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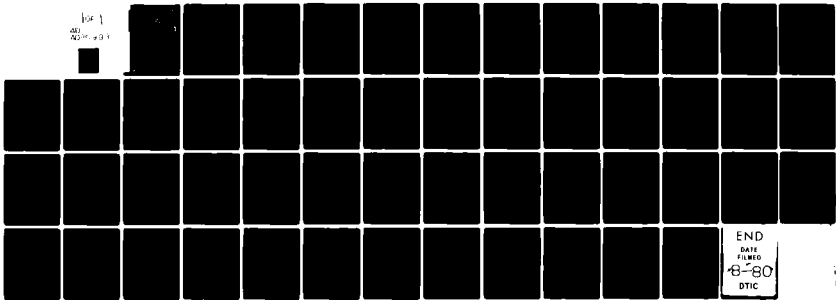
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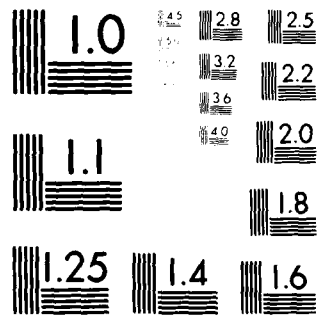
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BAYESIAN ESTIMATION IN THE ONE-PARAMETER
LATENT TRAIT MODEL

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HARIHARAN SWAMINATHAN
AND
JANICE A. GIFFORD

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Laboratory of Psychometric and
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School of Education
University of Massachusetts
Amherst, MA 01003

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latent trait model were obtained for two cases):

- (1) Conditional estimation of ability (for those situations when items are previously calibrated), and
- (2) Joint estimation of item and ability parameters.

For each of the two cases, a simulation study was carried out to study the efficacy of the two Bayesian procedures described and to compare the Bayesian estimates with the comparable maximum likelihood estimates. The Bayesian and maximum likelihood estimates were compared with respect to: (a) the mean value of the estimates, as compared with the mean values of the true values; (b) the mean squared error difference between true values and estimated values; and (c) the regression of the true value on the estimated value. Overall, the results favored the Bayesian estimates; the means of the estimates are closer to the means of the true values; the slopes and intercepts are in general closer to one and zero respectively; and the mean square deviations are dramatically smaller (in some cases, one-tenth the size of those for ML estimates).

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**Bayesian Estimation in the One-Parameter
Latent Trait Model^{1,2}**

INTRODUCTION

In recent years there has been considerable interest among measurement theorists and practitioners in latent trait theory since it offers the potential for improving educational and psychological measurement practices. However, before latent trait theory can be successfully applied to solve existing measurement problems, the problem of estimating parameters in latent trait models has to be addressed.

The literature in latent trait theory abounds with procedures for the estimation of parameters. The estimation procedures that have been developed over the past thirty years range from heuristic procedures such as those given by Urry (1974) and Jensema (1976) to conditional as well as unconditional maximum likelihood procedures (Andersen, 1970, 1972, 1973a, 1973b; Bock, 1972; Lord, 1968, 1974; Samejima, 1969, 1972; Wright & Panchapakesan, 1969; Wright & Douglas, 1977). With the exception of the "conditional" maximum likelihood procedure provided by Andersen (1970) for the one-parameter model, the maximum likelihood estimators of the parameters in the latent trait models are less than optimal as a result

¹The research reported here was performed pursuant to Grant No. N0014-79-C-0039 with the Office of Naval Research and to Grant No. FQ 7624-79-0014 with the Air Force Human Resources Laboratory. The opinions expressed here, however, do not reflect the positions or policies of these agencies.

²The author is grateful to the encouragement and support provided by Dr. Malcolm Ree of the Air Force Human Resources Laboratory, and to Dr. Charles Davis of the Office of Naval Research.

of the problem of estimating "structural parameters" in the presence of "incidental parameters" (Andersen, 1970; Zellner, 1971, pp. 114-154). The "structural parameters" in latent trait models are the item parameters while the "incidental parameters" are the ability parameters since these increase without bound as the number of examinees is increased to provide stable estimates of the parameters. Furthermore, as Novick, Lewis, and Jackson (1973) have remarked, "in the estimation of many parameters some, by chance, can be expected to be substantially overestimated and the others substantially underestimated."

When several parameters have to be estimated simultaneously, and when, as in the present case, both structural and incidental parameters have to be estimated, a Bayesian solution to the estimation problem may be appropriate (Zellner, 1971, pp. 114-119). This is particularly true if prior information or belief about the parameters is available, since in this case, the incorporation of this information will certainly increase the "accuracy" or the meaningfulness of the estimates. An example of this was encountered by Lord (1968), where in order to prevent estimates of the item discrimination parameter from drifting out of bounds, it was necessary to impose limits on the range of values the parameter could take. Although the estimation procedure employed by Lord (1968) was not Bayesian, this illustrates the role of prior information in obtaining meaningful estimates. A further argument that can be advanced in favor of a Bayesian approach is that the logic of the Bayesian inferential procedure is more appealing than the classical, sampling theoretic, inferential procedure. As Zellner (1971, p. 362) has pointed out, "...there is no need to justify inference procedures in terms of their

behavior in repeated, *as yet unobserved*,¹ samples as is usually done in the sampling theory approach." Consequently, it is possible to make probabilistic statements about the parameters themselves, based on the information that is available.

Bayesian Procedures

It may be instructive to review briefly the Bayesian estimation procedure. Let $p(\underline{y}, \underline{\theta})$ denote the joint probability density function (pdf) for a random observation vector \underline{y} and a parameter vector $\underline{\theta}$, also random. Then,

$$\begin{aligned} p(\underline{y}, \underline{\theta}) &= p(\underline{y}|\underline{\theta}) p(\underline{\theta}) \\ &= p(\underline{\theta}|\underline{y}) p(\underline{y}) \end{aligned}$$

where

$$p(\underline{\theta}|\underline{y}) = p(\underline{\theta}) p(\underline{y}|\underline{\theta})/p(\underline{y})$$

or,

$$[1] \quad p(\underline{\theta}|\underline{y}) \propto p(\underline{\theta}) p(\underline{y}|\underline{\theta})$$

since $p(\underline{y}) \neq 0$ is a constant. Equation [1] is the essence of Bayes' Theorem and is of primary importance in the estimation of parameters and for drawing inferences concerning the parameters. The probability density function $p(\underline{\theta}|\underline{y})$ is the *posterior pdf* for the parameter vector $\underline{\theta}$, given the sample information or data, and $p(\underline{\theta})$ is the *prior pdf* for the vector $\underline{\theta}$. The quantity $p(\underline{y}|\underline{\theta})$ is a proper pdf as long as \underline{y} is a random variable. However, the moment the vector \underline{y} is realized, $p(\underline{y}|\underline{\theta})$ ceases

¹The italics have been provided by the authors.

to have the interpretation as a pdf. In this case, $p(\underline{y}|\underline{\theta})$ is strictly a mathematical function of $\underline{\theta}$, well known as the *likelihood function*.

Since the notation $p(\underline{y}|\underline{\theta})$ can be mistaken for a pdf, the likelihood function is often written as $L(\underline{y}|\underline{\theta})$, and sometimes, to emphasize the fact that it is a function of $\underline{\theta}$, as $L(\underline{\theta}|\underline{y})$. Thus, the expression given in [1] can be written as

$$[2] \quad p(\underline{\theta}|\underline{y}) \propto L(\underline{\theta}|\underline{y}) p(\underline{\theta}) .$$

It is interesting to note that if $p(\underline{\theta})$ is assumed to be a constant, i.e., the prior belief about $\underline{\theta}$ is summarized via a uniform distribution, the posterior pdf of $\underline{\theta}$ is proportional to the likelihood function. In a sense, this interpretation constitutes a Bayesian justification of maximum likelihood principle.

Once the prior belief about the parameter $\underline{\theta}$ is specified, the joint posterior pdf of the vector $\underline{\theta}$ given the data can be readily obtained. The posterior pdf of $\underline{\theta}$ contains all the information necessary for drawing inferences concerning $\underline{\theta}$ (jointly or individually) and for obtaining estimates of $\underline{\theta}$ once a "loss function" is prescribed. For instance, if a squared-error loss function is deemed appropriate, then the mean of the posterior pdf of $\underline{\theta}$ can be taken as the estimator of $\underline{\theta}$. On the other hand, if a zero-one loss function is appropriate, then, the mode of the posterior pdf of $\underline{\theta}$ is the estimator of $\underline{\theta}$. Similarly, for the absolute deviation loss function, the median of the posterior pdf of $\underline{\theta}$ is the appropriate estimator.

The Bayesian procedure described above has been successfully applied in a variety of situations. For a sampling of these applications the reader is referred to Novick and Jackson (1974), and Zellner (1971).

However, Bayesian methods have found only a limited application in the area of latent trait theory. Birnbaum (1969) obtained Bayes estimates of the ability parameter in the one- and two-parameter logistic models under the assumption that the item parameters were known. He chose, for mathematical tractability, the prior pdf of θ_i , the ability of the i th examinee, to be the logistic density function, i.e.

$$p(\theta_i) = \exp(-D\theta_i) / [1 + \exp(-D\theta_i)]^2$$

where $D=1.7$ is a scaling factor. Owen (1975), in applying the latent trait model in an adaptive testing context, obtained Bayes estimates of ability, θ_i , under the assumption that the prior pdf of θ_i was normal with mean, zero, and variance, unity.

The Bayesian procedure suggested by Birnbaum (1969) and Owen (1975) require rather exact specification of prior belief.¹ An alternative and a more powerful procedure has been suggested by Lindley (1971). He has shown that if the information that is available can be considered *exchangeable*, then a hierarchical Bayesian model can be effectively employed for the estimation of parameters.

In order to illustrate the hierarchical model, let us consider the problem of estimating, say, the ability θ_i of an individual ($i=1, \dots, N$). If it can be assumed, a priori, that exchangeability holds, i.e., the information about θ_i is no different from the information about any other θ_j , observed or yet to be observed, then, θ_i can be assumed to be a random sample from some distribution, $p(\theta)$. For convenience, if $p(\theta)$ is taken

¹Meredith and Kearnes (1973) and Sanathanan and Blumenthal (1978) have obtained empirical Bayes estimators of the ability parameters for the one-parameter model. In these procedures the prior pdf is estimated from the data.

to be normal with mean μ and variance σ^2 , then this would constitute specification of the first stage of the hierarchical model. Since μ and σ^2 are unknown, specifying prior beliefs on these "hyperparameters" would constitute the second stage of the hierarchical model. Usually, the hyperparameter distributions are specified in such a way that they depend upon constants which can be determined from the prior belief the investigator has about the parameters, and hence the hierarchical model terminates at the second stage. With this two stage model, it is possible to estimate θ_i ($i=1, \dots, N$) without any reference to the nuisance parameters, μ and σ^2 .

Novick (1971) has described this hierarchical model as an analog of the empirical Bayes procedure advocated by Robbins (1955) and the simultaneous estimation procedure provided by Stein (1962). Furthermore, as Novick, Lewis, and Jackson (1973) have pointed out, this procedure not only employs the direct information gained through the observation of an individual, but also the collateral information contained in observations from other individuals. They further note that, "In effect, this collateral information is used to provide 'prior' information for the estimation.... Thus to some extent, the problem of selecting prior distributions for Bayesian analyses is neutralized, and this is effected from a strictly Bayesian approach."

The hierarchical Bayesian model has been successfully employed by Lindley and Smith (1972), Novick et al. (1973), and Zellner (1971), to name a few. However, this approach has not been employed for estimating parameters in latent trait models. The purpose of this paper, hence, is to provide a Bayesian estimation procedure, in the sense of Lindley, for estimating parameters in the one-parameter latent trait model.

Bayesian Estimation in the One-Parameter Logistic Model

The Model

Let X_{ij} denote a random variable that represents the binary response of an examinee i ($i=1, \dots, N$) on item j ($j=1, \dots, n$). If the examinee responds correctly to the item, $X_{ij}=1$, while for an incorrect response, $X_{ij}=0$. We assume that the complete latent space is unidimensional, and that the probability, $P[X_{ij}=1]$, that an individual with ability parameter θ_i will correctly respond to an item with difficulty parameter, b_j , is given by the logistic model,

$$[3] \quad P[X_{ij}=1|\theta_i] = \exp(\theta_i - b_j) / \{1 + \exp(\theta_i - b_j)\}.$$

On the other hand, the probability that the individual will respond incorrectly is given by

$$[4] \quad \begin{aligned} P[X_{ij}=0|\theta_i] &= 1 - P[X_{ij}=1|\theta_i] \\ &= 1 / \{1 + \exp(\theta_i - b_j)\} . \end{aligned}$$

The probabilities given in Equations [3] and [4] can be combined to yield

$$[5] \quad P[X_{ij} = x_{ij}|\theta_i] = \exp\{x_{ij}(\theta_i - b_j)\} / \{1 + \exp(\theta_i - b_j)\}$$

where $x_{ij}=1$ for a correct response and $x_{ij}=0$ for an incorrect response.

The above model, since it depends only on one item parameter, difficulty, is commonly known as the Rasch model or the one-parameter logistic model. For a detailed description of this model and its properties, the reader is referred to Wright (1977).

Conditional Estimation of Ability

In some situations it may be of interest to estimate the ability θ_i of an examinee who takes a test which has been calibrated, i.e., the difficulty parameters are known. Moreover, since the problem of estimating ability when the item parameters are known is simpler to deal with and provides an illustration of the basic ideas involved, this case will be dealt with in detail first.

The model given by Equation [5], should in the strict sense be expressed as

$$[6] \quad P[X_{ij} = x_{ij} | \theta_i, b_j] = \exp\{x_{ij}(\theta_i - b_j)\} / \{1 + \exp(\theta_i - b_j)\} .$$

Although there are several ways to write the model, the expression given by [6] is the most convenient for the present situation.

It follows, from the principle of local independence, that the joint probability of responses of the N examinees on n items is given by

$$[7] \quad P[X_{11}=x_{11}, X_{12}=x_{12}, \dots, X_{ij}=x_{ij}, \dots, X_{Nn}=x_{Nn} | \theta_1, \theta_2, \dots, \theta_N; b_1, b_2, \dots, b_n] \\ = \prod_{i=1}^N \prod_{j=1}^n \exp\{x_{ij}(\theta_i - b_j)\} / \{1 + \exp(\theta_i - b_j)\} .$$

Once the responses of the N examinees on the n items are observed, the above expression ceases to have the probability interpretation and becomes the likelihood function, $L(\underline{X} = \underline{x} | \underline{\theta}, \underline{b})$. Upon simplification,

$$[8] \quad L(\underline{X} = \underline{x} | \underline{\theta}, \underline{b}) = \exp\left\{ \sum_i \sum_j x_{ij}(\theta_i - b_j) \right\} / \prod_i \prod_j \{1 + \exp(\theta_i - b_j)\} \\ = \exp\left\{ \sum_i r_i \theta_i - \sum_j q_j b_j \right\} / \prod_i \prod_j \{1 + \exp(\theta_i - b_j)\}$$

where $r_i = \sum_j x_{ij}$, and $q_j = \sum_i x_{ij}$. Since the item parameters are

known constants, the likelihood function is strictly a function of $\underline{\theta}$ and, hence, can be expressed as

$$[9] \quad L(\underline{x}|\underline{\theta}, b) \propto \exp\left\{\sum_i r_i \theta_i\right\} / \prod \{1 + \exp(\theta_i - b_j)\} .$$

Returning to Equation [1], we see that in order to obtain the posterior density function of $\underline{\theta}$ given the observations and the item parameters, it is necessary to specify the prior distribution of $\underline{\theta}$. To this end, in the first stage of the hierarchical model, we assume that, apriori, the ability parameters, θ_i , are independently and identically normally distributed, i.e.,

$$[10] \quad \theta_i | \mu, \phi \sim N(\mu, \phi).$$

The assumption that the thetas are independently and identically distributed follows from the assumption of exchangeable prior information about the thetas. The assumption of normality also appears to be reasonable and has been made by numerous authors, e.g., Lord and Novick (1968).

In order to complete the hierarchical Bayesian model, we have to specify prior distributions for μ and ϕ . This is the second stage. At this level, we assume that, apriori, μ and ϕ are independently distributed, and that μ has the uniform distribution. Thus,

$$[11] \quad p(\mu, \phi) \propto p(\phi).$$

The uniform distribution is not a proper distribution, although this choice can be justified to some extent (Zellner, 1971, pp. 41-43). It may, however, be more appropriate to specify a "non-diffuse" prior and this possibility will be explored further in a later paper.

It now remains to specify the form of $p(\phi)$. Since ϕ is the variance of θ_1 , ϕ can be assumed to have the inverse chi-square, χ^{-2} distribution, i.e.,

$$[12] \quad p(\phi | \nu, s^2) \propto \phi^{-\left(\frac{\nu}{2}+1\right)} \exp(-\nu s^2 / 2\phi).$$

The quantities ν and s^2 are parameters of the inverse chi-square distribution, and have to be specified a priori. The inverse chi-square distribution can be expressed in different ways. Novick and Jackson (1974) prefer the form

$$p(\phi | \nu, \lambda) \propto \phi^{-\left(\frac{\nu}{2}+1\right)} \exp(-\lambda / 2\phi).$$

For this form, the mean of the distribution is $\lambda / (\nu - 2)$ and the mode is $\lambda / (\nu + 2)$. For the form given by Equation [12] the mean is $s^2 \nu / (\nu - 2)$ and the mode is $s^2 \nu / (\nu + 2)$, with both mean and mode approaching s^2 as ν increases. These two forms are clearly equivalent, but the form given by Equation [12] is employed in the sequel because it provides a direct interpretation of the parameter ν and s^2 . The quantity s^2 thus represents the investigator's belief about the "typical" value of the parameter ϕ while ν represents his/her degree of confidence.

The joint posterior distribution of $\underline{\theta}' = [\theta_1, \theta_2, \dots, \theta_N]$ given \underline{b} and the item responses is given by

$$[13] \quad p(\underline{\theta} | \underline{b}, \underline{x}) \propto L(\underline{x} | \underline{\theta}, \underline{b}) p(\underline{\theta} | \mu, \phi) p(\mu, \phi).$$

The likelihood function $L(\underline{x} | \underline{\theta}, \underline{b})$ is given by Equation [9], $p(\mu, \phi)$ by Equation [12], and

$$\begin{aligned}
 [14] \quad p(\underline{\theta}|\underline{\mu}, \phi) &= \prod_{i=1}^N \phi^{-\frac{1}{2}} \exp\{-\frac{1}{2}(\theta_i - \mu)^2/\phi\} \\
 &= \phi^{-\frac{N}{2}} \exp\{-\sum_{i=1}^N (\theta_i - \mu)^2/2\phi\}
 \end{aligned}$$

Combining these expressions, we have,

$$\begin{aligned}
 [15] \quad p(\underline{\theta}|\underline{b}, \underline{x}, \mu, \phi, \nu, s^2) &\propto [\exp\{\sum_i r_i \theta_i\} / \prod_i \prod_j \{1 + \exp(\theta_i - b_j)\}] \\
 &[\phi^{-\frac{N}{2}} \exp\{-\sum_i (\theta_i - \mu)^2/2\phi\}] [\phi^{-\frac{(\nu+1)}{2}} \exp(-\nu s^2/2\phi)]
 \end{aligned}$$

The above expression depends upon the "nuisance" parameters μ and ϕ and hence these have to be integrated out. Since $\sum (\theta_i - \mu)^2 = \sum (\theta_i - \theta_{.})^2 + N(\theta_{.} - \mu)^2$, and

$$\int_{-\infty}^{\infty} \exp\{-N(\theta_{.} - \mu)^2/2\phi\} d\mu \propto \phi^{\frac{1}{2}},$$

integration with respect to μ yields

$$[16] \quad p(\underline{\theta}|\underline{b}, \underline{x}, \phi, \nu, s^2) \propto L(\underline{x}|\underline{\theta}, \underline{b}) \phi^{-(N+\nu+1)/2} \exp[-\{\nu s^2 + \sum (\theta_i - \theta_{.})^2\}/2\phi].$$

Noting that

$$\int_0^{\infty} \phi^{-m} \exp(-k/\phi) d\phi \propto k^{-(m-1)}$$

and integrating with respect to ϕ , we obtain

$$[17] \quad p(\underline{\theta}|\underline{b}, \underline{x}, \nu, s^2) \propto L(\underline{x}|\underline{\theta}, \underline{b}) \{\nu s^2 + \sum_i (\theta_i - \theta_{.})^2\}^{-(N+\nu-1)/2}$$

$$\begin{aligned}
 [18] \quad &= [\exp\{\sum_i r_i \theta_i\} / \prod_i \prod_j \{1 + \exp(\theta_i - b_j)\}] \\
 &\cdot \{\nu s^2 + \sum_i (\theta_i - \theta_{.})^2\}^{-(N+\nu-1)/2}
 \end{aligned}$$

The joint posterior modes are obtained by differentiating $\log p(\theta|\underline{b}, \underline{x})$ with respect to θ , setting these derivatives equal to zero, and solving the resulting equations:

$$[19] \quad \sum_{j=1}^n P_{ij} = r_i - (\theta_i - \theta) / \sigma^2 \quad (i=1, \dots, N)$$

where

$$P_{ij} = \exp(\theta_i - b_j) / \{1 + \exp(\theta_i - b_j)\}$$

and

$$\sigma^2 = \{v s^2 + \sum_{i=1}^N (\theta_i - \theta)^2\} / (v + N - 1) .$$

Since this system of equations is non-linear, numerical procedures have to be employed. The Newton-Raphson iterative procedure is ideally suited for this situation. Let

$$[20] \quad f(\theta_i) = \sum_{j=1}^n P_{ij} + (\theta_i - \theta) / \sigma^2 - r_i .$$

Then

$$[21] \quad f'(\theta_i) = \sum_{j=1}^n P_{ij}(1 - P_{ij}) + \{\sigma^2(1 - \frac{1}{N}) - 2(\theta_i - \theta) / (v + N - 1)\} / (\sigma^2)^2 .$$

If $\theta_i^{(k)}$ is the value of θ_i at the k th iteration, then $\theta_i^{(k+1)}$ is given by

$$[22] \quad \theta_i^{(k+1)} = \theta_i^{(k)} - f(\theta_i^{(k)}) / f'(\theta_i^{(k)}) ,$$

with $\theta_i^{(0)}$, the starting value being given by (Wright & Douglas, 1977),

$$[23] \quad \theta_i^{(0)} = b . + \{1 + s_b^2 / 2.89\} \log(r_i / n - r_i)$$

where

$$b . = \sum b_j / n, \text{ and } s_b^2 = \sum (b_j - b .)^2 / (n - 1) .$$

Although the iterative scheme given in [22] is for estimating the ability θ_i for each individual, in reality, only the ability corresponding to each raw score r ($r=1, \dots, n-1$) need be estimated. The ability corresponding to raw score $r=0$ and $r=n$ cannot be estimated by virtue of [23]. Hence, individuals who obtain perfect score or zero score are eliminated from the analysis. It should also be pointed out the Newton-Raphson scheme given above is not the vector version of the procedure since for this procedure the matrix of derivatives $\{\partial f / \partial \theta_i \partial \theta_j\}$ has to be computed and inverted. The procedure described here worked sufficiently well, converging in as few as three to four iterations.

Joint Estimation of Item and Ability Parameters

The case considered above, where the item parameters were assumed to be known, provides the necessary background for the Bayesian estimation procedure. However, this situation may not be realistic and, hence, it is necessary to develop a procedure for the joint estimation of the item and ability parameters.

We proceed in the manner indicated for the case of known item parameters. Hence, in addition to making the assumptions about the ability parameters, we have to make assumptions regarding the item parameters. Again, as in the previous case, we specify prior beliefs about the parameters in two stages. In the first stage, for the model given in [5], we assume:

$$[24a] \quad \theta_i | \mu_\theta, \phi_\theta \sim N(\mu_\theta, \phi_\theta), \quad (i=1, \dots, N)$$

$$[24b] \quad b_j | \mu_b, \phi_b \sim N(\mu_b, \phi_b). \quad (j=1, \dots, n)$$

In addition, we assume that, apriori, θ_i and b_j are independent, θ_k and θ_l ($k \neq l$) are independent, and b_k and b_l are independent.

As for the ability parameters, the specification of prior belief about b_j seems reasonable, especially if an item bank is available. This assumption has been made by several authors (Lord & Novick, 1968; Wright & Douglas, 1977). Furthermore, as a result of the hierarchical Bayesian model, departures from this assumption appear to have a negligible effect on the estimates of b_j .

For the second stage, we assume that

$$\begin{aligned}
 [25a] \quad p(\mu_\theta, \phi_\theta) &\propto p(\phi_\theta) \\
 &\propto \phi_\theta^{-(v_\theta/2+1)} \exp(-s_\theta^2 v_\theta / 2\phi_\theta),
 \end{aligned}$$

and

$$\begin{aligned}
 [25b] \quad p(\mu_b, \phi_b) &\propto p(\phi_b) \\
 &\propto \phi_b^{-(v_b/2+1)} \exp(-s_b^2 v_b / 2\phi_b).
 \end{aligned}$$

We have thus assumed that, apriori, the hyperparameters are independent, and that the prior information about the parameters, μ_θ and μ_b , is "vague".

The joint posterior pdf of $\underline{\theta}$, and \underline{b} , is given by

$$\begin{aligned}
 [26] \quad p(\underline{\theta}, \underline{b} | \underline{x}, \mu_\theta, \phi_\theta, \mu_b, \phi_b, v_\theta, s_\theta^2, v_b, s_b^2) \\
 \propto L(\underline{\theta}, \underline{b} | \underline{x}) \left\{ \prod_{i=1}^N p(\theta_i) \prod_{j=1}^n p(b_j) \right\} p(\phi_\theta) p(\phi_b)
 \end{aligned}$$

where $L(\underline{\theta}, \underline{b} | \underline{x})$ is the likelihood function given by [8]. Now

$$[27] \quad \left\{ \prod_{i=1}^N p(\theta_i) \right\} p(\phi_\theta) \propto \phi_\theta^{-(N+v_\theta+2)/2} \exp(-v_\theta s_\theta^2 / 2\phi_\theta) \exp\{-(\theta_i - \mu_\theta)^2 / 2\phi_\theta\}.$$

Upon integrating with respect to ϕ_θ and μ_θ , we have, from [17]

$$\begin{aligned}
 [28] \quad & \int_{-\infty}^{\infty} \int_0^{\infty} \left\{ \prod_{i=1}^N p(\theta_i) \right\} p(\phi_\theta) d\mu_\theta d\phi_\theta \\
 & = [v_\theta s_\theta^2 + \sum_{i=1}^N (\theta_i - \theta)^2]^{-(N+v_\theta-1)/2}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 [29] \quad & \int_{-\infty}^{\infty} \int_0^{\infty} \left\{ \prod_{j=1}^n p(b_j) \right\} p(\phi_b) d\mu_b d\phi_b \\
 & = [v_b s_b^2 + \sum_{j=1}^n (b_j - b)^2]^{-(n+v_b-1)/2}
 \end{aligned}$$

Combining [26], [28] and [29], we obtain the joint posterior density of $\underline{\theta}$ and \underline{b} :

$$\begin{aligned}
 [30] \quad & p(\underline{\theta}, \underline{b} | \underline{x}, v_\theta, s_\theta^2, v_b, s_b^2) \\
 & = \left\{ \exp\left(\sum_{i=1}^N r_i \theta_i\right) \right\} \left\{ v_\theta s_\theta^2 + \sum_{i=1}^N (\theta_i - \theta)^2 \right\}^{-(N+v_\theta-1)/2} \\
 & \cdot \left\{ \exp\left(-\sum_{j=1}^n q_j b_j\right) \right\} \left\{ v_b s_b^2 + \sum_{j=1}^n (b_j - b)^2 \right\}^{-(n+v_b-1)/2} \\
 & \cdot \left[\prod_{i=1}^N \prod_{j=1}^n \{1 + \exp(\theta_i - b_j)\} \right]^{-1}.
 \end{aligned}$$

The quantity given as $L(\underline{\theta}, \underline{b} | \underline{x})$,

$$\exp(\sum r_i \theta_i) \exp(\sum q_j b_j) / \prod \prod \{1 + \exp(\theta_i - b_j)\} = \prod \prod \exp(\theta_i - b_j) / \{1 + \exp(\theta_i - b_j)\},$$

and, hence, is bounded. In fact,

$$|L(\underline{\theta}, \underline{b} | \underline{x})| \leq 1.$$

Therefore, it follows that

$$\int |p(\underline{\theta}, \underline{b})| \dots | d\underline{\theta} d\underline{b} < \int [v_{\theta} s_{\theta}^2 + \sum_{i=1}^N (\theta_i - \theta_{.})^2]^{-(N+v_{\theta}-1)/2} d\underline{\theta} \\ \cdot \int [v_b s_b^2 + \sum_{j=1}^n (b_j - b_{.})^2]^{-(n+v_b-1)/2} d\underline{b} .$$

The integrals on the right of the inequality clearly exist since the kernels are those of multivariate t densities. Hence, the posterior pdf, $p(\underline{\theta}, \underline{b} | \underline{x}, v_{\theta}, s_{\theta}^2, v_b, s_b^2)$, is a proper pdf although the normalizing constant cannot be evaluated explicitly.

The joint posterior modes may be taken as estimates of θ_i and b_j ($i=1, \dots, N$; $j=1, \dots, n$). These are obtained by setting equal to zero the derivatives of $\log p(\underline{\theta}, \underline{b} | \dots)$, and solving the resulting equations:

$$[31] \quad \sum_{j=1}^n P_{1j} = r_i - (\theta_i - \theta_{.}) / \sigma_{\theta}^2 \quad (i=1, \dots, N),$$

$$[32] \quad \sum_{i=1}^N P_{1j} = q_j + (b_j - b_{.}) / \sigma_b^2 \quad (j=1, \dots, n),$$

where

$$P_{1j} = \exp(\theta_i - b_j) / \{1 + \exp(\theta_i - b_j)\} ,$$

$$r_i = \sum_j x_{1j} ,$$

$$q_j = \sum_i x_{1j} ,$$

$$\sigma_{\theta}^2 = \{v_{\theta} s_{\theta}^2 + \sum_i (\theta_i - \theta_{.})^2\} / (v_{\theta} + N - 1) ,$$

and

$$\sigma_b^2 = \{v_b s_b^2 + \sum_j (b_j - b_{.})^2\} / (v_b + n - 1) .$$

Since the systems of equations is non-linear, the Newton-Raphson procedure is employed to solve the equations iteratively. In order to accomplish this, we let

$$[33] \quad f(\theta_i) = \sum_{j=1}^n P_{1j} + (\theta_i - \theta_{.}) / \sigma_{\theta}^2 - r_i$$

and

$$[34] \quad h(b_j) = \sum_{i=1}^N P_{ij} - (b_j - b_{\cdot}) / \sigma_b^2 - q_j \quad .$$

Then

$$[35] \quad f'(\theta_i) = \sum_{j=1}^n P_{ij}(1 - P_{ij}) + \{ \sigma_b^2 (1 - \frac{1}{N}) - 2(\theta_i - \theta_{\cdot}) / (v_{\theta} + N - 1) \} / (\sigma_{\theta}^2)^2 \quad ,$$

and

$$[36] \quad h'(b_j) = - \sum_{i=1}^N P_{ij}(1 - P_{ij}) - \{ \sigma_b^2 (1 - \frac{1}{n}) - 2(b_j - b_{\cdot}) / (v_b + n - 1) \} / (\sigma_b^2)^2 \quad .$$

As before, if $\theta_i^{(k)}$ and $b_j^{(k)}$ denote the values of θ_i and b_j at the k th iteration, then

$$[37] \quad \theta_i^{(k+1)} = \theta_i^{(k)} - f(\theta_i^{(k)}) / f'(\theta_i^{(k)}) \quad ,$$

and

$$[38] \quad b_j^{(k+1)} = b_j^{(k)} - h(b_j^{(k)}) / h'(b_j^{(k)}) \quad .$$

Starting with initial values $\theta_i^{(0)}$ ($i=1, \dots, N$), and $b_j^{(0)}$ ($j=1, \dots, n$), where $\theta_i^{(0)}$ is given by [23], and

$$b_j^{(0)} = \log [(N - q_j) / q_j]$$

θ is estimated. These values of θ are then used to obtain revised estimates of b . This process is repeated with the revised estimates of b being used to obtain revised estimates of θ . The process is terminated when the convergence criterion is reached. This procedure is not the full Newton-Raphson procedure and, in this case, is preferred to the full Newton-Raphson procedure since the latter requires obtaining an inverse of the

matrix of second derivatives at each stage of the iteration. In practice, the procedure outlined here converges rather rapidly.

As pointed out earlier, although the equations provided are for estimating $\theta_i (i=1, \dots, N)$, only $\theta_r (r=1, \dots, n-1)$ need be estimated. In order to carry this out, the quantities given in Equations [34] and [35] have to be computed as follows:

$$\sum_{i=1}^N P_{ij} \simeq \sum_{r=1}^{n-1} N_r P_j(\theta_r)$$

$$\sum_{i=1}^N P_{ij}(1-P_{ij}) \simeq \sum_{r=1}^{n-1} N_r P_j(\theta_r)\{1-P_j(\theta_r)\}$$

where N_r denotes the number of examinees who obtained raw score r and

$$P_j(\theta_r) = \exp(\theta_r - b_j) / \{1 + \exp(\theta_r - b_j)\} .$$

Large Sample Properties of the Posterior Distribution

The posterior pdf, $p(\underline{\theta}, \underline{b} | v_\theta, v_b, s_\theta^2, s_b^2, \underline{x})$, given by Equation [30] is a product of the likelihood function and a multivariate "double-t" distribution. The "double-t" distribution is a product of two multivariate t densities (Tiao & Zellner, 1964; Zellner, 1971, p. 101). As a result of its complex form, properties of the posterior pdf cannot be obtained. However, it is possible to obtain the asymptotic properties of the posterior pdf, and this will suffice, in most cases, for inferences to be drawn regarding the parameters.

Let \underline{t} be a vector of parameters, and \underline{y} a vector of observations. Then, the posterior pdf of \underline{t} , $p(\underline{t} | \underline{y})$, is

$$p(\underline{t}|\underline{y}) \propto p(\underline{t}) L(\underline{y}|\underline{t})$$

where $p(\underline{t})$ is the prior distribution of \underline{t} and $L(\underline{y}|\underline{t})$, the likelihood function. Then, for large samples,

$$[39] \quad p(\underline{t}|\underline{y}) \propto L(\underline{y}|\underline{t}) ,$$

and, in turn, $L(\underline{y}|\underline{t})$ is approximately multivariate normal centered at $\hat{\underline{t}}$, the maximum likelihood estimate, with dispersion matrix

$$[40] \quad \Sigma = [-\partial^2 \log L(\underline{y}|\underline{t}) / \partial t_i \partial t_j]_{\underline{t}=\hat{\underline{t}}}^{-1}$$

Thus, for large samples,

$$\underline{t}|\underline{y} \sim N(\hat{\underline{t}}, \Sigma_{\underline{t}=\hat{\underline{t}}}) .$$

For a detailed discussion of this result we refer the reader to Jeffreys (1961, p. 193) and Zellner (1971, p. 32).

This result clearly applies in the present situation when both n and N , the number of items and the number of examinees, are large.

Denoting the $[(n+N) \times 1]$ vector $[\underline{\theta}, \underline{b}]$ as

$$\underline{t}' = [\underline{\theta}' \quad \underline{b}'] ,$$

$$[41] \quad \underline{t}|\underline{x} \sim N(\hat{\underline{t}}, \Sigma) .$$

In order to evaluate Σ , we write

$$[42] \quad \Sigma = \begin{bmatrix} G_{\theta} & G_{\theta b} \\ G_{b\theta} & G_b \end{bmatrix}^{-1}$$

where

$$\begin{aligned}
 [43] \quad G_{\theta} &= \{-\partial^2 \log L(\underline{x}|\underline{\theta}, \underline{b}) / \partial \theta_{\ell} \partial \theta_m\}_{\underline{\theta}=\hat{\underline{\theta}}} \\
 &= \left\{ \sum_{j=1}^n P_{\ell j} (1 - P_{mj}) \right\} \delta_{\ell m}
 \end{aligned}$$

where $\delta_{\ell m}$ is the Kronecker delta,

$$\begin{aligned}
 [44] \quad G_b &= \{-\partial^2 \log L(\underline{x}|\underline{\theta}, \underline{b}) / \partial b_{\ell} \partial b_m\}_{\underline{b}=\hat{\underline{b}}} \\
 &= \left\{ \sum_{i=1}^N P_{i\ell} (1 - P_{im}) \right\} \delta_{\ell m},
 \end{aligned}$$

and

$$\begin{aligned}
 [45] \quad G_{\theta b} &= \{-\partial^2 \log L(\underline{x}|\underline{\theta}, \underline{b}) / \partial \theta_i \partial b_j\}_{\underline{\theta}=\hat{\underline{\theta}}, \underline{b}=\hat{\underline{b}}} \\
 &= P_{ij} (1 - P_{ij}) .
 \end{aligned}$$

Thus, the marginal distribution of θ_i has mean $\hat{\theta}_i$, the maximum likelihood estimate of θ_i , and variance, $\sigma_{\theta_i}^2$, given by the i th diagonal element of Σ , i.e.,

$$[46] \quad \sigma_{\theta_i}^2 = [G_{\theta} - G_{\theta b} G_b^{-1} G_{b\theta}]_{ii}^{-1} .$$

Similarly, the marginal distribution of b_j has mean \hat{b}_j , the maximum likelihood estimate of b_j , and variance, $\sigma_{b_j}^2$, given the j th diagonal element of Σ , i.e.,

$$[47] \quad \sigma_{b_j}^2 = [G_b - G_{b\theta} G_{\theta}^{-1} G_{\theta b}]_{jj}^{-1} .$$

This approximation to the posterior pdf of $\underline{\theta}$ and \underline{b} can be improved upon if we take into account the "double-t" distribution (see Equation [30]). For a sufficiently large sample, the multivariate t density approaches the normal density. Thus, in the expression

$$[v_{\theta} s_{\theta}^2 + \sum_{i=1}^N (\theta_i - \bar{\theta})^2]^{-(v_{\theta} + N - 1)/2}$$

if we write $v_{\theta} = Nk_{\theta}$ where $0 < k_{\theta} \leq 1$, for large N, we obtain

$$\begin{aligned} [48] \quad [Nk_{\theta} s_{\theta}^2 + \sum_{i=1}^N (\theta_i - \bar{\theta})^2]^{-N(k_{\theta} + 1)/2 + \frac{1}{2}} &\sim \exp\left[-\frac{(k_{\theta} + 1)}{2k_{\theta} s_{\theta}^2} \sum_{i=1}^N (\theta_i - \bar{\theta})^2\right] \\ &= \exp\left\{-\frac{1}{2} \underline{\theta}' A_{11} \underline{\theta}\right\} \end{aligned}$$

where

$$A_{11} = \frac{(k_{\theta} + 1)}{k_{\theta} s_{\theta}^2} \left[I_N - \frac{1}{N} \underline{1} \underline{1}' \right]$$

with I_N being the identity matrix and $\underline{1}' = [1 \ 1 \ 1 \ \dots \ 1]$. Similarly, for large n,

$$\begin{aligned} [49] \quad [v_b s_b^2 + \sum_{j=1}^n (b_j - \bar{b})^2]^{-(v_b + n - 1)/2} &= [nk_b s_b^2 + \sum_{j=1}^n (b_j - \bar{b})^2]^{-n(k_b + 1)/2 + \frac{1}{2}} \\ &\sim \exp\left\{-\frac{(k_b + 1)}{2k_b s_b^2} \sum_{j=1}^n (b_j - \bar{b})^2\right\} \\ &= \exp\left\{-\frac{1}{2} \underline{b} A_{22} \underline{b}\right\} \end{aligned}$$

where

$$A_{22} = \frac{(k_b + 1)}{k_b s_b^2} \left[I_n - \frac{1}{n} \underline{1} \underline{1}' \right]$$

Thus,

$$[50] \quad [v_b s_b^2 + \sum_{j=1}^n (b_j - b_{.j})^2]^{-(v_b + n - 1)/2} [v_\theta s_\theta^2 + \sum_{i=1}^N (\theta_i - \theta_{.i})^2]^{-(v_\theta + N - 1)/2}$$

$$\propto \exp \left\{ -\frac{1}{2} (\underline{\theta}' A_{11} \underline{\theta} + \underline{b}' A_{22} \underline{b}) \right\}$$

$$[51] \quad = \exp \left\{ -\frac{1}{2} \underline{t}' A \underline{t} \right\}$$

where

$$\underline{t}' = [\underline{\theta}' \quad \underline{b}'] ,$$

and

$$A = \begin{vmatrix} A_{11} & 0 \\ 0 & A_{22} \end{vmatrix}$$

Combining [41] and [51], we have

$$[52] \quad p(\underline{\theta}, \underline{b} | \underline{x}, v_\theta, v_b, s_\theta^2, s_b^2)$$

$$[53] \quad \propto \exp -\frac{1}{2} \{ (\underline{t} - \hat{\underline{t}})' \Sigma (\underline{t} - \hat{\underline{t}}) + \underline{t}' A \underline{t} \}$$

$$[54] \quad = \exp \left\{ -\frac{1}{2} (\underline{t} - \underline{\tau})' T (\underline{t} - \underline{\tau}) \right\}$$

where

$$[55] \quad T = \Sigma + A$$

and

$$[56] \quad \underline{\tau} = (\Sigma + A)^{-1} (\Sigma \hat{\underline{t}})^*$$

*This result follows from the fact that

$$(\underline{x} - \underline{a})' A (\underline{x} - \underline{a}) + (\underline{x} - \underline{b})' B (\underline{x} - \underline{b}) = (\underline{x} - \underline{t})' T (\underline{x} - \underline{t}) + \text{constant},$$

where

$$T = A + B,$$

and

$$\underline{t} = (A + B)^{-1} \{ A \underline{a} + B \underline{b} \} .$$

If the off diagonal matrix $G_{\theta b}$ in [42] can be ignored, then

$$[57] \quad \sigma_{\theta_i}^2 = \left[\sum_{j=1}^n P_{ij}(1-P_{ij}) \right]^{-1} + (N-1)(k_{\theta}+1)/Nk_{\theta}s_{\theta}^2$$

and

$$[58] \quad \sigma_{b_j}^2 = \left[\sum_{i=1}^N P_{ij}(1-P_{ij}) \right]^{-1} + (n-1)(k_b+1)/nk_b s_b^2 .$$

The expression [57] is useful when the item parameters are considered known. Similarly, [58] is applicable when the ability parameters are known. In general, however, when the item and ability parameters are estimated simultaneously, the off diagonal matrix, $G_{\theta b}$, cannot be ignored, and hence, in this case, the complete expression given by either [46] or [55] should be employed. With these results it is possible to construct "credibility intervals" (Novick & Jackson, 1974) for the parameters of interest.

COMPARISON STUDIES

In order to study the efficacy of the Bayesian procedure described above and to compare the Bayesian estimates with the maximum likelihood estimates, a simulation study was carried out. Although simulation studies may not be realistic in some situations, they can be justified in the present context since only through a simulation study can one estimation procedure be compared with another.

Artificial data, representing the responses of N individuals on n items, were generated using DATGEN (Hambleton & Rovinelli, 1973) according to the one-parameter logistic model. In generating the values of

θ_i and b_j ($i=1, \dots, N$; $j=1, \dots, n$), it was assumed that θ_i and b_j were independently and identically normally distributed with mean, zero, and variance, unity (we shall return to a discussion of this issue later). The design of the comparison study was conceptualized in terms of the following, completely crossed, factors: estimation procedure (Bayesian, maximum likelihood); number of examinees, N (20, 50); number of items, n (15, 25, 40, 50). This design was carried out for (i) conditional estimation of θ , and, (ii) joint estimation of θ and b .

The size of the examinee population, N , and the test length, n , were chosen to facilitate comparison of the maximum likelihood and the Bayesian estimates for small sample sizes and short tests, since the large sample behavior of the maximum likelihood estimates has been studied by Swaminathan and Gifford (1979). These authors have found that maximum likelihood estimates of θ_i and b_j approach the true values for N as large as 200 and n as large as 100. Since for these values of N and n , Bayesian estimates can be expected to be the same as maximum likelihood estimates, the study was focused on small values of N and n .

The Bayesian estimates and the maximum likelihood estimates were compared with respect to accuracy. The two sets of estimates were compared with respect to: (a) the mean value of the estimates, as compared with the mean value of the true values; (b) the mean squared error difference between the true values and the estimated values; and, (c) the regression of the true value on the estimated value.

It may be argued that since the joint modes of the posterior distribution were taken as estimates of the parameters, the criterion employed to determine the accuracy of the estimates is incompatible with the loss

function employed to arrive at the estimates. This is a valid argument. However, we are primarily interested in comparing the Bayesian estimates with the maximum likelihood estimates. Since, in one sense, the maximum likelihood estimates can be thought of as the modes of the posterior distribution derived under the assumption that the prior information is vague, comparison of two modal estimates using a different criterion other than that involved in deriving the estimates may be justifiable; particularly since this will not provide an "unfair" advantage to one set of estimates.

Comparison of the two estimation procedures in terms of the regression of true values on the estimates needs some explanation. If τ is the true value of the parameter and E , the estimate, then $E(T|E=e) = \beta_0 + \beta_1 e$. If $\beta_0=0$ and $\beta_1=1$, then, it can be concluded that the estimates are unbiased, and hence, the departure from the expected values of β_0 and β_1 can be taken as an indicator of bias. It should be pointed out here that the classical notion of bias is not critical in Bayesian analyses. Nevertheless, comparison of the regression lines will provide a further assessment of the accuracy of the two procedures.

The comparison of the maximum likelihood (ML) procedure and the Bayesian procedure for the conditional estimation of ability θ is provided in Table 1. The first column contains the means of the true values of θ , the ML estimates, and the Bayesian estimates. The second column provides an assessment of accuracy in terms of the mean squared deviation between the estimate $\hat{\theta}$ and the true value, θ_t . The correlations between θ_t and $\hat{\theta}$ for each estimation procedure is displayed in column four, while the regression of θ_t on $\hat{\theta}$ is given in column five.

Table 1
 Conditional Estimation of θ : Comparison of the Bayesian
 Estimate and Maximum Likelihood Estimate

Number of Examinees	Number of Items	$\bar{\theta}$		$\Sigma(\hat{\theta}-\theta_t)^2/N$		Correlation		Regression				
		True	ML	Bayes	ML	Bayes	ML	Bayes	B_0	B_1	B_0	B_1
20	15	-.334	-.321	-.319	.479	.137	.885	.912	.122	1.326	-.035	.853
	25	-.328	-.241	-.222	.175	.083	.951	.950	.171	1.254	.086	.941
	40	.087	.088	.088	.106	.056	.959	.955	-.020	1.246	-.002	1.039
	50	.292	.433	.393	.138	.048	.979	.981	.045	1.327	.069	1.109
50	15	-.000	.130	.115	.440	.117	.915	.928	.130	1.423	.115	.891
	25	-.159	-.218	-.177	.282	.129	.950	.944	-.016	1.271	-.024	.961
	40	-.089	-.167	-.152	.231	.099	.976	.977	-.040	1.428	-.044	1.218
	50	.171	.255	.257	.246	.091	.982	.985	.021	1.361	.057	1.167

An examination of the correlations between the true values and estimates reveals that, in general, the difference between ML and Bayes estimates, is negligible for relatively large values of N and n. However, for small values of N and/or n, the Bayes estimates correlate better with true values than the ML estimates.

The correlation coefficient, by itself, is not a sufficient indicator of the accuracy of estimation. Clear differences between the Bayesian and ML procedures emerge when we examine the other criteria.

In general, the means of the Bayesian estimates, in comparison with the ML estimates, are closer to the means of the true values. This result can be anticipated if we examine the estimating equations [19]. The estimating equations for ML estimates are:

$$\sum_{j=1}^n P_{ij} = r_i \quad (i=1, \dots, N).$$

The additional term in the Bayesian estimating equations, $(\theta_i - \theta) / \sigma^2$ contributes to the regression of the estimates towards the mean, and hence, the Bayesian estimates are closer to the means of the true values. The only exception occurs with N=20 and n=15, 25. At this point, there is no explanation for this anomolous result. Further replications are clearly necessary to establish this point conclusively.

The most dramatic difference between the Bayesian estimates and the ML estimates is with respect to the mean squared deviations of the estimates from the true values. In general, the mean squared deviations are much smaller for the Bayesian estimates than for the ML estimates. The difference is particularly noticable with small N and n. In these cases, the mean squared deviations for the ML estimates is almost four times as

large as that for the Bayesian estimates. This finding can again be explained by the fact prior information is most helpful in these cases. This, together with the regression effect described previously, results in an increase in the accuracy of the estimation procedure.

An examination of the regressions of true values on estimated values also provides some interesting results. In general, the intercepts and the slopes of the Bayesian regressions are closer to zero and one respectively, than the ML regressions. The trend for the intercepts is reversed for large n . In these cases, the intercepts for the ML regressions are closer to zero than the intercepts for the Bayesian regressions. This latter result is interpretable, since the maximum likelihood estimates of θ , for large N and n , approach the true values. However, the trend for small n and N is rather surprising since, as a result of regression towards the mean, the Bayesian estimates can be expected to be "biased." The only explanation for this finding is that the ML procedure is severely biased for small n and N , even more so than the Bayesian procedure.

The above findings, for conditional estimation of θ , appear to be valid for the joint estimation of θ and b (Tables 2 and 3). In fact, the results for the joint estimation of θ and b favor the Bayesian estimates on all counts for both θ and b : the means of the estimates are closer to the means of the true values; the mean squared deviations are much smaller (in some cases, one-tenth the size of those for ML estimates); the slopes and intercepts are closer to one and zero respectively (the only exception occurs for large N and n , in which case, the intercepts of the ML regression are closer to zero).

Table 2
 Joint Estimation of $\bar{\theta}$ and \underline{b} : Comparison of the Bayesian
 and Maximum Likelihood Estimates of Ability

Number of Examinees	Number of Items	$\bar{\theta}$		$\Sigma(\theta_i - \bar{\theta})^2 / N$		Correlation		Regression				
		True	ML	Bayes	ML	Bayes	ML	Bayes	B_0	B_1	B_0	B_1
20	15	-.334	-.237	-.281	.993	.139	.889	.912	.325	1.683	.004	.852
	25	-.224	-.173	-.224	.991	.134	.938	.938	.230	1.802	.007	1.034
	40	.049	.089	.064	.498	.077	.970	.970	.007	1.654	.009	1.115
	50	-.282	-.126	-.104	.453	.144	.969	.971	.373	1.765	.267	1.317
50	15	-.000	.160	.074	.956	.115	.914	.928	.160	1.792	.075	.954
	25	-.117	-.090	-.091	1.015	.209	.925	.925	.111	1.719	.037	1.089
	40	-.089	-.235	-.225	.664	.227	.977	.977	-.074	1.810	-.101	1.409
	50	.171	.302	.280	.776	.238	.984	.985	.006	1.730	.046	1.362

Table 3
 Joint Estimation of $\hat{\theta}$ and \hat{b} : Comparison of the Bayesian
 and Maximum Likelihood Estimates of Item Parameters

Number of Examinees	Number of Items	\hat{b}		$\Sigma(b-\bar{b})^2/N$		Correlation		Regression			
		True	ML	ML	Bayes	ML	Bayes	B_0	B_1	B_0	B_1
20	15	.050	.279	.083	.096	.946	.939	.187	1.827	.035	.944
	25	-.066	.264	.100	.153	.929	.954	.399	2.061	.176	1.162
	40	.080	1.008	.014	.307	.907	.909	-.130	1.723	-.075	1.109
	50	-.127	.005	.018	.190	.933	.933	.207	1.592	.148	1.033
50	15	.050	.132	.033	.099	.983	.985	.035	1.931	-.031	1.281
	25	-.030	.087	.051	.512	.982	.980	.147	1.978	.096	1.476
	40	.109	.166	.093	.307	.961	.960	-.025	1.741	-.056	1.357
	50	-.049	-.076	-.069	.303	.972	.974	.011	1.783	.001	1.431

DISCUSSION

The Bayesian procedure for estimating parameters in the one-parameter latent trait model is an attractive alternative to the maximum likelihood procedure. Bayesian procedures are conceptually more appealing since direct interpretations of probability statements involving the parameters are possible. Empirically, as the results of the comparison study indicate, the Bayesian estimates of the parameters are superior to the maximum likelihood estimates in terms of their accuracy.

Although the empirical results demonstrate the effectiveness of the Bayesian procedure, it may be argued, and correctly, that the simulation of the data favored the Bayesian procedure. The data were generated to meet the strong distributional assumptions required by the Bayesian procedure. In addition, in specifying prior belief about the distribution ϕ_θ and ϕ_b , s_θ^2 and s_b^2 were set equal to one with the corresponding v_θ and v_b being specified as 15. In the simulation ϕ_θ and ϕ_b were set at one, and the specification of s_θ^2 , s_b^2 and the relatively large values for v_θ and v_b reproduced the true state of affairs. It is not surprising, therefore, that the Bayesian procedure proved to be superior to the maximum likelihood procedure.

In fairness to the study, it should be pointed out that the simulation and the accurate specification of prior belief were deliberate in order to determine the applicability of the Bayesian procedure, at least, under ideal conditions. Preliminary investigations with non-normal data and also with poor specification of priors indicate that the Bayesian procedure, being based on a hierarchical model, is relatively robust and is superior to the maximum likelihood procedure. A detailed study of the effects of poor specification of priors and departures from underlying

assumptions is currently under way and we expect to report these results in the near future.

Despite the encouraging results obtained, a theoretical problem still remains with the estimation procedure. The procedure described in this paper requires the joint estimation of n structural parameters and N incidental parameters. If $N \rightarrow \infty$ while n remains fixed, the joint posterior pdf may not become concentrated about the estimated values. This trend is evident from Tables 2 and 3; with increasing N , the intercept and slope do not tend to zero and one respectively. This problem is similar to the one that exists with maximum likelihood estimates. Although from a Bayesian point of view asymptotic properties, such as consistency, are not critical, the lack of them, at least to some degree, is disconcerting. It appears that this situation can be remedied, if when estimating the n structural, or item, parameters, the ability parameters are considered nuisance parameters and can be integrated out to yield the marginal posterior pdf of \underline{b} . The marginal posterior pdf is currently not available as a result of the exceedingly complex form of the joint posterior pdf. Approximations, such as the one indicated (Equation [15]) may be employed to simplify the joint posterior pdf. Initial investigations reveal that this approximation is reasonably good, but further research in this area is clearly needed.

In summary, we note that the Bayesian procedure developed in this paper is relatively simple to implement, and computationally as efficient as the maximum likelihood procedure. Despite the issues raised above, the Bayesian procedure has the potential for greatly improving the accuracy of the estimates. Moreover, the maximum improvement in accuracy occurs

for small values of N and n , a result that can be expected, and this makes the Bayesian procedure more attractive than the maximum likelihood procedure.

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