_{-1}^{\gamma} = V_{-1}^{\delta}.$$

But

$$\delta \in \Delta(N_{s'}) \text{ so } \min_{\gamma \in \Delta(N_{s'})} V_{-1}^{\gamma} \leq V_{-1}^{\delta}$$

$$\text{so } V_{-1}^{\delta} = \min_{\gamma \in \Delta(N_{s'})} V_{-1}^{\gamma}.$$

□

#### 4. Extensions of Model to Erlang Service Times, etc.

In Sections 1, 2, and 3, exponential service times were assumed. As in Section IV.4, with component lifetimes in the Basic Model, service times also can be extended to Erlang distributions in the Non-instantaneous Repair Model, while still retaining the Markov decision chain structure. For many applications, an Erlang repair time is more realistic. Using it, one assumes the longer time the repair of a unit has been going on, the greater the probability of its completion. (IFR property).

Suppose now that service times are distributed  $G_{\frac{1}{m\sigma}, m}(t)$

an Erlang distribution with shape parameter  $m$  and mean  $\sigma$ , all servers being identical. Then if  $R$  is the service time,  $R = R_1 + \dots + R_m$  where  $R_m \sim \exp(1/\sigma)$ . As is standard procedure in queuing theory, think of the service as done in  $m$  consecutive parts with completion after the  $m^{\text{th}}$  stage is finished. For the general Non-instantaneous Repair Model with non-identical components, a state  $\underline{s} = \langle s_1, \dots, s_n \rangle$  where  $s_i$  is the state of the  $i^{\text{th}}$  component is generalized so that a component can be in one of  $m + 2$  states:

1. working
2. failed
3. failed, in  $1^{\text{st}}$  service stage
- .
- .
- $m+2$ . failed, in  $m^{\text{th}}$  service stage.

Decisions only need be made concerning a component if it is failed, not in service and there is a free server. The usual state simplifications occur if components are identical. If, in addition, we assume that a component can also be in  $\ell$  states of degradation, a Degradation/Non-instantaneous Repair Model is obtained. The total possible number of states,  $(\ell + 1 + m)^n$ , is now rather large and unworkable, even computationally, for all but very small values of  $m, \ell, n$ .

Example 5.6 Degradation/Non-instantaneous Repair Model  
with Erlang Repairs

$n = 2$  components  $L_1^0 \sim \exp(\lambda_1^0)$   $L_2^0 \sim \exp(\lambda_2^0)$

$l = 2$  degradation states  $L_1^1 \sim \exp(\lambda_1^1)$   $L_2^1 \sim \exp(\lambda_2^1)$

$s = 1$  single server, repair times  $R \sim G_{\frac{1}{3\sigma}, 3}(t)$   
(so, 3 repair stages)

states  $s = \langle s_1, s_2 \rangle$  where  $s_1, s_2 \in \{-3, -2, -1, 0, 1, 2\}$

-3 = first stage repair

-2 = second stage repair

-1 = third stage repair (last)

0 = new (repair completed)

1 = first degradation state

2 = second degradation state (failed) .

Note a new twist to this combined model - repair could be undertaken on a working component in a lower state of degradation and the component could become further degraded or fail before service is completed. All kinds of decisions on what to do in such cases can be treated under this formulation.

## CHAPTER VI

### COMPUTATIONAL METHODS OF PRODUCING OPTIMAL POLICIES

#### 1. Introduction

The past four chapters have treated four distinct types of coherent system repair models, coming up with theorems which either produce a general optimal policy given cost, system and lifetime parameters in simpler cases, or limit the number of possible optimal policies. Such general optimal policies, where possible to obtain, are clearly the most desirable results. However, for many cases the theorems developed in Chapters II - V do not apply. A partial listing of such cases would include:

- (1) Non  $k$ -of- $n$  systems with or without identical components.  
For this case, in the Basic Model, the  $V_{-1}$  optimal policy can be obtained using Theorem 2.4 if there is no fixed charge. Otherwise, nothing applies.
- (2) The case of a fixed charge  $L > 0$ .  
If components are identical or system is  $k$ -of- $n$ , then results can be obtained for the Basic Model.
- (3) Multi-server cases of non-instantaneous repair.
- (4) Cases where there are many degradation states, or one has Erlang service, component lifetimes.

Indeed, while Chapters II - V cover many cases of interest, there are many more possible cases not solved for which a user of

a coherent system repair model would want the optimal solution. In fact, given some possible application, chances are it would not be covered by any of the theorems. The aim of this chapter is to discuss methods of solving on the computer a given coherent system repair model which can be formulated as a Markov decision chain.

The next section discusses two general types of algorithms which could be used in optimizing such processes by computational methods: policy improvement and linear programming.

Section 3 describes a method of computing optimal policies for the Markov coherent system repair models (or any other Markov decision chain with suitable state or decision space) using MINOS, a non-linear programming code developed at Stanford.

Some test results using the MINOS solution technique will be presented in Section 4.

## 2. Possible Algorithms

There are two possible algorithms which can be used in computing a gain  $(V_{-1})$  optimal policy for a continuous time Markov decision chain (or Markov renewal program) with infinite horizon and no discounting. These are policy iteration and linear programming.

Suppose we are given a general Markov decision chain with

- (i) states:  $1, 2, \dots, N$  (indexed  $i$ )
- (ii) decisions:  $D_i =$  set of possible decisions in state  $i$  (indexed  $k$ ), finite

(iii) transition probabilities:

$p_{ij}^k = Q_{ij}^k(\infty) =$  probability  $i \rightarrow j$  given decision  $k$   
in state  $i$

$V_i^k = Q_{1,i}^k =$  expected holding time in state  $i$  | decision  $k$

(iv) cost structure:  $r_i^k =$  cost of decision  $k$  in state  $i$

(v) objective: minimize  $V_{-1}$ , the long run expected cost  
per unit time.

The preceding problem can be reexpressed as a linear program in terms of variables  $x_i^k$ , where  $x_i^k =$  probability {state =  $i$  and decision is  $k$ }. The optimal  $x_i^k$  values then give the optimal randomized policy for the given Markov decision problem, i.e., values of  $D_i^k = p\{\text{decision } k | \text{state } i\}$

$$D_i^k = \frac{x_i^k}{\sum_{k \in D_i} x_i^k}$$

The LP is as follows:

$$\text{minimize: } \sum_i \sum_k x_i^k r_i^k$$

$$\text{subject to: } \sum_k x_i^k - \sum_j \sum_k x_j^k p_{ji}^k = 0, \quad i = 1, 2, \dots, N$$

$$\text{and } \sum_i \sum_k x_i^k v_i^k = 1$$

$$x_i^k \geq 0 \quad \forall i, k$$

The objective function is just the expression for  $V_{-1}$  of a given policy  $\delta$  expressed in terms of  $x_i^k$  variables while the equality constraints are a rewrite of the stationary probability equations for a policy  $\delta$ :

$$\begin{bmatrix} \underline{x}^\delta (I - P_\delta) = \underline{0} \\ \sum x_i^\delta = 1 \end{bmatrix} .$$

Wagner [34] observed (by complementary slackness) that a nonrandomized policy is optimal over the class of stationary randomized policies in the case where each optimal policy has a single ergodic chain, (as is true for coherent system repair models). Thus, in an optimal solution to (P),  $x_i^k > 0$  for only one  $k \forall i = 1, \dots, N$ . This indicates an optimal decision of  $k$  given state  $i(\delta(i) = k)$ .

Not all states  $i$  need have  $x_i^k > 0$  for some  $k$ . Those that do not are transient states in a  $V_{-1}$  optimal policy and those for which it is, are ergodic. Thus,  $\mathcal{E} = \{\text{set of ergodic states in optimal policy}\} = \{i : \sum_k x_i^k > 0\}$ .

The second method for finding a  $V_{-1}$  optimal policy is policy iteration. This algorithm has the advantage that it can handle a countable state space ( $N = \infty$ ) while  $N < \infty$  is needed for the LP to be defined. Also, it will often find the optimal solution in fewer iterations than the LP will.

Policy iteration involves repeating two steps, value determination and policy improvement until a policy  $\delta$  with minimum  $V_{-1}^{\delta}$  is arrived at. Given the general Markov decision chain and the fact that every optimal policy defines a single ergodic chain, (in a coherent system repair model), the two steps are executed as follows. Suppose one starts with policy  $\delta$ . Let  $\underline{u} = \langle u_1, \dots, u_N \rangle$  and  $g$  be a scalar.

(1) value determination - solve the system (given  $\delta$ )

$$r_i^{\delta(i)} + \sum_{j=1}^N p_{ij}^{\delta(i)} u_j = u_i + v_i^{\delta(i)} \cdot g, \quad i = 1, \dots, N$$

and  $u_{i_0} = 0$  for some  $i \leq i_0 \leq N$  for  $g, \underline{u}$ . Since

$$g^{\delta} = V_{-1}^{\delta} = \frac{P^* R_0}{P^* Q_1} \quad \text{and} \quad (I - Q_0)V_{-1} = 0,$$

this  $g = g^{\delta}$ , the gain value for policy  $\delta$ .

(2) policy improvement - using the current values of  $g^{\delta}, \underline{u}^{\delta}$ , find another policy  $\gamma$  such that  $\gamma_i =$  what  $\gamma$  does in  $i$

$$\text{minimizes } \left\{ \frac{r_i^k}{v_i^k} + \frac{1}{v_i^k} \left( \sum_{j=1}^N p_{ij}^k u_j \right) - u_i \right\} \quad \forall i = 1, \dots, N.$$

If  $\gamma \equiv \delta$ , then terminate;  $\delta$  is optimal. Otherwise, go back to step (1) and replace  $\delta$  by  $\gamma$ . Continue the procedure until  $x, g$  are found satisfying the termination conditions:

$$(t_1) \quad r_i^{\delta(i)} + \sum_{j=1}^N p_{ij}^{\delta(i)} u_j = u_i + v_i^{\delta(i)} \cdot g \quad i = 1, \dots, N$$

$$(t_2) \quad r_i^k + \sum_{j=1}^N p_{ij}^k u_j \geq u_i + v_i^k \cdot g \quad \forall (i, k) \text{ pairs .}$$

Condition  $(t_1)$  implies  $g = V_{-1}^{\delta}$  while  $(t_2)$  implies  $\delta$  is  $V_{-1}^{\delta}$  optimal ( $V_{-1}^{\delta} = g^*$ ), i.e., there are no possible "improvements". Given a finite number of states, termination will occur in a finite number of iterations. In effect, the policy iteration method is computing  $V_{-1}^{\delta}$  for various  $\delta$ , but is choosing the successive  $\delta$ 's in an efficient manner so as to go through as few of them as possible before reaching an optimal one.

It should be noted that the linear program (P) is not using the same algorithm for solution as the policy improvement method mentioned. However, the two problems are intimately related in that if the policy improvement termination conditions and a desire to maximize  $g$  are put in linear program form, the LP so formed is the dual problem to LP (P). Although problem (P) is the more intuitive formulation, its dual (D) turns out to be more efficient computationally, especially in cases where information about higher

levels of optimality than  $V_{-1}$  is desired. The dual problem (D) is shown below in LP form with  $g, \underline{u} = \langle u_1, \dots, u_N \rangle$  as variables:

$$(D) \quad \text{minimize: } (-g)$$

$$\text{subject to: } u_i - \sum_{j=1}^N p_{ij}^k u_j + v_i^k \cdot g \leq r_i^k \quad \forall (i,k)$$

where  $g, u_i$ 's are unrestricted in sign.

Two well known results from duality theory will help relate the problems (P) and (D) as well as assist in developing a computational method in Section 3. These are:

Theorem 6.1: (Duality Theorem) [7] or [34]

(a) If both the primal (P) and dual (D) problems possess feasible solutions, the primal problem has an optimal solution  $\hat{x}_j^k, j = 1, 2, \dots, N, k \in \Delta(j)$ , the dual problem has an optimal solution  $\hat{g}, \hat{u}_i, i = 1, \dots, N$  and  $\sum_i \sum_k r_i^k \hat{x}_i^k = -\hat{g}$ .

(b) If either has a feasible solution with finite optimal objective value, then the other one has a feasible solution with the same optimal objective function value. [So the optimal objective is (P) or (D) is  $V_{-1}$  in a coherent system repair model by (b) since clearly (P) has a feasible solution with finite objective].

Proof: See [7], [34].

A corollary to this which is useful in relating optimal values of primal and dual variables is:

Corollary 6.2: (Complementary Slackness). Let (P), (D)

be expressed in the matrix form:

$$\begin{array}{ll}
 \text{(P)} & \text{maximize: } \underline{x} \\
 & \text{subject to: } \underline{Ax} \leq \underline{b} \\
 & \underline{x} \geq 0 \\
 \text{(D)} & \text{minimize: } \underline{y} \\
 & \text{subject to: } \underline{y}^t \underline{A} \geq \underline{c} \\
 & \underline{y} \geq 0
 \end{array}$$

where

$$\begin{array}{lll}
 |\underline{c}| = n, & |\underline{y}| = m & |\underline{A}| = m \times n \\
 |\underline{x}| = n & |\underline{b}| = m .
 \end{array}$$

Let  $\underline{x}^*$ ,  $\underline{y}^*$  be corresponding feasible solutions to (P) and (D).

Then both are optimal  $\Leftrightarrow \underline{y}^* \cdot \langle \underline{Ax}^* - \underline{b} \rangle = 0$ , (dot product of two vectors)

and  $\underline{x}^* \cdot \langle \underline{y}^{*t} \underline{A} - \underline{c} \rangle = 0$ .

Proof: See [30] or [31].

Corollary 6.2 implies that whenever a constraint in one of the problems holds with strict inequality, so that there is slack in the constraint, the corresponding variable in the other problem is zero. This result will allow us, in the next section, to know which of the  $x_1^k$  variables from (P) are positive from the solution of the dual problem.

Once a  $V_{-1}$  optimal policy has been determined for a given Markov decision chain using either policy iteration or linear programming, a bias optimal policy can be obtained by solving an altered Markov decision problem again by either policy improvement or linear programming. Given the original Markov decision problem

and its associated LP (D), the altered problem is defined as follows:

- (i) states: the same (1, ..., N)
- (ii) decisions: restricted to  $\Delta'$ , where  $\Delta'$  = set of policies  $\langle i, \delta(i) \rangle$  such that

$$u_i - \sum_{j=1}^N p_{ij}^{\delta(i)} u_j + v_i^{\delta(i)} g = r_i^{\delta(i)} \quad \forall i = 1, \dots, N$$

where  $u_1, \dots, u_N$  and  $g$  are the optimal values found solving the original problem. Clearly the optimal policy,  $\delta^*$  from the  $V_{-1}$  step is in  $\Delta'$ .

- (iii) transition probabilities:  $p_{ij}^k$ , the same
- (iv) cost structure:  $r_i^k$  is replaced by  $\bar{r}_i^k$ , where

$$\bar{r}_i^k = (-R_1)_i^k + ((Q_2)_i^k \cdot \frac{1}{2}) g^* - (Q_1)_i^k \cdot u$$

$$u = \langle u_1, \dots, u_N \rangle$$

being the solution to

$$(I - Q_0^{\delta^*}) u = R_0^{\delta^*} - Q_1^{\delta^*} g^*$$

and

$$u_N = 0$$

found for  $\delta^*$

in  $V_{-1}$  problem.

- (v) objective: minimum  $V_{-1} = g$  (which gives minimum  $V_0$  in original problem).

See Denardo [10] for verification of the fact that if  $u$  is chosen as shown, that the  $V_{-1}$  optimal solution to the altered Markov decision problem indeed gives a  $V_0$  optimal policy for the original problem in the case where every  $V_{-1}$  optimal policy defines an irreducible Markov chain. (All coherent system repair models have this property - see Section II.2).

For a general problem (arbitrary Markov decision chain) in which some  $V_{-1}$  optimal policies might generate multiple ergodic chains of states, the aforementioned altered problem may not yield a  $V_0$  optimal solution (a case never encountered in a coherent system repair model). In such cases, a  $V_0$  optimal solution can still be gotten by adding a third step which takes into account states which are transient under every possible policy.

The above 2 or 3 steps can be continually reapplied, modifying  $r_i^k$ 's and restricting decisions appropriately, to get  $V_1, V_2, \dots$  etc., optimal policies until a unique policy, "optimal", is reached. For reasons stated in Section I.4,  $V_0$  optimality is considered "optimal" in coherent system repair models so two is the maximum number of  $V_{-1}$  problems needed to be solved to determine an optimal decision for every state. In many cases, solving one problem will suffice in getting  $V_0$  optimality as will be seen in the following two sections.

### 3. Computing Optimal Policies Using MINOS - A Linear Programming Method.

In this section, a linear programming method for solving a coherent system repair model on the computer using MINOS is described. Although policy improvement algorithms may on the whole be more efficient and require fewer iterations to reach optimality, there has been little development of computer codes which might efficiently carry the method out. On the other hand, there has been a lot of work done by the Systems Optimization Laboratory (SOL) at Stanford on linear and nonlinear programming codes. See [23] and references listed there. It is for this reason that an LP algorithm is used.

MINOS is actually a code developed by Michael Saunders and Bruce Murtagh [23] for solving large scale nonlinear programming systems which have linear constraints. Of course, it works on linear programs as well. There are other codes which are designed for linear programs and can handle larger sized problems (e.g., MPS III [referenced in [23]]) but for the size problems tested here, MINOS is sufficient. Given an LP:

$$\begin{aligned} & \text{minimize: } cx \\ (P) \quad & \text{subject to: } Ax = b \\ & \quad \quad \quad x \geq 0, \end{aligned}$$

MINOS is most efficient if  $A$  is sparse (has lots of zeros) and the number of rows isn't too large ( $\leq 1500$ ). Coherent system

repair models fit these conditions for small numbers of components (under 10) or states of degradation. In general, they define LP's which have many more columns than rows, which means MINOS can handle larger problems than might otherwise be expected in solving (P). Of course, in a general nonidentical component model, even the number of rows (states) get large very quickly, e.g., if there are  $l$  degradation states and  $n$ -components, the number of states  $\approx (l + 1)^n$ . The number of columns is even larger by a factor of  $2^n$  since in each state  $s$ , one can in general decide to repair any subset of the failed components ( $2^{n-|s|}$  possibilities), and there is one variable per possible decision per state. For larger problems, the LP code MPS III could be used but clearly for any moderately large number of components (even  $\geq 15$ ), model simplifications to restrict the number of states and/or decisions must be undertaken before the problem can become of reasonable size. Such theorems as presented in Chapters II - V could provide such assistance in cases of identical components,  $k$ -of- $n$  systems, or certain specific model cases.

Certain of the models can be solved with less difficulty than others (due to smaller number of states). The Basic Model is easiest with the Noninstantaneous Repair Model requiring only a small increase in state space. In contrast, a Degradation Model requires a dramatic increase in states and for even small numbers of degradation states, would be too large to do even on the computer.

One good way of simplifying the problem computationally would be to break the system down into independent subsystems of components, the subsystems being in series, i.e.,  $s = \{1, \dots, n\} = B_1 \cup \dots \cup B_r$ ,  $r < n$  where  $B_i \cap B_j = \phi$ ,  $i \neq j$  and the system works if and only if the subsystem corresponding to  $B_i$  is working for every  $i$ . The idea is then to run the model on the smaller subsystems and then add the costs from each one to get the overall result. For the Basic Model and  $V_{-1}$ , this can be done as a consequence of Theorem 2.1. For other models or  $V_0$ , the truth or falsehood of this result is an open question, although for  $V_{-1}$  at least, I think it will be true.

Before stating and proving the aforementioned theorem, a clarification is needed on what is meant by independent subsystems. In the Basic Model, the states of the system depend on the status of each component as well as the status of the system.

Definition: An independent coherent subsystem  $B_i$  is a subset of components  $\{1, \dots, r\}$ ,  $r \leq n$  which form a coherent system such that the status of  $B_i$  is unaffected by components  $r + 1, \dots, n$ .

Theorem 6.3: Suppose we have a coherent system  $\mathcal{P}$  of  $n$  components which can be broken down into  $p$  independent coherent subsystems,  $B_i$ , which are in series. Then, given the Basic Model, the optimal long run expected cost for the system is the sum of

optimal long run expected costs for the subsystems, i.e.,

$$V_{-1, \mathcal{P}} = \sum_{j=1}^p V_{-1, B_j} . \quad \text{The policy is the same.}$$

Proof: To follow example and discussion.

The advantage of independent series subsystems is clear.

Suppose  $n = 20$ , i.e., there are twenty components. The total number of states in the Basic Model could be up to  $2^{20} = 1,048,576$  or over a million - clearly impossible by any reasonable computational standards. Just being able to break things down into two independent 10-component subsystems would reduce the total states needed to  $2 \times 2^{10}$  or 2,024 - a feasible number. Further reduction to four 5-component subsystems lowers the total states to  $4 \times 2^5$  or 128. An example where this proves useful is now given.

Example 6.1: Suppose a system consists of  $n$  independent different types of components which are in series, i.e., the system works if and only if each component is functioning. One possible way to improve the reliability or performance of the system is to add duplicates for each component type. The system is now composed of  $n$  parallel independent subsystems in series. If the level of redundancy of component  $i$  is denoted by  $r_i$ , then the total possible number of system states would be  $\prod_{i=1}^n (r_i + 1)$ ,  $r_i \geq 1$ . However, by modeling each subsystem separately, and adding  $V_{-1}$ 's, one gets

away with running  $n$  subsystems with  $r_1, r_2, \dots, r_n$  states respectively, almost a trivial problem. This type of subsystem is done in Markov reliability modeling of fault tolerant systems in [24]. The system reliabilities computed for each subsystem are then multiplied together to get the total system reliability. Fault tolerant systems are discussed further in Section 7.4 (Applications).

Proof: (Theorem 6.3) Let the independent subsystems of components  $1, \dots, n$  be reordered so that  $B_1 = \{1, \dots, i_1\}$ ,  $B_2 = \{i_1 + 1, \dots, i_1 + i_2\}$ ,  $\dots$ ,  $B_p = \{i_1 + \dots + i_{p-1} + 1, \dots, i_1 + \dots + i_p\}$ . Theorem 2.1 tells how to find the  $V_{-1}$  optimal policy in the Basic Model case for a coherent system by testing all subsets of components for which the system operates. The  $V_{-1}$  optimal policy is the one which keeps such a set of components operating by fixing one as soon as it fails at minimum cost. Let  $s$  be any such set and define  $V_{-1,s}$  as the  $V_{-1}$  for the policy as mentioned above which keeps only the set  $s$  of components working. Let  $s$  also represent the policy just mentioned. Suppose  $s_0$  is optimal for the whole system,  $\mathcal{P}$ . It is desired to show

$$V_{-1,s_0} = \sum_{j=1}^p V_{-1,s_0 \cap B_j}$$

By Theorem 2.1,

$$v_{-1, s_0} = \sum_{\substack{i \in s_0: \\ s_0 \sim i \text{ is} \\ \text{down in } \mathcal{S}}} \frac{K_i + p}{\mu_i} + \sum_{\substack{i \in s_0: \\ s_0 \sim i \text{ is} \\ \text{up in } \mathcal{S}}} \frac{K_i}{\mu_i} .$$

If state  $s_0 \sim i$  is up for  $\mathcal{S}$ , then  $(s_0 \sim i) \cap B_j$  is up for subsystem  $B_j$ ,  $j = 1, \dots, p$  since the subsystems are in series.

If state  $s_0 \sim i$  is down for  $\mathcal{S}$ , then  $(s_0 \sim i) \cap B_j$  is down for  $B_j$  for at least one  $j$  since the  $B_j$ 's are in series. So  $s_0$  has to be up in  $\mathcal{S}$  for the policy "keep  $s_0$  working" to be defined. Thus,  $s_0 \cap B_j$  is up in  $B_j \forall j = 1, \dots, p$ . Thus,  $(s_0 \sim i) \cap B_j$  is down for exactly one  $B_j$ , the one which  $i$  is in. Thus

$$\begin{aligned} v_{-1, s_0} &= \sum_{j=1}^p \sum_{\substack{i \in s_0 \cap B_j \\ (s_0 \cap B_j) \sim i \\ \text{is down in } B_j}} \frac{K_i + p}{\mu_i} + \sum_{j=1}^p \sum_{\substack{i \in s_0 \cap B_j \\ B_j \cap s_0 \sim i \text{ is} \\ \text{up in } B_j}} \frac{K_i}{\mu_i} \\ &= \sum_{j=1}^p v_{-1, s_0 \cap B_j} \quad \text{by Theorem 2.1.} \quad \square \end{aligned}$$

A simple example is now solved by hand using the LP method, following which the procedure for solving a general problem using MINOS is presented.

Example 6.2:  $n = 2$ , parallel system, different components,  $L = 0$ , Basic Model.

Markov decision problem:

States:        12        1        2        0

Decisions:    A        A, R<sub>2</sub>    A, R<sub>1</sub>    R<sub>1</sub>, R<sub>2</sub>, R<sub>12</sub>

$$\text{Transitions: } P_{12}^A = \begin{bmatrix} 0 & \frac{\lambda_2}{\lambda_1 + \lambda_2} & \frac{\lambda_1}{\lambda_1 + \lambda_2} & 0 \end{bmatrix} = P_{1\cdot}^{R_2} = P_{2\cdot}^{R_1} = P_{\cdot 0}^{R_{12}}$$

$$P_{1\cdot}^A = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} = P_{2\cdot}^A = P_{\cdot 0}^{R_1} = P_{\cdot 0}^{R_2}$$

$$v_0^{R_{12}} = v_{12}^A = v_1^{R_2} = v_2^{R_1} = \frac{1}{\lambda_1 + \lambda_2}; \quad v_1^A = v_0^{R_1} = \frac{1}{\lambda_1}; \quad v_2^A = v_0^{R_2} = \frac{1}{\lambda_2}.$$

Costs:

$$r_{12}^A = r_1^A = r_2^A = 0$$

$$r_1^{R_2} = K_2; \quad r_2^{R_1} = K_1; \quad r_0^{R_1} = K_1 + p; \quad r_0^{R_2} = K_2 + p$$

$$r_0^{R_{12}} = K_1 + K_2 + p.$$

Objective: minimize  $v_{-1}$ , then  $v_0$ .

Linear Programming Formulation for (P), primal problem.

8 variables  $x_i^k$  = probability {state = i and decision is k}

$$(x_{12}^0, x_1^0, x_1^2, x_2^0, x_2^1, x_0^1, x_0^2, x_0^{12}) .$$

5 constraints plus  $x_i^k \geq 0 \forall i, k$  plus objective function.

column row	12,0	1,0	1,2	2,0	2,1	0,1	0,2	0,12	RHS
OBJ	0	0	$K_2$	0	$K_1$	$K_1+p$	$K_2+p$	$K_1+K_2+p$	---
S12	1	0	0	0	0	0	0	0	= 0
S1	$\frac{-\lambda_2}{\lambda_1+\lambda_2}$	1	$1-\frac{\lambda_2}{\lambda_1+\lambda_2}$	0	$\frac{-\lambda_2}{\lambda_1+\lambda_2}$	0	0	$\frac{-\lambda_2}{\lambda_1+\lambda_2}$	= 0
S2	$\frac{-\lambda_1}{\lambda_1+\lambda_2}$	0	$\frac{-\lambda_1}{\lambda_1+\lambda_2}$	1	$1-\frac{\lambda_1}{\lambda_1+\lambda_2}$	0	0	$\frac{-\lambda_1}{\lambda_1+\lambda_2}$	= 0
S0	0	-1	0	-1	0	0	0	1	= 0
NORM	$\frac{1}{\lambda_1+\lambda_2}$	$\frac{1}{\lambda_1}$	$\frac{1}{\lambda_1+\lambda_2}$	$\frac{1}{\lambda_2}$	$\frac{1}{\lambda_1+\lambda_2}$	$\frac{1}{\lambda_1}$	$\frac{1}{\lambda_2}$	$\frac{1}{\lambda_1+\lambda_2}$	= 1

$$\begin{aligned} \text{Suppose } L &= 0 & K_1 &= 2 & \mu_1 &= 1 \\ p &= 1 & K_2 &= 6 & \mu_2 &= 2 \end{aligned}$$

Then (P) can be written as

$$\begin{aligned} &\text{minimize: } cx \\ &\text{subject to: } Ax = b \\ & \quad x \geq 0, \end{aligned}$$

where

$$x = \left\langle \overset{A}{x_{12}}, \overset{A}{x_1}, \overset{R_2}{x_1}, \overset{A}{x_2}, \overset{R_1}{x_2}, \overset{R_1}{x_0}, \overset{R_2}{x_0}, \overset{R_{12}}{x_0} \right\rangle$$

$$c = \langle 0, 0, 6, 0, 2, 3, 7, 9 \rangle \quad (\text{objective row})$$

$$A = \begin{array}{cccccccc|l} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{row} \\ -\frac{1}{3} & 1 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & S1 \\ -\frac{2}{3} & 0 & -\frac{2}{3} & 1 & \frac{1}{3} & 0 & 0 & -\frac{2}{3} & S2 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & S0 \\ \frac{2}{3} & 1 & \frac{2}{3} & 2 & \frac{2}{3} & 1 & 2 & \frac{2}{3} & \text{NORM} \end{array}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The dual problem (D)

can be written as follows:

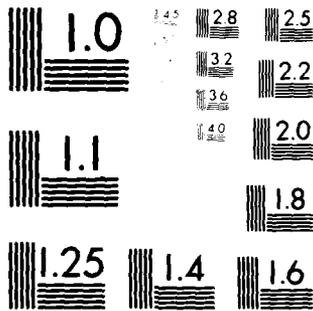
$$\begin{array}{l} \text{maximize: } g \\ \text{subject to: } u^t A \leq c \end{array}$$

where  $u = \langle u_{12}, u_1, u_2, u_0 \rangle$  and  $g$  are unrestricted in sign.

We know  $\max g = \min cx = V_{-1}$  (duality theorem) and  $x^* \cdot \langle u^t A - c \rangle = 0$  (complementary slackness) so,  $u^t A_{\cdot(i,k)} < c \Rightarrow x_i^{k*} = 0$  ( $i \notin \mathcal{E}$  under  $\delta^*$ ), ( $A_{\cdot(i,k)}$  is the  $i, k^{\text{th}}$  column of  $A$ ).

Example 6.2 will be solved as part of Section 4.





MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

The fundamental principle behind MINOS is an efficient and reliable implementation of the revised simplex method for linear programming (see [ 7 ]). This combines established sparse-matrix technology with stable numerical methods for computing and modifying a triangular factorization of the usual square basis matrix B [see references in [23], p. 8].

For usage in solving purely linear problems, the following two items must be supplied as input:

- (1) the SPECS file - to specify certain run time parameters
- (2) the MPS file - to specify the objective, constraints, and bounds on variables in standard MPS format.

MPS format is defined under the title "CONVERT DATA" in IBM document number SH20-0968-1, "Mathematical Programming System-Extended (MPSX), and Generalized Upper Bounding (GUB)", pp. 199-209.

The following SPECS are used in all test problems in Section 4:

SPECS FILE

BEGIN SPECS

MINIMIZE	OBJ	objective
OBJECTIVE	RHS	defines RHS vector
RHS	BND	defines bounds vector
BOUNDS	100	max number of rows
ROWS	500	max number of columns
COLUMNS	1500	max number of nonzero
ELEMENTS		matrix elements
INPUT FILE	31	defines input MPS file
ITERATIONS	500	maximum number of iterations
SOLUTION	yes	go to a solution.
END SPECS		

The input MPS file (labeled 31 in SPECS) will vary from problem to problem. It is uniquely specified given any Markov coherent system repair model. This is done by choosing values for each of the following parameters:

- States:         $n$  = number of components  
                    $\ell$  = number of degradation states/component  
                    $I_{id}$  = indicator function indicating whether one has an identical component, k-of-n system or not
- Repair:         $s$  = number of servers  
                    $\sigma \geq 0$  = mean repair time
- Components:  $\mu_i^d$  = mean holding time of component  $i$  in degradation state  $d$   
                    $\lambda_i^d = \frac{1}{\mu_i^d}, 1 \leq i \leq n, 0 \leq d \leq \ell - 1$
- System Type: - list mincut sets
- Penalty Cost:  $p \geq 0$  for system failure
- Fixed Charge:  $L \geq 0$  per repair decision
- Repair Costs:  $K_i^d$  to repair component  $i$  when in degradation state  $d, 1 \leq d \leq \ell, 1 \leq i \leq n$
- Objective:     $V_{-1}$  or  $V_0$  (minimize)

For large problems which might need to be tested many times, it would be worth writing a Fortran program to produce the MPS input file given the above model parameters. However, for the relatively small problems to be tested, it is easier to create a new MPS file directly for each separate example.

Output from MINOS includes:

- (1) a listing of iterations to solution
- (2) Rows section: indicates the numerical value taken on by various rows at optimality, including whether or not the row is at its upper limit (in the case of  $\leq$  constraints in problem (D)). Also the optimal value of the  $i^{\text{th}}$  dual problem variable corresponding to the  $i^{\text{th}}$  row is listed.
- (3) Columns section: gives optimal values of variables and the reduced costs for each, i.e., the coefficient of variable  $x_j$  in the objective row of the optimal simplex tableau.

Given information on the input, output and working efficiency of MINOS, the question remains as whether to use the LP (P) or (D) to solve the problem. One immediate point favoring (D) is that the  $u_i$  variables are given as output as well as the  $x_i$  while only the  $x_i$ 's appear in the output for (P). However, the efficiency of MINOS depends the most on the number of rows and less so on the columns. This favors use of (P), which has many fewer rows than columns, in cases where only  $V_{-1}$  optimality is desired or the problem (D) would have too many rows to be efficiently run under MINOS. The advantage of problem (D) is in cases where a  $V_0$  optimal solution is sought. To determine the restricted decision space  $\Delta'$  for the altered Markov decision problem, one must know

which of the rows  $(i, k)$  in  $(D)$  are at their upper bound at optimality. This information appears directly in the rows section output of problem  $(D)$ , while for  $(P)$ , these comparisons would have to be made separately in addition to the fact that values of  $u$  would have to be solved for on the side using a system of  $N$  linear equations using dual activity values given in "rows" section. In summary, a rule of thumb might be

- if small problem and want

$V_{-1}$  only  $\rightarrow$  use  $(P)$

$V_{-1}$  and  $V_0$   $\rightarrow$  use  $(D)$

- if large problem and want to use

MINOS  $\rightarrow$  use  $(P)$  if  $(D)$  has too many rows for MINOS

OR

if want  $(D)$  to be used, then recommend using some other code.

This rule will be followed in the small test problems presented in Section 4. It should be noted that for coherent system repair model type Markov decision chains where  $V_0$  optimality specifies a unique policy subject to an infinitesimal change in one of the cost or lifetime model parameters and the optimal ergodic chain is always irreducible, it will frequently happen that the only policy in  $\Delta'$  is the  $\delta^*$  determined by solving the initial LP. In such cases, only one LP need be solved to get a complete optimal solution. This occurs in Examples 6.2 and 6.3 of the next section.

In solving certain problems by LP, it may happen that

$$t_i^k = u_i - \sum_{j=1}^N p_{ij}^k u_j + v_i^k g - r_i^k < 0 \quad \forall k, \quad \text{some } i.$$

If so, then slight modifications in the state and decision spaces of the altered Markov decision problem must be made. Let

$B_i = \{k \in \Delta_i : t_i^k = 0\}$ . If  $B_i \neq \phi \quad \forall i$ , then there are no problems, define  $\Delta'$  as before. Suppose  $B_i = \phi$  for some states  $i$ . Let  $\Omega = \{i : B_i \neq \phi\}$ . Notice that  $i \notin \Omega$  can occur only if  $i$  is transient under all gain optimal policies, a condition which is usually satisfied by all transient states in a coherent system repair model, unless there are ties among some of the  $V_1^\delta$  values. Modify  $\Omega$  and  $B_i$  using the following algorithm:

Algorithm 1:

- (1) Look for some  $i \in \Omega$  and  $k \in B_i : p_{ij}^k > 0$  for some  $j \notin \Omega$ . If such a pair exists, go to step (2), otherwise stop.
- (2) Delete  $k$  from  $B_i$ . If doing so renders  $B_i$  empty, delete  $i$  from  $\Omega$ . Go to step (1). □

Now, let the state space in the altered program be restricted to  $\Omega$  (using its terminal definition from the above algorithm) and let decisions be restricted to the terminal definitions of  $B_i$ . The altered Markov decision problem will now find a  $V_0$  optimal policy which is independent of states  $i \notin \Omega$ . Examples of states  $i \notin \Omega$

in coherent system repair models would be ones which are inaccessible from the starting state  $i_0$  = all components working under a certain policy. The algorithm will never remove decisions corresponding to ergodic states.

There is one other difficulty which can arise. The terminal definition of  $\Omega$  using Algorithm 1 may delete some states that are accessible from the initial state, an unacceptable situation since in a coherent system repair model it is desired to specify an optimal decision in every state that is accessible from the state with all components working. Such a situation may be corrected by:

- (1) Run the LP (D) once and note which primal variables are  $> 0$  to get the optimal decisions in the ergodic states. If  $B_i = \phi$  for any  $i$  or the terminal definition of  $\Omega$  in Algorithm 1 includes all states accessible from the initial state, then proceed as described previously. If not, then
- (2) Rerun (D) but first delete rows corresponding to nonoptimal decisions in ergodic states. This should now produce a  $\Omega$  which includes all  $i$  accessible from  $\{1, \dots, n\}$ . Now proceed as before, (see Example 6.3;  $k = 1$   $p = 3$ , for example).

If policy improvement is used instead of linear programming, such difficulties never occur for  $t_i^k = 0$  for some  $k$  for each  $i$ .

#### 4. Some Test Results

In this section some test results for some small coherent system repair models will be presented. First, Example 6.2 was looked at to make sure the program was working, as its optimal policy can be computed using theory from Chapter II. Only the relevant output is presented. See Figures 6.1 and 6.2, 6.3 at the end of the chapter for sample input and output from MINOS.

Example 6.2: Basic Model,  $n = 2$ , parallel system.

$$\begin{array}{lll} K_1 = 2 & K_2 = 6 & p = 1 \\ \mu_1 = 1 & \mu_2 = 2 & L = 0 \end{array}$$

From theory

$$\begin{aligned} v_{-1} &= \min \left\{ \frac{K_1+p}{\mu_1}, \frac{K_2+p}{\mu_2}, \frac{K_1}{\mu_1} + \frac{K_2}{\mu_2} \right\} \\ &= \min \{3, 3.5, 5\} = 3 \end{aligned}$$

so,  $\mathcal{E} = \{0\}$        $\delta_0^* = R_1$

$$v_{-1} < \left(1 + \frac{\mu_2}{\mu_1}\right) \frac{K_1}{\mu_1} = 6 \quad \text{so} \quad \delta_1 = A \quad \delta_2 = A \quad \text{is optimal.}$$

optimal policy:       $\begin{array}{lll} 1 & 2 & 0 \\ A & A & R_1 \end{array}$

Results: (using (P) or (D) - both were tested).

$$\text{primal variables: } x_{12}^A = 0 \quad x_2^A = 0 \quad x_0^{R_2} = 0$$

$$\text{at optimality } x_1^A = 0 \quad x_2^{R_1} = 0 \quad x_0^{R_{12}} = 0$$

$$x_1^{R_2} = 0 \quad x_0^{R_1} = 1$$

$$\text{dual variables: } u_{12} = -7 \quad u_2 = -6 \quad g = 3$$

$$\text{at optimality } u_1 = -3 \quad u_0 = 0$$

rows at upper bound in (D):

12, A	2, A
1, A	0, R <sub>1</sub>

conclusions:

$$(1) \mathcal{E} = \{0\} \quad \delta_0^* = R_1$$

(2)  $\Delta' = \{12, A; 1, A; 2, A, 0, R_1\} = \text{single policy}$

so  $AAR_1$  is  $V_0$  optimal (no need to do second LP)

(3) Optimal gain is 3.0

(4) These results agree with the theory. □

Now, consider the following problem using the Basic Model which is not so trivial:

Example 6.3: Basic Model,  $n = 3$ ,  $L = 0$

$$\begin{array}{lll} K_1 = 1 & K_2 = 1.55 & K_3 = 3.2 \\ \lambda_1 = 3 & \lambda_2 = 2 & \lambda_3 = 1 \end{array}$$

$p$  varies

$k =$  type of system = 1 or 2 of  $n$ .

Theoretical results for  $V_{-1}$ :

$$k = 1, p = 1 \quad V_{-1} = \min\{\lambda_i(K_i + p), i = 1, 2, 3; \lambda_1 K_1 + \lambda_2 K_2\}$$

$$= \lambda_3(K_3 + p) = 4.2 \quad \mathcal{E} = \{0\} \quad \delta_0^* = R_3$$

$$k = 1, p = 2 \quad V_{-1} = \lambda_3(K_3 + p) = 5.2 \quad \mathcal{E} = \{0\} \quad \delta_0^* = R_3$$

$$\left. \begin{array}{l} k = 1, p = 3 \quad V_{-1} = \lambda_1 K_1 + \lambda_2 K_2 = 6.1 \\ k = 1, p = 10 \quad V_{-1} = \lambda_1 K_1 + \lambda_2 K_2 = 6.1 \end{array} \right\} \begin{array}{l} \mathcal{E} = \{1, 2\} \quad \delta_1^* = R_2 \\ \delta_2^* = R_1 \end{array}$$

$$k = 2, p = 2 \quad V_{-1} = \min\{\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3,$$

$$\lambda_i(K_i + p) + \lambda_j(K_j + p)_{i \neq j}\} \quad i, j \in \{1, 2, 3\}$$

$$= \lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 = 3 + 3.1 + 3.2 = 9.3$$

so,  $\mathcal{E} = \{12 \quad 13 \quad 23\} \quad \delta_{12}^* = R_3 \quad \delta_{13}^* = R_2 \quad \delta_{23}^* = R_1$

$k = 2, p = .9 \quad V_{-1} = \lambda_2(K_2 + p) + \lambda_3(K_3 + p) = 9.1$

$k = 2, p = .5 \quad V_{-1} = \lambda_2(K_2 + p) + \lambda_3(K_3 + p) = 7.8$

so,  $\mathcal{E} = \{2, 3\} \quad \delta_2^* = R_3 \quad \delta_3^* = R_2$

Computational results:

$k = 1, p = 1$

primal variables:  $x_0^{R_3} = 1.0 \quad x_i^k = 0, \quad \text{otherwise}$

dual rows at upper limit:

123,A	23,A	3,A
12,A	1,A	0,R <sub>3</sub>
13,A	2,A	

conclusions:

(1)  $\mathcal{E} = \{0\}, \delta_0^* = R_3$  (agrees with theory)

(2) optimal to do nothing in other states - no need for second LP, i.e., "A" in states 1,2,3.

Notes:

(1) In all of the variations in Example 6.2, to save time, decision variables for states  $s : |s| > k$  were fixed to "A" since known to be so by Chapter II results.

(2) The dual LP was used in all cases here to obtain solutions.

p = 2, k = 1

primal variables:  $x_0^{R_3} = 1.0$      $x_1^k = 0$ , otherwise

dual rows at  
 upper limit: 123,A    23,A    3,A  
                   12,A    1,R<sub>3</sub>    0,R<sub>3</sub>  
                   13,A    2,R<sub>3</sub>

conclusions:

(1)  $\mathcal{E} = \{0\}$ ,  $\delta_0^* = R_3$  (agrees with theory)

(2) optimal policy in states 1, 2, 3:

1 : R<sub>3</sub>

2 : R<sub>3</sub>

3 : A

no need for second LP.

p = 3, k = 1

primal variables:  $x_1^{R_2} = 2.0$      $x_1^k = 0$ , otherwise  
                                    $x_2^{R_1} = 3.0$

	<u>1<sup>st</sup> run of (D)</u>	<u>run (D) with</u>	
dual rows at	123,A    1,R <sub>2</sub>	123,A    1,R <sub>2</sub>	1, $\delta, \delta \neq R_2$
upper limit:	12,A    1,R <sub>3</sub>	12,A    2,R <sub>1</sub>	2, $\delta, \delta \neq R_1$
	13,A    2,R <sub>1</sub>	13,A    3,A	rows eliminated.
$B_0 = \phi$	23,A    3,A	23,A    0,R <sub>12</sub>	
$\Omega = \{12,1,2\}$	after use ALG 1.	Okay.	



conclusions: (1)  $\mathcal{E} = \{2, 3\}$      $\delta_2^* = R_3$      $\delta_3^* = R_2$

(2) optimal decisions in transient states:

$$\delta_{12}^* = \delta_{13}^* = \delta_{23}^* = A \quad \delta_1^* = R_3 .$$

Note: In the input of data, one must be very careful to be accurate and carry at least 4-5 decimal points in determining the correct  $V_0$  optimal solution. The set of dual rows which are at their upper limit can be very sensitive to small changes in the parameters. Realistically, if two policies are that close, using either would be optimal and the easier one to implement, practically speaking, could be chosen. If only interested in  $V_{-1}$ , the such accuracy is not as important.

p = .9, k = 2

primal variables:     $x_2^{R_3} = 1.0$      $x_i^k = 0, \text{ otherwise}$   
                            $x_3^{R_2} = 2.0$

dual rows at  
 upper limit:    123,A    23,A    3,R<sub>2</sub>  
                           12,R<sub>3</sub>    1,R<sub>23</sub>  
                           13,R<sub>2</sub>    2,R<sub>3</sub>

conclusion:     $\mathcal{E} = \{2, 3\}$      $\delta_2^* = R_3$      $\delta_3^* = R_2$

transient states:     $\delta_{12}^* = R_3$      $\delta_{13}^* = R_2$      $\delta_{23}^* = A$      $\delta_1^* = R_{23}$

p = 2, k = 2

primal variables:  $x_{12}^{R_3} = 1.0$        $x_i^k = 0$ , otherwise  
 $x_{13}^{R_2} = 2.0$   
 $x_{23}^{R_1} = 3.0$

dual rows at  
upper limit: 123,A    23,A    2,R<sub>3</sub>      B<sub>3</sub> =  $\phi$   
                  12,R<sub>3</sub>    23,R<sub>1</sub>       $\Omega = \{123,12,13,23\}$   
                  13,R<sub>2</sub>    1,R<sub>23</sub>      after ALG 1, okay.  
  states 1,2,3 inaccessible.

conclusions:

- (1)  $\mathcal{E} = \{12, 13, 23\}$   
          R<sub>3</sub> R<sub>2</sub> R<sub>1</sub>
- (2)  $\delta_1^*, \delta_2^*, \delta_3^*$  never needed since inaccessible given set of ergodic states. These values actually could be obtained by dropping rows corresponding to nonoptimal decisions in ergodic states and rerunning the problem.

Now, consider a noninstantaneous repair example, for which only the  $V_{-1}$  solution is sought. In this case, problem (P) is used for determining the solution.

Example 6.4: Noninstantaneous repair model, n = 4

L = varies      K = 1       $\sigma = 1$       parallel system  
p = varies       $\mu = 1$        $\ell = 0$   
s = number of servers varies

Theoretical results:

Unknown, except in  $s = 1$  cases with  $L = 0$ .

The following parameter values were tested:

$s = 4, p = 1$

L	0	1.4	9.5	49.5	72.5	74.5	76.5	78.5	89.5	99.5
---	---	-----	-----	------	------	------	------	------	------	------

$s = 4, L = 0$

p	1, 5, 7, 10, 20, 30
---	---------------------

$s = 2, p = 20, L = 0$
------------------------

$s = 1, p = 20, L = 0$
------------------------

Results:

$s = 4, p = 1$  (varying L)

L	optimal ergodic chain (optimal policy)											
0	0,0	1,0	[R <sup>(0)</sup> ]									
	R <sub>1</sub>	A										
1.4												R <sup>(0)</sup>
9.5												R <sup>(0)</sup>
49.5	(same as L = 0)											R <sup>(0)</sup>
72.5												R <sup>(0)</sup>
74.5												R <sup>(0)</sup>
76.5												R <sup>(0)</sup>
78.5	4,0	3,0	2,1	2,2	1,0	1,1	1,2	1,3	0,0	0,1	0,2	0,3
	A	A	A	A	A	A	A	A	R <sub>4</sub>	R <sub>3</sub>	R <sub>2</sub>	A
	A policy which says do nothing until the system fails, then repair all failed items possible subject to server availability.											
89.5	(same as L = 78.5)											
99.5	(same as L = 78.5)											

Clearly, the  $L = 78.5$  policy should be optimal for all  $L$  values large enough. The fact that the mean repair time is large (= mean component lifetimes) explains why the fixed charge must get so large to have any effect on optimal policy, (large  $\sigma$  favors repair only when necessary).

$s = 4, L = 0$  results

p	optimal policy							
1	1,0	0,0						
	A	$R_1$	$(R^{(0)})$					
5	2,0	1,0	0,1					
	A	$R_1$	A	$(R^{(1)})$				
7	3,0	2,0	1,0	1,1	0,1			
	A	$R_1$	$R_1$	A	A	$(R^{(2)})$		
10	4,0	3,0	3,1	2,1	2,2	1,2	0,3	[requires use of more than one server]
	A	$R_1$	A	$R_1$	A	$R_1$	A	
20	same as $p = 10$							

The  $p = 10$  case is another example of a noninstantaneous repair model for which it is optimal to use more than one server at a time (unlike instantaneous repair).

Clearly, if the  $s = 4$  optimal policy never uses more than  $s' < 4$  servers, then the  $s = s'$  optimal policy will be the same as for  $s = 4$ . Thus, the  $p = 20$  case is chosen as the one to vary  $s$  in.

p = 20, L = 0, s = 1, 2, 3, 4 results

s	V <sub>-1</sub> optimal policy							
4	4,0 A	3,0 R <sub>1</sub>	3,1 A	2,1 R <sub>1</sub>	2,2 A	1,2 R <sub>1</sub>	0,3 A	(uses 3 servers)
3	(same as s = 4)							
2	4,0 A	3,0 R <sub>1</sub>	3,1 A	2,1 R <sub>1</sub>	1,1 R <sub>1</sub>	1,2 A	0,2 A	(uses 2 servers)
1	4,0 A	3,0 R <sub>1</sub>	2,0 R <sub>1</sub>	2,1 A	1,0 R <sub>1</sub>	1,1 A	0,1 A	(R <sup>(3)</sup> , uses 1 server)

The previous examples are intended only to give a sampling of the kinds of new results obtainable computationally. Further experimentation could lead to formulation of new theorems or counterexamples to certain conjectures about optimal policy forms. Most importantly of all, these techniques could be implemented to solve a specific real-world problem which could be formulated as a coherent system repair model. Sample input and output from Example 6.2 are shown in Figures 6.1, 6.2, 6.3.

FIGURE 6.1 - MPS Input Data File for Example 6.2 (Dual LP)

```

00100  NAME          EX1U
00200  ROWS
00300  N  OBJ
00400  L  X12A
00500  L  X1A
00600  L  X1R2
00700  L  X2A
00800  L  X2R1
00900  L  X0R1
01000  L  X0R2
01100  L  X0R12
01200  COLUMNS
01300  U12          X12A          1.0
01400  U1          X12A          - .3333
01500  U1          X1R2          .6667          X1A          1.0
01600  U1          X0R12         - .3333          X2R1          - .3333
01700  U2          X12A          - .6667          X1R2          - .6667
01800  U2          X2A           1.0           X2R1          .3333
01900  U2          X0R12         - .6667
02000  U0          X1A           -1.0          X2A           -1.0
02100  G          OBJ           -1.0          X12A          .6667
02200  G          X1A           1.0           X1R2          .6667
02300  G          X2A           2.0           X2R1          .6667
02400  G          X0R1          1.0           X0R2          2.0
02500  G          X0R12         .6667$
*E

(EX1U.DAT.1)
*I2500
02600  RHS
02700  RHS          X1R2          6.0          X2R1          2.0
02800  RHS          X0R1          3.0          X0R2          7.0
02900  RHS          X0R12         9.0
03000  BOUNDS
03100  FR BND          U12
03200  FR BND          U1
03300  FR BND          U2
03400  FR BND          U0
03500  FR BND          G
03600  ENDATA$

```

8 decisions → 9 rows (counting objective)

4 states → 5 columns

The primal would have 5 rows and 9 columns.

**FIGURE 6.2 - MINOS Row and p = 1 Column  
Output for Example 6.2 (Dual)**

12500	NUMREP	COLUMN	AT	ACTIVITY	OBJ GRANTENT	LOWER	UNIT
UPPER	LIMIT	ACTIVITY	ACTIVITY				
12600							
12700	7	OBJ	RS	-3.00000	3.00000		NONE
	NONE						
12800	8	X12A	UL	0.00000	0.00000		NONE
	0.00000						
12900	9	X1A	UL	0.00000	0.00000		NONE
	0.00000						
13000	10	X1P2	RS	4.00020	1.99980		NONE
	6.00000						
13100	11	X2A	UL	0.00000	0.00000		NONE
	0.00000						
13200	12	X2P1	RS	1.00020	0.99980		NONE
	2.00000						
13300	13	XAP1	UL	3.00000	0.00000		NONE
	3.00000						
13400	14	XAP2	RS	6.00000	1.00000		NONE
	7.00000						
13500	15	XAP1>	RS	5.00010	3.99990		NONE
	9.00000						
13600							
13700	SECTION 2 - COLUMNS						
13800							
13900	NUMREP	COLUMN	AT	ACTIVITY	OBJ GRANTENT	LOWER	LIMIT
UPPER	LIMIT	ACTIVITY	ACTIVITY				
14000							
14100	1	U12	RS	-2.00020	0.00000		NONE
	NONE						
14200	2	U1	RS	-3.00000	0.00000		NONE
	NONE						
14300	3	U2	RS	-4.00000	0.00000		NONE
	NONE						
14400	4	U0	FP	0.00000	0.00000		NONE
	NONE						
14500	5	G	RS	3.00000	-1.00000		NONE
	NONE						
14600	6	PMS	FD	-1.00000	0.00000	-1.00000	
	-1.00000						

$\mathcal{E} = \{0\}$        $\delta_0^* = R_1$   
 $B_{12} = A$   
 $B_1 = A$       so,       $AAR_1$  is  $V_0$  optimal policy.  
 $B_2 = A$   
 $B_0 = R_1$

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**FIGURE 6.3 - MINOS Row and Column Output for  
Example 6.2 (Dual) p = 5**

13400	SECTION 1 - ROWS				...	ACTIVITY...	SLACK	ACTIVITY	...	LOWER	LIMIT.	
13500	NUMREP	...	ROW..	AT	...	ACTIVITY...	SLACK	ACTIVITY	...	LOWER	LIMIT.	
13600	UPPER	LIMIT.	DUAL	ACTIVITY	...	ACTIVITY...	SLACK	ACTIVITY	...	LOWER	LIMIT.	
13400	NONE	7	OBJ	RS	1	-4.99955	4.99955			NONE		
13500	NONE	8	X12A	UL	2	0.00000	0.00000			NONE		
13600	0.00000	9	X1A	RS	3	-0.99955	0.99955			NONE		
13700	0.00000	10	X12	UL	4	6.00000	0.00000			NONE		
13800	6.00000	11	X2A	UL	5	0.00000	0.00000			NONE		
13900	0.00000	12	X2P1	UL	6	2.00000	0.00000			NONE		
14000	2.00000	13	X2R1	RS	7	4.99955	2.00045			NONE		
14100	2.00000	14	X2R2	RS	8	9.99910	1.00090			NONE		
14200	11.00000	15	X2R12	RS	9	8.66590	4.33410			NONE		
14300	13.00000											
14400	1	SECTION 2 - COLUMNS				... <td>ACTIVITY... <td>OBJ</td> <td>GRADIENT. <td>... <td>LOWER</td> <td>LIMIT</td> </td></td></td>	ACTIVITY... <td>OBJ</td> <td>GRADIENT. <td>... <td>LOWER</td> <td>LIMIT</td> </td></td>	OBJ	GRADIENT. <td>... <td>LOWER</td> <td>LIMIT</td> </td>	... <td>LOWER</td> <td>LIMIT</td>	LOWER	LIMIT
14500	NUMREP	... <td>COLUMN.</td> <td>AT</td> <td>... <td>ACTIVITY... <td>OBJ</td> <td>GRADIENT. <td>... <td>LOWER</td> <td>LIMIT</td> </td></td></td></td>	COLUMN.	AT	... <td>ACTIVITY... <td>OBJ</td> <td>GRADIENT. <td>... <td>LOWER</td> <td>LIMIT</td> </td></td></td>	ACTIVITY... <td>OBJ</td> <td>GRADIENT. <td>... <td>LOWER</td> <td>LIMIT</td> </td></td>	OBJ	GRADIENT. <td>... <td>LOWER</td> <td>LIMIT</td> </td>	... <td>LOWER</td> <td>LIMIT</td>	LOWER	LIMIT	
14600	UPPER	LIMIT.	REDUCED	COST.	M+J	ACTIVITY... <td>OBJ</td> <td>GRADIENT. <td>... <td>LOWER</td> <td>LIMIT</td> </td></td>	OBJ	GRADIENT. <td>... <td>LOWER</td> <td>LIMIT</td> </td>	... <td>LOWER</td> <td>LIMIT</td>	LOWER	LIMIT	
14700	1	U12	BS	10	-11.99910	0.00000				NONE		
14800	NONE	2	U1	RS	11	-5.99910	0.00000			NONE		
14900	NONE	3	U2	BS	12	-9.99910	0.00000			NONE		
15000	NONE	4	U0	FP	13	0.00000	0.00000			NONE		
15100	NONE	5	A	BS	14	4.99955	-1.00000			NONE		
15200	NONE	6	PHS	EO	15	-1.00000	0.00000			-1.00000		
15300	-1.00000											

$$\mathcal{E} = \{1, 2\}$$

$$\delta_1^* = R_2,$$

$$\delta_2^* = R_1$$

$$B_{12} = A$$

$$B_1 = R_2$$

$$B_2 = R_1, A$$

$$B_0 = \phi$$

→ altered  
decision  
space

$$B_{12} = A$$

$$B_1 = R_2$$

$$B_2 = R_1$$

$V_0$  is  
optimal  
policy

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**\* FIGURE 6.4 - SOLVER Program (Fortran)  
Used to Implement MINOS**

```

00101 SOLVER.FOR.6
00102
00270      FERR48 = 20000
00300      DPTH = INTCORE / 20000
00310      C
00320      CALL GO (2,INTCORE)
00330      RETURN
00340      C
00350      END
00360      C
00370      C
00380      C
00390      C
00400      SUBROUTINE GO (2,INTCORE)
00410      C
00420      IMPLICIT REAL*8(A-H,O-Z)
00430      FERR48 = 20000
00440      COMMON /FILES/ ISCF,INPUT,LOGDB,INEND,INERT,IPRCH,ILONG,LOOMP
00450      COMMON /SOLN/ IPSOLN,ISOLN,MSOLN,ISTRIC,LSHIFP
00460      COMMON /FACOR/ FES,INF,INTOBJ,INTINC,INFX,INIF,IOBJ,NPFOB
00470      C
00480      C
00490      ISCRCH = 30
00500      REWIND ISCRCH
00510      ISPECS = 32
00520      OPEN(UNIT=32,FILE='PECS.DAT',DEVICE='DISK')
00530      OPEN(UNIT=31,DIRLOG,DEVICE='DISK')
00540      REWIND ISPECS
00550      OPEN(UNIT=6,FILE='MINOUT',DEVICE='DISK')
00560      IPSOLN = 1
00570      CALL MINOS(2,INTCORE,ISPECS,ISCRCH,IPSOLN,
00580      IERROR,INROWS,NCOLS,LAS,LAL,LPI,LIS,LFREE,IFREE)
00590      WRITE (ISCRCH,9000) IERROR,IC
00600      9000 FORMAT(2I4)
00610      100 IF (IERROR.NE.0) STOP
00620      CLOSE(UNIT=6)
00630      RETURN
00640      C
00650      END OF GO

```

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\* Obtained from Robert Condap, Department of Operations Research,  
Stanford University

```

02400 SUBROUTINE HPSIN( 2,IMCORE,ISPEC,ISCRCH,IPSOLN,
02410 I, IERR,NKONS,NCOLS,MELNS,LE,LT,L1,LHS,LFREE,NFREE)
02420 IMPLICIT REAL*8(C-G,O-Z)
02430 REAL*8 IMCORE,KPUS
02440 LOGICAL COMU,RESTR
02450 COMMON /BGCOM / TOLSNP,MBUMP,MSPIKE,MSPK
02460 COMMON /LGCOM / CGBETA,ITNCG,MSGCG,NUDCG,RESTR
02470 COMMON /CONCOM / ETASH,ETASK,CONK(4),LUTOL,ICIFNS,NFHIL
02480 COMMON /CORE / KZ1,KZ2,KZ3
02490 *P2530150
02500 COMMON /DJCOM / TOLDJ1,TOLDJ2,TOLDJ3,TOLDJ
02510 COMMON /EPSCOM / EPS,EPS0,EPF1,EPF2,EPF3,EPF4,EPF5,PLINFY
02520 COMMON /FILES / ISCR,INPUT,TOLDB,INENB,INSRT,IPRCH,ILOAD,IOUMP
02530 COMMON /FPEMS / KCHK,KINP,KSPU,KLOG,ITFREQ,I2FREQ
02540 COMMON /FRCOM / FX,SINF,ITOBJ,MININ2,INF,NINF,TOBJ,INPROB
02550 COMMON /INTCOM / ITN,ITNLIN,IFNS,KHULLU,KHUPI
02560 COMMON /INUCOM / INUKG,INUITN,INUMOD,INBELEN,KAL,KUL,KHU,KDU,NFF
02570 COMMON /ITHLOG / UJU,THETA,PIROT,CONU,NONOPT,JP,JO,KHUB,RSUB
02580 COMMON /LPLUM / KMS,INST,MMK,TEER,IDEBUG
02590 COMMON /LUFIL / INBED,NETAL,NETAR,NETAU,KL,KU,LBGN,ULBGN
02600 COMMON /MPCOM / RIJTOL,BSTRUC(2),MINMAX,MLST,MR,
02610 NAME(2),MOBJ(2),MKMS(2),MKNS(2),MBND(2)
02620 COMMON /MULCOM / MPARM(9),IPARM(9)
02630 COMMON /PRCCOM / MPARPR,MPRC,MPRC,KPRC,MREJ,MIKEJ,JKJ(2)
02640 COMMON /PRCCM / MMULPM,NEWSB
02650 COMMON /RGTOLS / XTOL(3),FTOL(3),BTOL(3),FINORM,RSNORM,TOLRG
02660 COMMON /SOLNCH / ISOLN,KSOLN,MSOLN,NSTATE,LCHKUP
02670 COMMON /TOLS / TOLN,TOLPIU,TRPIU1,TRPIU2,TOLROW,XNORM
02680 COMMON /XUMUS2 / HMOPDR,HMURDI,HMURDH
02690 IVERS = 330278
02700 C
02710 MHKZ = IMCORE
02720 ISCP = ISCRCH
02730 IERR = 0
02740 NKONS = 0
02750 NCOLS = 0
02760 IF (.ISPEC.EQ.0) IERR = -1
02770 IF (.ISPEC.EQ.ISCRCH) IERR = -1
02780 IF (.IMCORE.LT.100) IERR = 30
02790 IF (.IERR.NE.0) GO TO 900
02800 CALL INITLZ
02810 C
02820 READ PROBLEM SPECIFICATIONS FROM FILE 'ISPEC'
02830 AND OUTPUT IN SPECIAL FORMAT TO FILE 'ISCRCH'
02840 C
02850 NSPEC = 0
02860 CALL SPECS( 2,ISPEC,ISCRCH,IVERSN,NSPEC )
02870 IF (.NSPEC.LT.0) IERR = -1
02880 IF (.IERR.NE.0) GO TO 900
02890 C
02900 RE-READ SPECS FROM FILE 'ISCRCH'
02910 C
02920 CALL SPECS2( NSPEC,NKONS,NCOLS,MELNS,
02930 I, H,M2,MN,MIN,N,NE,NPT,HR,HN,HNQ,HNKS,NS )
02940 C
02950 READ HPS DATA FROM FILE 'INPUT'
02960 C
02970 CALL HPSIN( 2,MHKZ,NKONS,NCOLS,MELNS,
02980 I, H,M2,MN,MIN,N,NE,NPT,HR,HN,HNQ,HNKS,NS )
02990 C
03000 IF (.INPUT.NE.0) REWIND INPUT
03010 REWIND ISCRCH
03020 IF (.LE.0) IERR = 40
03030 IF (.IERR.NE.0) GO TO 900
03040 CALL DRIVER( 1,MHKZ,IPSOLN,
03050 I, H,M2,MN,MIN,N,NE,NPT,HR,HN,HNQ,HNKS,NS,
03060 2, ZKMR,ZKME,ZKMH,ZKMX,ZKBL,ZKBU,ZKMB,ZKRN,
03070 3, ZKMS,ZKMF,ZKMU,ZKMF,
03080 4, ZKML,ZKHU,ZKKS,ZKLU,ZKLU,ZKPI,
03090 5, ZKMG,ZKMY,ZKYI,ZKGI,ZKGB,ZKGB )
03100 C
03110 REWIND ISCRCH
03120 IF (.GT.4) IERR = 4
03130 IF (.IERR.NE.0) GO TO 900
03140 CALL DRIVER( 1,MHKZ,IPSOLN,
03150 I, H,M2,MN,MIN,N,NE,NPT,HR,HN,HNQ,HNKS,NS,
03160 2, ZKMR,ZKME,ZKMH,ZKMX,ZKBL,ZKBU,ZKMB,ZKRN,
03170 3, ZKMS,ZKMF,ZKMU,ZKMF,
03180 4, ZKML,ZKHU,ZKKS,ZKLU,ZKLU,ZKPI,
03190 5, ZKMG,ZKMY,ZKYI,ZKGI,ZKGB,ZKGB )
03200 REWIND ISCRCH
03210 IF (.GT.4) IERR = 4
03220 900 IERROR = IERR

```

This work is based on the MATICARIN

```

03590      RETURN
03600 C      END OF MINUS
03610      END
03710      SUBROUTINE CALCP6(MODE,ITR,K,F,G,ISTATE,IPROB)
03810      RETURN
03910      END

```

Page 2

```

00100 C
00200 C      SUBROUTINE SOLN( ICHLL, H, HNE, HPI, H, HND, N, MS,
00300 I, HM, HB, HE, HS, H, BL, BU, BT, C, S, PI, Y, HST, ISTATE, UNDISK )
00400 C      IMPLICIT REAL*8(C-F, I-D)
00500 C      COMMON /CYCLE/ KC, NMOD, HOSOLN, HOCR
00600 C      COMMON /DJOUR/ TOLDJ1, TOLDJ2, TOLDJ3, TOLDJ
00700 C      COMMON /EPS/ EPS, EPS0, EPS1, EPS2, EPS3, EPS4, EPS5, PLIIFY
00800 C      COMMON /FILES/ ISCP, INPUT, IOLDB, IHEMB, INSR1, IPNCH, ILOD, IOUNP
00900 C      COMMON /FICOR/ FIC, SINF, INT00, HINIME, HIC, IINF, IOB0, IPRUB
01000 C      COMMON /INTCOM/ ITH, ITHL, IN, IPMS, KMODL, KMODPI
01100 C      COMMON /LPCOR/ KMS, INI, HMR, IERR, IDEBUG
01200 C      COMMON /MPSCOR/ HJ, TOL, EST, IOLC2, HINMH, HJST, HEP,
01300 I, HANE(2), NOBJ(2), MRHS(2), MPNG(2), HEND(2)
01400 C      COMMON /SOLN/ ISOLN, KSOLN, HSOLN, HSTATE, LCH, GR
01500 C      COMMON /TOLS/ TOLS, TOLP10, TRP101, TRF102, TOLP0W, ANORM
01510!50
01600 C      INTEGER HANE(2), HEANPI(2), HSCD(2), HBMS(2), HSTC(2)
01700 C      INTEGER ISTATE(2)
01800 C      REAL HANE, BL(2), BU(2)
01900 C      REAL*8 HINIME(2), CONNO(2), KMS(2), PI(2), Y(2)
02000 C      INTEGER IOLC2
02100 C      LOGICAL UNDISK, IHSBL
02200 C
02300 C      THE FULL SOLUTION VECTOR GETS STORED IN THE ARRAY Y AS FOLLOWS:
02400 C
02500 C      Y(1) ... Y(KMS+1) VALUES OF EACH VARIABLE IN THE H-MATRIX
02600 C      Y(KMS+2) ... Y(N) THE RHS, ALWAYS = -1.0
02700 C      Y(KMS+1)...Y(N) VALUES OF THE LOGICAL VARIABLES (SLACKS)
02800 C
02900 C      -----
03000 C      LPR = 6
03100 C      IF (UNDISK) LPR = ISOLN
03200 C      BPLUS = 0.14*PLIIFY
03300 C      IHSBL = HINF.61.C
03400 C      IF (ICHLL.NE.1) GO TO 200
03500 C
03600 C      HVOID OVERWRITING STATE VECTOR
03700 C
03800 C      DO 50 J=1, H
03900 C      50 HSTC(J) = HSCD(J)
04000 C
04100 C      DO 100 J=1, H
04200 C      IF (HSTC(J).GT.1) GO TO 100
04300 C      B1 = BL(J)
04400 C      B2 = BU(J)
04500 C      Y(J) = B1
04600 C      IF (HSTC(J).EQ.1) Y(J) = B2
04700 C      IF (B1.EQ.B2) HSTC(J) = 4
04800 C      IF (HMS<Y(J).LT.BPLUS) GO TO 100
04900 C      Y(J) = 0.0
05000 C      HSTC(J) = 5
05100 C
05200 C      100 CONTINUE
05300 C
05400 C      DO 150 J=1, MS
05500 C      K = HBA(J)
05600 C      I = KCU(J)
05700 C      Y(K) = I
05800 C      IF (I.LT. BL(K)+TOLK) HSTC(K) = 6
05900 C      IF (I.GT. BU(K)+TOLK) HSTC(K) = 7
06000 C
06100 C      150 CONTINUE
06200 C      IF (ANN.EQ.0 .OR. IHSBL .OR. KSOLN.EQ.0) RETURN
06300 C
06400 C      USE ZH INSTEAD OF X IN CASE LAST FUNGD RESET ZH
06500 C
06600 C      DO 180 K=1, H
06700 C      I = KCK(K)
06800 C      Y(K) = I
06900 C
07000 C      180 CONTINUE
07100 C      RETURN
07200 C
07300 C      CALL Z -- OUTPUT SOLUTION IN APPROXIMATELY HNS FORMAT
07400 C
07500 C      200 CONTINUE

```

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08000 C
08100 C   EXTRACT PMS AND OUTPUT ROWS
08200 C
08300 CALL UIMPAC( M,NE,NPI,MR,ME,MAX,KRMS )
08400 OBJ = SINP
08500 IF (NINF.EQ.0) OBJ = MINIMZ#FX
08600 REWIND ISCR
08700 READ(ISCR, 1050) IO
08800 WRITE(6, 1000) IO,OBJ,ISTATE,MPMS,ITM
08900 WRITE(6, 1005) MOBJ,MINMAX,PKMS,MRNG,MBND
09000 WRITE(6, 1010)
09100 C
09200 DO 240 I=1,M
09300   J = KRMS+I
09400   IA = HST(J)+1
09500 C   TURN LOGICALS INTO ROWS
09600   IF (IA.LT.3) IA = 3-IA
09700   IF (IA.GT.6) IA = 15-IA
09800   KMS = X(I)
09900   B = KMS
10000   SLK = Y(J)
10100   ROW = KMS - SLK
10200   B1 = B - BU(J)
10300   B2 = B - BL(J)
10400   PY = PI(I)
10500   READ(ISCR, 1050) IO
10600   CALL SOLFRT( J,IO,IR,ROW,SLK,B1,B2,PY,1,BPLUS,ONDISK )
10700 C240 CONTINUE
10800 C
10900 C   OUTPUT COLUMNS
11000 C
11100 TOLD = 0.14TOLDJ3
11200 WRITE(6, 1020)
11300 C
11400 DO 300 J=1,KRMS
11500   K1 = HEC(J)+1
11600   K2 = HEC(J)+1
11700   CJ = 0.0
11800   DJ = 0.0
11900   DO 250 K=K1,K2
12000     IR = HCK(K)
12100     IR = IABS(IR)
12200     DJ = DJ - PI(IR)*ACK
12300     IF (IR.EQ.10BJ) CJ = HCK(K)
12400 C250 CONTINUE
12500   IF (INFSBL) GO TO 280
12600   IF (J.GT.NM) GO TO 270
12700   CJ = CJJ + CJ
12800   DJ = MINIMZ#CJJ + DJ
12900 C270 DJ = MINIMZ#DJ
13000 C280 IF (DABS(DJ).LE.TOLD) DJ = 0.0
13100   IA = HST(J)+1
13200   BJ = Y(J)
13300   B1 = BL(J)
13400   B2 = BU(J)
13500   K = M+J
13600   READ(ISCR, 1050) IO
13700   CALL SOLFRT( J,IO,IR,KJ,CJ,B1,B2,DJ,K,BPLUS,ONDISK )
13800 C300 CONTINUE
13900 C
14000 REWIND ISCR
14100 RETURN
14200 C
14300 1000 FORMAT(12,' PROBLEM NAME:', 3X, 2H4, 13X,
14400 1) OBJECTIVE VALUE:', 1P20.10//,
14500 2) STATUS:', 3X, 2H4, 9X,
14600 3) PHASE:', 19, ' ITERATION:', 16//
14700 1005 FORMAT(
14800 1) OBJECTIVE:', 6X, 2H4, ' (', 4S, ')/',
14900 2) PMS: ', 6X, 2H4//,
15000 3) RANGES: ', 6X, 2H4//,
15100 4) BOUNDS: ', 6X, 2H4//
15200 1010 FORMAT(12,' SECTION 1 - ROWS'//,
15300 1) NUMBER ...ROW.. AT ...ACTIVITY... SLACK ACTIVITY',
15400 2) ..LOWER LIMIT. ..UPPER LIMIT. ..DUAL ACTIVITY ...I//
15500 1020 FORMAT(12,' SECTION 2 - COLUMNS'//,
15600 1) NUMBER ..COLUMN.. AT ...ACTIVITY... OBJ GRADIENT.',
15700 2) ..LOWER LIMIT. ..UPPER LIMIT. ..REDUCED COST. M+J//
15800 1050 FORMAT(2H4)
15900 C   END OF SOLT
16000 END
.P0

```

CHAPTER VII  
CONCLUSIONS, APPLICATIONS AND EXTENSIONS

1. Summary of Model Results - Chapters II-V

This chapter will present a summary of results from previous chapters in this section. Following that in Section 2, two basic theorems which relate to all coherent system repair models are drawn out from the results. Under what parameter assumptions they do or do not apply is discussed. Section 3 looks into some conclusions one might draw from Sections 1 and 2 results about how changes in various model parameters affect the optimal policy or the minimum cost per unit time attainable. Possible applications of these models or the results of such are discussed in Section 4. A listing of possible model extensions and topics for future research concludes the chapter and thesis.

The summary of previous results along with references to the chapter section, and theorem from which each was drawn will be presented as a series of tables. This allows for a clear and concise categorization of the numerous items.

Classification will be done according to model type and particular parameter assumptions made for it. Table 7.1 summarizes the categories:

TABLE 7.1 - Categorization of Results for Coherent System Repair Models

<u>I. Basic Model</u>	<u>Found in</u>
A. system type: general coherent, ≠ series objective: $V_{-1}$	] Table 7.2
B. system type: k-of-n, k ≠ n (series) objective: $V_{-1}$	] Table 7.3
C. system type: k-of-n, k ≠ n (series) objective: $V_0$	] Table 7.4
<u>II. Degradation Model</u>	
A. system type: series or parallel objective: $V_{-1}$	] Table 7.5
<u>III. Noninstantaneous Repair Model</u>	
A. system type: k-of-n, identical components (nonseries) objective: $V_{-1}$	] Table 7.6

Tables 7.2 → 7.6 will comprise the rest of this section.

TABLE 7.2 - Basic Model, General Coherent System,  $V_{-1}$  Results

		Chapter	Section	Theorem(s)
	For any policy $\delta$ , $V_{-1}^\delta$ can be written as a convex combination of certain $V_{-1}^{\gamma_s}$ where $\gamma_s$ are policies which never repair more than one unit simultaneously.	II	2	2.1, 2.2
L = 0	Never repair more than one unit at a time in an ergodic state.	II	2	2.3
No fixed charge	$V_{-1}$ optimal policy: keep a certain subset of components working - repair any that fail immediately. $s$ determined by $s = \operatorname{argmin}_{t \in \{1, \dots, n\}} \hat{V}_t$ where	II	2	2.4
	$\hat{V}_t = \sum_{\substack{i \in t \\ i: t-i \in \mathcal{P}_p}} \frac{K_i}{\mu_i} + \sum_{\substack{i \in t \\ i: t-i \in \mathcal{P}_p}} \frac{K_i+p}{\mu_i}$			
	If one can break the system down into independent subsystems in series, the optimal system $V_{-1}$ is the sum of the optimal $V_{-1}^s$ for each subsystem.	VI	3	6.3
L > 0 fixed charge	Still have $V_{-1}^\delta$ written as a convex combination of certain quantities but these are now not all $V_{-1}^s$ of simpler policies.	II	3	2.5

TABLE 7.3 - Basic Model, k-of-n System,  
 $V_{-1}$  Results ( $k \neq n$  series)

All results from Table 7.2 for  $\begin{pmatrix} L = 0 \\ L > 0 \end{pmatrix}$  plus:

	nonidentical components	identical components
$L = 0$	$V_{-1}$ optimal policy form: Keep cheapest set of $k-1$ or $k$ components working.	Ch. II Sec. 2 Ex.2.4 Optimal form: do nothing until $k$ units left working Repair one unit in $k$ or $k-1$ , depending on $p$ .
	Parallel case: if have sufficiently large spread of $\lambda_i K_i$ 's compared to $\lambda_i$ 's, then $V_{-1}$ optimal policy will repair the items with cheapest $\lambda_i K_i$ , no matter <u>what</u> $p$ is.	Ch. III Sec. 2 Lem 3.12 Policy form: do nothing until $k$ items left working -among policies which re- pair when $k$ left, $R_{\ell-1}$ optimal $\Leftrightarrow f_{k+1}(\ell-1) < L/K$ $< f_{k+1}(\ell)$
$L > 0$	Policy form: too compli- cated in general for a nice result.	Ch. II Sec. 3 -among policies which re- pair when $k-1$ left, $R_{\ell}$ optimal $\Leftrightarrow f_k(\ell-1) < \frac{L+p}{K}$ $< f_k(\ell)$
	If $p=0$ ; do nothing until system fails	Ch. II Sec. 3 -optimal policy is the better of the previous two where $f_k(i) = (i+k) \left[ \frac{1}{k} + \dots + \frac{1}{i+k-1} \right]$ $- i$
	identical components	
$L > 0$	If $n_1 < n_2$ , number of com- ponents repaired in an optimal policy for $k$ -of- $n_2$ is $\geq$ number repaired for a $k$ -of- $n_1$ . If number repaired for $n_1$ is less than all possible, then the number repaired for $n_2$ is the <u>same</u> .	Ch. II Sec. 3 Thm 2.8 If $k_1 < k_2$ , then an optimal $k_2$ -of- $n$ policy will re- pair more components than a $k_1$ -of- $n$ .  If $n, k \rightarrow \infty$ , $n-k$ fixed, an optimal policy will be to repair <u>all</u> failed compo- nents any time repair is undertaken.

TABLE 7.4 - Basic Model,  $V_0$  Results, k-of-n System

nonidentical components		identical components
Do nothing until k units left working.	Ch. II Sec. 3 & Ch.III,1 Thm. 3.3	
Parallel system; $L = 0$ : never repair up to more than two components working simultaneously.	Ch. III Sec. 1 Thm. 3.4	NOT NEEDED
If $p < \frac{\sum_{i=1}^k \lambda_i K_i}{\lambda_1 + \dots + \lambda_k} p$ , then	Ch. III Sec. 1 Lem.3.5	$V_{-1}$ gives
never repair up to more than k + 1 units working simultaneously.		unique optimal
Parallel case; $L = 0$ $\lambda_i > \lambda_j$ and $K_i > K_j \Rightarrow$ never repair unit i.	Ch. III Sec. 2 Lem.3.6	decision in
Parallel case: restrictions on possible optimal policies given $V_{-1} = \lambda_\ell (K_\ell + p)$ , any $\ell$ .	Ch. III Sec. 2 L.3.7,3.8 Th. 3.9 Cor.3.10	each accessible
Parallel case: restrictions on optimal policies given a specific $\ell$ in $V_{-1} = \lambda_\ell (K_\ell + p)$ .	Ch. III Sec. 2 Lem.3.11	state

TABLE 7.5 - Degradation Model Results (and Conjectures)

	nonidentical components	identical components
series system	<p><u>Optimal policy:</u> Let <math>i^{\text{th}}</math> component get down to a certain state of degradation, say <math>\ell_i</math> - then repair, where <math>\ell_i</math> is determined by the <math>K</math>'s and <math>\lambda</math>'s [get using <math>V_{-1}</math> only].</p>	<p><u>Optimal policy:</u> When any component reaches <math>k^{\text{th}}</math> state of degradation, repair it. [get using <math>V_{-1}</math> only].</p>
parallel system	<p>Too complicated to analyze theoretically.</p>	<p><u>Conjectures:</u> (These results still need to be verified)</p> <p>Never optimal to repair <math>&gt; 1</math> unit at a time in an ergodic state.</p> <p>Never repair until one unit or less working.</p>
general system	<p>Can modify degradation model slightly to get an extension of the Basic Model allowing for Erlang component lifetimes.</p>	<p><math>V_{-1}</math> optimal solution to <math>M_{(n,\ell)}</math> same as that to <math>M_{(2,\ell)}</math>, <math>\ell \geq 1</math>, <math>n \geq 2</math>, where <math>M_{n,\ell}</math> is a Degradation Model with <math>n</math> components, <math>\ell</math> degradation states.</p>

TABLE 7.6 - Noninstantaneous Repair Model Results

		Chap- ter	Sec- tion	Theorem
	<u>V<sub>-1</sub> optimal policy form: repair whenever the server is free and the number of working components is below a certain number, say j. (Call this policy R<sup>(j)</sup>).</u>	V	2	Th. 5.1 Lem.5.2 Th. 5.3
s = 1	The optimal policy is			
	R <sup>(j)</sup> , if $u(j) < \frac{p\mu}{f_k(K+l\sigma)} < u(j+1)$			
single	R <sup>(k-1)</sup> , if $0 < \frac{p\mu}{f_k(K+l\sigma)} < u(k)$			
	where			
server	$z_1 = 1; \frac{z_{j+1}}{j+1} = \frac{\mu^{j+1}}{j+1} + \sigma z_j$			
L = 0	$f_1 = 1, f_k = \frac{(k-1)! \sigma^{k-1}}{\mu^{k-2} (\mu + (k-1)\sigma)}; k \geq 2$			
	$u(j) = \frac{j+1}{\sigma^{j-1} (j-1)!} \left[ \frac{z_j}{j} - \frac{\mu z_{j-1}}{j+1} \right]$			
	Relationship of optimal policy to k, f <sub>k</sub> : number of components repaired could increase or decrease with k, depending on μ, σ.	V	2	Lem.5.4 Lem.5.5
s > 1	Not true that no fixed charge means never repair more than one unit at a time.	V VI	3 4	Ex. 5.5 Ex. 6.3
multiple servers	If a policy which uses s' servers at most is optimal for a model which has s > s', then it is optimal for the model with any s' ≤ t ≤ s.	V	3	Lem.5.6

## 2. Two Basic Theorems

Section 1 presented a summary of results obtained in Chapters II through V for three different categories of coherent system repair models. Due to significant differences in state space, decision space, and component lifetime distributions, results for each model had to be obtained separately and in general, optimal policy forms are quite different. Even results of similar type had to be proved in a different fashion for each model. Thus, the allocation of separate chapters to each model, even though all came under the heading of a coherent system repair model.

Despite these differences in model structure and, thus, methods of obtaining results, one would expect similarities between many of the models since they are all modeling the same type of activity - the maintenance of a deteriorating system of components. Two basic policy "types" appear throughout. These are:

- (1) If one has a  $k$ -of- $n$  system and instantaneous repair, then never repair until there are  $k$  units left in the system.
- (2) If there is no fixed charge for repair (only component and labor costs), then one never repairs more than one unit at a time in an optimal policy in an ergodic state.

The applicability of statements (1) and (2) to the various coherent system repair models discussed up to now appears in Tables 7.7 and 7.8.

TABLE 7.7 - Application of Statement:

"If have a k-of-n system with instantaneous repair, then never repair when more than k units in the system are up"

to various coherent system repair models.

	<u>Basic Model (k-of-n)</u>		<u>Notes</u>
L = 0	True	True	for proof see Theorems 2.3, 3.3
L > 0	True	True	

	<u>Degradation Model (L = 0)</u>		
	<u>series</u>	<u>parallel</u>	
identical components	True	conjecture True (true n = 2, l = 2)	series is obvious
nonidentical components, L > 0	conjecture to be true (future research)		

Noninstantaneous Repair Model - False

see Lemma 4.1 (s = 1)  
for counterexample.

TABLE 7.8 -- Application of Statement:

"If there is no fixed charge for repair, then one never repairs more than one unit at a time in an optimal policy in an ergodic state"

to various coherent system repair models

	<u>Basic Model</u>	<u>Notes</u>
general coherent system	True	Theorem 2.3 also, false if extended to Erlang case (see Example 4.3)
k-of-n system/ identical components	True	

	<u>Degradation Model</u>		
	nonidentical components	identical components	
Series	True	True	Series - Lemmas 4.2, 4.3 plus Theorem 4.1
parallel	conjecture true (future research)	conjecture true n=2, l=2	

Noninstantaneous Repair Model

False	see Examples 5.5, 6.4
-------	-----------------------

Statement (1) is intuitive, although nontrivial to prove in many cases. It applies to all instantaneous repair models considered and I suspect it applies to others as well. The second statement is a much more interesting result. It seeks conditions on model parameters under which if there is no fixed cost per repair decision, then no more than one unit ever will be repaired at one time once the set of ergodic states has been reached.

Given statement (2) concerning the ergodic states, the next natural question to ask is, what about the transient ones, i.e., does a similar result to (2) hold in states not in  $\mathcal{E}$ ? The answer is definitely no if all transient states are considered. Even in the simplest parallel case, it can be optimal to repair two units at once in state 0 if it happens to be transient, (for specific example see Example 6.3,  $p = 3, k = 1$  case). However, if statement (2) is reworded to say:

(3) "If there is no fixed charge, then it is never optimal to repair more components than are needed to bring the number working to level  $\ell$ ",

then it becomes applicable in certain cases.

If the system is parallel and  $\ell = 2$ , then (3) holds, (see Table 7.4). In the  $k$ -of- $n$  case with  $\ell = k + 1$  and

$$p < \frac{\sum \lambda_{1p} K_{1p}}{\sum \lambda_{1p}},$$
 (3) also holds (see Table 7.4). Whether or not it

holds for any  $k$ -of- $n$  system is still up in the air. For a general coherent system,  $\ell$  would be the number of units in the minimum path set which has the largest number of components.

### 3. Effects on and Sensitivity of Models to Various Parameters

Section 2 discussed relationships between various coherent system repair models with respect to the applicability or not of two basic theorems restricting the policy space. In this section conclusions concerning parameter-related interrelationships are drawn and these are compared to the expected reality in a real-world situation. Effects on the optimal decisions as well as on the minimum cost per unit time attainable will be noted. Parameters will be treated in the following order:

- (a) system type (k-of-n, varying k or n)
- (b) penalty cost for system breakdown, p
- (c) fixed charge for repair, L
- (d) mean repair time,  $\sigma$
- (e) component lifetimes (means  $\mu_i$  or distributions)
- (f) objective: when  $V_{-1}$  is sufficient to determine optimality.

Most of these effects will be intuitively expected ones, supporting the validity of the coherent system repair model.

Consider a coherent system repair model for a k-of-n system which has instantaneous repair. If k is increased to  $k_1$ , then the decision space is reduced (since one will never reach states  $s : |s| < k_1 - 1$  and  $\delta_s \neq A$  if  $|s| = k_1 - 1$ . Meanwhile, the costs per decision are either left the same or raised (for  $s : |s| = k - 1$ , repair cost now includes p, system breakdown cost). Thus, the minimum  $V_{-1}$  (or  $V_0$ ) attainable is larger for higher k given fixed n. Similarly, if k is fixed and n is raised,

costs on previous decisions remain the same while some new states and decisions are added. Thus, the minimum  $V_{-1}$  ( $V_0$ ) attainable is lower. These results coincide with what might be expected given the reliability of a k-of-n system goes down as k increases and goes up as n increases.

The noninstantaneous repair model is the only one for which this does not occur. This behavior is due to two facts:

- (1) With the noninstantaneous repair model structure, it is still possible to enter states  $s$  where  $|s| < k - 1$ .
- (2) The penalty cost is applied upon entering a state  $s : |s| = k - 1$  only. No  $p$  is incurred in states  $s : |s| < k - 1$ . If  $p$  were incurred in all states  $|s| \leq k - 1$ , then the previous argument would apply and minimum  $V_{-1}$  would increase with  $k$ .

Lemmas 5.4 and 5.5 show that, for  $s = 1$ , that given

$k^* = \frac{1 + \sqrt{1 + 4\frac{\mu}{\sigma^2}}}{2}$ , that for  $k < k^*$  increasing  $k$  raises the minimum  $V_{-1}$  but for  $k > k^*$ , increasing  $k$  lowers the minimum  $V_{-1}$ .

As this second behavior is unrealistic, one concludes that the model works best for cases where  $\sigma$  is enough larger than  $\mu$  to make  $k^* \geq n$ . For large  $k$ , the assumption of  $p$  being incurred only at entry into states  $|s| = k - 1$  becomes more unrealistic as the possibility increases of being able to keep moving from state to state while the system is failed while incurring no penalty. In such cases, it might be better to assume the penalty

is incurred in all states  $|s| \leq k - 1$  even though the system has only broken down once, or to restrict the decision space to exclude policies which have only "system-failed" states in their ergodic structure.

Behavior of optimal decisions with respect to  $k$  are harder to characterize due to the many other parameter effects other than to say as  $k$  increases with fixed  $n$ , the number of components repaired or kept working goes up due to an earlier penalty incurrence. For the Basic Model with  $L = 0$ , the general  $V_{-1}$  optimal policy for a  $k$ -of- $n$  system is to keep the cheapest set of  $k$  or  $k - 1$  components working. In the  $L > 0$  identical component Basic Model, Theorem 2.7 states that as  $k$  increases, more components (or the same number) will be repaired in an optimal policy. Lemma 5.5 gives the same result for  $k < k^*$  in the single server noninstantaneous repair model while for  $k > k^*$  (unrealistic case) the reverse becomes true.

Now, take the parameter  $p$ , the penalty cost for system breakdown. Clearly, all other variables fixed, increasing any single cost can only raise the possible minimum  $V_{-1}$ . As would be naturally expected, the higher the cost of potential system failure, the more likelihood of a policy being optimal which avoids or minimizes the chance of such failure. Example 6.3 illustrates such policy changes well for varying  $p$  in the case of a Basic Model with 3 components and no fixed charge. In example 6.4, a noninstantaneous repair case shows up. For the instantaneous repair models, a policy can always be chosen so as to avoid the penalty

entirely and if  $p$  is high enough, one of these will be chosen. If  $p$  is zero, there is no reason to avoid system failure and, indeed, it will never be optimal to repair before such happens in any of the models. The same results hold for noninstantaneous repair except that there is no policy which can guarantee no system failures.

The fixed charge  $L$ , has a clear effect on policies. The larger it gets, the greater the pressures to start multiunit repair simultaneously as opposed to repairing single units on different occasions. At least in exponential component lifetime cases if there is little or no fixed charge, the natural incentive is to keep as few items working or under repair as possible because such states have the longest holding times, creating a longer time between repairs (costs incurred), thus the prevalence of  $L = 0$  theorems stating under what conditions one never will repair more than one unit at a time. The identical component Basic Model with  $L > 0$  described in Section 2.3 and Example 6.4 best illustrate policy behavior as  $L$  increases. Obviously, as  $L$  increases, minimum  $V_{-1}$  will increase, with  $V_{-1}^{\delta}$  increasing the fastest for those policies which undertake repair the greatest number of times.

The mean repair-time effect is harder to judge. Most distinguishable of all differences are those between  $\sigma = 0$  and  $\sigma > 0$ . In the former, the items in service do nothing to affect system behavior since time in service is always zero. The decision maker can always instantaneously put the system into any state he

pleases. In contrast, if  $\sigma > 0$ , no matter how many units are sent to repair at once, the system may fail before repair on some or all is completed. As  $\sigma$  is increased, repair must be started "sooner", i.e., when fewer components have failed to have the same effect of preventative repair of failed components to prevent system failure. The fact that in the noninstantaneous repair model it is no longer true that repair is never done when more than  $k$  units of a  $k$ -of- $n$  system are working points out this effect. If  $s = 1$  and the components are identical, the optimal policy form is  $R^{(j)}$ , where  $j$  is the number of components working when repair is to be initiated. Countering this is the natural system tendency to want to have as few items under repair or working as possible to maximize holding times between states. In the single server case of Theorem 5.3, as  $\sigma$  gets large,  $u(k) \rightarrow \infty$  and the optimal policy approaches  $R^{(k-1)}$ , i.e., "it takes so long to repair why not let the system fail anyway".

Previously described parameters help determine the optimal number to repair or whether to repair or not. It is the mean lifetime and repair cost of the separate components which determine the attractiveness of different components in deciding which to repair over others. In the Basic Model, a component's "repair attractiveness" depends on its cost/mean life ratio, both with or without failure,  $\frac{K_i+p}{\mu_i}$  or  $\frac{K_i}{\mu_i}$ , as is shown by Theorem 2.4. If there are degradation states, then the key indicators of "desirability of repairing component  $i$  when in its  $j^{\text{th}}$  degradation state" are

$\frac{\kappa_i^j}{\sum_{\ell=0}^{j-1} \mu_i^\ell}$  . In the series Degradation Model, whatever  $j$  produces the minimal  $\frac{\text{cost to repair at } j}{\text{expected time spent in degradation states } 0, 1, \dots, j - 1}$  is the optimal state in which to

repair component  $i$ . These same ratios are a large but not the only determining factor in the parallel case.

The final remark concerns the choice of optimality criterion up to level  $V_{-1}$  or  $V_0$ . For most Markov coherent system repair models tested,  $V_{-1}$  was not sufficient to determine an optimal decision in each state. The few notable exceptions were:

- (1) Basic Model, identical components
- (2) Degradation Model, series case.

$V_{-1}$  results were much easier to obtain and over an infinite planning horizon are the only important ones to know except in cases where it is expected that a long time might be spent in the transient states before entering the set of ergodic ones, thus, making the optimal decisions there more worth knowing.

Some possible applications for these models now follow in Section 4.

#### 4. Applications

In this section some possible applications of coherent system repair models or some variations of them are looked at.

Consider a multiitem inventory model. The inventory manager is seeking to control inventories of  $n$  products over an infinite continuous time horizon so as to minimize either his long run expected cost per unit time or his total expected costs. The demand for product  $i$  is a random variable with exponential distribution  $= 1 - e^{-\lambda_i t}$  = probability {product  $i$  is demanded before time  $t$ }. The continuous probability distribution implies that only one unit will ever be demanded at a single time. Suppose that each time an item is demanded (and sold if available), the manager (who always knows the stock of various items) can decide to either do nothing about product  $i$  for any  $i$  below initial stock level, or to order it up to the initial stock level. If a demanded item is not available in stock then a penalty cost "p" is incurred, the demand is lost, and the item must be ordered up to the initial stock level. There is a cost  $K_i$ /unit to order the  $i^{\text{th}}$  product and a fixed charge  $L \geq 0$  per order placed. If orders are assumed to arrive immediately, then this inventory model is just the Degradation Model for a series system (Section 4.2) with parameters:

- 1)  $n$  = number of products keeping track of = number of components
- 2)  $l_i$  = space allotted in warehouse for storing item  $i$  (in terms of number of items capacity) = number of degradation states for  $i$

- 3)  $\mu_i$  = mean time between demands for  $i$  = mean component lifetimes (independent of degradation state).
- 4)  $K_i^\ell = \ell \cdot K_i$  = cost to order  $\ell$  items = cost to repair  $i$  in degradation state  $\ell$ .
- 5)  $p$  = penalty cost for unsatisfied demand of an item = system failure cost.
- 6)  $L$  = fixed cost to order = fixed cost to repair.

If  $L = 0$ , using the theorems of Section 4.2 for  $V_{-1}$  optimality, the results are quite interesting. The  $V_{-1}$  optimal decision on component  $i$  is to repair it whenever its degradation state gets up to

$$j_i = \operatorname{argmin}_j \frac{K_i^j}{\sum_{\ell=0}^j \mu_i^\ell} = \frac{j \cdot K_i}{j \cdot \mu_i} = \frac{K_i}{\mu_i} \text{ for all } j .$$

Thus, every policy is  $V_{-1}$  optimal and to find the optimal ordering policy,  $V_0$  results for the series degradation model for the special case described above would have to be studied. For the maintenance model application, the  $V_{-1}$  solution was sufficient because of flexibility in changing cost or other parameter data at some insignificant decimal point to break "ties" in

$$\frac{K_i^j}{\sum_{\ell=0}^j \mu_i^\ell} .$$

This procedure is clearly not allowable in the inventory application. The important point, however, is the fact that certain inventory problems can be formulated and solved as coherent system repair models and vice-versa.

Although useful in solving certain specialized inventory problems, the most useful applications of coherent system repair models would be in more standard system-maintenance type situations. These would include:

- (1) maintenance and/or surveillance of complex electronic and/or mechanical systems
- (2) maintenance of the human body
- (3) inspection and control of pollutants in the environment
- (4) maintenance of ecological balance in populations of plants and animals.

Given the specific model structure and assumptions of coherent system repair models, they would be most useful in modeling maintenance problems where policy decisions are to be made over a long ( $\approx$  infinite time) horizon, and decisions are not likely to be made at times other than the instants of a component failure. To use the model directly, the components should have approximately an exponential (constant failure rate) or Erlang distribution. However, there are model extensions which could approximately handle the case of nonexponential or Erlang components, (see Section 5).

Some specific examples of possible applications of various coherent system repair models or extensions of them are now presented.

Example 7.1 (A Replacement Model): Suppose one has a complex electronic or mechanical unit which consists of  $n$  distinct (some could be identical) components which form a coherent system and have approximately a constant failure rate. If failed, a component  $i$  can either be left untouched, as long as the system is still operating or replaced at a cost  $K_i$ . There are an unlimited supply of spares for each component. If the system fails, there is a penalty  $p$  incurred. This is an example of a Basic Model problem. If the components can be in degradation states, then one has a Degradation Model.

Example 7.2: Suppose one has a deteriorating system of components for which maintenance decisions like in Example 7.1 can be made at the times of component failures. In this case, however, the operator/manager is concerned not so much with costs, but with keeping the expected fraction of time the system will be failed, at a minimum. Repair of components is no longer instantaneous, otherwise the system can obviously be kept always operating. The Noninstantaneous Repair Model can be used to model this situation if

- (1) set  $l = K_1 \equiv 0$
- (2) set  $p = \sigma = \text{mean repair time}$
- (3) assume  $p$  is incurred in every state where the system is down.

Example 7.3: Suppose we have to decide, in constructing a system of  $n$  identical units in parallel, how large to make  $n$  so as to insure that the system will be operating on the average at least 95% of the time, if system is to be maintained by a single server who has a mean service time of a) .2, b) .3, or c) 1 unit of time. The mean life of a component is 1 unit of time and failure rate is constant.

Using the results of Section 5.2, run the model for  $\sigma = .2, .3, 1.0$  for increasing values of  $n$  in a parallel system until minimum  $V_{-1}$  is  $\leq .05$ . Use  $K_1 = 0, p = \sigma, l = 0$ .

$\sigma = .2$  (policy  $R^{(n-1)}$  does it)

$n$	2	3
$\min V_{-1}$	.06	.025

$n = 3$  does it.

$\sigma = .3$

$n$	2	3	4
$\min V_{-1}$	.101	.062	.047

$n = 4$  does it.

$$\sigma = 1$$

n	2	3	4	
min $V_{-1}$	.4	.375	.37	impossible for <u>any</u> n, can not use such a slow server!

This same type analysis could be done for very complicated systems with components having Erlang lifetimes or degradation states.

Example 7.4: Suppose an electric power company is trying to maintain  $n$  - load generators, which form an electrical system network. The system is working if and only if electrical demand is satisfied for every customer in the region covered. The generators can be observed in a number of degradation states and repair is noninstantaneous. A combination of Noninstantaneous Repair and Degradation Models can be used to model this situation. The solution would have to be found on the computer. Even had the model been simple enough to apply some of the results of this thesis, they would not apply, because the system defined here is noncoherent. The nature of load generators is such that demand might not be satisfied given a certain subset working at a certain degradation level, but it might be satisfied with some subset of those generators working.

Example 7.5 Fault Tolerant Systems [24]: Fault tolerant computing is a rigorous discipline covering the design, analysis and maintenance of highly reliable computer systems. Having started in the 1960's due to the space programs, the subject now embraces a wide spectrum of problems concerning the reliability of computer systems. A Markov model for evaluating the reliability of such systems has been established in [24]. It assumes the fault tolerant computer to be made up of  $n$  independent modules which can each have a given number of spares. This is the parallel subsystem in series setup mentioned in Example 6.1. Given a level of spare redundancy, the Markov process is assumed to start with all components working and it runs until everything fails. The system is assumed to be nonrepairable or closed, thus, all the system-failed states can be lumped into a single state. The system reliability at any time  $t$  is then just the sum of the probabilities of being in the working states at time  $t$  given that one started with everything working.

One new concept utilized by Ng in [24] is the idea of coverage. See Arnold [1] or Bouricius et al [4] for a discussion and its effects. The coverage of a component is the probability that if it fails and there are spares available, that the system will detect the failure and switch in a spare. Perfect coverage ( $\equiv 1$ ) has been assumed throughout in coherent system repair models. Imperfect coverage, even .98 or .99 has been shown in [4] to significantly effect system reliability. With minor modifications

to the state space, imperfect coverage can be introduced and incorporated into coherent system repair models. The process then becomes a semi-Markov process because the transition probabilities now depend on which unit has failed (thus, the next state) as well as on the current state. Veinott's solution technique still works in this case (see Denardo [11]).

Examples 7.1 through 7.5 have given some indication of possible uses and applications of coherent system repair models. There are many possible real-world maintenance problems which could be formulated as such a model, but given one, it is unlikely that its system parameters would be such that one of the theorems would apply directly to solve the problem. In such cases, optimal solutions would be found computationally.

#### 5. Extensions and Topics for Future Research

In this, the final section, some possible topics for future research in the coherent system repair model area are covered. These fall into three categories:

- (1) Additional theoretical results for models falling within the given coherent system repair model structure.
- (2) Extensions of coherent system repair model structure and possible theoretical results forthcoming.

### (3) Computational techniques for obtaining solutions.

Probably the most important topics to be worked on at this point are the computational techniques. There were lots of theoretical results obtainable for the simplest coherent system repair model, the Basic Model. The Degradation and Noninstantaneous Repair Models, being more complex, yielded fewer theoretical results. Clearly, to get many more results for the previously treated models or extensions of them, computational techniques will be necessary.

The method in Chapter VI using MINOS is okay for small to moderate sized problems (up to 10-15 components for a general system). For any large sized Basic, Degradation, or Noninstantaneous Repair Model or any more complex extension of these, more efficient algorithms will be needed as well as more theory which would allow one to restrict the decision or state space initially before going to the computer. Extensions of Theorem 6.3 on the independent series subsystem technique to other cases besides  $V_{-1}$  and a Basic Model would be most desirable.

I would expect that the development of an efficient code implementing the policy improvement algorithm for finding minimum  $V_{-1}$ ,  $V_0$  might do the job better than a linear program would for the same size problem. This would allow one to solve problems which have countable state spaces such as the time-dependent extension of the Basic Model to be introduced later in this section.

Once an efficient computational technique has been set up for solving at least moderate sized problems, then not only will the possibility of finding the optimal solutions to more complex real-world problems exist, but also the possibility of observing solution behaviors which would lead to formulation of more theorems restricting the policy space. Then, perhaps even larger problems could be solved.

More immediately, with the assistance of the MINOS LP-procedure outlined in Chapter VI, the optimal solution structures, both  $V_{-1}$  and  $V_0$  need to be more thoroughly studied for the Degradation and Noninstantaneous Repair Models or combinations of the two. Open problem gaps in Tables 7.7 and 7.8 need to be resolved concerning cases when it is never optimal to repair when more than  $k$ -units are up in a  $k$ -of- $n$  system or never optimal to repair more than one unit at a time in an ergodic state. Due to complexity of calculations, no  $V_0$  results have been yet obtained for either model and very little for  $V_{-1}$  in the case of more than one server or nonidentical components. Some examples of possible problems to be studied in these areas are:

- (1) If repair is instantaneous, then in many models it is never optimal to repair more than one unit at a time in an ergodic state, so, one server is all that is needed. In the noninstantaneous repair case, two examples were given where this is false. However, in both examples

even though more than one server was needed, repair was never initiated on more than one unit at a time. Would the rewording of statement (2), Section 7.2, make it true in the noninstantaneous repair case?

- (ii) If  $p$  is small enough in a  $k$ -of- $n$  system, then one never repairs up to more than  $k + 1$  components working from any transient state. Is this true for larger  $p$  (yes, if parallel,  $k = 1$ ) or more generally, is it true that in a general coherent system that one never repairs up to more than  $\ell + 1$  components working where  $\ell$  is the number of components in the largest minimum path set?

As far as possible model extensions go, there are many possible. The Markov decision chain formulation is very general and many of the maintenance models for deteriorating units in categories mentioned in Chapter I, Section 2, can be extended to include systems of  $n$  independent components as with coherent system repair models. Of course, some would probably have too many states to be solvable, even on the computer, not being of much use in that case. Two possible extensions which, although are quite complicated statewise, would be very important from the standpoint of applications to be able to formulate and solve, would be a time-dependent model and an uncertain information model.

There are many maintenance situations where failure rates of components are not constant and a time factor needs to be accounted for.



Transitions still occur at the instants of component failures, but now depend on the "ages" of the components as well as just "which" components are working and the decision taken. So,

$$P_{1_0 2_0, t}^A = \left\{ \begin{array}{ll} 1/4, & t = 1_1, 0, 2_2, 2_1 \\ 0, & \text{otherwise} \end{array} \right\}$$

$$P_{2_2, t}^{R_1} = \left\{ \begin{array}{ll} 1/2, & t = 1_1, 0 \\ 0, & \text{otherwise} \end{array} \right\}, \text{ etc.}$$

If desired, the decision space could be expanded to include repair of a nonfailed but "aged" component. Also, the distributions of  $L_1$  and  $L_2$  can be adjusted to be IFR, DFR or whatever failure rate assumption is desired.

In the uncertain information area, the model of Rosenfield [26] could be applied directly and modified to the case of a coherent system repair model. Components can either be assumed to have exponential or discrete lifetimes. The state of the system can be represented as  $(s, t)$  where  $s$  = working configuration of components last known with certainty and  $t$  = time units since perfect information.

Decisions available in a state  $(s, t)$  are:

- A) do nothing
- B) repair some subset of known failed components in  $s$  at usual coherent system repair model costs.

- (C) pay an inspection cost  $M > 0$  to ascertain exact state of system and then repair or not, based on true value of  $s$ .

A possible application of this extension is the following:

The control rods in a nuclear power generating system form a k-of-n system. The cost of inspecting is high but the cost of system failure could be even higher. This model could help determine the optimal time intervals between inspections as well as the optimal decision given inspection results.

Other topics for possible future research on coherent system repair models include:

- (1) the introduction of a discount rate,  $\alpha$ , in which case none of the theory presented applies but policy improvement can be used to find an optimal policy with little more work than required for the  $\alpha = 0$  case.
- (2) changing to a finite time horizon in which case neither the theory nor the computational methods presented in this thesis apply. The types of results obtainable here would have to be computational or theoretical but with limited prospects of obtaining useful results compared to the infinite horizon case. The following is a discrete time example:

Example 7.7: Consider a single-item inventory model whose time horizon is finite over  $n$  periods. The objective is to minimize the total expected  $n$ -period costs. Observe: stock at the beginning of each period  $(x_i)$  and then order a nonnegative amount up to the storeroom limit  $N - 1$  for instantaneous delivery to get the starting stock  $y_i$  at the beginning of period  $i$ . The demand occurs according to the distribution

$$\phi_i(\cdot | y_i) = \begin{cases} \text{initial stock } x_{i+1} & \text{in period } i + 1 \text{ given} \\ \text{starting stock } y_i & \text{in period } i = \begin{cases} 1, & \cdot = y_i - 1 \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

$$\text{Cost} = c_i(z) \text{ of ordering } z \text{ units} = z \cdot K$$

$$\text{Storage cost, } G_i(y_i) = G(y_i) = \begin{cases} p, & y_i = 0 \\ 0, & \text{otherwise} \end{cases}$$

Demand is "backlogged" and if a shortage of an item occurs, ordering is required. This is the finite horizon discrete time version of a parallel-system,  $N$ -identical component Basic Model so results obtained by Veinott [32] for the single item inventory model apply to this case.

Let  $f_i(x)$  = minimum expected cost in periods  $1, \dots, n$  given  $x$  = initial inventory in period  $i$ . Under suitable regularity conditions,  $f_i(x) = \min_{N-1 \geq y \geq x \vee 0} \{c_i(y-x) + G_i(y) + E[f_{i+1}(x_{i+1}) | y]\}$ .  
 $\forall i = 1, \dots, n$  and  $f_{n+1}(\cdot) = 0$  where  $y = y_i(x) \geq x$  is the starting stock in period  $i$  and  $x \vee 0 = \max\{x, 0\}$ .

Two results are obtained:

- A) If  $c_i(\cdot)$  is convex, then  $y_i(x) \uparrow x$
- B) If  $G_i(\cdot)$  is convex, unsatisfied demands are backlogged, and demands independent, then  $x - y_i(x) \uparrow x$ , i.e.,  $y_i(x)$  does not increase as fast as  $x$ .

These results, which hold in this case, follow from Veinott's theory of minimizing subadditive functions on a sublattice. [See [32]].

To prove these results, a lot of work (by Veinott) was necessary but yields are poor from a practical maintenance standpoint. Result A) states that for any time period  $i$ , if  $\delta_i^*(k) = R_\ell$  = optimal decision in period  $i$ ,  $k$  units working,  $R_\ell$  denoting repair of  $\ell$  units, then if  $k' > k$ ,  $\delta_i^*(k') = R_{\ell'}$ ,  $\ell' \leq \ell - k'$  [cannot repair more than one less in state  $k + 1$  than was repaired in state  $k$ ].

Result B) states that, however many you repair in state  $k$ , you cannot get a policy to repair more than that in any state  $k' > k$  in any one time period and still be optimal. This result, which somewhat limits the possible set of optimal policies, is weak when one considers that in the stationary infinite horizon case, an exact optimal policy is easily found. Of course, considering that this is a time-dependent model, which could prove to be much more difficult to solve even for infinite horizon problems, it may be significant. Given a continuous time coherent system repair problem with finite

planning horizon  $T$ , similar techniques - using  $f_t(s)$  and the model structure to define a recursion relation and then somehow proving some results using the relation so obtained - should be employed.

Example 7.8: Parallel system, Basic Model. Finite

horizon,  $T$ .

Let  $f_t(s)$  = minimum expected cost given you are in state  $s$  with " $t$ " amount of time to go.

Let  $\delta_t(s)$  = optimal policy decision in state  $s$  with  $t$  units of time to go.

Denote  $R_{\Omega_s}$  as the decision to repair the subset  $\Omega_s$  of failed components in state  $s$ . Also  $R_\phi = A$

$$f_t(s) = \min_{\Omega_s} f_t^{R_{\Omega_s}}(s)$$

where

$$f_t^{R_{\Omega_s}}(s) = \int_0^{T-t} \left( \sum_{i \in s \cup \Omega_s} \lambda_i f_{t-\tau}^{(s \cup \Omega_s - i)} \right) e^{-\tau \sum_{i \in s \cup \Omega_s} \lambda_i} d\tau + \sum_{i \in \Omega_s} K_i + L(+p, \text{ if } s = 0) .$$

This whole thesis has considered three variations within the coherent system repair model structure presented in Chapter I.

Although each gave different optimal policies and involved different theorems to get them, all were formulated as a continuous time Markov Decision Model with a finite state and decision space. Results were always based on minimizing long run expected cost per unit time or total expected cost in the case of  $V_{-1}$  ties, the solution technique being to enumerate the policies and either find an optimal one or to eliminate certain ones from being optimal given the specific Markov transition probabilities and costs for the given model.

The key assumption on the system which allowed Markov chain formulation was that of exponential component lifetimes and repair times (if non-instantaneous). This is what guaranteed the Markov property, that for every state of the system (which is a function of the component states), the probability distribution of the length of time spent in that state was independent of what states the system had been in previously. By breaking up component lifetimes or repair times into stages, we were able to extend the possible distributions to Erlang, while keeping the Markov property. However, for other distributions, the Markovian property is lost and the Markov decision chain approach is no longer applicable, as can be seen in the following simple example:

Example 7.9: Consider the Basic Model, a parallel system with  $n = 2$  components and no fixed charge except that now assume  $L_i$ , the lifetime of the  $i^{\text{th}}$  component, has a general continuous distribution  $F_i(t)$ .

So,

$$L_1 \sim F_1(t), \text{ mean } \mu_1$$

$$L_2 \sim F_2(t), \text{ mean } \mu_2 .$$

Suppose we define the states as before, being the configuration of working components. Start with both components new, state 12. It is easy to compute the holding time distribution in state 12 as well as the probability of going to states 1 or 2 (depending on which component fails first). However, then problems start. The holding time in state 1 is just the distribution of  $L_1$  given it has lasted through the first transition, a quantity which depends on how long component 1 has been up for, information which is not kept track of by the simplest time-free definition of states. To have time-dependent states would require an uncountably infinite number of states given continuous component lifetimes, getting into the realm of diffusion processes. If component lifetimes had a discrete distribution, then we have the time-dependent model of Example 7.6.

It would be useful to develop some kind of a model which allows for general component lifetime distributions while still retaining the same state and decision spaces defined in the Basic Model. This would be a very interesting but probably difficult problem.

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by Randall E. Fleming

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Four types of costs can be incurred: component replacement charge, penalty cost for system breakdown, fixed charge for repair, and per-hour labor charge. Decisions to repair some subset of failed components or to do nothing are made immediately following a component failure. If the system fails, some repair must be performed. A policy consists of specification of a decision in every state of the Markov chain accessible from the assumed initial state of all components new. Structures of and restrictions on optimal policies are studied.

Three types of coherent system repair models are treated: (I) a Basic Model in which components are either working or failed, (II) a Degradation Model in which working components can degrade before failure, and (III) a Noninstantaneous Repair Model in which repair times are exponential, there being a finite number of servers. The first two models assume instantaneous repair and unlimited service.

Two types of results are obtained for these models - ones which obtain a unique optimal policy for a special case and ones which restrict the number of possible optimal policies for a more general case. Two new policy types are encountered in the case of no fixed charge. These are ones which "never repair until a certain number of components are left working" and ones which "never repair up to more than a certain number of components at once" or "never repair more than one component simultaneously in an ergodic state". The first type of policy appears in  $k$ -of- $n$  systems with Instantaneous Repair Models. Precise optimal policies are computed for the series degradation model and numerous other cases in which the components are identical. Optimal policy restriction theorems and optimal policy computations are based on policy enumeration with comparisons between long run expected costs, and in the case of ties, total expected costs.

Computational procedures using linear programming and policy improvement routines are discussed and an LP routine is implemented on some simple test cases. Effects on and sensitivity of the minimum cost values and optimal policies of the models to various component and system parameters is discussed. The paper concludes with a look at the numerous possible applications and related topics for future research.

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