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THE THEORY OF SCREEN

Report to

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by

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ABSTRACT

This report documents the theory which underlies the SCREEN program, a computer program designed to evaluate acoustic detection and localization performance of an anti-submarine protective force about a Naval task force or other shipping. A companion user's manual for SCREEN supplements this report. The measures used to evaluate the SCREEN performance are: cumulative detection probability against specific target approach tracks and cumulative localization performance against these same tracks. In addition to cumulative measures, "snapshot" detection and localization measures are also computed, which provide an indication of the detection and localization coverage of the defensive screen at a specified time.

The SCREEN program operates on data files which contain moderately detailed descriptions of the acoustic environment (propagation, noise, etc.), the sensor parameters and tactics, and the screen penetrator (target) parameters and tactics. These descriptions include both deterministic and stochastic parameters . The data file contents can be created, altered, and displayed by the user under program control. Once the data files have been created, subsequent use of SCREEN is straightforward and concise, involving user-selectable program options and machine prompts for input.

The underlying detection process is a modified (λ, σ)-jump process. The target process is a modified Integrated Ornstein-Uhlenbeck (IOU) process. The basic localization algorithm is an "Information Flow" Kalman filter. Bayesian updating techniques are used to evaluate search effort, along lines similar to techniques found in computer assisted search programs which are currently being implemented in the Fleet.

PREFACE

This is a report to the Chief of Naval Operations (Op-961) under Contract No. N00014-76-C-0811, which gives the theoretical basis for the algorithms and procedures contained in the SCREEN program, a computer program designed to evaluate acoustic detection and localization performance of an anti-submarine protective force or other shipping. This is a companion volume to reference [a], the user's manual for the SCREEN program.

The theory of SCREEN incorporates the results of several lines of analysis developed largely by this firm over recent years. These lines include cumulative detection probability and other acoustic detection modeling, Kalman filtering and other localization techniques, stochastic target motion models, and the methodology of computer assisted search.

The acoustic detection model is based on algorithms for cumulative detection probability (cdp) involving the (λ, σ)-jump process that have a long history of development beginning with reference [b] in 1964. A commentary on the validity of this line of model development is found in reference [c]. In Chapter II of this report, the present theory extends the development to the case of a randomly sampled jump process.

Models for target localization and target motion analysis (TMA) also have a long history of development. The SCREEN program uses Kalman filter techniques developed in reference [g], based on the "information flow" approach to the Kalman iteration technique. This work is representative of a number of generically similar approaches to TMA and should provide a reasonable expression of expected localization performance of bearings-only and active sensors. The present theory, Chapter III, extends the basic Kalman iteration to include correlated observations, which is an important improvement over previous algorithms.

The SCREEN program evaluates screen performance against targets that follow penetration strategies as dictated by various target files created by the SCREEN user. Each target file describes in essence a target diffusion process which is a discrete-time analog of the Integrated Ornstein-Uhlenbeck (IOU) process, references [d] and [e]. The IOU process, and other processes for target motion have received substantial study in recent years, see for example reference [m]. The present work, described in Chapter III, is related in reference [q] to this earlier work. Recent work on the statistical analysis of these stochastic processes is reported in reference [y].

The methodology of computer-assisted search (CAS) also has a long history, and since the SCREEN program is representative of one line of CAS development, an extended discussion of its development seems merited at this point.

The use of computers in planning search effort goes back to the H-Bomb search off of Palomares, Spain in 1966, and a subsequent extension of the technique in the Scorpion search in 1968 (see reference [u]). This early methodlogy was further developed and used in the first real-time computer program for Bayesian search planning, the U. S. Coast Guard CASP program (reference [x]). This has been operational since 1972.

The essential innovative features of the early search programs were the inclusion of target "scenarios" to define prior target distributions and move them through time, and Bayesian updating to show the effects of search effort as an aid in subsequent search planning. These programs are Monte Carlo simulations and have subsequently been developed into a line of standardized CAS programs whose development is sponsored by ONR.

In 1975, as a direct precursor of SCREEN, Dr. T. L. Corwin, then attached to the COMSUBPAC staff, devised an analytic search program (ASP) for use on a desktop calculator. The ASP program utilizes target scenarios, described as stochastic diffusion processes, and evaluates search against them in an analytic fashion. Useful features of the ASP program which are retained in SCREEN include the capability to operate the program in real time and to modify the program to account for positive contact information, remove such contact information, and adjust search tactics in real time for actual, versus planned, operations. The principal documentation of these programs is in reference [v].

The analytic search methodology of SCREEN is an outgrowth of Corwin's work. Work on SCREEN commenced in 1976 under the initial sponsorship of CAPT W. Mitchell of Op-96. The initial structure was developed by Dr. Bossard and the first working programs were produced in early 1977 with the substantial assistance of Dr. W. H. Barker. The first working program provided only detection performance measures. These were subsequently extended to include localization measures in the summer of 1978.

In mid-1978, the SCREEN program was used for the first time as an important analysis tool in the Submarine Alternatives Study (SAS) which was conducted by CNO (Op-02) at the request of the Secretary of the Navy and led by CAPT James Van Metre. In that study, the SCREEN program has been utilized extensively to analyze alternative U. S. submarine designs in the Anti-Surface Warfare (ASUW) and direct support roles. In addition to this use of SCREEN, it was used in various short analyses by CNO (Op-96) in the winter of 1978-1979. Based on these uses, many improvements have been incorporated into the program and increased confidence in its utility and general model accuracy has been gained.

A number of analysts, in addition to the authors of the present report, were involved in the development and testing of the SCREEN program. The principal additional contributors were: Dr. W. H. Barker (1976-1977), Dr. L. K. Arnold (winter 1977-1978), and Dr. B. E. Scranton (since Fall, 1978). Mr. B. M. McDaniel was extensively involved in programming aspects during this period of time and Mr. R. L. Andersson developed a number of computer "bookkeeping" routines useful to the program. Dr. Scranton was responsible for much of the testing and improvements to SCREEN included during the SAS and Op-96 analyses noted above. Dr. D. P. Kierstead also provided a useful critique of the program algorithms.

We wish to acknowledge the excellent support and cooperation of Op-961 from successively CAPT William Mitchell, USN (ret.), Mr. Robert A. Hallex, and most recently CAPT Raymond Wyatt, USN. Without their continuing support and constructive help, this project would never have achieved the present state of development. We wish to also acknowledge the support of CAPT Van Metre and of Dr. David Stanford of Science Applications, Incorporated, who sponsored the use of the SCREEN program in the recent Submarine Alternatives Study.

SUMMARY

This report gives the theorical basis for the algorithms used in the SCREEN program. This program was developed for the Chief of Naval Operations (Op-961) and designed to evaluate the acoustic detection and localization performance of an anti-submarine protective force about high-value shipping in a Naval convoy or task force scenario.

The ultimate use of the SCREEN program is in the assessment of alternative ASW designs. To do such an analysis properly, it is necessary to display enough detail to show the effect of proposed design changes. It is intended that SCREEN provide adequate detail for such assessment, without being overweighted with too much detail.

The screen performance measures encompass both detection and localization. Performance is assessed against various opposing target strategies. Although the program is designed primarily for use in CNO analyses, the program design also allows for interactive use in a real time situation. Such a use would require the services of a fairly powerful computer--more capable than is generally available at sea at the present time. Work to modularize SCREEN and thus reduce the computer requirements is in progress.

This is one of two reports on the SCREEN program. The present volume discusses the theoretical aspects of the program but does not enter into the details of its use. The second volume, reference [a], is a user's manual describing the actual operation of the SCREEN program.

The first chapter gives a general introduction to the type of problems that SCREEN addresses. In addition, this chapter attempts to show the place of SCREEN in the spectrum of similar types of analyses. Chapter II discusses the definition of a screen formation, describes what is involved in defining a sensor and sensor platform, and then turns to a description of the detection process assumed in the model. Chapter III discusses the localization measures developed for the SCREEN program and shows how they are related to a classical Kalman filter target motion analysis. The final chapter turns to the target's screen penetration strategy, and describes how the target tactics are modeled and how they can be modified to yield a flexible capability in the SCREEN program for designing a broad class of target strategies.

Each chapter refers to appendices which provide various background material as well as calculations of a more detailed nature that would disrupt the flow of the text but are required for a full understanding of the theory. A glossary is included at the end of the report.

In the remainder of the summary, we will briefly discuss where the SCREEN program fits in Navy analyses, and then briefly address sensors, the detection algorithms, the localization process, and the target strategies.

The Scope of SCREEN Analyses

The level of detail in the SCREEN program places it somewhere between most of the analyses of screen formations performed in the past, and the detailed modeling that characterizes engineering or technical descriptions of the various components of the detection and localization process. The SCREEN program is probably more detailed than any overall screen evaluation model previously developed. A consequence of this is that the parameters needed for complete description of the sensors, the environment, and the targets include quantities usually omitted in similar kinds of analysis. On the other hand, it should be noted that, for any one of the components of the SCREEN program, there exist models which carry the level of detail far beyond the level found in SCREEN.

To pick one example, the SCREEN program assumes that the total signal excess as observed at the beamformer output of a sensor is describable by a simple process which involves only two parameters: the standard deviation in the random component of signal excess and the rate at which independent samples of the signal excess may be observed. Now, one undoubtedly could consider individual random processes for each component of the signal excess equation--the target radiated noise, the propagation medium, the background noise, the signal processor, etc.--but a model which included all of these processes would require at least a dozen parameters for its description. In comparison with such a model, then, the random signal excess process used in SCREEN appears to be very simple. Further examples of the level of detail found in SCREEN are given in Chapter I.

Thus, the modeling assumptions made in the SCREEN program represent a judgment on the part of the developers as to the minimal level of detail required to reflect the distinctions among the various screen components that the program is designed to evaluate. In support of these assumptions, we note that the program has been used heavily in an extended study in the past year and the richness of detail in the model appears to have been justified; at the same time, the model seemed to be simple enough to be used without discouraging the participants.

Indeed, the SCREEN program is designed for convenience of operation. All of the environmental, sensor, and target information is preserved in data files which, once initialized, can be used repeatedly in different combinations. Moreover, either batch or interactive operation is possible. In the usual research task, the batch mode would be invoked, in which a sensor file, including the tactics for each screen unit throughout an entire tactical engagement, is specified in advance, together with a predetermined environment and one or more target files to model penetration tactics. However, at any time, even when working basically in a batch mode of operation, it is possible to enter into an interactive mode of operation, in which the penetration scenario is advanced a few steps at a time, pausing as necessary to make changes in the screen formation, screen unit tactics or penetrating target strategy. This capability makes it possible to convert SCREEN into a two-sided game in which the adaptive strategies for each side are input to the program based on presumed "intelligence" or tactical assessment.

SCREEN operates in discrete time with a uniform time step. Essentially, any input data can be modified at any program time step, with the subsequent problem evolution based on the modified data. The data files thus constructed will contain the total time history of the encounter, and future reference to these files will reflect the choice of different parameter values for the different time steps in the problem. In this way, complex screen formations (including detailed tactics for each platform) may be developed and then reused in subsequent analyses.

There are essentially two types of sensors modeled by SCREEN: active and passive. Active sensors provide both range and bearing information whereas passive sensors provide only bearing information. Omnidirectional sensors are modeled by equating them to directional sensors with a large bearing standard deviation. Line arrays which give ambiguous bearings (the correct bearing and its reflection about the array axis) are also modeled as a special type of passive sensor.

A sensor is assumed to detect a target by accumulating the signal from the target over a period of time controlled by a parameter called the integration time. Sensors are assumed to scan azimuthally over their coverage region at an average rate determined by the scan time. The scan mechanism is random rather than systematic so that it will not have a regular period that could be exploited by a target. During an encounter, the performance of a sensor is degraded by an availability factor which is a measure of the likelihood that the sensor is operable during the engagement.

A screen formation consists of sensors placed on sensor platforms which are in turn positioned to protect high-value units (HVUs) against attack by enemy submarines. The term "sensor" includes everything--the array, beamformer, signal processing and display equipment--directed toward detecting, classifying, and localizing a target based on a particular narrowband tonal, broadband noise level or active echo. Each different combination of processing gear, sensor and target noise level (a specific tonal, broadband level, or active echo) is a different "sensor." A sensor platform, or even a particular sonar, may thus correspond to several screen "sensors."

Sensors are combined into sensor groups. The formation of groups is under the user's control. Groups exist for two principal reasons. First, sensors within a group may behave in a correlated fashion, as determined by a group correlation parameter (between groups, sensors operate independently). Second, the summary performance measures may be calculated and displayed for individual groups or for the entire screen; thus, it is possible to see how various segments of the screen contribute to overall performance by grouping sensors in different ways.

Screen performance--the depth and quality of the coverage provided by the screen--depends on the environment, the target penetration tactics (noise levels, speed, evasive maneuvers, etc.), and individual sensor performance. The SCREEN program evaluates screen coverage using short-term, or snapshot, coverage diagrams and longer-term, or cumulative, detection and localization measures, the latter applied against specific target penetration strategies. Maps of the posterior target distribution (showing target distribution given no detection by the screen) can also be displayed. These SCREEN performance measures, on which we will elaborate momentarily, apply to the specific environments, sensor line-ups, and screen placements as declared during the program operation. Generally, both detection and localization performance measures are computed individually for each sensor in the screen, and then combined to yield screen performance measures.

The Detection Model

The starting point in the detection performance evaluation is the classical sonar equation for passive or active sensors. The sonar equation determines the mean signal excess evaluated at the signal processing equipment of the specific sensor. This determination involves parameters derived from the environmental data, the sensor files, and the target files, including the target radiated noise level (or target strength in the case of active sonars), the propagation loss between the target and sensor, the ambient noise, interference from radiated noise by other screen units and HVUs, self-noise, recognition differential, etc. The sonar equation as used in SCREEN is discussed in Appendix A.

The actual detection probability evaluated for each sensor depends on the mean signal excess, as determined from the sonar equation, and five additional parameters that characterize the sonar: (1) the sensor integration time; (2) the sensor scan time; (3) the sensor availability; (4) the signal excess standard deviation; and (5) the signal excess relaxation rate. The use of these five parameters will now be described briefly.

Each sensor is assumed to declare detections based on the accumulation of signal over time, by a process in which the "old" signal is weighted and summed with the current signal. The specific detection function, $\mathscr{D}$, and weighting scheme assumed is as follows (see text equation (II-1) of Chapter II and ensuing discussion):

$$\mathcal{D}(t) = \frac{\int_{-\infty}^{t} (I_S(s)+I_N(s))dW_t(s)}{E[\int_{-\infty}^{t} I_N(s)\, dW_t(s)]} - 1, \qquad \text{(S-1)}$$

where

$I_S(s)$ = intensity of target signal of the beamformer output

$I_N(s)$ = intensity of interfering noise at the beamformer output

$W_t(s)$ = historical weight function,

and E[·] denotes expected value. The numerator in (S-1) is the total received weighted signal. The denominator is the expected background noise. The weight function used in SCREEN corresponds to "exponential decay" and is given by:

$$W_t(s) = \exp[-(t-s)/\omega], \qquad \text{(S-2)}$$

where ω is the sensor integration time mentioned above. The integration time can be viewed as the average time window over which signal history is accumulated.

The snapshot detection probability for a sensor at time t, p(t), is a function of the expected value of $\mathcal{D}$, the recognition differential RD for the sensor, and the signal excess standard deviation σ mentioned above. Its defintion is:

$$p(t) = \int_{-\infty}^{E[\mathcal{D}(t)]} n(y;RD,\sigma^2)dy \qquad \text{(S-3)}$$

where $n(\cdot\,;\mu,\Sigma)$ denotes the Gaussian density function with mean μ and covariance matrix Σ--see equation (H-1) in Appendix H. Because of the integration over the past signal history, the snapshot probability at time t depends in theory on the target and sensor parameters at all times up to t. However, it is usually considered that the integration time ω is small compared with the time span of the total engagement, so that for practical purposes, p(t) depends on the behavior of sensor and target only in the immediate vicinity of time t; hence, the term "snapshot," which is intended to convey the idea of a short time interval (a "glimpse") around time t. Snapshot probabilities are distinguished from cumulative probabilities in that the latter involve longer-term time correlations.

Snapshot probability calculations are the basis for "snapshot coverage maps" which show the coverage area about a sensor, a sensor group, or the entire screen. These maps provide a convenient way to show the performance of individual sensors, and can be the basis of a rough initial placement of screen sensors to provide uniform coverage about the HVUs.

The cumulative detection probability (cdp) for a sensor against a specific penetrating target track is computed using the discrete (λ, σ) jump process to model the random signal excess process, i.e., the random component of $\mathcal{D}$. This process, originally proposed in the ASW context by J. D. Kettelle in 1959, is described in references [b] and [c] as well as in Chapter II. In this process, samples of the random component of $\mathcal{D}$ at two times t_1 and $t_2 > t_1$ are either completely correlated (with probability $\exp-[\lambda(t_2-t_1)]$) or completely independent (with probability $1-\exp[-\lambda(t_2-t_1)]$). The relaxation rate λ is the expected number of independent samples per hour. The standard deviation of the distribution from which the samples are drawn is σ, the signal excess standard deviation mentioned in connection with equation (S-3).

The random process used in SCREEN for cdp calculations carries the jump process one step farther than in references [b] and [c] by including a sampling rate π based on the sensor scan time r. The concept behind scan time is the notion that the sonar operates somewhat like a searchlight sweeping through its azimuthal coverage area. The scan time is the mean time between successive looks at a given portion of the coverage area. Each look is assumed to be adequate to give a detection opportunity if one exists. The relationship between scan time r and the sampling probability π is:

$$\pi = \exp(-\Delta t/r), \qquad \text{(S-4)}$$

where Δt is the length of the program's uniform time step. (We remark that the theory allows π to vary with the time step, which would be the case if, for example, r did.) Equation (S-4) expresses the assumption that the scan process is completely random, in contrast to a systematic search. At each discrete sample time, an independent sample of the scan process is taken to determine whether a sample of the signal excess process occurs.

The equations for computing cdp for this "jump process with random sampling" are fairly complicated, although computer implementation is quite straightforward. The equations are given in Appendix B.

Finally, the sensor availability p_a is the probability that the given sensor is operating during the encounter. The concept of availability is intended to cover such matters as equipment failure and repair, and thus to reflect equipment reliability. It is assumed that the cycle for equipment "down time" is long compared to that of a tactical engagement. Hence, the probability p_a is used to reduce cumulative (as opposed to snapshot) probabilities to reflect these events.

The Localization Model

Chapter III develops the localization model used in SCREEN. The localization measures are based on target motion analysis techniques involving the "information flow" Kalman filter. These techniques are described in references [g], [h], and [l], as well as in Chapter III. The basic feature of the approach is to relate the target state vector X and its covariance matrix P = Var(X) to an information matrix $\mathcal{I}$ and an information vector $\mathcal{X}$ via the relations:

$$\mathcal{I} = P^{-1} \quad , \quad \mathcal{X} = P^{-1}X. \tag{S-5a}$$

or, equivalently,

$$P = \mathcal{I}^{-1} \quad , \quad X = \mathcal{I}^{-1}\mathcal{X}. \tag{S-5b}$$

Working in the information domain--i.e., with $\mathcal{I}$ and $\mathcal{X}$ as opposed to X and P--has attractive computational features, most notably that information is "additive:" if prior information quantities $\mathcal{I}_0$ and $\mathcal{X}_0$ are constructed from prior mean X_0 and covariance P_0 by (S-5a) and information quantities ($\mathcal{I}_n$, $\mathcal{X}_n$) are constructed analogously for each of m independent observations against the prior distribution, then the posterior mean and covariance are given by (S-5b), where

$$\mathcal{I} = \mathcal{I}_0 + \sum_{n=1}^{m} \mathcal{I}_n, \tag{S-6a}$$

$$\mathcal{X} = \mathcal{X}_0 + \sum_{n=1}^{m} \mathcal{X}_n. \tag{S-6b}$$

In case the observations are correlated, this formulation must be modified in accordance with reference [l]. The essential change is that to each new observation a "net information gain" is determined and added to current information, so that, e.g., equation (S-6a) takes the form

$$\mathcal{I} = \mathcal{I}_0 + \sum_{n=1}^{m-1} \mathcal{I}_{n,n+1}. \tag{S-7}$$

If the observations are, in fact, independent, then $\mathscr{I}_{n,n+1}$ is just $\mathscr{I}_{n+1}$ as per equation (S-6a). In fact, $\mathscr{I}_{n,n+1}$ involves the same quantities that form $\mathscr{I}_n$ and $\mathscr{I}_{n+1}$; of course, it also is a function of the correlation between the n^{th} and $(n+1)^{th}$ observations. Details behind equations (S-5), (S-6), and (S-7) appear in Chapter III.

The information gain quantities are computed for each sensor, according to its type. Both passive and active sensors provide bearing information. The bearing process is a (λ, σ) jump process (such as is defined for the detection model) whose parameters are a function of signal-to-noise ratio (SNR) and the sensor's beamwidth in the direction of the target. These quantities are part of the sensor's description. The process is assumed to be unbiased--i.e., the true bearing is the mean--and to have a correlation structure of the form required for the use of equation (S-7). Active sensors provide bearing and range information. The range process is also an unbiassed jump process whose parameters are functions of SNR. In addition, the standard deviation of the process is proportional to the true range. Of course, it is assumed to have the required correlation structure. The construction of the information gain quantities $\mathscr{I}_{n,n+1}$ from these data is detailed in Appendix E.

After the quantities $\mathscr{I}_{n,n+1}$ have been computed and $\mathscr{I}$ obtained by equation (S-7), the resulting posterior covariance matrix P may be obtained by equation (S-5b). An ideal localization measure would be obtained by taking the average of this covariance matrix over all possible combinations of screen unit detections; that is to say, an expected covariance matrix. As this involves an almost prohibitively cumbersome computational process, the SCREEN model employs the following substitute.

Let p(n) be the probability that observation n occurs. (This will, in practice, be the product of the snapshot probability, equation (S-3), and the sampling probability, equation (S-4.) Then define "expected" information by

$$\hat{\mathscr{I}} = \mathscr{I}_0 + \sum_{n=1}^{m+1} p(n+1)\, \mathscr{I}_{n,n+1}, \tag{S-8}$$

and then define the "expected" covariance matrix $\hat{P}$ by invoking equation (S-5b) formally. The quotation marks about the word "expected" refer to the fact that $\hat{\mathscr{I}}$ and $\hat{P}$ are not actual probabilistic expectations--see the discussion in Chapter III. The matrix $\hat{P}$ serves as the basis for the SCREEN localization measures. Although it is not the true expected covariance matrix, it is easier to obtain algorithmically and, at the same time, is felt to provide a useful measure of screen localization.

The Target Motion Model

The target motion model in SCREEN is used to develop sample screen penetration tracks which are used in turn to determine cumulative sensor performance measures. The model in fact allows the program user to define several target approach strategies. A general (cumulative) performance measure may be obtained by averaging measures of performance against several such strategies.

The idea behind consideration of several approach strategies is the representation of the full (or as full as possible) spectrum of penetrator tactics against the screen being analyzed. Of course, the tactics followed by a target during an approach depend on a number of factors, such as his position relative to the base task force track, his operating noise characteristics, his degree of knowledge of the screen unit positions and base track, and his assessment of his own vulnerability and the screen's protective capability. In most cases, due consideration of these factors results in a limited number of viable approach strategies, which may vary in sophistication from a "damn-the-torpedoes" flank-speed intercept course to a cautious approach that attempts to detect and evade screen units. Generally speaking, only a few carefully-selected such strategies are considered in an analysis using SCREEN, rather than a large number of arbitrary approach strategies.

A given approach strategy, as modeled in SCREEN, reflects assumptions about penetrator's motion as perceived by the screen. From an analytical viewpoint (the viewpoint of the program), these assumptions include:

(i) the initial location distribution of the target when the approach begins;

(ii) the mean and covariance of the target's course-speed distribution, as a function of time into the approach;

(iii) mean time between independent (random)course and speed selections;

(iv) positive contacts (bearing lines and SPAs) from intelligence sources outside the screen (these will be called "posits" in the sequel); and

(v) marginal constraints which reflect target objectives or restraints on the target's motion.

The target motion model is designed to encompass these five assumptions.

Each target approach strategy in SCREEN is described by a separate target file. The strategy is represented and stored in that file as a multivariate Gaussian distribution for which the state variable is the position of the target at each time step throughout a penetrator-screen encounter. Gaussian distributions are used because of their analytical features. The bulk of the target motion model consists of techniques used to define and modify the target strategy files--that is, the mean vector and covariance matrix of the representative Gaussian distribution--in accordance with the assumptions (i)-(v) above. The underlying intent is to have these techniques reflect operationally reasonable parameters and procedures. We will now summarize the nature of the techniques. Specific details are contained in Chapter IV and Appendices G, H, I, J, and K.

The mathematical process at the heart of the model is a generalization of the Ornstein-Uhlenbeck process, which was originally developed to describe Brownian motion (see references [d] and [e]). The generalized process, developed for SCREEN, is reported here for the first time. It is used to describe the target's velocity and may be described as follows. The target velocity at any time $t > 0$ is given by

$$V_t = \nu_t + \epsilon_t$$

where ϵ_t is a Gaussian random variable with mean 0 and covariance matrix Γ_t; and there is a nonnegative function μ such that

$$\int_t^\infty \mu(x)dx = 0$$

for all t, and if $s \leq t$, then ϵ_s and ϵ_t have a joint multivariate Gaussian distribution with cross-covariance given by

$$\mathrm{Cov}(\epsilon_s, \epsilon_t) = \exp[-\int_s^t \mu(x)dx]\Gamma_s. \qquad \text{(S-9)}$$

The corresponding position process is, of course, obtained by integration.

The process actually employed by SCREEN involves a discretization of the above process: a velocity obtains for each time step and is assumed constant between time steps. The resulting position process is called a Discrete Integrated Ornstein-Uhlenbeck (DIOU) process. The mean velocity vectors ν_t at each time step are chosen by the program user to reflect the basic target approach strategy desired. (The representative track will thus consist of one or more linear track segments.) These choices, together with the chosen initial position and the length of the program time step, determine the (mean) target positions at each time step during the evolution of the penetration. These positions are the components of the mean vector stored in the corresponding target file. The covariances among these positions, which comprise the covariance matrix stored in the file, are derived from equation (S-9) (see text equations (IV-6)). The (Gaussian) distribution for which these are the moments is called the unconstrained prior distribution (UPD).

The UPD alone reflects only the assumptions (i)-(iii) above. Without further conditioning, the target location distributions would be continually expanding, due to the contribution of the random velocity. This would not reflect properly either the target objective of a restricted attack position or other considerations which have the effect of "focusing" target motion (such as passing between two screen units). Therefore, marginal constraints and posits (see assumptions (iv) and (v) above) are used to constrain the UPD. The resulting distribution is called the constrained prior distribution (CPD) if no posits are present; otherwise, the term "modified distribution" is used.

The incorporation of a constraint (a generic term used in this report for a marginal constraint on a posit) into the modified distribution has its simplest expression in the "information domain." That is, let γ_τ and C_τ be the mean vector and covariance matrix of the modified distribution--τ is the number of time steps beyond 0 over which the penetration strategy is studied; C_τ is a $(2\tau) \times (2\tau)$ matrix--and define corresponding information quantities in accordance with equation (S-5a):

$$\mathcal{I}_\tau = C_\tau^{-1} \quad , \quad \mathcal{X}_\tau^{\cdot} = C_\tau^{-1}\gamma_\tau . \tag{S-10}$$

Then (analogously to equation (S-6)) to each constraint there is made to correspond a matrix D and vector d so that, if $\tilde{\mathcal{I}}_\tau$ and $\tilde{\mathcal{X}}_\tau^{\cdot}$ denote the information quantities after incorporation of the constraint,

$$\tilde{\mathcal{I}}_\tau = \mathcal{I}_\tau \ (+)\ D, \tag{S-11a}$$

$$\tilde{\mathcal{X}}_\tau = \mathcal{X}_\tau^{\cdot} \ (+)\ d. \tag{S-11b}$$

Since a constraint will generally pertain to a given time step, the dimensions of D and d will usually be 2 x 2 and 2 x 1, respectively. The summation (+) in equations (S-11) means that D and d are to be added to the submatrices of $\mathcal{I}_\tau$ and $\mathcal{X}_\tau$ respectively corresponding to the time step to which the constraint pertains. (Appendix G contains a more detailed definition of the (+) operation.)

The exact forms of D and d depend on the nature of the constraint involved. Chapter IV presents and discusses these forms in detail. The "additivity of information" mentioned above in connection with the localization model also applies here. One important implication of this is that a constraint may be removed by subtracting the appropriate D and d from $\mathcal{I}_\tau$ and $\mathcal{X}_\tau^{\cdot}$.

The post-constraint mean vector $\tilde{\gamma}_\tau$ and covariance matrix $\tilde{C}_\tau$ may be obtained from $\tilde{\mathscr{I}}_\tau$ and $\tilde{\mathscr{X}}_\tau$ by equation (S-5b):

$$\tilde{C}_\tau = \tilde{\mathscr{I}}_\tau^{-1} \quad , \quad \tilde{\gamma}_\tau = \tilde{\mathscr{I}}_\tau^{-1} \tilde{\mathscr{X}}_\tau . \qquad \text{(S-12)}$$

However, in most SCREEN analyses, τ will be so large that the inversion of $\tilde{\mathscr{I}}_\tau^{-1}$ required by equations (S-12) is computationally impractical. Therefore, the SCREEN model contains a method, deriving from equations (S-11) and (S-12), whereby the quantities D and d may be used to obtain $\tilde{C}_\tau$ and $\tilde{\gamma}_\tau$ directly from C_τ and γ_τ without inverting a matrix any larger in dimensions than D. (Typically, said matrix will be 2 x 2.) This method is described by text equations (IV-12).

Finally, it is desirable from the standpoint of the user to be able to increase the number of time steps over which the screen analysis is done--which is tantamount to increasing the dimension of the UPD--even though constraints have already been incorporated. The SCREEN model is designed so that this is possible without removing and reapplying the constraints. This feature is particularly handy in analyses involving the interactive mode, less so in those for which the batch mode is appropriate.

THE THEORY OF SCREEN

CHAPTER I

THE SCREEN PROGRAM AND ITS USE

This report gives the theoretical basis for the algorithms used in the SCREEN program, which is a program developed for the Chief of Naval Operations (Op-961), designed to evaluate acoustic detection and localization performance of an anti-submarine protective force about high value shipping in a naval convoy or task force scenario. Such a protective force is often called a screen; hence, the name for the SCREEN program.

The need for a program such as SCREEN arises in the analysis of alternative future Navy ship and aircraft weapon system designs, including both the platform itself and the various weapon system components that it carries. Such designs and design improvements must compete for their survival not only against one another, but also against existing systems; furthermore, competition is not confined to a platform type (e.g., airplane, ship, or submarine), but must include cross-platform comparisons. Hence, the need exists for an analysis tool which can fairly compare current and proposed weapon systems, including cross-platform comparisons, and with enough detail to reflect the important design features. SCREEN is such a tool.

In designing a program such as SCREEN, it is necessary to walk a narrow line between oversimplification and excessive detail. The pull toward simplification comes from the desire to make SCREEN into a tool which is usable by the general analyst without discouraging complexity. On the other hand, many design improvements involve technical details which, if literally transliterated into an analysis program, would swamp the analysis in complexity. The problem is to produce a tool that is not difficult to use, but which at the same time reflects the important design features of the weapon systems involved.

The SCREEN program has answered the program design problems by several means, which are spelled out in some detail in the next section. First, specific decisions were made to limit the descriptive parameters. Still, a lot of detail remains which is not normally found in multi-platform analyses. Second, the data preparation allows the user a large range of flexibility as to the level of detail which he desires to pursue: it is possible to create data files very quickly with default or constant values, or they can be constructed in careful detail. Data files, once created, can be used repeatedly in different analyses, without requiring regeneration. Third, once the data files have been prepared, they are easily reviewed and corrected: in most cases, it is possible to verify exactly what the data files say. Fourth, the operation of SCREEN is very simple once the basic data files have been established. Most of the program features exist in a logical structure of program options with a few clear machine prompts to guide the user in specifying what the program should do.

The remainder of this chapter gives an overview of the SCREEN program. In the next section, the program parameters are described, and the level of detail of the program is explained. The final section describes the typical uses of SCREEN. Chapter II describes screen sensor formations, the detection process, and detection performance measures. Chapter III treats localization performance measures and their relation to Kalman filtering target motion analysis techniques. Chapter IV considers the target motion model and describes how target penetration tactics are reflected in the SCREEN program. Appendices and notes contain various background discussions which are too detailed to include in the text. The various notational conventions employed in this report are summarized in the glossary.

A companion to this volume is the SCREEN User's Manual, reference [a]. There, the actual operation of the SCREEN program is described in detail. Hence, only those aspects of the program which directly relate to the theory underlying it will be discussed in this volume.

The Components of SCREEN

In this section, we given an overview of SCREEN, designed to present the general features of the program and the level of detail used. The SCREEN program models an encounter between an attacking submarine and the defensive acoustic screen placed about high value shipping. The study of this type of encounter has been conducted for many years using many different analysis approaches. SCREEN is distinguished from similar types of analysis by the level of detail used to describe the environment, sensor performance, the detection process, the localization process, and by a target motion model whose assumptions may be modified by the presence of contact information and assumed attack objectives.

The basic starting point for the detection process is the classic sonar equation. This is described in detail in Appendix A. The sonar equation, whether for active or passive sonars, is an expression for the mean or expected signal excess at the output of the sonar processing equipment. To obtain an accurate expression for this signal excess quantity, it is necessary to know the level of the emitted signal by the target or active sonar, the loss encountered during propagation through the ocean medium, the total degradation at the receiver due to background noise (including self-noise and other unwanted acoustic interference), and the gain for the processing equipment used. To express these components accurately requires a detailed description of the total detection process.

In past analyses of screen performance, the capability to model these components of the sonar equation was generally limited by the resources available for the analysis task. It was usually impossible--due to the time frame or the resources available--to describe with any accuracy each of these components, and so various approximations were made in order to focus more detailed effort on the actual purpose of the study in question. SCREEN attempts to describe these components in greater detail than previously done in most analyses. Even so, no component of the detection process is described in "ultimate" detail, because even SCREEN is forced to make simplifying assumptions. However, SCREEN is felt to be sufficiently accurate in its description of the components of the sonar equation and of the sensors used in the screen formation, and the SCREEN detection model is believed to incorporate enough detail to reveal the salient properties of the detection process.

We will now briefly summarize the level of detail found in the SCREEN program.

The environment. The environment is described* by specifying propagation loss in tabular form at one* mile intervals out to 120* miles from each sensor. Multiple propagation loss curves are permitted, corresponding to different frequencies, depths of operation, etc. The program can store up to ten* distinct propagation loss curves.

If the corresponding sensor is active, a reverberation curve is also described along with the propagation curve. It is formally similar in appearance to a propagation loss curve. The reverberation curve corresponds to "total reverberation" and is the composite effect of surface, bottom, and volume reverberation.

Associated with propagation loss is a value for omnidirectional ambient noise for the applicable ocean area. Similarly, associated with a reverberation curve is a corresponding active source level, since the reverberation experienced is a function of the source level.

In general, one will expect the propagation loss curve and the reverberation curve for a given sensor to change depending on the target and source depth, frequency of operation, and the operating area. One does not normally expect the propagation loss or reverberation curves to be smooth or monotone but rather irregular in appearance due to such things as convergence zones in the case of propagation loss curves and the various reverberation components in the case of reverberation curves.

The environmental description does not provide explicitly for directional ambient noise. The directional noise field due to the task force itself is treated separately, as described below. It is possible to introduce directional ambient noise in a limited fashion--for example, the vertical directionality can be factored into the propagation loss curve. In general, though, if directional ambient noise becomes an important factor in screen performance, then the SCREEN program should be modified to incorporate this added feature.

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* Throughout this report, the asterisk after a numerical value denotes that the number value is declared in a FORTRAN parameter statement and can be altered if necessary at the cost of recompilation.

Targets. Targets are described in SCREEN in terms of generic types of approach strategy. An intelligent enemy will not barrel straight into its target, particularly if he has some knowledge of the screen disposition. He will, instead, choose from among a variety of screen penetration tactics.

Consider some examples. From a starting position ahead of the screen, a penetrator may choose to approach directly (a forward approach) or he may instead choose to skirt around forward screen elements to approach from the flanks or the rear. He may attempt to pass midway between two active escorts. In the case of Figure II-1 (see Chapter II), he may attempt to skirt behind the forward submarine escorts but ahead of the sonobuoy field (assuming he has knowledge of the approximate location of the various screen elements).

A screen can be tested against a variety of such target penetration strategies, which are described in the target data files. A target file describes a base strategy about which a target distribution is placed. The distribution may reflect either the screen's uncertainty in target strategy or it may reflect the target's position uncertainty relative to the screen unit locations. This distribution is basically the subject of Chapter IV. The actual construction of target data files is deferred to reference [a].

Target radiated noise is expressed as an omnidirectional average value for each sensor detection mode considered in a particular problem. The directionality of radiated noise (or active target strength) is not modeled, nor are other matters such as target depth. These could be inserted in the SCREEN model by moderate reprogramming if the analysis warrants the necessary effort.

Sensor descriptions. Sensor descriptions are quite detailed and can be divided roughly into two parts, one addressing detection information and the other addressing localization information.

Detection. For detection information, a sensor is identified as either being active, passive, or a passive line array. An active sensor is assumed to provide both range and bearing information upon detecting a target. A passive sensor is assumed to provide bearing information (although an omnidirectional sensor can be modeled crudely). A line array is distinguished from other passive sensors by the fact that it provides ambiguous bearing information; namely, its beam response pattern is such that it cannot distinguish between a given bearing and its reflection about the array axis. This fact is primarily used in determining the effect of interfering noise on detection performance.

The beam pattern of a sensor is described by the single parameter, beamwidth. The main lobe is a cosine function of the bearing deviation from the center bearing, scaled to achieve half power (3db down) at the stated beamwidth. Sidelobes are not modeled except that the reflected beam is modeled for line arrays. The beam pattern is used to determine the off-axis effect of interfering noise when the beam is pointing at the target and the interference arrives on the main beam. A more detailed model would incorporate more detailed beam patterns, including sidelobes.

The effect of background noise is handled in three separate ways. First, each sensor is characterized by its ability to discriminate against omnidirectional background noise. This quantity, the directivity index, is applied against the omnidirectional ambient noise number that accompanies a propagation loss curve. Second, the performance of most of the sensors is degraded by noise produced by the sensor platform itself. This is called self-noise. Self-noise is described in the model by a nominal value (the mean effective self-noise) together with a contour which shows the change in self-noise as a function of relative bearing. By means of this contour it is possible to distinguish, for example, between detection on beams which are aimed away from the sensor platform and beams which are aimed toward the sensor platform. It should be noted that the self-noise modeled is the net self-noise after the signal has passed through the beamformer--i.e., the directivity of the sensor is included. This quantity is sometimes called L_E.

The third component of interfering noise is noise due to other ships in the screen. Every high value unit and sensor platform is permitted to radiate noise interference at the frequencies or in the frequency bands of the screen sensors. The model calculates the net effect at the receiving array of this interfering noise by tracing the interference through the propagating medium and through the simplified beampattern of the receiving array described above.

The description of environment, targets, and sensors contained in SCREEN is not as detailed as could be developed, but it is sufficiently detailed to allow one to investigate such things as mutual interference, performance in different environments, the use of mixed detecting modes by the various platforms, etc.

Random signal excess. All the parameters we have described thus far have to do with the mean signal excess, i.e., the components of the sonar equation. In addition, there are random fluctuations in the signal excess which must also be described because they enter heavily into the question of the detection event. The model assumed for these random fluctuations is the (λ, σ) jump process in which the signal at an array is assumed to have a Gaussian random component with standard deviation σ which assumes new independent values at exponentially distributed intervals at the rate λ per hour. This jump process is described in a number of places, for example in references [b] and [c]. The version used in SCREEN is presented in detail in Chapter II.

The (λ, σ) jump process is one of the simplest models one can have for random fluctuations in the signal which takes into account the time correlation in those fluctuations. It only involves two parameters. Various modeling efforts in the past have realized that the actual signal process is undoubtedly more complicated than a simple (λ, σ) model, although nobody can describe it accurately. In fact, every component of the sonar equation undoubtedly has its own process and in addition, there is some degree of correlation along the various processes induced by the common medium, or other phenomena. Thus, the model chosen for the detection process in the SCREEN program is relatively simple among the class of models that have been used in the past to describe that process. Nonetheless, it is believed to be adequate for the present purposes of screen analyses. The time may come when it is desired to put more detail into the stochastic process, but the analyst is cautioned to note that it is virtually impossible, due to the small data base and the high cost of conducting operational experiments, to actually estimate more than one or two parameters for the random process, and so there is some question as to whether greater detail is merited.

In addition to the individual sensor detection process, one must describe how a sensor group or sensor screen combines its efforts to accumulate detections. We will not enter into this here other than to say that a very simple model for correlation between sensors has been used, a model which is also used in other analyses, notably recent work on the Pacific coast in describing sonobuoy performance. The reader is referred to the detailed discussion of this model in Chapter II.

Localization. It has been noted by various analysts that evaluating the detection performance of a screen is not enough: the real need is to describe localization as well as detection. The motivation behind this is that if detection capability becomes the ultimate measure of effectiveness (MOE), then sensors that are able to detect at extremely long range but perhaps without performing much localizing, will appear to be star performers--whereas, in fact, they may provide information that is tactically unusable in prosecuting a contact.

One of the difficulties in evaluating the localization performance of a screen is that in the past, localization measures have required extremely detailed descriptions of detection events and the tactical responses to those events. Basically, the only way to "really" analyze localizations is by numerous Monte Carlo replications in each of which the tactical interaction is played and the final result of an encounter is then summarized. To evaluate localization for a screen over a wide variety of target penetration tactics (or to develop localization "coverage" maps as described in later chapters) would require a prohibitive number of Monte Carlo replications in order to smooth out the normal variations due to the Monte Carlo sampling process itself.

The SCREEN program represents to our knowledge the first attempt to provide a screen localization measure which avoids excessive computational complexity. This approach, based on a Kalman filter target motion analysis (TMA) technique, is described in Chapter III. Because this capability of the program has not yet been exercised to any extent, it is impossible to say whether the SCREEN approach is the ultimate answer to localization performance evaluation. However, it does produce measures which are easily compared and does relate in a crude way the ability of a screen to localize targets.

To describe the screen localization performance, it is necessary to provide information about the statistical properties of every sensor, specifically: 1) the standard deviations in the solution range and/or solution bearing as a function of the mean signal-to-noise ratio, and 2) information regarding the mean time between independent bearing and range samples. As remarked before, the attempt has been made to keep these statistical parameters as few in number as possible. Those chosen appear to be the minimum required to obtain a reasonable description of screen localization performance.

Level of detail: a final comment. It can be seen from the discussion above that the amount of information required to describe a sensor completely is fairly extensive--even though it can be at the same time argued that every type of information could have been expanded in detail to an almost limitless extent. As a companion to the SCREEN program, it will be necessary to provide standard sensor descriptions to a level of detail hitherto unknown except in specialized engineering studies. However, once this information has been collected and compiled, the subsequent use of such information is quite straightforward; this is the power of the SCREEN program. With time--and if the SCREEN program is used--it is expected that standard data bases will be established for the standard SCREEN components and standard environments, and that these details of analysis can be henceforth taken for granted as the first step in other SCREEN analyses.

The Uses of SCREEN

In this section, the uses of SCREEN are examined. The discussion begins with a survey of the types of problems that can be addressed. This is followed by comments on creation of a defensive screen. Finally, some overall remarks are made on the possible analyses that can be performed with the SCREEN program.

The classes of problems addressed by SCREEN. The SCREEN program is nominally designed to aid in the analysis of an ASW defensive screen placed about high value shipping. However, the program design lends itself to a broader range of applications.

At one extreme, the SCREEN program can be used to analyze isolated tactical problems. Examples of these are:

(1) One-on-one engagement. A screen consisting of one sensor platform (possibly with several sensors) can be the basis for an analysis of a one-on-one engagement when a sufficiently short time step is chosen so that specific approach/evasion maneuvers can be modeled for both the sensor platform and the target. The Kalman filter target motion analysis built into the localization portion of SCREEN can estimate the refinement of a fire-control solution. Performance in different environments can be analyzed. The user can analyze performance in different environments as well as the use of several sensors simultaneously, including use of active sensors.

(2) One-on-many engagement. The SCREEN program can assess the performance in detection/localization by a single platform against multiple simultaneous targets considering the sensitivity to environment, the effect of relative bearings holding contact, etc.

(3) Many-on-one, many-on-many engagements. The SCREEN program can also analyze various tactical engagements between several search platforms against one or several targets including differing degrees of task force communications. (This would probably require the interactive rather than the batch option of SCREEN; the distinction is addressed in a later subsection.)

(4) Use of reactive screen elements. The SCREEN program can examine an engagement between a reactive screen platform (such as a deck-launched weapon or search platform) and a target. This would require two separate runs of SCREEN: one run to model the general task force screen or the performance of the host platform's search--which is used to generate the detection and localization opportunities (location and size) for the reactive forces--and the second run to model the reactive units' performance (redetection and localization). Undoubtedly, the second run would use a much tighter time step and spatial resolution than the first run would. This type of analysis would aid in proper placement of reactive forces within a screen.

(5) Placement of screen units and modules. The effects of task force noise, the possibilities for mutual exchange of information, and the placement of localizing reactive forces can be analyzed. A typical example of this analysis is placement of a flanking sonobuoy field, taking into consideration the effect of task force noise interference.

At the opposite extreme, the SCREEN program can assist in analyzing the performance of a task force in an extended scenario. A Hunter-Killer (HUK) Group evaluation may extend over days and include a number of environments. The evaluation of task force transit vulnerability includes different environments which imply different defensive screen configurations and different attack tactics. The SCREEN program can be used ashore by fleet staffs to plan such engagements, as well as used on-site in real time to provide interactive assistance in planning, execution, and assessment.

Designing a screen: background. In the remainder of this section, we will discuss the thought processes that are involved in designing a defensive screen and how they interface with the use of SCREEN. Any analysis involving the SCREEN program will utilize some basics in the screen design and perhaps use the SCREEN program to modify the design from time to time.

A task force ASW screen is primarily designed to counter the likely penetration tactics of opposing submarines. The opposition presumably has the mission to attack the high value units in the task force. Possibly, the attack of the escort units themselves is a secondary objective. Success for the target may be measured by penetration to a designated weapon launch region. Success as defined by the screen is to minimize the likelihood of this event.

Before we discuss the question of what measure of effectiveness the screen should use in assessing its performance, we note that a real encounter between a defensive screen and opposing submarines is a two-sided game where each side has partial information about the other side and each side is operating under certain constraints that limit its options. Theoretically, optimal screen design should be a solution to this two-sided game. In practice, however, the solution possibilities are so huge that no computable solution to the game is possible. Hence, the usual procedure is to postulate reasonable tactics, evaluate them, and modify them thus searching for configurations that yield improved performance. This is an inexact science, subject to errors in application.

The smart target. Usually we assume that the penetrating target has some information about the screen composition and that it will attempt to evade screen units to the extent of its knowledge subject to the primary objective of accomplishing its mission. This being the case, the task of the screen designer is to produce a screen which is sufficiently nonporous that the anticipated intelligent target will find it difficult to exploit gaps in coverage. Such gaps might appear, for example, between an outer screen and an inner screen. The penetrating target could exploit a gap by skirting around the outer units and then passing ahead into an advantageous position. Of course, for each modified screen configuration, a new set of intelligent target penetration tactics is probable. At some point, small changes in tactics produce negligible changes in screen configuration and the user considers the screen to be fixed. It is important to realize, however, that in reality this is a dynamic two-sided problem and that design of an optimal screen against fixed penetrating tactics may result in an optimistic assessment of performance when in fact the penetrators may be able to adapt their tactics. Conversely, the design of optimal penetrator tracks against a fixed screen configuration may result in an optimistic assessment of the penetrator's capability when, in fact, the screen commander may be able to adapt his tactics.

Operating constraints. In addition to considering the two-sided game with the target, the design of a screen must take into account various operating constraints. The next few paragraphs are some examples.

Perhaps the most obvious operating constraint is that the speed of advance dictated by the PIM (Position and Intended Motion) which the screen defensive units may be obliged to maintain may not be optimal for their search functions. We do not wish to discuss the question here, but experience indicates that aircraft carriers (CVs) do not slow down to give screen units opportunities to search for or prosecute attackers.

Thus, sometimes sensor platforms must engage in sprint and drift or similar search tactics which result in incomplete or less than optimal coverage of the search area. Sometimes the penetrating target submarine can take advantage of these gaps in coverage, particularly after it detects the noise radiated by the search platform during the sprint phase.

Other constraints on the screen design include collateral duties of some of the search units, especially in the case of surface platforms in the inner screen about a CV. Such screen units may have to perform point defense against cruise missile attacks and retrieve downed aircraft and pilots lost at sea on top of the usual screen detection functions. These other duties not only detract from the general search, but also force the escorts to maintain a position very close to the high radiated noise of the high value units, which may further degrade their detection performance.

Another common source of constraint are the operating characteristics of the search units. Helicopters, for example, can maintain sonobuoy fields only within a limited distance from the launching platform. This is due to the endurance of the aircraft. This limitation again may result in sonobuoy fields being placed in high background noise regions and in portions of the search area where their contribution is less effective.

As a final example, communications requirements may enforce another kind of constraint on screen units. Direct support submarines are an obvious example of screen units whose performance may face several limitations because of the communications requirement. VP aircraft may not be available because of assignment to communications relay.

Screen effectiveness measures. It has already been indicated that the SCREEN program contains both detection and localization measures. Historically, analysis of screens has been restricted almost exclusively to detection because it has been difficult in the past to obtain general localization performance measures apart from limited simulations. Such simulations can perhaps assess the efficacy of a given fixed screen but they lack in flexibility. The SCREEN program allows the potential for evaluating screens from the viewpoint of their localization capability.

The detection and localization capabilities of different types of screen units vary widely, according to the type of unit. Passive escorts which utilize the towed arrays frequently are characterized by very long range detection capability. However, the ability of these sensors to localize the target is often very poor. In addition, the passive platforms are not necessarily quick-response platforms, and so even if they could localize the targets, they might not be in a position to launch a weapon. Other screen elements, such as helicopters using dipping sonar, are extremely effective in localizing contacts once a detection has been made, but are fairly ineffective in achieving detections if the search area is very large. Active sonars generally provide good detection and localization capability but suffer the disadvantage that the target can almost always hear the active sonar before the sensing platform can detect the target. Hence, there is the serious possibility that the target may evade active screen elements.

A screen well-designed in terms of both detection and localization performance would place its detecting units so that a target would be localized with high probability by the time it arrived at its weapons-launch position. If continuous full coverage is not possible, then at least early warning might enable those platforms which are better able to react to a contact to position themselves in order to prosecute. This type of mutual assistance among search units may provide the best overall screen design. While it is difficult enough to design a static screen if the two-sided game is allowed, a dynamic screen in which screen units react to their own detections is probably an order of magnitude more difficult. Such questions lead to challenging and stimulating analysis.

Real-time vs. batch design. The logical approach to designing a static screen is with the batch mode of operation of the SCREEN program. Candidate screen designs can be played against appropriate penetrating tactics and the results analyzed, using as the measure of performance, for example, the probability of a successful attack by the penetrating submarine. That is, the screen which minimizes this probability might be considered the best screen.

In the past, the SCREEN program has also been used in real-time analysis to plan a dynamic reactive type of screen. In the work of T. L. Corwin at COMSUBPAC which formed the basis for SCREEN, the programs were used during actual exercises involving a task force with direct support submarines; this reactive screen design was the principal application of the early versions of the program. Further remarks along this line are made in the Preface and on page 14.

Designing penetration tracks. The design of penetration tracks for the targets is, of course, simply the other side of the two-sided game. The principal parameters in this design involve the target's understanding of the screen it is trying to penetrate.

In recent analyses of submarine alternatives, the SCREEN program was used to analyze penetration tactics against screens consisting of both passive and active screen elements. The considerations involved in the design of target penetration tracks included such things as:

(1) the relationship between target speed and radiated noise (which, of course, affected the performance of passive sensors in the screen and the set of feasible approach tracks);

(2) the requirement imposed on the penetrating submarine to perform a rough localization of the screen elements in order to design evasive maneuvers;

(3) a total time budget during which the approach to attack had to be achieved;

(4) the distribution of initial arrival from which the task force penetration was assumed to commence;

(5) a selection among alternative weapons with attendant launch range requirements;

(6) a selection among alternative launch positions for a given weapon; and

(7) the ability to maintain sufficient reserve propulsion capacity to escape after the initial attack (this is primarily applicable to diesel submarines).

This analysis of target penetration tactics was probably more elaborate than would typically be the case in most screen analyses, but it turned out that the care involved in target track design was needed to show the true effects on performance of the differing candidate submarine capabilities. Reference [t] is the summary report of these analyses.

It should be noted that the target track data designed for SCREEN use include uncertainty information in the placement of the target track. This uncertainty usually reflects the screen's uncertainty as to the target's location on an assumed approach tactic. It can also reflect the target's assumed uncertainty in the location of screen units: the placement of his track on a relative motion board about the screen is subject to error due to his uncertainty concerning the location of screen units.

Performance assessment. The performance assessment that is done by the SCREEN program serves two purposes: viz., designing screens and/or penetrating tactics, and assessing the performance of these well-designed screens. As mentioned before, there are two general measures of performance provided--detection and localization--and each of these has both a short-term (snapshot) and a long term (cumulative) level of performance. We will discuss each of these in turn.

Snapshot coverage maps. The short term performance of screens is used to construct so-called coverage maps which basically provide the pictorial representation of how well the screen searches out the coverage area of its sensors. For both detection and localization coverage maps, the basic setup is as follows. A gridwork is superimposed over the operating area selected. For an individual sensor or sensor group, it would be a region about that sensor or group; it could also be the coverage area about the entire screen. (The size of the coverage area is an input provided by the user.) At every gridpoint of this coverage area, the performance of the sensor, sensor group or screen is evaluated, conditioned upon the presence of a submarine target at that gridpoint, and a corresponding number is placed there.

Typical coverage maps are shown in Figures II-2 and III-1. The detection map in Figure II-2 is basically derived from the sonar equation. Postulating a target at each of the gridpoints, the detection probability for the screen is evaluated and a number is placed on the gridpoint which is ten times the probability of detection computed for that gridpoint. A blank (no entry) indicates that the detection probability was less than .05. A star indicates that a value greater than .95 was obtained and a numeral between one and nine indicates that a value within .05 of that numeric quantity was obtained. By superimposing coverage maps for sensors or by calculating coverage maps for complete screens, it is possible to see at a glance where the high coverage areas and low coverage areas exist. As a first step in design for optimal coverage, the screen units would be shifted about so as to provide a "pleasing" appearance of the corresponding coverage maps. We have deliberately avoided stating what "pleasing" is because it may be a function of a number of things. One may, for example, choose to allow gaps in detection coverage maps in order to provide overlapping coverage which would give better localization. One might also desire to provide increased coverage in the regions of most likely approach tracks and provide only limited coverage where the approach tracks are less likely.

A given coverage map is applicable to a specific set of radiated target noise levels. Since different approach tracks may entail different target noise patterns (because they entail different speeds and hence different radiated noise), it may be necessary to produce several coverage maps for different radiated noise levels in order to obtain a good composite coverage picture.

Localization coverage maps are analogous to detection coverage maps except that they provide information about snapshot localization. Since the snapshot localization maps consider short periods of time, localization is only achieved if there is an opportunity for a cross-fix between passive detection sensors or if there is a possibility for active detection (in which case, both range and bearing are obtained and, therefore, a localization occurs immediately). In the design of a screen, the localization snapshot maps may be useful if it is believed that one may capitalize on the opportunities for cross-fixes or active coverage in placing active forces or achieving attacks. Implicit in this is the requirement for communications, which has been mentioned briefly in an earlier subsection on operating constraints.

Cumulative measures. The ultimate performance measures for the screen are the cumulative performance measures. Snapshot coverage maps are only an indication of detection and localization at one fixed time and thus do not take into account the kinematic aspects of the problem, which are important in the final analysis.

There are two types of cumulative detection measures. One is based upon the calculation, as a function of time, of the cumulative probability of detection against specified target penetration tracks. This probability can be used to calculate such measures as the probability of detecting a target prior to the arrival of the target at an attack launch position. Such a cumulative detection measure would be used in batch mode of operation.

In addition, it is possible to use the cumulative measures in the dynamic (reactive) design of a screen by examining the posterior distribution of targets given search by the screen with no positive detection results. In this circumstance, the region where the coverage was heavy would have been well searched and therefore less likely to contain the target, given that no detection was in fact observed. On the other hand, lightly covered areas would tend to be more likely spots for targets to be. Examination of these posterior maps may reveal target "hotspots," i.e., localized areas on the posterior map where the posterior location probability is significantly higher than elsewhere on the map. When the SCREEN program is operated interactively, it is logical to place reactive forces so as to cover these hotspots as they develop.

In actual use of a related program, DENS, in the Pacific, these hotspots arose in a number of tactically significant ways. Frequently, parts of the planned coverage area of a screen would remain uncovered due to such happenstances as unanticipated equipment outages. The effect of these outages would be revealed in the posterior maps generated once these outages were incorporated into the screen evaluation. By examining these posterior maps, it was possible to take collective action to search some of the gapped coverage area. By this means, it was possible to partially recover from system malfunctions (sonar outages, screen units out of position, etc.) in the course of the dynamic evolution of the screen.

Cumulative localization is in essence a Kalman filter TMA using the assumed target tactic as an input to the Kalman filter. As time evolves, it is possible for a passive screen to develop a very accurate target localization, and this is reflected in the localization measure. Thus, it is in principle possible to design a passive screen about a task force so as to localize a target by the time it penetrates to a predesignated critical region about the high value units. It would be appropriate then to place reactive forces or other attack weapon systems so that they could prosecute localizations achieved by the outer screen elements. To the best of our knowledge, no screen to date has been designed using this type of analysis, and so the true value of the SCREEN program in support of such a design is untested at this time (late 1979).

During some recent analysis, the principal use of the SCREEN program has been in the batch mode of operation, analyzing alternative designs of screen and penetrating submarine platforms. It is expected that the batch mode will probably be the principal use of the SCREEN program. However, we emphasize that it is designed with batch or interactive use in mind, and the theory has been modified to allow for certain features one needs when operating the program interactively in real time. One such feature is the capability to incorporate contact information on the targets as they might be received by the on-scene commander. Such contact information may be delayed in arrival, may be altered at subsequent time steps based on new information, etc. The capability to modify target processes to reflect these realistic considerations is one of the strong points of the SCREEN program.

CHAPTER II

SCREEN FORMATIONS AND THE DETECTION PROCESS

This chapter outlines the nature of a screen formation, discusses the model used in SCREEN to represent the detection processes of the sensors in a screen, and derives the detection performance measures computed by the program.

Screen Formations

This section identifies the basic components of a screen, indicates the sort of principles which govern its formation, and describes how these are represented by the SCREEN program. Figure II-1 illustrates a typical screen formation which might be the subject of a SCREEN analysis. It is only a sample, in that not every such formation needs to have all the components or the exact geometry illustrated in this figure.

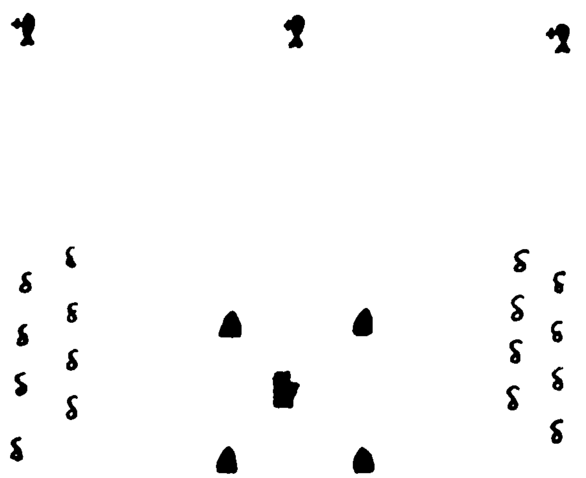
Position and intended motion (PIM). The base motion of the task force is described by the PIM. The PIM course and speed determine the base track of the task force. The PIM coordinates are selected for convenience; they might describe the conceptual center of the screen, or they may coincide with the coordinates of one of the protected units. The PIM coordinates help to define the center of various maps which the SCREEN program is capable of producing.

High value units (HVUs). The purpose of a screen is to defend HVUs. In Figure II-1, the carrier plays the role of the HVU. The particular nature of an HVU is not important; indeed, HVU need not be synonymous with "carrier," since other ships--e.g., oilers, merchant vessels under protection--may be classified as HVUs. Evidently, a screen may--and generally will--contain more than one HVU. For SCREEN purposes, the HVUs represent the objectives of attacking submarines, as well as possible interfering noise sources for the screen's defensive sonars. The motion of the HVUs is assumed to be the same as the task force PIM.

Sensor platforms. The screen itself consists of one or more sensor platforms, each of which in turn hosts one or more sensors. In general, several distinct types of sensor platforms may be included.

FIGURE II-1

A TYPICAL SCREEN FORMATION



LEGEND

= submarine

= surface escort

= sonobuoy

= carrier (HVU)

In Figure II-1, for example, surface escorts surround the carrier--they are placed so as to serve a number of different functions, including plane guard (retrieving plane crews of downed aircraft) and cruise missile defense. Their proximity to the carrier may degrade their passive sonar performance, because of interfering noise emanating from the carrier.

The figure also shows two flanking sonobuoy fields. These may be placed and monitored by helicopters operating from the carrier deck. Placement of these fields in the screen may be determined by several competing objectives:

(1) To be removed from the sound field of the carrier and its surface escorts, it is desired to place the sonobuoy fields as far from the task force center as possible.

(2) Servicing of the buoys by the helicopters implies a maximum range from the carrier for field placement, due to helicopter endurance limitations and task force motion.

(3) The desire to detect targets with enough advance warning to allow attack before coming under fire makes placement away from the carrier desirable.

Submarine escorts in the forward portion of the screen illustrated in Figure II-1 are intended to provide advance warning of approaching submarines. They may also be placed to take part in an attack on any penetrators.

The various platforms have physical capabilities and limitations that affect screen performance. Sonobuoys, for example, are stationary in the water. As a result, a buoyfield will eventually fall behind the task force PIM. At some point, the field must be relaid in a position forward of the old field. Thus, the coverage provided by the field has a cyclic nature, sometimes ahead of and sometimes behind PIM. As another example, the submarine escorts may have speed constraints imposed by their sonar suites. That is, it may be necessary for them to slow to a speed less than PIM in order to be able to detect at long range. This means that a submarine escort will generally be required to "sprint and drift," alternating sprint cycles to overtake PIM with drift cycles that lag PIM.

The foregoing is intended to demonstrate the sort of screen formation and assumptions about it which may be analyzed using SCREEN. With this as background, the representation of a screen's sensors by the program may now be considered.

Sensors. A sensor is a fundamental entity used in the search (detection and localization) process of the screen. A sensor consists of all components of detection gear that contribute to holding contact on a target, including:

(1) the sonar array,

(2) the beamformer,

(3) the signal processor,

(4) the display device and display mode,

(5) the detection/classification mechanism, and

(6) the localization mode.

All controllable parameters for these various devices are assumed to be specified in advance of any search. In the case of the SCREEN model, they generally will be program inputs or else indirectly affect input parameters. Examples of such parameters are:

(1) for the sonar array--sensor depth, array geometry;

(2) for the beamformer--the method of phasing and shading used, self-noise discrimination;

(3) for the signal processor--its integration time, whether it operates narrowband or broadband, and its analysis bandwidth;

(4) for the display device--the frequency band displayed and the degree of sensitivity, scan rate (number of bearings displayed);

(5) for the detection mechanism--the specific frequency of interest, whether the mechanism is aurally, visually, or automatically activated; and

(6) for localization--Kalman filter parameters (see Chapter III).

An example of a sensor, then, would be a DIFAR sonobuoy at an operating depth of 300 feet, used with an Advanced Signal Processor (ASP) and an Automatic Line Integration (ALI) display with a five-minute integration time, set to detect a specific narrowband tonal.

Each possible frequency of interest corresponds to a "detection mode," which is specified in SCREEN by a "noise index." A target (screen penetrator) may radiate in a number of detection modes, but a given sensor may search in only one. That is, each sensor is assigned one and only one detection mode. Up to 10* distinct detection modes may be handled simultaneously in one SCREEN run.

Sensors in SCREEN are of three types: active, passive, and passive line arrays. An active sensor is one which emits a signal and provides both bearing and range information about a target upon detecting it. A passive sensor, on the other hand, does not emit a signal and provides only bearing information. A line array is distinguished from other passive sensors by the fact that the array, due to the nature of its beam response pattern, cannot distinguish between a given bearing and its reflection about the array axis.

A single search unit may require several sensors (in the terminology of this report) for its description in SCREEN. For example, if a submarine's narrowband sonar is used to attempt to detect any of several distinct tonals, then each tonal search is viewed within the SCREEN model as a distinct sensor, with each tonal corresponding to a detection mode. Moreover, several sensor types may operate on a single sensor platform. For example, in Figure II-1, each surface escort might operate an active/passive hull-mounted sonar as well as a towed passive line array. (Note that at least three sensors would be required per escort, in this example.) Thus, to use SCREEN to model a screen will, in general, require more sensors than there are sonars or search platforms in the actual screen. This leads naturally to the concept of sensor groups.

Sensor groups. When the screen is set up for analysis by SCREEN, the sensors are assigned to sensor groups. The prime purpose for defining sensor groups is to identify logical collections of sensors which may be assumed, for purposes of performance evaluation, to operate in a "correlated" fashion. For example, the various detection modes of a narrowband sonar could be viewed as a group. For other examples, the sonars and processors of a given search platform could be viewed as a group, as could also the sensors in an entire sonobuoy field. In general, then, groups are collections of sensors whose detections are or may be correlated, whereas the groups themselves are stochastically independent of each other. The nature of the correlation and other distinctions between groups and sensors, will be given later in this chapter, when group detection performance measures are discussed.

It is worth noting that sensor groups also can play a role in the functional operation of SCREEN. The sensors (or, more precisely, the sensor parameters) for a given SCREEN run are stored in a "sensor file." Being able to group sensors eases the mechanics of constructing such a file by taking advantage of the capability, built into the program, to copy entire groups of sensors by simply stating the position of a particular sensor (the "kingpin") of the group. For example, if one of the vertical rows of four sonobuoys in Figure II-1 were defined as a group with the top buoy as the "kingpin" (note that this group may involve a multiple of four sensors if each sonobuoy has several associated sensors), then the remaining three rows of buoys could be defined by "copying" the first row at the kingpin positions represented by the top buoys in each row. This use of sensor groups is actually less important than the use for correlation of detection performance. The reader is referred to the SCREEN User's Manual, reference [a], for further discussion. In typical uses of SCREEN, "sensor files" are established and then reused repeatedly without being rebuilt.

The SCREEN Detection Model

The detection events depicted in SCREEN are dependent upon signal excess histories governed by the model to be presented in this section. The purpose of the model is to develop two basic detection performance measures for a screen and its sensors: a snapshot detection probability and a cumulative detection probability. Snapshot detection probability is the probability that a given sensor holds contact on a target at a particular point in time. The "snapshot" nature refers to the fact that factors such as sensor motion, target motion, and temporal relaxation times (time intervals between independent samplings of random signal excess) are not significant contributors. These factors do become significant in the computation of cumulative detection probability, which is the probability that a target will be detected during some extended period of time. The SCREEN detection model for sensors will be described in the context of describing these two performance measures.

The detection process; integration time. A sensor achieves detections by processing the signal received from a target. The usable signal, expressed in decibels, is called the signal excess or signal-to-noise ratio. The mean value of signal excess is a function of time, given by the sonar equation which is described for both passive and active sensors in Appendix A.

The detection mechanism which is modeled in SCREEN is an energy detector. The total received energy (signal plus noise) is integrated over time and compared to the expected energy that would be received if noise alone were present. A detection is declared when this ratio exceeds the detection threshold. In some mathematical descriptions of the detection process, an attempt is made to model the thresholding process, and then derive appropriate expressions for the probability of detection based on the assumed model. For our purposes, such an approach would lead to an undesirable side excursion into the problems of calculating snapshot probability because of non-linearities that are implicit in ratio thresholding and the use of logarithmic parameters. To avoid this, we will simply define snapshot probabilities in terms of the expected value of the total received energy, and omit the detailed modeling of the thresholding process. A justification for this approach is that the mathematical results reduce to the standard thresholding results for "instantaneous" detections, in the limit as the integration time for the received energy approach zero. We will now give a mathematical model for this mechanism.

Define a detection function, $\mathcal{D}$ as follows:

$$\mathcal{D}(t) = \frac{\int_{-\infty}^{t} (I_S(s)+I_N(s))dW_t(s)}{E[\int_{-\infty}^{t} I_N(s)dW_t(s)]} - 1, \qquad \text{(II-1)}$$

where:

$I_S(s)$ = intensity of target signal at the beamformer output.

$I_N(s)$ = intensity of interfering noise at the beamformer output.

$W_t(s)$ = weight function.

E[·] denotes the expected value. The integrand in the numerator of equation (II-1) is the total received signal, including noise. The denominator is the expected contribution due to background noise and serves as a reference level. The weight function reflects the type of energy integrator used. In the SCREEN model, the weight function is given by

$$dW_t(s) = \frac{1}{\omega} \exp\left(- \frac{(t-s)}{\omega}\right) ds, \qquad \text{(II-2)}$$

for $s \leq t$.

The quantity ω is known as the integration time for the sensor in question. Note that ω is given by:

$$\omega = \int_{-\infty}^{t} (t-s) dW_t(s). \qquad \text{(II-3)}$$

The weight function given by equation (II-2) is called "exponential decay" and is very commonly used with digital signal processor. Another common weight function is the "moving window," which is the uniform distribution on $(t-\omega, t]$:

$$dW_t(s) = \begin{cases} \frac{1}{\omega} ds & \text{for } t-\omega < s \leq t \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(II-4)}$$

A true "instantaneous" detection mechanism would have $W_t(s) = \delta(t-s)$, where $\delta(\cdot)$ denotes the Dirac delta function; so

$$\int_{-\infty}^{t} \phi(s) dW_t(s) = \phi(t) \qquad \text{(II-5)}$$

for a general function ϕ.

Snapshot detection probability. Since the expected value of an integral is the integral of the expected value, the expected value of equation (II-1) is easily seen to be:

$$E[\mathcal{D}(t)] = \frac{\int_{-\infty}^{t} E[I_S(s)]dW(s)}{\int_{-\infty}^{t} E[I_N(s)]dW(s)} . \tag{II-6}$$

The snapshot probability of detection at time t, p(t), for the sensor whose detection mechanism is given by (II-1) is defined to be

$$p(t) = \int_{-\infty}^{10 \log_{10} E[\mathcal{D}(t)]} n(y;RD,\sigma^2)dy, \tag{II-7}$$

where $n(\cdot\,;RD,\sigma^2)$ is the (one-dimensional) Gaussian density function with mean RD and variance σ^2 (see equation (H-1)), RD is the sensor recognition differential (in decibels), and σ is the standard deviation of the random component of signal excess.

The combination of equations (II-2), (II-6), and (II-7) leads to a very compact iteration when time is discretized. If J_S and J_N denote the numerator and denominator of (II-6):

$$J_S(t) = \int_{-\infty}^{t} E[I_S(s)]dW(s) \tag{II-8a}$$

$$J_N(t) = \int_{-\infty}^{t} E[I_N(s)]dW(s), \tag{II-8b}$$

then for δt sufficiently small:

$$J_S(t+\delta t) \cong e^{-\delta t/\omega}\, J_S(t) + (1-e^{-\delta t/\omega})\, E[I_S(t+\delta)] \tag{II-9a}$$

$$J_N(t+\delta t) \cong e^{-\delta t/\omega}\, J_N(t) + (1-e^{-\delta t/\omega})\, E[I_N(t+\delta)] . \tag{II-9b}$$

The values of $E[I_S]$ and $E[I_N]$ are obtained directly from the signal and noise components of the sonar equation. Note that in the limit as $\omega \to 0$,

$$10 \log_{10} E[\mathcal{D}(t)] = 10 \log_{10} E[I_S(t)] - 10 \log_{10} E[I_N(t)].$$

which is the sonar equation. Here $E[I_S(t)]$ is target strength minus propagation loss, and $E[I_N(t)]$ incorporates the noise terms. Thus, equation (II-7) reduces to the standard method of computing instantaneous detection probability given, e.g., in reference [c].

The snapshot detection process--single sensor. Each sensor in the task force screen has its own detection mechanism of the form described in the preceeding paragraphs. Thus, a sensor j has probability $p_j(z(t), t)$ that it holds contact at time t on a target located at z(t). Because of the integration over the past signal history, as shown in (II-1), this probability depends on the target and sensor parameters at all times up to t. However, it is usually considered that the integration time ω given by (II-3) is small compared with the time span of the total engagement, so that for practical purposes, $p_j(t) = p_j(z(t), t)$ depends on the behavior of sensor and target only in the immediate vicinity of time t; hence, the term "snapshot" is intended to convey the idea of a short time interval around time t. In future references to snapshot probabilities, target motion is suppressed. Snapshot probabilities are distinguished from cumulative probabilities in that the latter involve longer term time correlation properties, including target motion, that will be discussed in the next subsection.

The concept of snapshot probability leads naturally to a representation of the detection coverage of a sensor. A snapshot coverage map shows the snapshot detection probability $p_j(z, t)$ as the target location z varies over the region surrounding the sensor j. To compute a coverage map, a gridwork is placed over a map centered at the sensor, a (hypothetical) target is placed at each gridpoint, and the snapshot probability is computed over a timespan which is on the order of the sensor integration time ω. The result is a map which indicates the detection coverage of that sensor at the specified time. Figure II-2 shows a typical snapshot coverage map produced by SCREEN. Table II-1 defines the numerical symbols used in this figure.

Note that other snapshot detection performance measures may be derived from the snapshot detection probabilities. For example, the expected number of contacts held on the target by the entire screen at time t is given by

$$E_c(t) = \sum_j p_j(t), \qquad \text{(II-10)}$$

where the sum is taken over all the sensors in the screen. Analogous measures for groups of sensors may be obtained by restricting the sum in (II-10) to the sensors in the group or groups in question.

Cumulative detection probability--single sensor. The cumulative detection probability (cdp) for a sensor j is the probability that a target following a given track for a time interval $t_0 \leq s \leq t$ is detected by the sensor at some time during that interval. In the discussion of cdp which follows, the sensor, the target track, and the initial time t_0 will be viewed as fixed. Furthermore, since the SCREEN detection model uses discrete time, the discussion will be in that context. Hence, cdp will be viewed as a function of a sequence of times t_i, $i = 0, 1, 2, \ldots$; typically, $t_i = t_0 + i(\delta t)$, where δt is

FIGURE II-2

TYPICAL SENSOR DETECTION COVERAGE MAP (PDSEN OPTION)

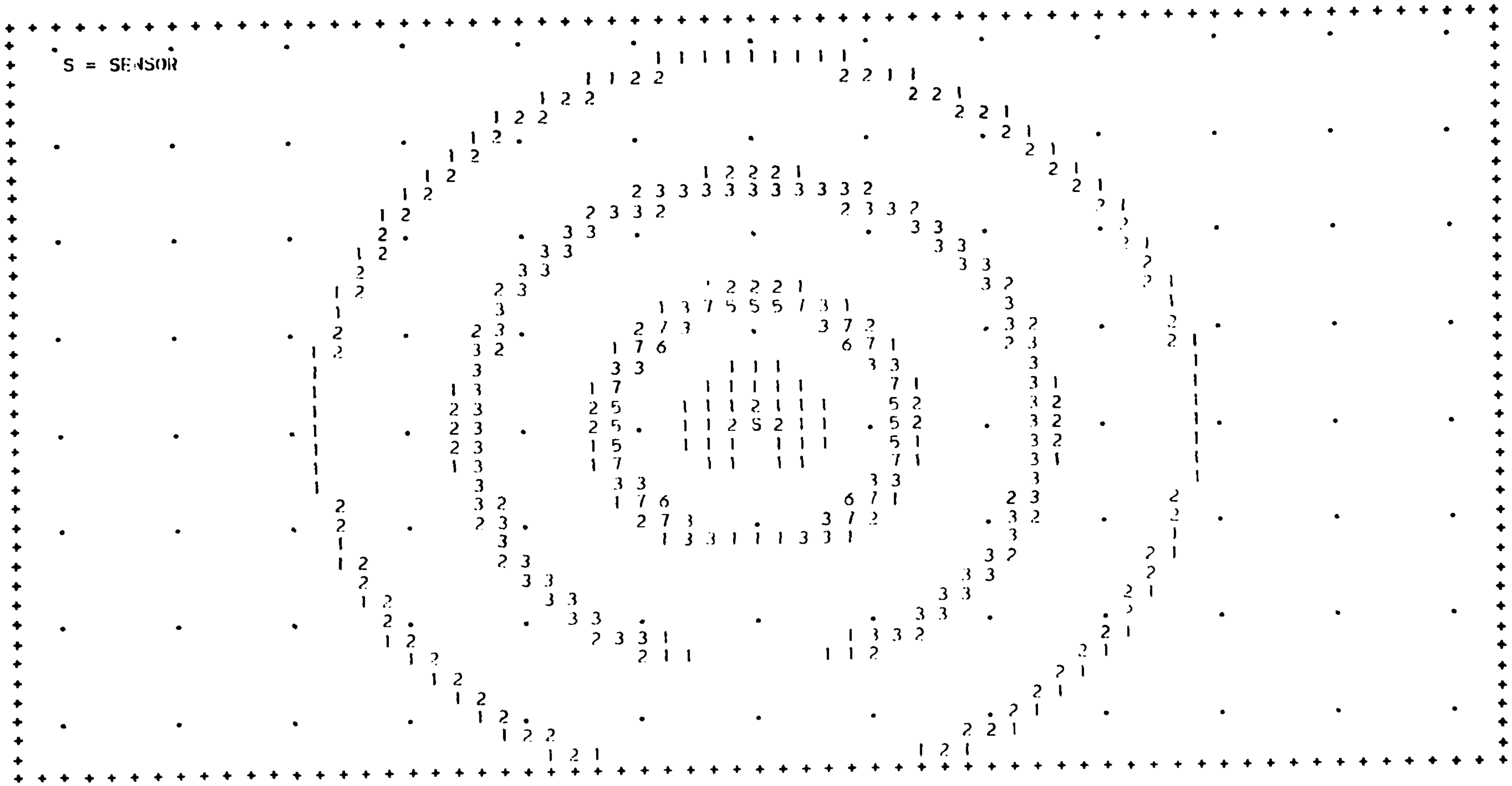


TABLE II-1

SYMBOL DEFINITIONS IN PDSEN AND PDSTEP

Symbol	Probability Range
*	0.95-1.00
9	0.85-0.95
8	0.75-0.85
7	0.65-0.75
6	0.55-0.65
5	0.45-0.55
4	0.35-0.45
3	0.25-0.35
2	0.15-0.25
1	0.05-0.15
blank	0.00-0.05

the time interval. The snapshot detection probability for sensor j at time t_i and the cumulative detection probability at time t_i will be denoted by p_i and cdp_i, respectively. When specific reference to sensor j is desired, we will write p_{ji} and cdp_{ji}.

Before proceeding, we remark that SCREEN allows for the reduction of a sensor's cdp to reflect the sensor's availability during an encounter. This reduction is accomplished by multiplying the cdp by a probability of availability $p_a(j)$, which is a program input (and hence represents a subjective assessment on the part of the program user). The "availability" concept is intended to reflect equipment reliability. In using p_a to degrade cumulative vice snapshot detection performance, it is assumed that the equipment's periods of operation and any "down" time it may incur are long compared to the length of time of a typical SCREEN tactical engagement.

In order to discuss cdp's, it is necessary to consider the temporal behavior of the detection process. Each sensor in the SCREEN program has three parameters which describe the statistical behavior of its detection mechanism:

RD = the recognition differential,

σ = the signal fluctuation standard deviation, and

λ = the detection sampling rate.

The use of RD and σ in determining p_{ji} has already been discussed in equation (II-7); it remains to consider the rate λ at which the detection mechanism obtains independent looks at the signal process.

The detection process assumed in SCREEN is the (λ, σ) jump process, described, e.g., in references [b] and [c]. The essence of the jump process is that new independent detection opportunities arrive at exponentially distributed times, with parameter λ. Specifically, for any two times t_i and $t_j > t_i$, there are associated snapshot detection probabilities p_i and p_j computed as in (II-7). With probability $\exp[-\lambda(t_j-t_i)]$, these two detection events are completely correlated and with probability $(1-\exp[-\lambda(t_j-t_i)])$ the two events are completely independent. Thus, the cumulative failure probability (PF) for these two times is:

$$PF = \exp[-\lambda(t_j-t_i)] \min\{\bar{p}_i, \bar{p}_j\} + (1-\exp[-\lambda(t_j-t_i)])\bar{p}_j \bar{p}_i.$$

where $\bar{p} = 1-p$. The value of cdp is then given by 1-PF.

The term "jump" arises from the notion that "independence" is itself a tangible thing which occurs at exponentially distributed times. Thus, using the notation of the preceding paragraph, p_i and p_j are independent if a "jump" occurred in the interval (t_j-t_i)--recall that this event has probability $(1-\exp[-\lambda(t_j-t_i)])$. This is a convenient way to visualize the process.

The jump process is the simplest (nontrivial) analytic model which includes the notion of time correlation in the detection mechanism. Indeed, this is its main appeal. Undoubtedly, the "correct" model is much more complex; for one thing, the true model should distinguish between the random components introduced by the signal, the noise, and the detection mechanism itself. However, the simple model represented by the jump process has proved to be analytically useful in many situations, a fact which warrants its use in SCREEN.

Various formulas have been developed for computed cdp under different assumptions, as is also discussed in references [b] and [c]. One of the most useful of these is the so-called unimodal formula, which is given in detail in Appendix B; see equation (B-1). The unimodal formula assumes that the time sequence p_1, p_2, ... of snapshot probabilities for a given sensor is unimodal; that is, the snapshot probabilities are nondecreasing in magnitude until some "modal" time, after which they are nonincreasing. Monotone increasing or decreasing sequences are a special case of the unimodal formula. The unimodal assumption leads to a very compact iterative formula to compute cdp which requires very little memory of the past. This formulation is as follows. Initialize variables used in the iteration by:

$$\left.\begin{aligned} PF(0) &= 1 \\ p_0 &= 0 \\ \phi_0 &= 0 \end{aligned}\right\} . \qquad \text{(II-11a)}$$

Then for $n \geq 1$:

$$\left.\begin{aligned} \phi_n &= \max\{\phi_{n-1}, p_n\} , \\ \psi_n &= \min\{p_{n-1}, p_n\} , \\ PF(n) &= \frac{\bar{\phi}_n}{\bar{\phi}_{n-1}} \left[\bar{\psi}_n + \psi_n e^{-\lambda(t_n - t_{n-1})}\right] PF(n-1). \end{aligned}\right\} \qquad \text{(II-11b)}$$

Here, of course, PF(n) is the cumulative failure probability at step n, and p_n is the snapshot probability at step n. Search is presumed to begin with step one. Note that only the quantities p_{n-1}, ϕ_{n-1}, and PF(n-1) must be preserved in the transition from step n-1 to step n; earlier computed values do not need to be retained in the iteration.

Random sampling. The cdp algorithm in SCREEN is a generalization of equations (II-11). The idea behind this generalization is that some sensors cannot search for, or be responsive to, cues to a target's presence in an omni-directional fashion, but rather can only focus their attention, so to speak, upon a small part of the screen coverage region at any given time. For example, a long-range active sonar may search one particular bearing sector at any given time. Hence, while the sonar is scanning one such bearing sector, a target in a different bearing sector--even one with an equal or stronger signal excess history than any target in the sector being searched--may go undetected. Thus, a snapshot detection may only be viewed as a probability of detection given that a detection opportunity exists, i.e., given that the sensor has a "glimpse" of the target. This "glimpse" event is governed by the sensor's scan rate. The implication for cdp computations is that the computation of PF(n) must not only involve the signal excess history, as reflected by the quantities $p_1, \ldots, p_n$, but also the "glimpse" process, which will be reflected by quantities $\pi_1, \ldots, \pi_n$, where π_i is the probability that a detection opportunity, or "glimpse," occurs at time t_i, and is related to the scan rate. An alternative description of this "glimpse" event is that the signal excess process is sampled at time t_i; in this context, the "glimpse" process will be called the "sampling process."

The sampling process is assumed to be independent of the random signal excess process. The SCREEN sampling process assumes that a sensor's search in a given portion of its coverage area occurs at random times, according to a Poisson process. This "random sampling" is in contrast to a systematic search, in which the same sector would be searched at uniformly spaced times. Its use represents the intent that a search plan not be defeated by a target which can take advantage of a systematic search schedule, as it might in the case of an active search, for example. Specifically, the probability that a sensor has a glimpse of the target during the interval between times t_{j-1} and t_j is given by:

$$\pi_j = 1 - \exp[-(t_j - t_{j-1})/r], \qquad \text{(II-12)}$$

where r is the expected time between successive looks by the sensor. Typically in SCREEN, $t_j - t_{j-1}$ is equal to the length of the program's uniform time step, and hence is independent of j. The quantity r is a required input parameter for each sensor. Finally, the model assumes that if the sensor glimpses in the direction of the target, the glimpse lasts for whatever integration time is required to set up a detection opportunity (otherwise, the scanning would be nonsensical).

At this point, before discussing the SCREEN cdp algorithm, a short example might help illustrate the roles which the various processes and components just described play in determining cdp. Suppose a sensor is assumed to scan a given sector of its coverage area on the average of once every fifteen minutes, and consider a sequence of four time steps (numbered 1 through 4) which are a half-hour apart. (Thus, equation (II-12) implies that $\pi = 1-e^{-2} = .865.$) Now suppose that the (snapshot) probability that the sensor will detect the target in question at time step i, given that the sensor scans the sector containing the target, is equal to p_i, where $p_1 = .1$, $p_2 = .4$, $p_3 = .7$, and $p_4 = .5$. In practice, these probabilities are computed according to equation (II-7). The sampling process in this example is whether or not the sensor scans the target's sector. There are sixteen possible outcomes for this process. One such outcome would be that the target's sector is scanned at each of the four time steps: that is to say, the signal excess process, from which the p_i ultimately arise, is sampled at each of the four time steps. This is the case covered by references [b] and [c]. The cdp for this outcome, according to equation (B-1), is

$$1-(.3)(1-(.1)e^{-.5\lambda})(1-(.4)e^{-.5\lambda})(1-(.5)e^{-.5\lambda}),$$

where λ is the relaxation rate for the signal excess process. Another possible outcome of our sampling process would be that the target's sector is scanned only at the first, second, and fourth time step. The cdp according to this outcome is

$$1-(.5)(1-(.1)e^{-.5\lambda})(1-(.4)(e^{-\lambda})).$$

Note that $p_3 = .7$ does not "play," so the modal probabilility has changed (to $p_4 = .5$) as have the time intervals between the signal excess samples (so that .4 is now multiplied by $e^{-\lambda}$). The true cdp for the sensor in this example is the weighed sum of the cdp's for all sixteen possible outcomes. (The weights for the two specific outcomes just considered would be $\pi^4 \simeq .56$ and $\pi^3(1-\pi) \simeq .087$, respectively. It is this probability which is multiplied by the availability probability p_a. More details on cdp computations are given in Appendix B.

Appendix B derives the cdp formulas which are used in SCREEN for a jump process with random sampling. The resultant algorithms form an interation somewhat more complex than (II-11), requiring three additional variables to iterate. The formulation is as follows. Initialize by:

$$PF(0) = 1 \qquad QQQ = 0$$

$$p_0 = 0 \qquad RRR = 0 \qquad \text{(II-13a)}$$

$$\bar{\phi}_0 = 1 \qquad SSS = 1.$$

To go from step n-1 to step n, assume that PF(n-1), p_{n-1}, $\bar{\phi}_{n-1}$, QQQ, RRR, and SSS have been preserved from the previous step; compute p_n from its own iteration mentioned previously, compute π_n by equation (II-12), and compute $e^{-\lambda(t_n - t_{n-1})}$. Then the iteration is given in a FORTRAN syntax by the following steps:

$$\bar{\varphi}_n = \min\{\bar{\varphi}_{n-1}, p_n\}$$

$$\psi_n = \min\{p_{n-1}, p_n\}$$

$$QQQ = \psi_n \, e^{-\lambda(t_n - t_{n-1})} QQQ$$

$$RRR = \bar{\psi}_n RRR$$

$$TEMP1 = \pi_n [\bar{p}_n \, SSS + \bar{\phi}_n (RRR + QQQ)]$$

$$SSS = \bar{\pi}_n SSS \qquad \text{(II-13b)}$$

$$QQQ = \frac{\bar{\pi}_n}{p_n} QQQ + \frac{1}{\bar{\phi}_n} TEMP1$$

$$RRR = \frac{\bar{\pi}_n}{\bar{p}_n} RRR + \frac{1}{\bar{\phi}_n} TEMP1$$

$$PF(n) = \bar{\pi}_n \, PF(n-1) + TEMP1.$$

It should be noted that the foregoing algorithm gives only an approximation to the value of PF(n), even in the unimodal case. (In the event of unimodal probabilities and complete sampling ($\pi_i = 1$, all i) however, the iteration gives the same result for PF(n) as (II-11).) On the other hand, studies such as reference [j] have shown the approximation to be a good one, even in the nonunimodal case. Given this fact, the algorithm of equations (II-13) is to be preferred to an algorithm whose increased precision in computing PF(n) must be purchased with unwieldly storage requirements.

Snapshot and cumulative detection performance--sensor groups. In the SCREEN program, cdp is separately determined for each sensor in the screen. These cdps are then combined to yield the detection probability for a group of sensors or for the entire screen. The SCREEN program allows the user to compute cumulative or snapshot performance measures for any specified group, or for the screen as a whole.

One parameter required in definition of a group is the group correlation, ρ_g, $0 \leq \rho_g \leq 1$. This parameter is used to obtain the group detection performance as an interpolation between the two extreme cases of complete correlation and complete independence.

In the case of complete correlation (ρ_g=1), the "best" sensor in the group determines the group performance, and so the group cumulative failure probability is given by

$$PF_g = \min_{j \in g} PF_j,$$

where PF_j is the cumulative failure probability for sensor j, which may be obtained from the iteration (II-13). (Note: In this formula and the ones which follow, if a group g consists of sensors $j_1, \ldots, j_n$, then g will be identified with the index set $\{j_1, \ldots, j_n\}$.) In the case of complete independence (ρ_g=0), the group failure probability is simply the product over all sensors:

$$PF_g = \prod_{j \in g} PF_j.$$

In the intermediate case, the formula used is an interpolation between these extremes:

$$PF_g = \rho_g \min_{j \in g} PF_j + (1-\rho_g) \prod_{j \in g} PF_j. \qquad \text{(II-14)}$$

This equation may be used both for cumulative performance and for snapshot performance. In the latter case, PF_j should be replaced by $1-p_j$ for each sensor in the group, where p_j is the snapshot detection probability for sensor j at the time of interest. Figure II-3 shows a typical snapshot map for the entire screen. Table II-1 defines the numerical symbols used in this figure.

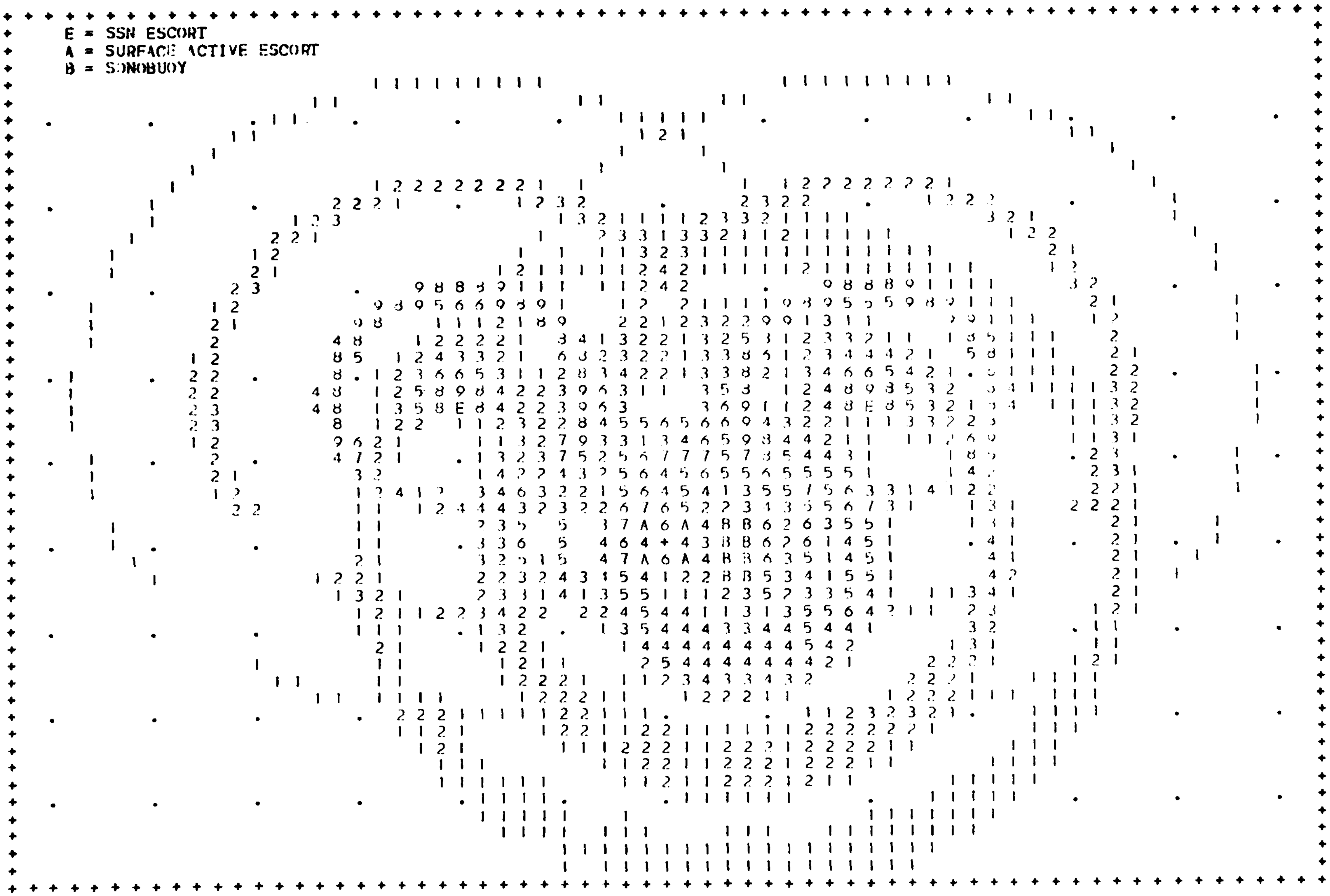
Screen detection performance. If G is a collection of groups--e.g., the entire screen--then the failure probability for G is the product of the failure probabilities for the groups in the collection:

$$PF_G = \prod_{g \in G} PF_g. \qquad \text{(II-15)}$$

This equation reflects the assumption that distinct groups--that is, the random components of the detection events associated with distinct groups--are assumed to be independent.

FIGURE II-3

TYPICAL SCREEN DETECTION COVERAGE MAP (PDSTEP OPTION)



CHAPTER III

THE LOCALIZATION PROCESS

This chapter describes the SCREEN localization model and, in particular, the measures of effectiveness computed by the program for use in assessing a screen's capability to localize targets. The localization measures are based on the Information Flow Kalman Filter algorithms of references [g], [h], and [l] and hence assume that targets follow constant velocity tracks. However, this assumption underlies only the construction of the localization measures. As will be seen in Chapter IV, the SCREEN target motion model is a more general process; its accompanying localization algorithm (called there the "incorporation of contact data") is less developed than the one to be described here. The basic operational difference between the two is that this chapter's algorithm is intended to focus on a simplified measure of the screen's localization capabilities (namely, the capability to localize a constant velocity target), whereas the algorithm of Chapter IV is a tool for modifying a target location distribution which ultimately is to be used to compute cumulative detection probabilities (see also Chapter II). Although it is worth noting that both algorithms have the same ultimate source in the theory of Bayesian analysis with Gaussian densities, exploring the connection is beyond the scope of this report.

The first section of this chapter presents the basic Information Flow Kalman Filter. The second section shows how these must be modified when correlated information is considered. The third section introduces the concept of "expected information" and describes the SCREEN localization measures. A fourth section outlines how the measures are used by the program. Appendices to this chapter address various analytical details.

The Information Flow Kalman Filter

This section discusses the basic Kalman filter algorithm underlying the localization process. It is based on the algorithm of § 6.3.4 of reference [h] and is also used in reference [g].

The problem. The object of the algorithm is to estimate a state vector X which satisfies a set of observations taking the form

$$z_n = l_n X + \epsilon_n \qquad n = 1, \ldots, m, \tag{III-1}$$

where

z_n is a d-dimensional observation vector,

l_n is a d by dim(X) measurement coefficient matrix,

ϵ_n is a d-dimensional measurement noise vector.

Specifically, it is desired to obtain the triple (X, P, R), where X is the state vector estimate, P is the covariance matrix of the error in the state vector estimate, and R is the residual, which is defined as follows. Let w_n be a weight matrix (to be determined momentarily). Then

$$R = \sum_{n=1}^{m} (w_n \epsilon_n)^T (w_n \epsilon_n). \tag{III-2}$$

In the SCREEN applications, X is the 4-vector of target position and speed components with respect to a rectangular coordinate system. Substituting (III-1) into (III-2) and collecting terms gives

$$R = X^T \left\{ \sum_{n=1}^{m} l_n^T W_n l_n \right\} X - 2X^T \{ l_n^T W_n z_n \} + \{ z_n^T W_n z_n \}, \tag{III-3}$$

where $W_n = w_n^T w$. The Kalman filter state vector estimate X is that which minimizes the residual, (III-3).

Generally, the weight matrices are chosen so that $W_n = \mathrm{Var}(\epsilon_n)^{-1}$. For then, assuming that each ϵ_n is normally distributed, it follows from (III-2) that R has a chi-square distribution with m by dim(X) degrees of freedom. Consideration of the residual may thus be helpful in determining a goodness-of-fit parameter. SCREEN does not do this within the framework of the localization process, but it is mentioned here for the sake of completeness and independent interest.

In addition to the observations (III-1), a prior triple (X_0, P_0, R_0) may also be specified.

<u>The information flow algorithm</u>. The three bracketed terms in (III-3) help to form the <u>information matrix</u> $\mathcal{J}$, the <u>information vector</u> $\mathcal{X}$, and the <u>information residual</u> $\mathcal{R}$, respectively. (The term "information" stems from the correspondence between $\mathcal{J}$ and Fisher information in the case of Gaussian densities.) When P is nonsingular, these

three quantities are, by definition, related to the triple (X, P, R) by

$$\mathscr{I} = P^{-1}, \qquad P = \mathscr{I}^{-1},$$
$$\mathscr{X} = P^{-1}X, \qquad X = \mathscr{I}^{-1}\mathscr{X},$$
$$\mathscr{R} = R + X^T P^{-1} X, \qquad R = \mathscr{R} - \mathscr{X}^T \mathscr{I}^{-1}\mathscr{X}. \tag{III-4}$$

The information flow algorithm may now be described. If

$$Z_n = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad L_n = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix}, \quad E_n = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix},$$

so that the equations (III-1) take the form

$$Z_m = L_m X + E_m, \tag{III-5}$$

and if $N_n = \mathrm{Var}(E_n)$, then the information quantities are given by

$$\mathscr{I} = \mathscr{I}_0 + L_m^T N_m^{-1} L_m, \tag{III-6a}$$

$$\mathscr{X} = \mathscr{X}_0 + L_m^T N_m^{-1} Z_m, \tag{III-6b}$$

$$\mathscr{R} = \mathscr{R}_0 + Z_m^T N_m^{-1} Z_m, \tag{III-6c}$$

where $(\mathscr{I}_0, \mathscr{X}_0, \mathscr{R}_0)$ are related to a prior (X_0, P_0, R_0) through equations (III-4), and are set to zero if no prior is specified. If the observations are uncorrelated, i. e., $\mathrm{Cov}(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$, then the equations (III-6) take the form

$$\mathscr{I} = \mathscr{I}_0 + \sum_{n=1}^{m} l_n^T (\mathrm{Var}\ \epsilon_n)^{-1} l_n, \tag{III-7a}$$

$$\mathscr{X} = \mathscr{X}_0 + \sum_{n=1}^{m} l_n^T (\mathrm{Var}\ \epsilon_n)^{-1} z_n, \tag{III-7b}$$

$$\mathscr{R} = \mathscr{R}_0 + \sum_{n=1}^{m} z_n^T (\mathrm{Var}\ \epsilon_n)^{-1} z_n. \tag{III-7c}$$

The relationship between the information quantities and the bracketed terms in (III-3) is evident from equations (III-7).

The equations (III-4) and (III-7) accomplish the information flow Kalman filter algorithm. For a proof, the interested reader is referred to reference [h]. The main feature of the algorithm is apparent by comparing (III-1) and (III-7) or (III-5) and (III-6): observations entail an "additive adjustment to information." These adjustments may take place even if P is singular.

Correlated Observations

In the applications addressed by reference [g], the dimensions of the observations are at most 2; moreover and more importantly, the observations are independent. Thus, equations (III-7) may be used and are computationally very tractable: the inverses involved are those of 2 x 2 matrices. In SCREEN localization process applications, however, observations will generally be correlated, requiring use of the equations (III-6), which because of the requirement to produce the inverse of the md x md matrix, N_m, is less desirable computationally. The algorithm described in this section is intended to be a computationally tractable implementation of the equations (III-6), and is taken from reference [l]. Appendix E will describe the algorithm for the types of observations considered in SCREEN. Appendix D contains the facts about correlation matrices that are relevant to the following discussion.

Suppose then that the measurement noise quantities are correlated: i.e., if $\sigma_i^2 = \mathrm{Var}(\epsilon_i)$, then there are d x d matrices ρ_{ij} such that

$$\mathrm{cov}(\epsilon_i, \epsilon_j) = \sigma_i \rho_{ij} \sigma_j .$$

(In the terminology of Appendix D, $\rho_{ij} = \rho(\epsilon_i, \epsilon_j)$ and $\sigma_i = \sigma(\epsilon_i)$. Note also that ρ_{ii} is the identity matrix, and $\rho_{ji} = \rho_{ij}^T$.) Thus, the (i, j)-element of $N_n = \mathrm{Var}(E_n)$ is $\sigma_i \rho_{ij} \sigma_j$.

The heart of the algorithm is the following assumption:

$$\rho_{ij}\, \rho_{jk} = \rho_{ik}; \qquad \text{for } i \leq j \leq k. \tag{III-8}$$

For example, if for a constant matrix ρ, ρ_{ij} is defined to be ρ^{j-i} for $i \leq j$ (and to be ρ_{ji}^T if $i > j$), then the conditions (III-8) will be satisfied. Other examples appear in Appendix E, in which the specific instances of the algorithm occurring in SCREEN are discussed. Since the indices are usually viewed in these applications as corresponding to successive sampling times, the conditions (III-8) may be described as "correlation in time." See also the note at the end of Appendix D.

Now, N_{n+1} can be written as

$$N_{n+1} = \begin{bmatrix} N_n & V_n \\ V_n^T & \sigma_{n+1}^2 \end{bmatrix},$$

where

$$V_n = \begin{bmatrix} \sigma_1 \rho_{1,n+1} \sigma_{n+1} \\ \vdots \\ \sigma_n \rho_{n,n+1} \sigma_{n+1} \end{bmatrix}.$$

If $S = N_n^{-1} V_n$ and $T = \sigma_{n+1}^2 - V_n^T S$, then by Proposition G-1,

$$N_{n+1}^{-1} = \begin{bmatrix} N_n^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1} [-S^T \quad I], \qquad \text{(III-9)}$$

where I is the d x d identity matrix.

It is evident, using the conditions (III-8), that

$$S = \begin{bmatrix} 0 \\ \\ \sigma_n^{-1} \rho_{n,n+1} \sigma_{n+1} \end{bmatrix}$$

solves $N_n S = V_n$. The symmetry of the matrices σ_i imples that

$$T = \sigma_{n+1} [I - \rho_{n,n+1}^T \rho_{n,n+1}] \sigma_{n+1}.$$

Equation (III-9) then becomes

$$N_{n+1}^{-1} = \begin{bmatrix} N_n^{-1} & 0 \\ & \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\sigma_n^{-1} \rho_{n,n+1} \\ \sigma_{n+1}^{-1} \end{bmatrix} \left[I - \rho_{n,n+1}^T \rho_{n,n+1} \right]^{-1} \left[0 \;\vdots\; -\rho_{n,n+1}^T \sigma_n^{-1} \;\vdots\; \sigma_{n+1}^{-1} \right]. \tag{III-10}$$

Now let

$$\mathscr{I}_{n,n+1} = \left(\sigma_{n+1}^{-1} l_{n+1} - \rho_{n,n+1}^T \sigma_n^{-1} l_n\right)^T \left[I - \rho_{n,n+1}^T \rho_{n,n+1}\right]^{-1} \left(\sigma_{n+1}^{-1} l_{n+1} - \rho_{n,n+1}^T \sigma_n^{-1} l_n\right), \tag{III-11a}$$

$$\mathscr{X}_{n,n+1} = \left(\sigma_{n+1}^{-1} l_{n+1} - \rho_{n,n+1}^T \sigma_n^{-1} l_n\right)^T \left[I - \rho_{n,n+1}^T \rho_{n,n+1}\right]^{-1} \left(\sigma_{n+1}^{-1} z_{n+1} - \rho_{n,n+1}^T \sigma_n^{-1} z_n\right). \tag{III-11b}$$

$$\mathscr{R}_{n,n+1} = \left(\sigma_{n+1}^{-1} z_{n+1} - \rho_{n,n+1}^T \sigma_n^{-1} z_n\right)^T \left[I - \rho_{n,n+1}^T \rho_{n,n+1}\right]^{-1} \left(\sigma_{n+1}^{-1} z_{n+1} - \rho_{n,n+1}^T \sigma_n^{-1} z_n\right). \tag{III-11c}$$

Then the following equalities are evident from equation (III-10):

$$L_{n+1}^T N_{n+1}^{-1} L_{n+1} = L_n^T N_n^{-1} L_n + \mathscr{I}_{n,n+1}. \tag{III-12a}$$

$$L_{n+1}^T N_{n+1}^{-1} Z_{n+1} = L_n^T N_n^{-1} Z_n + \mathscr{X}_{n,n+1}, \tag{III-12b}$$

$$L_{n+1}^T N_{n+1}^{-1} Z_{n+1} = Z_n^T N_n^{-1} Z_n + \mathscr{R}_{n,n+1}. \tag{III-12c}$$

If we define $\rho_{01} = 0$, then it is evident from equations (III-11) and (III-12) that the equations (III-6) may be rewritten as

$$\mathscr{I} = \mathscr{I}_0 + \sum_{n=0}^{m-1} \mathscr{I}_{n,n+1}, \tag{III-13a}$$

$$\mathscr{X}^{\cdot} = \mathscr{X}_0^{\cdot} + \sum_{n=0}^{m-1} \mathscr{X}_{n,n+1}^{\cdot}, \tag{III-13b}$$

$$\mathscr{R} = \mathscr{R}_0 + \sum_{n=0}^{m-1} \mathscr{R}_{n,n+1}. \tag{III-13c}$$

Note that equation (III-8) is still valid if $i = 0$.

The equations (III-11) may be cast in a slightly simpler form by introducing a few auxiliary terms. First of all, let

$$R_{n+1} = [I - \rho_{n,n+1}^T \rho_{n,n+1}]^{-\frac{1}{2}}.$$

(Note: in this report, if a positive definite symmetric matrix M is written

$$M = O \Lambda O^T,$$

where O is orthogonal and Λ is a diagonal matrix, then $M^{\frac{1}{2}}$ denotes the matrix

$$O \Lambda^{\frac{1}{2}} O^T,$$

where $\Lambda^{\frac{1}{2}}$ is obtained by taking the (nonnegative) square root of each component of Λ.)
Then let

$$\tilde{l}_{n+1} = \sigma_{n+1}^{-1} l_{n+1}, \tag{III-14a}$$

$$\tilde{z}_{n+1} = \sigma_{n+1}^{-1} z_{n+1}, \tag{III-14b}$$

$$L_{n,n+1} = R_{n+1}(\tilde{l}_{n+1} - \rho_{n,n+1}^T \tilde{l}_n), \tag{III-15a}$$

$$Z_{n,n+1} = R_{n+1}(\tilde{z}_{n+1} - \rho_{n,n+1} \tilde{z}_n). \tag{III-15b}$$

Then equations (III-11) take the form

$$\mathscr{I}_{n,n+1} = L^T_{n,n+1} L_{n,n+1}, \tag{III-16a}$$

$$\mathscr{X}_{n,n+1} = L^T_{n,n+1} Z_{n,n+1}, \tag{III-16b}$$

$$\mathscr{R}_{n,n+1} = Z^T_{n,n+1} Z_{n,n+1}. \tag{III-16c}$$

Equations (III-13) through (III-16) form the Kalman algorithm for correlated observations. The form of equations (III-13) shows that the quantities described by equations (III-16) may be used to adjust the information quantities as the observations are made, for the only correlation needed for an adjustment is that between the current observation and the previous one. Storage of the quantities σ_n, l_n, and z_n is also required to update from observation n to observation n+1. (Alternatively, $\sigma_n^{-1} l_n$ and $\sigma_n^{-1} z_n$ may be stored.) The inversion of the larger matrix N_m is avoided. The inverses required in the implementation of equations (III-13) are those of d x d matrices. Generally, d will be small--e.g., in the SCREEN context, d will be either 1 or 2--so these inversions will be computationally more tractable than those involved in equations (III-6). Table III-1 indicates the flow of the algorithm described in this section. Appendix E specializes the algorithm to bearing, bearing/range, and SPA observations, which are the observations of interest in SCREEN.

As a final note, if the observations are independent, then $\rho_{ij} = 0$ for all i and j; in this case, it is apparent from equations (III-11) and (III-13) that the algorithm reverts to that of equations (III-7). Thus, the present algorithm extends that of references [g] and [h].

Localization Measures Based on Kalman Filters

One important measure of effectiveness for an ASW screen relates to its ability to localize a target once it has been detected. One way to express this ability is to determine the expected value of the Kalman filter triple (X, P, R). To calculate such a quantity directly, however, requires complex and highly nonlinear operations. Indeed, the only practical way to determine it directly is by the use of Monte Carlo methods. But Monte Carlo methods are not feasible when it is desired to obtain two-dimensional pictures of performance such as SCREEN snapshot coverage maps: the number of replications which would be needed to insure that the figures on such a map lie within tolerable limits on the Monte Carlo "noise" would be astronomical.

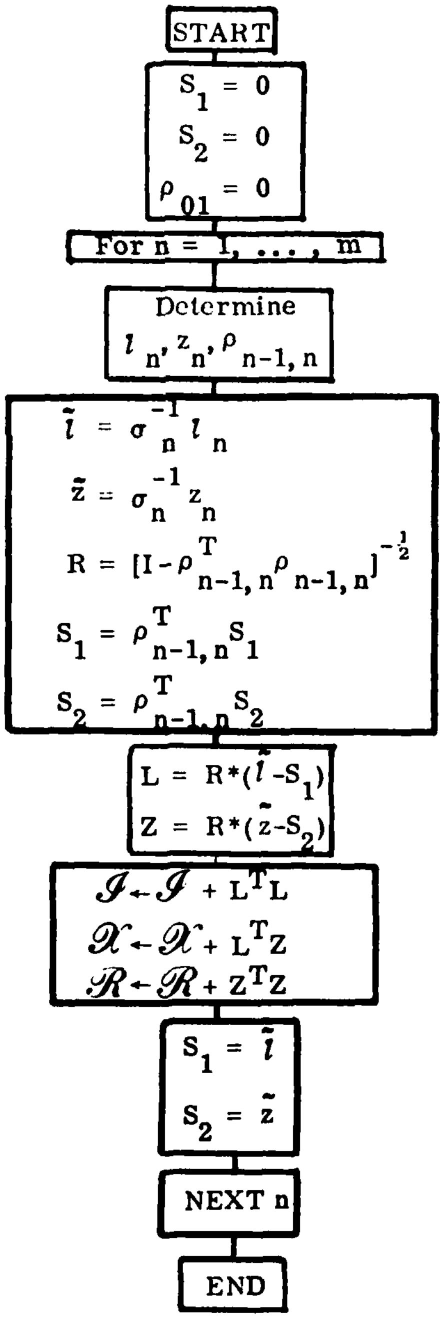
TABLE III-1

FLOW OF REVISED KALMAN ALGORITHM

Notes: 1) Algorithm is designed to process the m observations of equations (III-1) in the text.

2) As algorithm begins, the information quantities, $\mathcal{I}$, $\mathcal{X}$, $\mathcal{R}$, have been either initialized or computed via earlier information processing.

3) S_1 and S_2 denote storage arrays with dimensions d x dim X and d x 1, respectively. These arrays carry information between observations.



The approach taken in SCREEN is essentially to define a localization measure within the information domain. As is seen in the above treatment of Kalman filter information, information tends to be simply additive--even in the case of correlated measurements--and so easily lends itself to an expected value treatment. After defining expected information quantities, we then use the inverse (as in equations III-4) to obtain a measure of localization. The scheme which results is computationally very straightforward. However, as mentioned in Chapter I, the ultimate test of how precisely it depicts localization performance will only take place in a screen design analysis in which this type of performance is considered as a factor. Such an analysis at this time (late 1979) remains to be done.

"Expected" information. Suppose the observations underlying the equations (III-1) have respective probabilities p_n of being made*. The idea behind "expected" information is to incorporate this uncertainty into the algorithm of the preceding section.

First, consider the special case where the observations are independent. In this case, equations (III-7) show that each information quantity is a sum of information quantities computed for each observation and "prior information" ($\mathscr{I}_0, \mathscr{X}_0, \mathscr{R}_0$). Viewing each observation as a random variable with probability p_n, to each of which has been assigned three information quantities as given by the summands in equations (III-7), expected information quantities may then be defined in the classical fashion:

$$E(\mathscr{I}) = \mathscr{I}_0 + \sum_{n=1}^{m} p_n l_n^T (\mathrm{Var}(\epsilon_n))^{-1} l_n^T, \tag{III-17a}$$

$$E(\mathscr{X}) = \mathscr{X}_0 + \sum_{n=1}^{m} p_n l_n^T (\mathrm{Var}(\epsilon_n))^{-1} z_n, \tag{III-17b}$$

$$E(\mathscr{R}) = \mathscr{R}_0 + \sum_{n=1}^{m} p_n z_n^T (\mathrm{Var}(\epsilon_n))^{-1} z_n. \tag{III-17c}$$

The key to generalizing equations (III-17) to the case of correlated observations lies in viewing each term multiplied by p_n as being incremental information provided by the n^{th} observation. Indeed, if $\rho_{n,n+1} = 0$, then

$$\mathscr{I}_{n,n+1} = l_{n+1}^T (\mathrm{Var}(\epsilon_{n+1}))^{-1} l_{n+1}$$

---

* The quantities p_n are the product $\pi_n p_n$ of the snapshot probability, p_n, and the sampling probability, π_n, discussed in Chapter II: for economy of notation, we just write p_n.

and similarly for the other quantities in equations (III-11). Therefore, in the case of correlated observations, define ($\hat{\mathscr{I}}, \hat{\mathscr{X}}, \hat{\mathscr{R}}$) thus:

$$\hat{\mathscr{I}} = \mathscr{I}_0 + \sum_{n=0}^{m-1} p_{n+1}\, \mathscr{I}_{n,n+1} \tag{III-18a}$$

$$\hat{\mathscr{X}} = \mathscr{X}_0 + \sum_{n=0}^{m-1} p_{n+1}\, \mathscr{X}_{n,n+1} \tag{III-18b}$$

$$\hat{\mathscr{R}} = \mathscr{R}_0 + \sum_{n=0}^{m-1} p_{n+1}\, \mathscr{R}_{n,n+1}. \tag{III-18c}$$

These are not expectations in the true sense of the word, as the quantities of equations (III-17) are. The reason for this is that the incremental information quantities $\mathscr{I}_{n,n+1}$, etc., assume samples at both n and n+1; if this is not the case, then the correlation ρ will be different. Although the precise formula for expected information for correlated measurements may not be exceptionally difficult to derive, we will simply view the quantities (III-18) as the "expected" information, since they are reasonable generalizations of (III-17) and are computationally more tractable than the precise formulas.

"Expected" localization. The localization measure used in SCREEN is based on the inverses of the quantities (III-18), as given by equations (III-4), viz.,

$$\hat{P} = \hat{\mathscr{I}}^{-1}$$

and

$$\hat{X} = \hat{\mathscr{I}}^{-1}\hat{\mathscr{X}}. \tag{III-19}$$

These are called the "expected" localization parameters. From $\hat{P}$, in turn, is derived a scalar localization measure (in units of nautical miles), namely

$$L = \sqrt{\hat{P}_{11} + \hat{P}_{22}} \tag{III-20a}$$

where

$$\hat{P} = \begin{pmatrix} \hat{P}_{11} & \hat{P}_{12} & \hat{P}_{13} & \hat{P}_{14} \\ \hat{P}_{21} & \hat{P}_{22} & \hat{P}_{23} & \hat{P}_{24} \\ \vdots & \vdots & \vdots & \vdots \\ & & & \hat{P}_{44} \end{pmatrix} . \qquad \text{(III-20b)}$$

Since the Kalman filter vector X has position as its first two components, L is interpreted as the 1-sigma radius of a circular SPA which has the same area as the "expected" localization ellipse defined by (III-20b).

The quantity described by equations (III-20) is the basis for the snapshot localization measures in SCREEN. However, a technical issue arises in applying (III-20) when the matrix $\hat{\mathcal{J}}$ is singular. Although one's initial impulse is to simply consider L undefined, it is, on the other hand, quite possible for useful localization to be contained in $\hat{\mathcal{J}}$ even though it is singular. For example, cross-fixes between passive sensors at a single time will produce a position fix but no speed estimate. Appendix E contains the method used to perform a partial inversion of $\hat{\mathcal{J}}$ and resolve this dilemma.

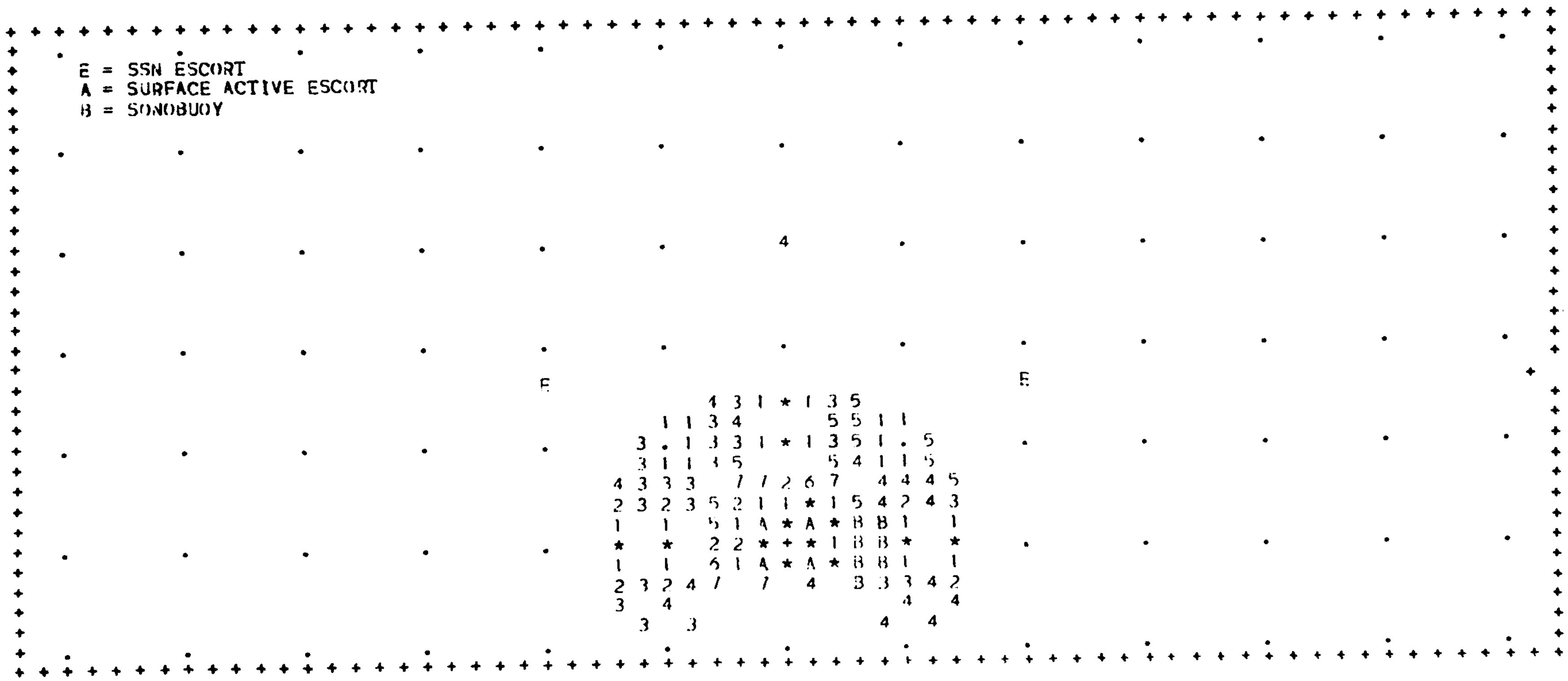
SCREEN Localization Performance Measures

The SCREEN program illustrates a screen's localization capabilities in two ways: by coverage maps and by target motion analysis (TMA) against penetrators. These are analogous to and serve as companions to snapshot and cumulative detection performance measures, respectively. Detection performance measures were described in Chapter II.

Localization coverage maps. Snapshot localization performance is illustrated by SCREEN by localization coverage maps such as displayed in Figure III-1. These are analogs of the snapshot detection coverage maps of which Figure II-2 is an example. The procedure for their construction is also similar: evaluate the "snapshot localization measure" given by equation (III-20) at every point of a gridwork superimposed over the screen coverage area. Since a snapshot map refers to a particular time (i.e., program time step) there is no correlation between observations, so that $\hat{\mathcal{J}}$ has the form of equation (III-17a), the sum being over the sensors playing at the time. More precisely, to obtain a given value on the map, a target is postulated to be positioned at the point of interest and the corresponding quantity L is computed as just described. If the resulting value of L is less than 9.5, a digit from 1 to 9 or an asterisk is displayed. An asterisk indicates that L is less than 0.5; a digit N indicates that $N-.5 \le L < N+.5$. A blank indicates either that L is greater than 9.5 or that L is undefined because $\hat{\mathcal{J}}$ is singular and contains insufficient localization information. As illustrated by Figure III-1, localization coverage (evidenced by nonblank grid points) occurs generally where cross-fixes between passive sensors or detections by active sensors are must likely to occur. The convergence zone intersection of the two forward passive sensors produces the

FIGURE III-1

LOCALIZATION MAP FOR SENSOR SCREEN
(POSTEP -- COMPANION TO FIGURE II-2)



localization '4' forward of the main body.

The numbers displayed in the snapshot localization map can be viewed as expressing the expected radius in miles of a 1-sigma circular SPA at that location, given detection.

Cumulative localization. The cumulative localization performance (clp) of a sensor is assessed by accumulating the "expected" information (III-18) for target penetration tracks*. This may be done at the same time that cumulative detection probability is determined. To obtain group or screen clp, the quantity $\hat{\mathscr{I}} - \mathscr{I}_0$ is accumulated for each sensor in the group or screen over the time span in question, these quantities are combined for the sensors in the group or screen (appealing again to the additivity of information), and finally this result is added to $\mathscr{I}_0$ and inverted to yield $\hat{P}$ and ultimately L, via (III-19) and (III-20). The localization performance is summarized in two ways: first, in a table stating the distribution of L-values given detection, and second, in a map similar to the snapshot coverage maps, showing localization given detection.

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* Recall that target penetration strategies are represented by averaging over a number of specific target tracks against which search and localization are performed.

CHAPTER IV

THE TARGET MOTION MODEL

This chapter describes the target motion model used in SCREEN, which is intended to incorporate the assumptions made about the penetration tactics used by submarines approaching the screen.

Introduction

The tactics followed by a target during an approach depend on a number of factors, for example: his position relative to the task force base track, his operating characteristics (speed and radiated noise), his state of knowledge about the screen unit locations and the base track of the task force, and an assessment of his vulnerability and the screen's protective capability.

In most cases, due consideration of these factors results in a limited number of viable approach strategies, which vary in sophistication from a 'damn-the-torpedoes' flank speed intercept course to a cautious approach that attempts to detect and evade screen units.

The SCREEN program allows the user to build and store a variety of target files, each of which corresponds to a particular approach strategy. The purpose behind this is to provide the user the capability to evaluate screen performance against various penetration tactics. The program can also be used from the opposite viewpoint, i.e., as a tool for finding optimal penetration tactics based on assumed levels of information about the task force. In most any screen evaluation, it is generally expected that to describe the full range of likely target penetration tactics will require multiple target files. Thus, the capability exists for running the program with up to 99 distinct target files. (The exact nature of a target file is discussed in reference [a].)

The target strategy as defined by a target file is simply a multidimensional Gaussian distribution for the joint location of target position at a specified sequence of times. A sample from this distribution is a particular target track. The mean of this distribution is the base track which characterizes this particular strategy.

Much of the remainder of this chapter concerns the question of how the probability distribution representing a target strategy is established. The intent is to provide a constructive approach to defining this distribution so that it can be related to tactically meaningful concepts. Specifically, the approach reflects such items as:

(1) the initial location distribution of the target;

(2) initial target course and speed distribution;

(3) mean time between new random course and speed selections;

(4) positive contacts (bearing lines and SPAs); and

(5) marginal constraints which reflect target objectives or motion restraints.

The mathematical process at the heart of the model is based on a generalized Ornstein-Uhlenbeck process. The first part of this chapter presents the background behind this process and its use to model target motion. Following a discussion of the physical and mathematical models to be considered, the various operations in the mathematical model are discussed. The chapter closes with an overview of the model's use. Of course, detailed discussions and derivations are deferred to notes and appendices as indicated.

Background: The Ornstein-Uhlenbeck Process and Generalizations

The Ornstein-Uhlenbeck (OU) process is a one-dimensional process on the velocity distribution of a particle moving along the real line. It was first considered by Uhlenbeck and Ornstein in 1930 (reference [d]) to provide an alternative model for Brownian motion. Doob, in reference [e], considered this process in the light of modern probability, gave the process its name, and showed it to be characterized by the following properties:

(1) the process is homogeneous in time (that is to say, the distribution is unaffected by the choice of time zero);

(2) the process is Markov; and

(3) the joint distribution at every two arbitrarily chosen times is bivariate Gaussian.

Specifically, Doob's theorem shows that if the process $u(t)$ with $E(u(t)) = m$ and $Var(u(t)) = \sigma^2$ satisfies (1), (2), and (3), then either

(i) the process is a "white noise" process: for every set of times $t_1 < \ldots < t_n$, the random variables $u(t_1), \ldots, u(t_n)$ are mutually independent and Gaussian, with mean m and covariance σ^2; or

(ii) the process is an <u>OU process:</u> there is a positive constant β such that if $t_1 < \ldots < t_n$, then $u(t_1), \ldots, u(t_n)$ have an n-variate Gaussian distribution with common mean m, common variance σ^2, and covariances given by

$$E[(u(t)-m)(u(s)-m)] = \sigma^2 \exp(-\beta |t-s|).$$

The process (i) might be considered a "pathological" OU process, with $\beta = \infty$.

Reference [e] also shows that the sample paths u(t) of the OU process are continuous, with probability 1. Therefore, integration is admissible, and the corresponding displacement process, the Integrated Ornstein-Uhlenbeck (IOU) process x(t) can be formed:

$$x(t) - x(0) = \int_0^t u(s)\, ds.$$

Because u(t) is continuous, x(t) has differentiable sample paths--(i.e., the particle has a finite velocity)--an improvement over Brownian motion. Even so, the IOU is not physically realizable because there is no acceleration: the sample paths in velocity space are nondifferentiable. (However, this <u>is</u> preferable to the random walk model for Brownian motion, whose sample <u>position</u> paths are nondifferentiable--velocity is nonexistent.)

Generalization of the OU process to two or more dimensions is straight-forward, and requires no further comment.

The application of the IOU process as a target motion model in search problems was initiated by reference [m], where it is argued that, despite its physical nonrealizability, it is still close enough to a physical process to have utility as a model. Indeed, it is demonstrated there that the IOU process gives the best fit among all Gaussian diffusions to the motion of a randomly touring target. Subsequent investigations into the IOU process as a target motion model revealed other advantages, including the removal of the "hour-glass" effect inherent in models whose sample tracks are formed from constant-velocity interpolations between uncorrelated draws from sequences of position distributions.

The target motion model developed in this chapter is based upon a generalized Ornstein-Uhlenbeck process. The idea of the generalization is to have the process be multi-dimensional and to allow the process parameters (mean and covariance) to vary with time. The generalized process in question is described as follows. Let ν_t be the mean target velocity at time t, as expressed in terms of a rectangular coordinate grid. (In the case of the SCREEN model, ν_t is two-dimensional.) The actual target velocity is a random quantity,

$$V_t = \nu_t + \epsilon_t, \tag{IV-1}$$

where ϵ_t has the following properties:

(1) For each t, ϵ_t is Gaussian with mean 0 and covariance Γ_t.

(2) There is a nonnegative function μ such that

(a) for all t, $\int_s^\infty \mu(x)dx = \infty$, and

(b) if $s \leq t$, then ξ_s and ξ_t have a joint multivariate Gaussian distribution with cross-covariance given by

$$\text{cov}(\epsilon_s, \epsilon_t) = \exp(-\int_s^t \mu(x)dx)\Gamma_s.$$

There is a unique Gaussian stochastic process satisfying the properties (1) and (2). This process will henceforth be called the GOU (for generalized Ornstein-Uhlenbeck) process. A one-dimensional GOU in which Γ_t and $\mu(x)$ are constant functions is an OU process as described by Doob's theorem (with m=0).

The Discrete IOU Process

Basically, the target motion model in SCREEN is a computationally attractive approximation to another model which is more desirable from a physical standpoint. It will be seen that both models--the physical model and its analytic counterpart--exhibit the same second order behavior; therefore, the two are "close enough" (in a sense already mentioned in connection with reference [m]) so that the computationally more attractive one may be implemented in SCREEN as representing the desired physical process. Both models will be described presently.

The physical model. The idea behind the physical model is as follows. To obtain an actual sample track, one adds to a base velocity (which is a function of time) random velocity vectors which are independently drawn from Gaussian distributions at exponentially distributed times; integration of the sample velocity path so generated will give the sample target track. The base velocity function is the "track plan"--for example, in a screen penetration, it reflects the general tactics to be used to move from a position outside of the screen to an attack position. The random component reflects variations in the base plan and uncertainties from the viewpoint of the screen defense.

The velocity process at work here may be called an exponential correlation process. This model was first proposed by D. C. Bossard and W. H. Barker in reference [o] in the form of a discrete time process, but will be described below in the framework of continuous time processes. There is a close connection between this process and the (λ, σ) jump process (see Chapter II or reference [b]), which is widely used as a model for signal excess fluctuations.

Let ν_t be the mean target velocity at time t, as expressed in a (two-dimensional) rectangular coordinate system. The actual target velocity at time t is a random quantity

$$V_t = \nu_t + \epsilon_t, \qquad \text{(IV-2)}$$

where ϵ_t has the following properties:

(1) For each t, ϵ_t is Gaussian with mean 0 and covariance Γ_t.

(2) There is a nonnegative function μ such that

  (a) for all t, $\int_t^{\infty} \mu(x)dx = \infty$, and

  (b) if $s < t$, then the probability that $\epsilon_s = \epsilon_t$ is $q = \exp[-\int_s^t \mu(x)dx]$ and the probability that ϵ_s and ϵ_t are independent Gaussian samples is 1-q.

In the terminology of reference [o], property (2) says that the target has "exponential memory." An alternate description of the model may be given in terms of Monte Carlo paths. A Monte Carlo sample (velocity) path for the exponential correlation process would be obtained by the following algorithm:

(1) Set $t = 0$ (or to the desired initial time).

(2) Select a random velocity ϵ_t from the Gaussian distribution with mean 0 and covariance Γ_t, independent of our prior draws.

(3) Draw a random number r from the uniform distribution on the unit interval [0, 1].

(4) Find a time t_r such that

$$r = \exp[-\int_t^{t_r} \mu(x)dx].$$

(That such a t_r exists follows from property (2)(a).)

(5) The actual target velocity between times t and t_r is given by:

$$V_s = \nu_s + \epsilon_t, \quad t \leq s < t_r.$$

(6) Set $t = t_r$.

(7) Return to Step 2.

Integration of the sample velocity path constructed as above will yield a sample target track.

This process is not Gaussian, as may be demonstrated by computing the characteristic function of the joint distribution of V_{t_1} and V_{t_2} for $t_1 < t_2$. This is done in Note 1 at the end of the chapter. It is very desirable to work within the framework of Gaussian processes if possible, because of the considerable computational advantages--for one thing, a Gaussian process is completely characterized by its second order statistics (mean and covariance structure); in addition, Bayesian updating of the type considered later presumes that the process is Gaussian. The next subsection will define a Gaussian model that will be used to approximate this physical model.

The mathematical model. The basic target motion model used in SCREEN is called the discrete IOU (DIOU) process. This computational model is a displacement process on the Cartesian plane in which the velocity is a step function obtained by sampling the GOU process at discrete times. This idea will now be made precise.

Let $0 = t_0 < t_1 < \ldots < t_\tau$ be specified sampling times. Let V_t be the GOU process defined in (IV-1), where μ is the step function such that $\mu(s) = \mu(t_{l-1})$ for $t_{l-1} \leq s < t_l$. Then, where $\delta_i = t_{i+1} - t_i$, define z_j for j = 1, 2, ... by

$$z_j = z_0 + \sum_{i=0}^{j-1} \delta_i V_{t_i}. \tag{IV-3}$$

The 2-vector $z_j - z_0$ is a displacement from an initial position z_0 obtained by sampling the GOU velocity process at times preceding t_j and integrating. The quantity z_0 is assumed to be drawn from an initial target location distribution, which is independent of the GOU process. Define the $(2\tau+2)$-dimensional vector Z_τ by

$$Z_\tau = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_\tau \end{bmatrix}, \qquad \text{(IV-4)}$$

where each component z_j for $j > 0$ is given by (IV-3). The random vector Z_τ defines the target track through time t_τ. Note that these tracks are not sample paths of the integrated GOU process, although they approach such tracks as $\tau \to \infty$ and the δ_i approach zero uniformly. See reference [q] for a direct demonstration of this using the relations (IV-6) below.

A multivariate Gaussian distribution for Z_τ is uniquely determined by that random vector's mean β_τ and covariance matrix B_τ. If $\beta_n(n)$ is the 2-vector consisting of the $(2n+1)^{th}$ and $(2n+2)^{th}$ components of β_n, then it is evident from (IV-3) that the mean vectors are related by

$$\beta_0 = z_0 \qquad \text{(IV-5a)}$$

$$\beta_{n+1} = \begin{bmatrix} \beta_n \\ \\ \beta_n(n) + \delta_n \nu_n \end{bmatrix}. \qquad \text{(IV-5b)}$$

(Here and in the following, ν_n, μ_n, V_n, and Γ_n denote ν_{t_n}, μ_{t_n}, V_{t_n}, Γ_{t_n}, respectively.) The covariance matrices are related by the recursion relations below, in which b_{ij} and h_{ij} are 2 x 2 matrices with the interpretations:

$$b_{ij} = \text{cov}(z_i, z_j),$$

$$h_{ij} = \text{cov}(z_i, \delta_{j-1}\epsilon_{j-1}),$$

$$B_\tau = [b_{ij}], \qquad i, j = 0, \ldots, \tau, \qquad \text{(IV-6a)}$$

$$b_{ij} = b_{i,j-1} + h_{ij}, \qquad 0 \le i \le j, \qquad \text{(IV-6b)}$$

$$b_{ij} = (b_{j,i})^T, \qquad 0 \le j \le i, \qquad \text{(IV-6c)}$$

$$h_{ij} = e_{j-1} h_{i,j-1}, \qquad 0 \le i < j, \qquad \text{(IV-6d)}$$

$$e_j = \frac{\delta_j}{\delta_{j-1}} \exp(-\delta_j \mu_j), \qquad j = 1, 2, \ldots, \qquad \text{(IV-6e)}$$

$$h_{ii} = h_{i-1,i} + (\delta_{i-1})^2 \Gamma_{i-1}, \qquad i = 1, 2, \ldots . \qquad \text{(IV-6f)}$$

The proof of the relations (IV-6) appears as Note 2 at the end of this chapter.

The recursion embodied by the relations (IV-6) may be described as follows. With $b_{00} = \mathrm{Var}(z_0)$ and $h_{00} = 0$, construct h_{01} by (IV-6a), h_{11} by (IV-6f) and b_{11} by (IV-6b). Now suppose that B_τ as in (IV-6a) and h_{ij} for $0 \leq i \leq j \leq \tau$ have been constructed. (Actually, at this stage, only $h_{i,\tau}$ for $0 \leq i \leq \tau$ are required.) Then, $B_{\tau+1}$ may be constructed by following these four steps:

(1) Use (IV-6a) and (IV-6f) to construct $h_{i,\tau+1}$ for $0 \leq i \leq \tau+1$.

(2) Use (IV-6b) to construct

$$H_\tau = \begin{bmatrix} b_{0,\tau+1} \\ \vdots \\ b_{\tau,\tau+1} \end{bmatrix}.$$

(3) By (IV-6c), $[b_{\tau+1,0}, \ldots, b_{\tau+1,\tau}] = H_\tau^T$.

(4) Construct $b_{\tau+1,\tau+1}$ by (IV-6b).

Thus, H_τ, H_τ^T and $b_{\tau+1,\tau+1}$ are constructed in that order, and

$$B_{\tau+1} = \begin{bmatrix} B_\tau & H_\tau \\ H_\tau^T & b_{\tau+1,\tau+1} \end{bmatrix}.$$

To summarize: the DIOU process, which is the basic target motion model in SCREEN, is the process Z_τ derived from the GOU velocity process in the manner indicated by (IV-3) and (IV-4), and has mean β_τ as described by (IV-5) and covariance matrix B_τ governed by the relations (IV-6).

<u>Relationship between the physical model and the mathematical model.</u> As noted before, the physical model is not Gaussian, whereas the DIOU is Gaussian. However, the two processes have the same second order behavior: they clearly have the same mean, and if $s < t$ and $q = \exp[-\int_s^t \mu(x)dx]$, then for ξ_s, ξ_t and $\mu(x)$ as in the exponential correlation model,

$$\text{cov}(\xi_s, \xi_t) = q\,\text{cov}(\xi_s, \xi_t \mid \xi_s = \xi_t) + (1-q)\,\text{cov}(\xi_s, \xi_t \mid \xi_s \text{ and } \xi_t \text{ are independent}),$$

$$= q\,\text{cov}(\xi_s, \xi_s)$$

$$= \exp[-\int_s^t \mu(x)dx]\Gamma_s,$$

which defines the same cross-covariance structure as the GOU. Thus, the GOU, although not physically realizable, provides the best fit in the sense of reference [m] among the Gaussian diffusions--which are to be preferred from a computational point of view--to the physical model given by exponential correlation.

Actually, the similarity between the two processes extends further: it is also the case that all odd-ordered central moments vanish for both the physical process and for the DIOU. Thus, the first disagreement in models occurs at the fourth-order moments.

In the subsequent analysis, then, the DIOU process will be used to model target motion, and will be interpreted as representing the physical model described above. There are two possible viewpoints regarding interpretations:

(1) The "true" model can be viewed as the Gaussian process described by the DIOU and the physical model an appropriate interpretation.

(2) The "true" model can be viewed as the physical process described above, and the mathematical treatment viewed as a technique which keeps track of the first and second moments of the process--a second order approximation to the true process.

Either viewpoint may be adopted, and the interested SCREEN user is encouraged to adopt one of them, according to his preference.

Operations on the DIOU

The DIOU process is the starting point, the raw material, so to speak, from which the actual target motion is synthesized. To arrive at the final target process, the DIOU must submit to a series of operations. The conceptual sequence of operations is as follows:

(1) The DIOU is constructed as in the last section for the times of interest--one refers to the operation of extending the time interval over which the process is considered as time-stepping, or as performing a time step increase.

(2) The DIOU process constructed in (1) is constrained to satisfy certain a priori conditions--more precisely, the marginal location distributions at certain times are specified.

(3) Weighted combinations of constrained DIOU processes in (2) are used to synthesize the underlying target motion.

(4) Positive contact information (if present) is used to modify the process in (3).

The first three steps build up the assumed target prior distribution for each time step of interest. This reflects assumptions concerning the target tactics and uncertainties in those tactics. Each DIOU component may reflect a specific scenario, a generic kind of tactic--for example, approaching a moving task force from different starting positions may involve essentially different tactics: e.g., a tail chase from a rear position versus a loiter from a position ahead of the base track. The constraints may reflect geographic limitations on target motion--for example, the passage from an ocean basin into a strait or channel. The weighting of distinct DIOU processes may reflect a subjective assessment of the likelihood of the different scenarios.

Normally, one would expect the construction process to proceed in the logical order given in the steps (1) through (4) above: first time step, then constrain, etc. However, in actual use, this may be an extremely inconvenient thing to do. In a real-time application, it would be desirable to time-step only as far as necessary to reflect the situation up to the present or near future; as time passes, the time window could expand accordingly. As well, it may be desirable to apply and remove constraints at will while the problem is in progress (e.g., in order to test various hypotheses). Finally, contacts may be reported after varying time delays; subsequent information could cancel or modify earlier contact reports.

Thus, although it is very desirable to think of the various operations on the DIOU as occurring in the logical order given above, one must, in fact, make provisions for applying and repealing these operations in any sequence, and to do so (if possible) without unravelling the process back to its raw components and rebuilding it, each time some new variation in the operations is desired. This is the motivation for the remaining treatment in this chapter: it is desired to perform operations directly on the modified DIOU process--even though the operations should logically be performed in the order given above--and the net result must be the same as though the operations were in fact performed in that logical order.

With this as background, the operations which will be applied to the DIOU can now be presented. The DIOU process itself, over the time interval $[0, t_T)$ is described completely by the pair (β_T, B_T) described previously: β_T represents the mean or base track and B_T is the covariance matrix of the position uncertainty about the base track. The Gaussian distribution determined by (β_T, B_T) will be called the unconstrained prior distribution (UPD). The operations on the DIOU which were described above will involve three different operations on the UPD:

(i) Marginal constraint - the marginal distributions for some components of Z_T are to have a specified Gaussian distribution.

(ii) Incorporation of contact data - the distribution for the track is modified to reflect information gained by sensor contacts on the target at particular times.

(iii) Time step increase - the time interval of interest $[0, t_T)$ is to be replaced by $[0, t_{T+1})$.

It will also be desirable to be able to undo operations (i) and (ii), for reasons indicated previously. The specific natures of these operations will be discussed in later subsections. The effect of each operation will be to transform the UPD into another Gaussian distribution, which will be called the modified target distribution in the sequel. Technically speaking, operations (iii) and (i) should be applied in that order to the UPD, obtaining a modified distribution to which operation (ii) may be applied: this puts the steps (1) through (4) mentioned earlier in the context of what follows. However, to reiterate the philosophy of this section in that context, the present goal will be to show how these operations may be accomplished--without regard to order--by directly operating on the modified distribution.

DIOU operations and the information domain. To distinguish the modified distribution from the UPD (although note that the two will coincide if operations (i) and (ii) are not performed), the mean and covariance matrix of the modified distribution over $[0, t_T)$ will be denoted by γ_T and C_T, respectively. Thus, according to the modified distribution,

$$\begin{aligned} \mathrm{pr}\{Z_T = Z\} &= n(Z;\gamma_T, C_T) \\ &= \exp\left\{-\frac{1}{2}\left[(Z-\gamma_T)^T C_T^{-1}(Z-\gamma_T)+K_T\right]\right\}, \end{aligned} \tag{IV-7}$$

where $n(\cdot\,;\sigma_T, C_T)$ is thus the Gaussian density with mean γ_T and covariance matrix C_T and K_T is a normalization constant which insures that the density integrates to unity over all space. See equation (H-1).

Assuming that C_T is nonsingular, the distribution may be specified in either one of two ways:

(a) By the pair (C_T, γ_T), whose components are defined by equations (IV-7). This will be called the covariance domain representation of the distribution.

(b) By the pair $(\mathscr{I}_T, \mathscr{X}_T)$, where

$$\mathcal{I}_\tau = C_\tau^{-1}, \tag{IV-8a}$$

$$\mathcal{X}_\tau^{\cdot} = C_\tau^{-1} \gamma_\tau . \tag{IV-8b}$$

This will be called the <u>information domain</u> representation of the distribution, in analogy with terminology of Chapter III. The matrix $\mathcal{I}_\tau$ will be called the "information matrix," and $\mathcal{X}_\tau^{\cdot}$ will be called the "information vector." If C_τ is nonsingular, then there is a one-to-one correspondence between the two domains, as indicated by equations (III-4). In the present case, this correspondence is given by the equations (IV-8) and the following relations:

$$C_\tau = \mathcal{I}_\tau^{-1}, \tag{IV-9a}$$

$$\gamma_\tau = \mathcal{I}_\tau^{-1} \mathcal{X}_\tau^{\cdot} . \tag{IV-9b}$$

Now, it will be demonstrated in later subsections that the marginal constraint and contact datum operations involve certain additive adjustments to the information matrix and information vector. In particular, each operation will pertain to a set $u = \{u_1, \ldots, u_k\}$ from the index set $\{1, \ldots, \tau\}$, and will give rise to a k x 2k matrix D and a 2k-vector d. The adjustments will then take the form

$$\tilde{\mathcal{I}}_\tau = \mathcal{I}_\tau \,(+)\, D, \tag{IV-10}$$

$$\tilde{\mathcal{X}}_\tau^{\cdot} = \mathcal{X}_\tau^{\cdot} \,(+)\, d, \tag{IV-11}$$

where $\mathcal{I}_\tau$ and $\mathcal{X}_\tau^{\cdot}$ ($\tilde{\mathcal{I}}_\tau$ and $\tilde{\mathcal{X}}_\tau^{\cdot}$) are the information matrix and vector of the modified distribution for Z_τ before (after) the operation has been performed. In equation (IV-10), the (+) sign means that D should be added to the 2k x 2k submatrix of $\mathcal{I}_\tau$ corresponding to the index set u. The sign has the analogous meaning in equation (IV-11). The (+) operation is described in more detail by the discussion of equations (G-4) and (G-5) in Appendix G.

With D and d in hand, the covariance domain representation may be adjusted according to the following equations which are derived in Appendix I.

$$\tilde{C}_\tau = C_\tau - C_\tau(u)\, D(I+C_{uu}D)^{-1}\, C_\tau(u)^T, \qquad \text{(IV-12a)}$$

$$\tilde{\gamma}_\tau = \gamma_\tau + C_\tau(u)\,(I+DC_{uu})^{-1}\, d - C_\tau(u)\, D(I+C_{uu}D)^{-1}\, \gamma_\tau(u), \qquad \text{(IV-12b)}$$

$$\tilde{K}_\tau = K_\tau - \log\det(I+C_{uu}D). \qquad \text{(IV-12c)}$$

In the equations (IV-12), I is the 2k x 2k identity, $\gamma_\tau(u)$ is the subvector of γ_τ corresponding to the index set u, $C_{uu} = (C_\tau)_{uu}$ is the submatrix of C_τ corresponding to the index set u, and $C_\tau(u)$ is the submatrix consisting of those columns of C_τ which contain C_{uu}. Also, as shown in Appendix I, $I+C_{uu}D$ and $I+DC_{uu}$ are nonsingular, so that the equations (IV-12) make sense. Moreover, 2k will generally be a modest number, so that taking the inverses will be computationally reasonable.

Thus, to apply an operation involving a marginal constraint or a contact datum according to the methodologies mentioned below, one need only construct the appropriate matrix D and vector d and invoke the equations (IV-10)-(IV-12). Although the time step increase operation is similar in form, it is handled separately, for reasons that will be made explicit later. As far as the other operations are concerned, the equations (IV-10)-(IV-12) may be viewed as a subroutine which may be called to accomplish the operation, given the appropriate D and d.

The "information adjustment" view made explicit by equations (IV-10) and (IV-11) will later play a central role in showing that the operations to be discussed shortly may in fact be performed in any order, in accordance with the requirements made at the outset of this section.

Now, the different operations will be discussed in turn, with technical details consigned to appropriate appendices.

Marginal constraint. A marginal constraint refers to the assertion that some marginal distribution of the UPD is equal to a specified (Gaussian) distribution. This operation is intended to reflect a priori suppositions about target motion. Its application will take the following form. If $u = \{u_1, \ldots, u_k\}$ is a set of indices, let z(u) be the random vector

$$\begin{bmatrix} z_{u_1} \\ \vdots \\ z_{u_k} \end{bmatrix};$$

thus, z(u) is a "subvector" of the random vector Z_T defined by (IV-4). The marginal distribution for z(u) is desired to have first and second moments given by

$$E(z(u)) = \phi_u, \quad Var(z(u)) = \Phi_u. \tag{IV-13}$$

Thus, it is desired to constrain Z_T, in the sense of Appendix H. so that z(u) has the Gaussian distribution with mean ϕ_u and covariance matrix Φ_u. The distribution for Z_T obtained by constraining the UPD in this manner will be called the constrained prior distribution (CPD).

The "information adjustment" quantities needed to accomplish the marginal constraint (IV-13) are

$$D = \Phi_u^{-1} - (B_{uu})^{-1}, \tag{IV-14a}$$

$$d = \Phi^{-1}\phi_u - (B_{uu})^{-1}\beta_T(u), \tag{IV-14b}$$

where $B_{uu} = (B_T)_{uu}$ and $\beta_T(u)$ are defined analogously to C_{uu} and $\gamma_T(u)$ in equations (IV-12). Equations (IV-14) assume that no other marginal constraints are already incorporated in the modified distribution. These quantities may be used if contact data has been incorporated; however, if other marginal constraints have been applied, then the quantities given by equations (IV-16) below must be used.

Suppose it is desired to remove the marginal constraint (IV-13). A typical and important instance when this may happen is when Φ_u is a block diagonal matrix representing all the constraints which have been applied (and not subsequently removed). If no other constraints are to be "left behind," then form the quantities

$$D = (B_{uu})^{-1} - \Phi_u^{-1}, \tag{IV-15a}$$

$$d = (B_{uu})^{-1}\beta_T(u) - \Phi_u^{-1}\phi_u, \tag{IV-15b}$$

which are the negatives of those in equations (IV-14). Applying the equations (IV-10)-(IV-12) will remove the constraint's effects from (the representations in each domain of) the modified distribution. It is worth reemphasizing that (IV-13) is assumed to be the only marginal constraint (or all of the constraints, viewing Φ_u as described above) reflected by the modified distribution.

Note that the adjustments that are described by equations (IV-14) and (IV-15) are functions not only of the constraint parameters ϕ_u and Φ_u but of the UPD parameters β_T and B_T. In fact, Φ_u^{-1} and $(B_{uu})^{-1}$ are the information matrices for the marginal distribution for z(u) in the CPD and UPD, respectively. In applying equation (IV-10), the difference between these matrices is added to (or subtracted from, in the case of removal) that part of the "prior" information matrix corresponding to the index set u. The reader should beware that this is not a replacement operation. Similar remarks apply to the information vectors. This dependence on the UPD parameters underscores the fact that, theoretically, the marginal constraint operation is an operation on the UPD.

However, the operation suggested by equations (IV-12) and (IV-14) could, in fact, be applied to any (Gaussian) distribution, in particular the CPD. In the latter case, B_{uu} and β_T (u) should be replaced in (IV-14) by C_{uu} and γ_T (u), respectively. It is to be emphasized that doing this after applying a constraint via (IV-14) does not apply both constraints to the UPD.

If no constraints are present in the modified distribution--i.e., if the distribution is the UPD or only contact data have been incorporated--then the constraint (IV-13) may be applied by using the quantities (IV-14). However, if constraints are already present, then proceed as follows. Let $u = \{u_1, \ldots, u_k\}$ and $v = \{v_1, \ldots, v_l\}$ be (disjoint) index sets and, for notational simplicity, suppose that $u_i < v_j$ for all i and j. Suppose a constraint corresponding to u has already been entered, and it is desired to add a constraint corresponding to v. Let $u' = \{u_1, \ldots, u_k, v_1, \ldots, v_l\}$,

$$\tilde{S} = (B_{uu})^{-1} B_{uv},$$

$$\tilde{T} = B_{vv} - B_{uv}^T \tilde{S},$$

$$\tilde{L} = \begin{bmatrix} -\tilde{S} \\ I \end{bmatrix} (-\tilde{T}^{-1}) [-\tilde{S}^T \quad I],$$

where $B_{uv} = (B_T)_{uv}$ is defined according to equation (G-6). Then the required "information adjustment" quantities are:

$$D = \tilde{L} \ (+) \ \Phi_v^{-1}, \qquad \text{(IV-16a)}$$

$$d = \tilde{L} \beta_T \ (u') \ (+) \ \Phi_v^{-1} \phi_v, \qquad \text{(IV-16b)}$$

where the (+) sign has the same essential meaning as before, only this time v is the pertinent index set. It should be noted that using equations (IV-16) is equivalent to applying equations (IV-15) with u and then applying equations (IV-14) with u'; i.e., removing all constraints and then putting back all constraints, old and new.

If the constraints corresponding to u and v have been incorporated into the modified distribution and it is desired to remove the constraints corresponding to v, then the pertinent information adjustment quantities are the negatives of those in (IV-16).

The relations in this subsection are discussed in Appendix I.

Incorporation of contact data. A contact datum is a numerical quantity or set of quantities assigned to an observation against a target at one of the sampling times. Such an observation may come from within or from outside of the screen. Two types of observations are considered in the model, a position estimate (or "SPA") and a line-of-bearing (LOB) contact. Different information quantities are assigned to each.

A position estimate at time t refers to an estimate δ of the target position at time t together with a 2 x 2 positive definite symmetric matrix Δ. The uncertainty in the position estimate is presumed to be bivariate Gaussian with mean zero and covariance matrix Δ. Hence, if Z_τ is the target's track and $D_t(\delta)$ denotes the event that an observation at time t culminates in the position estimate δ for the target at time t, then

$$\mathrm{pr}\{D_t(\delta) \mid Z_\tau = Z\} = n(\delta; z_t, \Delta), \qquad \text{(IV-17)}$$

where z_t denotes that component of Z corresponding to target position at time t (see (IV-4)). According to equation (IV-7),

$$\mathrm{pr}\{Z_\tau = Z\} = n(Z; \gamma_\tau, C_\tau). \qquad \text{(IV-18)}$$

To obtain the posterior distribution,

$$\mathrm{pr}\{Z_\tau = Z \mid D_t(\delta)\} = n(Z; \tilde{\gamma}_\tau, \tilde{C}_\tau),$$

the distributions of (IV-17) and (IV-18) must be combined according to Proposition (H-4). From that proposition it follows that the information adjustment quantities required to implement equations (IV-10) and (IV-11) are

$$D = \Delta^{-1}, \qquad \text{(IV-19a)}$$

$$d = \Delta^{-1}\delta. \qquad \text{(IV-19b)}$$

The development of the necessary information adjustment quantities for a LOB contact, whose geometry is illustrated by Figure IV-1, proceeds a bit differently. The development is parallel to that which yielded the information quantities for LOB contacts in Chapter III.

A LOB contact is the assertion that at time t, the target was observed at bearing B (measured from north) from a sensor at position (u_t, v_t). Let R be a nominal range from the sensor to the target (cf. equation (E-4)); such a range might be a "range of the day." Let the uncertainty in the bearing measurement be denoted by ϵ_B, so $\epsilon_B = B - B_0$, where B_0 denotes the true bearing from the sensor to the target. Finally, denote the components of z_t, the target position at time t, by (x_t, y_t).

Begin by arguing, as in Appendix E, that if ϵ_B is small, then

$$\epsilon_B \cong \sin \epsilon_B = \sin(B-B_0)$$

$$= \sin B \cos B_0 - \cos B \sin B_0$$

$$= \sin B \frac{y_t - v_t}{R} - \cos B \frac{x_t - u_t}{R} ,$$

the last equality being evident by Figure IV-1, and implying that

$$R\epsilon_B = y_t \sin B - v_t \sin B - x_t \cos B + u_t \cos B. \qquad \text{(IV-20)}$$

Now if

$$D_t(\epsilon) = u_t \cos B - v_t \sin B$$

and

$$A_B = [0, \ldots, \cos B, -\sin B, \ldots, 0],$$

where A_B has zeros in all component slots except those corresponding to $z_t = (x_t, y_t)^T$, then (IV-20) can be rewritten as

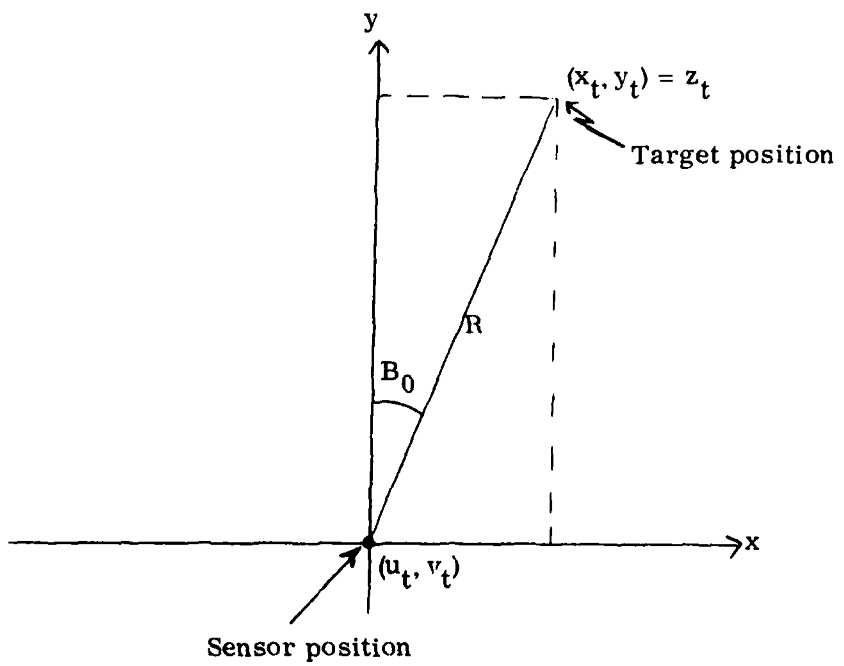
FIGURE IV-1

GEOMETRY OF A LOB CONTACT

Notes: 1) Grid is in nautical mile units, with positive y-axis as north.

2) B_0 is true bearing, as opposed to estimated bearing B.

3) Bearing error $\epsilon_B = B - B_0$ is Gaussian with mean 0 and variance σ_B^2.



$$D_t(\epsilon) = A_B Z_\tau + R\epsilon_B, \qquad \text{(IV-21)}$$

where Z_τ is, of course, given by (IV-4). If ϵ_B is viewed as having a one-dimensional Gaussian distribution with mean 0 and variance σ_B^2, then the version of equation (IV-17) corresponding to the LOB contact is

$$\mathrm{pr}\{D_t(\epsilon) = \delta \mid Z_\tau = Z\} = n(\delta; A_B Z_\tau, R^2\sigma_B^2). \qquad \text{(IV-22)}$$

(Note how the "range of the day" merely becomes a weight for the bearing uncertainty distribution.) Finally, combining the distributions of (IV-18) and (IV-22) according to Proposition (H-4) implies that the information adjustment quantities corresponding to the LOB contact are

$$D = A_B^T \frac{1}{R^2\sigma_B^2} A_B = \frac{1}{R^2\sigma_B^2}\begin{bmatrix} \cos^2 B & -\sin B \cos B \\ -\sin B \cos B & \sin^2 B \end{bmatrix} \qquad \text{(IV-23a)}$$

$$d = A_B^T \frac{1}{R^2\sigma_B^2}(u_t \cos B - v_t \sin B) = \frac{u_t \cos B - v_t \sin B}{R^2\sigma_B^2}\begin{bmatrix} \cos B \\ -\sin B \end{bmatrix}. \qquad \text{(IV-23b)}$$

A contact datum which has been incorporated into the modified distribution may be removed by using the negatives of the corresponding information adjustment quantities. Note that the information adjustment quantities do not depend on any prior covariance structure; hence, contact data may be incorporated in any order, without having to resort to special forms as in the case of the marginal constraints.

Time step increase. The purpose of this operation is to extend the time interval over which the target track is studied from $[0, t_\tau)$ to $[0, t_{\tau+1})$. Its effect on the distribution will be to increase the dimensions of the covariance matrix from $(2\tau+2) \times (2\tau+2)$ to $(2\tau+4) \times (2\tau+4)$ and the dimension of the mean vector from $(2\tau+2)$ to $(2\tau+4)$. Like the marginal constraint operation, the time step increase is technically applicable to the UPD. However, it will be seen shortly that the adjustments to the UPD may be used to update the modified distribution directly. The formulas below apply to $\tau > 1$; the cases $\tau = 0$ and $\tau = 1$ differ only slightly and are made explicit in Appendix J, which contains the derivations of all the formulas.

The UPD will be considered first. From the relations (IV-5) comes

$$\beta_{\tau+1} = \left[\frac{\beta_\tau}{\beta_\tau(\tau) + \delta_\tau \nu_\tau}\right]. \qquad \text{(IV-24)}$$

Next,

$$B_{\tau+1} = \begin{bmatrix} B_\tau & H_\tau \\ H_\tau^T & b_{\tau+1,\tau+1} \end{bmatrix}, \qquad \text{(IV-25)}$$

where

$$H_\tau = \begin{bmatrix} b_{1,\tau+1} \\ \vdots \\ b_{\tau,\tau+1} \end{bmatrix},$$

and $b_{\tau+1,\tau+1}$ may be constructed from B_τ according to the recursion of equations (IV-6). In order to describe how the time step increase affects the normalization constant and the information domain representation of the UPD, some auxiliary quantities must be derived and discussed.

Let $S = B_\tau^{-1}H_\tau$ and $T = b_{\tau+1,\tau+1} - H_\tau^T S$. As shown in Appendix J, the relations (IV-6) imply that these have the particularly simple forms:

$$S = \begin{bmatrix} 0 \\ -e_\tau I \\ (1+e_\tau)I \end{bmatrix}, \qquad \text{(IV-26a)}$$

where I is the 2 x 2 identity, and 0 is a $(2\tau-4) \times 2$ zero matrix; and

$$T = b_{\tau+1,\tau+1} + e_\tau b_{\tau+1,\tau-1} - (1+e_\tau)b_{\tau+1,\tau}. \qquad \text{(IV-26b)}$$

This may be rewritten, using equations (IV-6), as

$$T = \delta_\tau^2 \Gamma_\tau - e_\tau^2 \delta_{\tau-1}^2 \Gamma_{\tau-1}. \qquad \text{(IV-26c)}$$

The derivation of equation (IV-26c) is given in Note 3 at the end of this chapter.

Now, the normalization constants satisfy

$$K^{o}_{\tau+1} = K^{o}_{\tau} + 2 \log 2\pi + \log \det T. \tag{IV-27}$$

(Here and henceforth, a superscript o signifies that the quantity pertains to the UPD.) Next, let

$$L = \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1} [-S^T \quad I].$$

It follows from the equation (IV-26a) that the $(2\tau+4) \times (2\tau+4)$ matrix L may be written as

$$L = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix},$$

where Q is the 6 x 6 matrix defined by

$$Q = \begin{bmatrix} e_\tau I \\ -(1+e_\tau)I \\ I \end{bmatrix} T^{-1} [e_\tau I \; -(1+e_\tau)I \quad I].$$

Thus, due to the simplifications of (IV-26), L is essentially a 6 x 6 matrix, a fact not necessarily true of the analogous quantity $\tilde{L}$ employed in equation (IV-16). (These various simplifications say that the time step increase $\tau \to \tau+1$ involves only the time steps $\tau-1, \tau$ and $\tau+1$. This will be seen to hold for the modified distribution as well.)

Now the information domain counterparts to (IV-24) and (IV-25) are

$$\mathscr{I}^{o}_{\tau+1} = \begin{bmatrix} \mathscr{I}^{o}_{\tau} & 0 \\ 0 & 0 \end{bmatrix} + L, \tag{IV-28a}$$

$$\mathscr{X}^{o}_{\tau+1} = \begin{bmatrix} \mathscr{X}^{o}_{\tau} \\ \hline 0 \end{bmatrix} + L \, \beta_{\tau+1}, \tag{IV-28b}$$

Now, let (C_τ, γ_τ) and $(\mathscr{I}_\tau, \mathscr{X}_\tau^{\cdot})$ be respectively the covariance domain and information domain representation of the modified distribution for Z_τ. In particular all constraints and contact data reflected by this distribution are presumed to pertain to time preceding t_τ. The following formulas give the corresponding parameters for $Z_{\tau+1}$, subject to the same modifications.

$$C_{\tau+1} = \begin{bmatrix} C_\tau & C_\tau S \\ (C_\tau S)^T & T+S^T C_\tau S \end{bmatrix}. \quad \text{(IV-29a)}$$

where S and T are given by equations (IV-26);

$$\gamma_{\tau+1} = \left[\frac{\gamma_\tau}{(1+e_\tau)\gamma_\tau(\tau) + \delta_\tau \nu_\tau - e_\tau(\gamma_\tau(\tau-1) + \delta_{\tau-1}\nu_{\tau-1})} \right]. \quad \text{(IV-29b)}$$

$$\mathscr{I}_{\tau+1} = \begin{bmatrix} \mathscr{I}_\tau & 0 \\ 0 & 0 \end{bmatrix} + L, \quad \text{(IV-30a)}$$

where L is taken from equation (IV-28a);

$$\mathscr{X}_{\tau+1}^{\cdot} = \begin{bmatrix} \mathscr{X}_\tau^{\cdot} \\ \hdashline 0 \end{bmatrix} + L\beta_{\tau+1} = \begin{bmatrix} \mathscr{X}_\tau^{\cdot} \\ \hdashline 0 \end{bmatrix} + \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1}(\delta_\tau \nu_\tau - e_\tau \delta_{\tau-1}\nu_{\tau-1}). \quad \text{(IV-30b)}$$

The normalization constants are related by

$$K_{\tau+1} = K_\tau + 2 \log 2\pi + \log \det T. \quad \text{(IV-31)}$$

The equations (IV-29), (IV-30), and (IV-31) are derived in Appendix J. It is also shown there that if the modified distribution coincides with the UPD (i.e., no constraints or contact data have been incorporated), then the above formulas reduce to the formulas (IV-24), (IV-25), (IV-27), and (IV-28). The special cases $\tau = 0$ and $\tau = 1$ have also been consigned to Appendix J.

Note that the methodology of equations (IV-10)-(IV-12) is not applicable in the case of a time step increase, since dimensions are being increased in this case. The resemblance of (IV-30a) and (IV-30b) to (IV-10) and (IV-11) is worth noting, however, if only for independent interest.

Overview

According to the modified distribution which was the subject of the preceding section, the probability (density) of a particular target track over $[0, t_\tau)$ is given by

$$\mathrm{pr}\{Z_\tau = Z\} = \exp\left\{ -\frac{1}{2}\left[(Z-\gamma_\tau)^T \mathscr{J}_\tau (Z-\gamma_\tau) + K_\tau \right] \right\} \qquad \text{(IV-32)}$$

(cf. equation (IV-7)). Thus, only knowledge of γ_τ, $\mathscr{J}_\tau$, and K_τ are needed to make probabilistic statements about the target's track over the time interval in question. However, it is evident from equations (IV-12b) and (IV-12c) that knowledge of C_τ is required to modify γ_τ and K_τ so as to reflect the effects of a marginal constraint or contact on the target. Moreover, it is clear from the discussion of the time step increase operation which appeared in the previous section that B_τ must first be subjected to the time step increase before the latter is applied to the modified distribution; see equations (IV-26b) and (IV-29a). (Actually, only the last four columns of B_τ, together with the values of $\nu_{\tau-1}$ and ν_τ are needed for the time step increase.) B_τ and β_τ also play a role in the marginal constraint operation.

Therefore, in order to be able to invoke equation (IV-32) in its proper form when desired, it is necessary to keep proper track of the following parameters:

$$\beta_\tau, B_\tau, \gamma_\tau, C_\tau, \mathscr{J}_\tau, K_\tau .$$

Table IV-1 shows the equations from the previous section which accomplish the appropriate observation for each of these quantities. In Table IV-1, a "modification" refers to an operation related to a marginal constraint or a contact datum; it is assumed that the appropriate quantities D and d have been constructed. Recall also that the time step increase requires that certain quantities S, T be constructed from B_τ. Furthermore, equation (IV-29a) can be somewhat simplified, as pointed out in Appendix J (cf. equation (J-16)ff.).

TABLE IV-1

EQUATIONS GOVERNING OPERATIONS ON THE DIOU FOR REQUIRED DISTRIBUTION PARAMETERS

Note: Equation numbers are those appearing earlier in this chapter

	Modification with (D, d)	Time Step Increase
β_τ	---	(IV-24)
B_τ	---	(IV-6)
C_τ	(IV-12a)	(IV-29a)
γ_τ	(IV-12b)	(IV-29b)
$\mathcal{I}_\tau$	(IV-10)	(IV-30a)
K_τ	(IV-12c)	(IV-31)

Finally, suppose that $(D_1, d_1), \ldots, (D_n, d_n)$ are information adjustment quantities corresponding to a sequence of modifications to the modified distribution within a given time step. Then equations (IV-10) and (IV-11) imply

$$\tilde{\mathcal{I}}_\tau = \mathcal{I}_\tau (+) D_1 (+) \cdots (+) D_n, \qquad \text{(IV-33)}$$

$$\tilde{\mathcal{D}}'_\tau = \mathcal{D}'_\tau (+) d_1 (+) \cdots (+) d_n. \qquad \text{(IV-34)}$$

In equation (IV-33), each quantity to the right of a (+) sign is to be incorporated into a $(2\tau+2) \times (2\tau+2)$ matrix; e. g.,

$$\mathcal{I}_\tau \ (+) \ D_1 \ (+) \ D_2 = (\mathcal{I}_\tau \ (+) \ D_1) \ (+) \ D_2.$$

A similar remark applies to equation (IV-34). However, it is evident that this "addition" is commutative. A corollary of this fact is that the order of modification operations within a given time step--i. e., for a fixed interval $[0, t_\tau)$ under study--is immaterial.

(Cf, the discussion of equations (I-12) through (I-16).) In fact, more may be asserted. The derivations of equations (J-14) and (J-18) in Appendix J show that a modification may be applied either before or after a time step increase $\tau \rightarrow \tau+1$, so long as the sampling times involved in the modification precede t_τ. Therefore, as required, the operations described in the previous section may be executed without regard to order.

Relationship to detection and localization measures. This chapter closes by indicating briefly how the elements of the target motion model figure into the detection and localization measures described in the previous chapters.

The detection probabilities derived in Chapter II are computed against "sample paths" of the target motion process which has been the subject of this chapter. More precisely, such a sample path is a draw from a normal distribution with mean γ_τ and variance C_τ; cf. equations (IV-4) and (IV-7). To reconcile notation between the two chapters, if

$$Z_\tau = \begin{bmatrix} z_0 \\ \vdots \\ z_\tau \end{bmatrix}$$

is such a track, then snapshot probabilities against this track are computed for times $t_0, \ldots, t_\tau$ and are denoted in the notation of Chapter II by $p_j(z_{t_i}, t_i)$ for $i = 0, \ldots, \tau$ (see p. 23).

These snapshot probabilities serve to construct the observation probabilities p_n used in Chapter III to construct the "expected" information quantities and concomitant localization measures. (See the footnote on p. 42.) Note that for the purpose of defining the localization measures (and only for this purpose), the model assumes that target's course has constant velocity (course and speed). Thus, despite apparent similarities, the information adjustment quantities for LOB contacts and position estimate constructed in this chapter are fundamentally different from the information quantities constructed in Chapter III.

Notes

We conclude with some miscellaneous notes.

Note 1. This note demonstrates that the process given by (IV-2) is not Gaussian. If τ_1 and τ_2 are 2-vectors and $q = \exp\left[-\int_{t_1}^{t_2} \mu(x)dx\right]$, and if ϕ denotes the characteristic function, then

$$\phi(\tau_1, \tau_2) = \int \int \exp\left[i(\tau_1^T \xi_{t_2} + \tau_2^T \xi_{t_2})\right] d\,PR(\xi_{t_1}, \xi_{t_2})$$

$$= q \int \exp\left[i(\tau_1 + \tau_2)^T \xi_{t_1}\right] d\,Pr(\xi_{t_1})$$

$$+ (1-q) \left[\int \exp(i\,\tau_1^T \xi_{t_1}) d\,Pr(\xi_{t_1})\right] \left[\int \exp(i\,\tau_2^T \xi_{t_2}) d\,Pr(\xi_{t_2})\right],$$

so

$$\phi(\tau_1, \tau_2) = q \exp\left[-\frac{1}{2}(\tau_1 + \tau_2)^T \Gamma_{t_1} (\tau_1 + \tau_2)\right] + (1-q) \exp\left[-\frac{1}{2}(\tau_1^T \Gamma_{t_1} \tau_1 + \tau_2^T \Gamma_{t_2} \tau_2)\right],$$

using the fact (from reference [p], e. g.) that the characteristic function of a Gaussian distribution with mean m and covariance Λ is

$$\phi(\tau) = \exp(i\, m^T \tau - \frac{1}{2} \tau^T \Lambda \tau).$$

That the expression for $\phi(\tau_1, \tau_2)$ above cannot be put in this form (the former being a weighted sum of exponentials) implies that ϵ_{t_1} and ϵ_{t_2} are not jointly Gaussian. Hence, neither is the exponential correlation process, which serves as the physical model for target motion.

Note 2. The purpose of this note is to show that the covariance matrix $B_\tau = Var(Z_\tau)$ satisfies the relations (IV-6). To begin, set $b_{00} = cov(z_0, z_0) = Var(z_0)$ and $h_{00} = 0$, and define:

$$b_{ij} = cov(z_i, z_j) \qquad i, j = 0, 1, \ldots, \tau$$

$$h_{ij} = cov(z_i, \delta_{j-1} V_{j-1})$$

$$= cov(z_i, \delta_{j-1} \epsilon_{j-1}).$$

Now,

$$b_{0,j} = \text{cov}(z_0, z_0 + \sum_{k=0}^{j-1} \delta_k V_k) = \text{cov}(z_0, z_0)$$

since the initial distribution is independent of the GOU process. Thus, if $h_{0j} = 0$ for all j, then the quantities b_{0j} and h_{0j} satisfy the relations (IV-6b) and (IV-6d) for all j.

Next, note that $\text{cov}(V_i, V_j) = \text{cov}(\epsilon_i, \epsilon_j)$ for all i and j. Hence,

$$h_{11} = \text{cov}(z_1, \delta_0 V_0) = \text{cov}(\delta_0 V_0, \delta_0 V_0)$$

$$= \delta_0^2 \Gamma_0 = h_{01} + \delta_0^2 \Gamma_0,$$

and, if $i > 1$,

$$h_{ii} = \text{cov}(z_i, \delta_{i-1} V_{i-1})$$

$$= \text{cov}(z_{i-1} + \delta_{i-1} V_{i-1}, \delta_{i-1} V_{i-1})$$

$$= h_{i-1,i} + \delta_{i-1}^2 \Gamma_{i-1},$$

which is (IV-6f). Now, if $k < l$, then

$$\text{cov}(\epsilon_k, \epsilon_l) = \exp\left[-\int_{t_k}^{t_l} \mu(x)dx\right] \Gamma_k$$

$$= \exp\left[-\int_{t_k}^{t_{l-1}} \mu(x)dx\right] \exp\left[-\int_{t_{l-1}}^{t_l} \mu(x)dx\right] \Gamma_k$$

$$= \exp\left[-\int_{t_{l-1}}^{t_l} \mu(x)dx\right] \text{cov}(\epsilon_k, \epsilon_{l-1})$$

$$= \exp(-\delta_{l-1}\mu_{l-1}) \text{cov}(\epsilon_k, \epsilon_{l-1});$$

so if $1 \leq i < j$, then

$$h_{ij} = \mathrm{cov}(z_i, \delta_{j-1}\epsilon_{j-1})$$

$$= \sum_{k=0}^{i-1} \delta_k \, \delta_{j-1} \, \mathrm{cov}(\epsilon_k, \epsilon_{j-1})$$

$$= \sum_{k=0}^{i-1} \delta_k \, \delta_{j-1} \exp(-\delta_{j-2}\mu_{j-2}) \, \mathrm{cov}(\epsilon_k, \epsilon_{j-2})$$

$$= \frac{\delta_{j-1}}{\delta_{j-2}} \exp(-\delta_{j-2}\mu_{j-2}) \sum_{k=0}^{i-1} \delta_k \, \delta_{j-2} \, \mathrm{cov}(\epsilon_k, \epsilon_{j-2})$$

$$= e_{j-1} \, h_{i,j-1},$$

which is (IV-6d).

Finally,

$$b_{ij} = \mathrm{cov}(z_i, z_j)$$

$$= \mathrm{cov}(z_i, z_{j-1} + \delta_{j-1}V_{j-1}) \qquad \text{(by IV-3)}$$

$$= \mathrm{cov}(z_i, z_{j-1}) + \mathrm{cov}(z_i, \delta_{j-1}V_{j-1})$$

$$= b_{i,j-1} + h_{ij};$$

giving (IV-6b); and (IV-6c) holds because

$$b_{ij} = \mathrm{cov}(z_i, z_j)$$

$$= \mathrm{cov}(z_j, z_i)^T$$

$$= (b_{ji})^T.$$

Thus, the relations (IV-6) are established.

Note 3. This note uses the relations (IV-6) to derive equation (IV-26c) from (IV-26b). The derivation is due to Dr. D. P. Kierstead.

A simple induction argument using equations (IV-6) shows that the matrices b_{ij} are symmetric for all i, j. Then, beginning with equation (IV-26b),

$$T = b_{\tau+1,\tau+1} + e_\tau b_{\tau+1,\tau+1} - (1+e_\tau)b_{\tau+1,\tau}$$

$$= b_{\tau+1,\tau} + h_{\tau+1,\tau+1} + e_\tau b_{\tau+1,\tau-1} - (1+e_\tau)b_{\tau+1,\tau}$$

$$= h_{\tau+1,\tau+1} + e_\tau (b_{\tau-1,\tau+1} - b_{\tau,\tau+1})$$

$$= h_{\tau+1,\tau+1} + e_\tau (b_{\tau-1,\tau} + h_{\tau-1,\tau+1} - b_{\tau,\tau} - h_{\tau,\tau+1})$$

$$= h_{\tau+1,\tau+1} + e_\tau (b_{\tau-1,\tau} + h_{\tau-1,\tau+1} - b_{\tau,\tau-1} - h_{\tau\tau} - h_{\tau,\tau+1})$$

$$= h_{\tau+1,\tau+1} + e_\tau (h_{\tau-1,\tau+1} - h_{\tau\tau} - h_{\tau,\tau+1})$$

$$= h_{\tau,\tau+1} + \delta_\tau^2 \Gamma_\tau + e_\tau h_{\tau-1,\tau+1} - e_\tau h_{\tau\tau} - e_\tau h_{\tau,\tau+1};$$

applying equation (IV-6d) to the first, third, and fifth terms gives

$$T = \delta_\tau^2 \Gamma_\tau + e_\tau^2 h_{\tau-1,\tau} - e_\tau^2 h_{\tau\tau}.$$

Applying equation (IV-6f) to the middle term and making the evident cancellation gives

$$T = \delta_\tau^2 \ {}_\tau - e_\tau^2 \delta_{\tau-1}^2 \Gamma_{\tau-1}$$

which is equation (IV-26c).

Symbol	Definition
Var(X)	Covariance matrix of the random variable X
Cov(x, y)	Covariance between x and y
$\dim(\nu)$	Dimension of the vector ν
o	As a superscript, implies that quantity pertains to the UPD
0	As a subscript, symbolizes an initial or reference value
M^T	Transpose of the matrix M
M_{uu}	Submatrix of the matrix M corresponding to index set u (see equation (G-2))
M_u	Matrix of columns of M containing elements of M_{uu} (see equation (G-3))
I	Identity matrix
I_u	Matrix M such that M_{uu} = I and all other elements are zero (see Appendix G)
x(u)	Subvector of the vector x corresponding to the index set u (see equation (G-1))
$\oplus$	Power sum (see equation (A-3))
(+)	Submatrix or subvector addition (see equations (G-4) and (G-5))
Σ	Sum notation (is zero if lower index exceeds upper index)
π	Product notation (is one if lower index exceeds upper index)

Variables and Abbreviations

Symbol	Chapter/Appendix	Definition or Use
A(B)	E	Rotation
AA	A	Array attenuation
AN	A	Omni-directional ambient noise

Symbol	Chapter/Appendix	Definition or Use
B	E, IV	Bearing
B_n	E	Orientation of SPA n
B_τ	IV	UPD covariance matrix at time t_τ
BB	A	Broadband
BL	A	Background level
BN	A	Background noise
b_{ij}	IV	$Cov(z_i, z_j)$
C_τ	IV	Covariance matrix at time t_τ of the modified target distribution
CPD	IV	Constrained prior distribution
c'	B	Index of time of peak instantaneous or snapshot probability (unimodal formula)
$c'(\xi)$	B	c' for particular sampling outcome ξ
cdp	II	Cumulative detection probability
clp	III	Cumulative localization performance
D	IV	Information adjustment matrix
$D_t(\delta)$	IV	Detection event at time t yielding position estimate δ
$D_t(\epsilon)$	IV	Detection event at time t yielding bearing with error ϵ
DI	A	Directivity index
DIOU	IV	Discrete IOU Process
D(t)	II	Detection function
d	IV	Information adjustment vector

Symbol	Chapter/Appendix	Definition or Use
$E_c(t)$	II	Expected number of contacts at time t
E_n	III	Aggregate measurement noise vector
e_j	IV	See equations (IV-6)
$F_{n,n+1}$	E	Used to construct $L_{n,n+1}$, $Z_{n,n+1}$ for bearing observation
FOM	A	Figure of merit
$G_{n,n+1}$	E	Companion to $F_{n,n+1}$
GOU	IV	Generalized Ornstein-Uhlenbeck Process
H_τ	IV	$b_{1,\tau+1}$ $b_{\tau,\tau+1}$
HVU	II	High Value Unit
h_{ij}	IV	$\text{Cov}(z_i, \delta_{j-1}\epsilon_{j-1})$
$\hat{I}$	G	The matrix $[I \quad 0]$
I_N	II	Intensity of interfering noise
I_S	II	Intensity of target signal
IFKF	-	Information Flow Kalman Filter
IFKF (CO)	-	IFKF for correlated observations
IOU	IV	Integrated Ornstein-Uhlenbeck Process
$\mathscr{I}$	III, IV	Information matrix
$\hat{\mathscr{I}}$	III	"Expected" information matrix
$\mathscr{I}_{n,n+1}$	III	Information matrix increment in IFKF (CO)
J_N	II	$E(I_N)$
J_S	II	$E(I_S)$
K_τ	IV	Normalization constant

Symbol	Chapter/Appendix	Definition or Use
L	IV	Information matrix increment for time step increase
L_n	III	Aggregate measurement coefficient matrix
$L_{n,n+1}$	III	Auxiliary quantity used in IFKF(CO)
$\tilde{L}$	IV	Information matrix increment in marginal constraint operation
$l_j(\xi)$	B	Lapsed time since last sample before one at t_j, in outcome ξ
l_n	III	Measurement coefficient matrix
$\tilde{l}_n$	III	Auxiliary quantity in IFKF (CO) related to l_n
N_n	III	$\mathrm{Var}(E_n)$
NB	A	Narrowband
NL	A	Noise level
O	D	Orthogonal matrix
OU	IV	Ornstein-Uhlenbeck Process
P	III	Var(X)
P	G	Permutation matrix
PF	II, B	Cumulative failure probability, 1-cdp
PIM	II	Position and Intended Movement
PL	A	Propagation loss
p_a	II	Probability of availability
p_n	III	Probability that n^{th} observation is made
p_i	B	Snapshot or instantaneous detection probability (for unimodal formula)

Symbol	Chapter/Appendix	Definition or Use
$p_j(z, t)$	II	(Snapshot) probability that sensor j is in contact with target at position z at time t
Q	IV	Matrix to which L reduces
QQQ	II, B	Variable, related to Q(n), used in SCREEN iteration for PF
Q(n)	B	Auxiliary variable used in actual iteration for PF (unimodal case)
R	E, IV	Nominal range from sensor to target
RL	A	Reverberation level
RRR	II, B	Variable, related to R(n), used in SCREEN iteration for PF
R_n	III	Square root matrix in IFKF (CO)
R_n	E	Range n
R(n)	B	Auxiliary variable used in actual iteration for PF (unimodal case)
$\mathbb{R}$	III	Kalman filter residual
$\mathscr{R}$	III	Information residual
$\hat{\mathscr{R}}$	III	"Expected" information residual
$\mathscr{R}_{n, n+1}$	III	Information residual increment in IFKF (CO)
r_j	II	Expected time between looks
$S, \tilde{S}$	III, IV, G	Auxiliary quantities used in relating inverses of partitioned matrices
SPA	E, IV	Search Probability Area
SSS	II, B	Variable, related to S(n), used in SCREEN iteration for PF
S(n)	B	Auxiliary variable used in actual iteration for PF (unimodal formula)

Symbol	Chapter/Appendix	Definition or Use
S(n, j)	B	Subset of sampling outcomes
SE	A	Signal excess (FOM-PL)
SL	A	Source level
SNR	A	Signal-to-noise ratio
s_x	E	x-component of target velocity
s_y	E	y-component of target velocity
T, $\tilde{\tilde{T}}$	III, IV, G	Auxiliary quantities used in relating inverses of partitioned matrices
TEMP1	II	Variable used in SCREEN iteration for PF
TMA	I, III	Target Motion Analysis
TS	A	Target Strength
$U_{n,n+1}$	E	Used to construct $L_{n,n+1}$, $Z_{n,n+1}$ for SPA
UPD	IV	Unconstrained prior distribution
u, u_n, u_t	E, IV	x-component of sensor position
u	IV, App. I	Index set
$u_j(\xi)$	B	Time until next sample after one at t_j, in outcome ξ
u(t)	IV	OU velocity process
V_n	III	Submatrix of N_{n+1}
$V_{n,n+1}$	E	Companion to $U_{n,n+1}$
V_t, V_n	IV	Target velocity
v, v_n, v_t	E, IV	y-component of sensor position
W	A	Bandwidth
W_n	III	$w_n^T w_n$

Symbol	Chapter/Appendix	Definition or Use
$W_t(s)$	II	Historical weight function
w_n	III	Weight matrix for observation n
X	III	State vector
$\mathscr{X}$	III, IV	Information vector
$\hat{\mathscr{X}}$	III	"Expected" information vector
$\mathscr{X}_{n,n+1}$	III	Information vector increment in IFKF (CO)
x(t)	IV	$\int_0^t u(s)ds$
x_n, x_t	E, IV	x-component of target position
y_n, y_t	E, IV	y-component of target position
Z_n	III	Aggregate observation vector
$Z_{n,n+1}$	III	Auxiliary quantity used in IFKF (CO)
Z_τ	IV	Target track through time t_τ
z(t)	II	Target position at time t
z_n	III	n^{th} observation vector
$z_i = z_{t_i}$	IV	Position at time t_i
β_i	B	$1-\exp[-\lambda(t_i - t_{i-1})]$
β_τ	IV	UPD mean vector at time t_τ
Γ_n, Γ_t	IV	Covariance function in GOU
γ_t	IV	$E(Z_T)$ according to modified target distribution
Δ	IV	Covariance matrix of position estimate error
δ	IV	Position estimate
$\delta(\cdot)$	II	Dirac delta function

Symbol	Chapter/Appendix	Definition or Use
δt	II	Interval length
ϵ_{i}	III	Measurement noise vector
ϵ_t	IV	Random component of target velocity
ϵ_B	E, IV	Error in bearing measurement
ϵ_R	E	Error in range measurement
Λ	D	Diagonal matrix
λ	II	Detection sampling rate
$\mu_i, \mu(x)$	IV	Function used to describe exponential memory
ν_t, ν_n	IV	Mean of velocity process
ξ_i	B	Sampling process
π_i	II	"Look" probability
ρ_g	II	Correlation constant for group g
ρ_{ij}	III	Correlation matrix between i^{th} and j^{th} observations
ρ_1, ρ_2	E	SPA correlations
ρ_B	E	Correlation between bearing measurements
ρ_R	E	Correlation between range measurements
σ^2	II	Signal fluctuation variance
$\sigma^2_{1,n}, \sigma^2_{2,n}$	E	Principal variances of n^{th} SPA
σ^2_i	III	$Var(\epsilon_i)^2$
σ^2_B	IV	Variance in bearing measurement
Φ_u	IV	Covariance matrix for marginal constaint distribution
ϕ	IV, Note 1	Characteristic function

Symbol	Chapter/Appendix	Definition or Use
ϕ_u	IV	Mean of marginal constraint distribution
ϕ_n	B	$\max_{1\leq i\leq n} p_i$
ψ_n	II, B	$\min\{p_{n-1}, p_n\}$
ω	II	Mean integration time

APPENDIX A

SONAR EQUATION USED IN SCREEN

This appendix describes the form of the sonar equation used in SCREEN. The formulation given here is compatible with the Acoustic Baseline, reference [i]. The active and passive equations are described in the following sections, and have been implemented in FORTRAN on a PRIME 400 minicomputer as parts of SCREEN.

Detailed descriptions of target, sensor, and environment files beyond anything mentioned in this appendix may be found in the User's Manual, reference [a].

Passive Sonar Equation

The equation used for narrowband figure of merit, FOM (NB), and broadband figure of merit, FOM (BB), correspond to equations (2-11) and (2-12), respectively, of reference [i]. The equation is:

$$FOM = SL - BN - \begin{cases} RD & \text{for narrowband} \\ RD - 10 \log W & \text{for broadband,} \end{cases} \qquad \text{(A-1)}$$

where the terms of (A-1) are defined and stored as follows. (Note: the SCREEN program is interested in signal excess, SE, which equals FOM-PL where PL is the propagation loss.)

SL = target source level. This is stored in the target file, TARGXX, as a function of time. Each target file corresponds to a specific target type and screen penetration tactic.

RD = recognition differential. This is stored in the sensor file, SENSXX. Each sensor in the file corresponds to a specific sonar type and frequency or frequency band of operation. If narrowband, RD is stored; if broadband, RD-10 log W is stored.

BN = background noise. The formula for background noise is somewhat more complicated than appears in reference [i] because all interfering noise sources are explicitly considered. The equation used is the power sum:

$$BN = (AN-DI) \oplus (LN-SNR) \oplus (RN-AA-PL) \bigoplus_{i=1}^{N} (RN_i-AA_i-PL_i), \quad \text{(A-2)}$$

where $\oplus$ denotes power sum

$$10^{a \oplus b} = 10^{\log_{10} a} + 10^{\log_{10} b}, \quad \text{(A-3)}$$

the first three quantities correspond to the formula for BN which appears below equation (2-11) in reference [a], and the remaining quantities are the interference due to task force noise sources such as HVUs.

The two quantities AN (omnidirectional ambient noise) and DI (directivity index) are stored in the environmental file. The first, AN, is indexed to correspond to a specific propagation environment; and the second, DI, is indexed to correspond to a particular set of sonar aspect contours. Note that a given set of sonar contours is for a specific sonar, operating frequency and operating mode.

A nominal value of (LN-SNR) $\oplus$ (RL-AA-PL) is stored in the sensor file SENSXX. This is a function of time. Changes in this value for different beam directions are stored in the environmental file as the aspect contour labeled "DELTA (LN-SNR)." The actual value of this quantity at a given beam direction is the algebraic sum of the nominal value stored in the sensor file SENSXX and the value given by the aspect contour.

The values $(RN_i-AA_i-PL_i)$ are computed separately for each HVU unit. The radiated noise of the i^{th} HVU, RN_i, is stored in the sensor file, and is frequency dependent. The array attenuation against this noise, AA_i, and the propagation loss, PL_i, are determined from the beam pattern of the array and the convoy geometry. The received energy is reduced by $\cos\left(\frac{x}{BW} \times 60\right)$ for up to 3/2 times the beamwidth, where x is the angle between the bearing of target (signal) and the bearing of the interference (noise).

Active Sonar Equation

The equation used for Active Figure of Merit corresponds to that given by equation (2-7) of reference [i]. The equation is:

$$FOM = SL + TS - BL. \quad \text{(A-4)}$$

SL = sensor source level. db//1μ PA at 1 yard. This is stored in the environmental file along with the RL curve described below.

TS = target strength. db//1μ PA at 1 yard. This is stored in the target file.

Only the Background Level requires special comments.

BL = background level. db//1μ PA. The background level depends on whether the system is reverberation limited or noise limited. The equation for BL is (2-8) of reference [i].

$$BL = RML \oplus NML$$

where $\oplus$ denotes power sum,

$$RML = RL \oplus RDr,$$

and

$$NML = NL \oplus RDn;$$

RDr = recognition differential in a reverberation background. This is incorporated in the RL curve, described below.

RDn = recognition differential in a noise background. This is stored as RD in the sensor file.

NL = noise level. This is given by (2-1) of reference [i]. The form of equation is identical to the passive equation for BN, equation (A-2), and is evaluated in precisely the same way.

RL = reverberation level. This is provided for specific environments in much the same form as the propagation loss. See reference [i], Figure 3-1 for an example. The quantity RL + (RDr-RDn) is tabulated in the environment file much as is the PL curve.

With the data stored as indicated, the effective calculation of BL is:

$$BL = (RL + (RDr - RDn)) + (NL) + RDn,$$

which can be seen to be equivalent to the above expression for BL.

APPENDIX B

THE JUMP PROCESS WITH RANDOM SAMPLING

This appendix develops and discusses the cumulative detection probability algorithm for the jump process with random sampling. Most of the material in what follows appeared originally in reference [k].

For the purposes of this appendix, it is assumed that snapshot detection probabilities, $p_1, \ldots, p_n$ and sampling probabilities $\theta_1, \ldots, \theta_n$ have been determined as discussed in the main text. The underlying detection process is a (λ, σ) jump process. It is desired to compute the cumulative failure probability, PF(n).

The following notational conventions are used in this appendix. Sums, Σ, and products, Π, are zero and unity, respectively, whenever the lower bound exceeds the upper bound. The symbol $\bar{x}$ denotes (1-x).

The Unimodal Formula

According to reference [b], Theorem IV-1, if a unimodal jump process is sampled at a subset of times t_i, p_i is the detection probability at t_i, the jump probability β_i is defined by

$$\beta_i = 1 - e^{-\lambda (t_i - t_{i-1})},$$

and c is the index such that $p_c = \max_{1 \le j \le n} p_j$, then the cumulative detection probability at time t_n, P_n, is given by

$$P_n = 1 - \bar{p}_c \prod_{i=1}^{c-1} (1 - \beta_{i+1} p_i) \prod_{i=c+1}^{n} (1 - \beta_i p_i). \tag{B-1}$$

Note the obvious fact that if the process is unimodal over all the sampling times $\{t_i\}$, then it is also unimodal over a subset of sampling times.

A consequence of the mathematical development given below is that (B-1) is equivalent to the following iteration. Let $PF(n) = 1-P_n$. Then

$$\left.\begin{aligned}
PF(0) &= 1, \\
PF(1) &= \bar{p}_1, \\
PF(n) &= \frac{PF(n-1)}{\bar{\phi}_{n-1}} \bar{\phi}_n \left[\bar{\psi}_n + \psi_n e^{-\lambda(t_n - t_{n-1})}\right] \quad \text{for } n \geq 2, \\
\text{where} & \\
\phi_n &= \max_{1 \leq i \leq n} p_i, \\
\psi_n &= \min(p_{n-1}, p_n).
\end{aligned}\right\} \quad \text{(B-2)}$$

It is believed that this is a new result for the unimodal jump process.

To see that (B-1) and (B-2) are equivalent, note the following enumeration of special cases:

(a) $\phi_n = p_{n-1}$. In this case, $c = n-1$ and $p_n \leq p_{n-1}$ so that $\psi_n = p_n$. Thus, the bracketed term in (B-2) is the term $(1-\beta_n p_n)$ in (B-1). (Note that $1-\beta_n p_n = \bar{p}_n + p_n e^{-\lambda(t_n - t_{n-1})}$).

(b) $\phi_n = p_n$ and $\phi_{n-1} = p_{n-1}$. In this case $c = n$ and $\psi_n = p_{n-1}$. The bracketed term in (B-2) is the term $(1-\beta_n p_{n-1})$ in (B-1).

(c) $\phi_{n-1} \geq p_{n-1}$. In this case $c < n-1$ and unimodality requires that $p_n \leq p_{n-1}$, so $\psi_n = p_n$ and the bracketed term in (B-2) is the term $(1-\beta_n p_n)$ in (B-1).

Random Sampling

The SCREEN cdp algorithm is designed to combine snapshot detection probabilities p_i at times t_i into a cumulative failure probability PF(n) in the event that the detection process is sampled randomly, i.e., not all p_i's play. We will now derive the formula underlying the algorithm.

The following conventions and assumptions are in force. The snapshot probabilities are assumed to be unimodal; that is, there is some index c' such that

$$p_j \leq p_{j+1} \quad \text{if } j < c',$$

$$p_j \geq p_{j+1} \quad \text{if } j \geq c'.$$

The sampling process is denoted

$$\xi_j = \begin{Bmatrix} 1 & \text{if a detection opportunity occurs at time } t_j \\ 0 & \text{if not} \end{Bmatrix}.$$

For a given set of times $t_1, \ldots, t_n$, there are 2^n possible outcomes of this process. Hence, for notational simplicity, denote this set by 2^n. It is evident that the probability of a given outcome is given by

$$\prod_{j=1}^{n} (\xi_j \pi_j + \bar{\xi}_j \bar{\vartheta}_j), \tag{B-3}$$

where π_j is the probability that $\xi_j = 1$. For a given outcome ξ, let $l_j(\xi)$ be the lapsed time since the last sample prior to the one at t_j and let $u_j(\xi)$ be the time until the next sample after the one at t_j. If no sample occurs at t_j, then set $l_j = u_j = 0$. Finally, let $c(\xi)$ be that index c such that

$$p_{c(\xi)} = \max\{p_i \mid \xi_i = 1\}.$$

It is possible that $c(\xi)$ is not unique. However, assuming unimodality, the development which follows is not dependent on the exact choice of $c(\xi)$. A convention might be set according to

$$c(\xi) = \min\{k : p_k = \max\{p_i \mid \xi_i = 1\}\} .$$

To begin the development, note that equations (B-1) and (B-3) imply that

$$PF(n) = \sum_{\xi \in 2^n} PF(n, \xi), \tag{B-4}$$

where

$$PF(n,\xi) = \prod_{i=1}^{n} \left[\xi_i \, \theta_i + \bar{\xi}_i \, \bar{\theta}_i\right] \bar{p}_c \left[\prod_{i=1}^{c-1} (\bar{p}_i + p_i \, e^{-\lambda u_i})\right] \left[\prod_{i=c+1}^{n} (\bar{p}_i + p_i \, e^{-\lambda l_i})\right]$$

and where it is to be understood that u_i, l_i, and c are functions of ξ. (Recall that the formula (B-4) assumes unimodality in the probabilities $\{p_i\}$ as described above. Clearly, any subset of these probabilities is also unimodal.) The essence of our approach is to consider equation (B-4) in two cases--$n \leq c'$ and $n > c'$-- and to obtain recursive expressions in each case. The actual SCREEN cdp algorithm will be obtained in the next section by splicing the two recursions.

Let $\xi \in S(n, j)$, where $1 \leq j < n$ and

$$S(n,j) = \{\xi \in 2^n : \xi_n = 1; \ \xi_j = 1; \ \xi_i = 0, \ j+1 \leq i \leq n-1\} .$$

Then

$$
PF(n,\xi) = \begin{cases}
\theta_n \left[\prod_{i=j+1}^{n-1} \bar{\theta}_i \right] \theta_j \left[\prod_{i=1}^{j-1} (\xi_i \theta_i + \bar{\xi}_i \bar{\theta}_i) \right] \\
\quad \bar{p}_n (\bar{p}_j + p_j e^{-\lambda (t_n - t_j)}) \left[\prod_{i=1}^{j-1} (\bar{p}_i + p_i e^{-\lambda u_i}) \right], & \text{if } n \leq c' \\
\theta_n \left[\prod_{i=j+1}^{n-1} \bar{\theta}_i \right] \theta_j \left[\prod_{i=1}^{j-1} (\xi_i \theta_i + \xi_i \theta_i) \right] (\bar{p}_n + p_n e^{-\lambda (t_n - t_j)}) \\
\quad \bar{p}_c \left[\prod_{i=1}^{c-1} (\bar{p}_i + p_i e^{-\lambda u_i}) \right] \left[\prod_{i=c+1}^{j} (\bar{p}_i + p_i e^{-\lambda l_i}) \right], & \text{if } n > c',
\end{cases}
$$

where the quantities u_i, l_i, and c in the two products are functions of ξ. It follows that

$$
\sum_{\xi \in S(n,j)} PF(n,\xi) = \theta_n \left[\prod_{i=j+1}^{n-1} \bar{\theta}_i \right] \theta_j \begin{cases}
\bar{p}_n (\bar{p}_j + p_j e^{-\lambda (t_n - t_j)}) \\
\quad \sum_{\xi \in 2^{j-1}} \left[\prod_{i=1}^{j-1} (\xi_i \theta_i + \bar{\xi}_i \bar{\theta}_i) \right] \left[\prod_{i=1}^{j-1} (\bar{p}_i + p_i e^{-\lambda u_i}) \right], & \text{if } n \leq c' \\
(\bar{p}_n + p_n e^{-\lambda (t_n - t_j)}) \sum_{\xi \in 2^{j-1}} \left[\prod_{i=1}^{j-1} (\xi_i \theta_i + \bar{\xi}_i \bar{\theta}_i) \right] \\
\quad \bar{p}_c \left[\prod_{i=1}^{c-1} (\bar{p}_i + p_i e^{-\lambda u_i}) \right] \left[\prod_{i=c+1}^{j} (\bar{p}_i + p_i e^{-\lambda l_i}) \right], & \text{if } n > c'.
\end{cases}
$$

Note that the restriction of the sum to the outcomes on $t_1, \ldots, t_{j-1}$ does not affect the values of u_i, l_i, and c, with this exception: if, for some outcome $\xi \in S(n,j)$, t_i is the last time a sample occurs before t_j, then $u_i = t_j - t_i$, rather than zero. With this understanding, there follows that

$$\sum_{\xi \in S(n,j)} PF(n,\xi) = \begin{cases} \theta_n \left[\prod_{i=j+1}^{n-1} \bar{\theta}_i\right] \dfrac{\bar{p}_n}{\bar{p}_j} \left(\bar{p}_j + p_j e^{-\lambda(t_n - t_j)}\right) \theta_j \, P\{F(j) \mid \xi_j = 1\}, & \text{if } n \le c' \\ \theta_n \left[\prod_{i=j+1}^{n-1} \bar{\theta}_i\right] \left(\bar{p}_n + p_n e^{-\lambda(t_n - t_j)}\right) \theta_j \, P\{F(j) \mid \xi_j = 1\}, & \text{if } n > c' \end{cases} \tag{B-5}$$

where F(j) denotes the event that no detection was made through time t_j. Note that

$$PF(j) = \bar{\theta}_j \, PF(j-1) + \theta_j \, P\{F(j) \mid \xi_j = 1\}, \tag{B-6}$$

and that

$$PF(n) = \bar{\theta}_n \, PF(n-1) + \theta_n \bar{p}_n \left[\prod_{i=1}^{n-1} \bar{\theta}_i\right] + \sum_{j=1}^{n-1} \sum_{\xi \in S(n,j)} PF(n,\xi). \tag{B-7}$$

Now, for k = 1, ..., n, let

$$\phi_k = \max\{p_i : 1 \le i \le k\}.$$

Then equations (B-5), (B-6), and (B-7), together with the unimodality assumption, imply that

$$PF(n) = \bar{\theta}_n \, PF(n-1) + \theta_n \bar{p}_n S(n) + \theta_n \bar{\phi}_n [R(n) + Q(n)], \tag{B-8a}$$

where

$$Q(n) = \begin{cases} \sum_{j=1}^{n-1} \frac{p_j}{\bar{\phi}_j} e^{-\lambda(t_n - t_j)} \left[\prod_{i=j+1}^{n-1} \bar{\theta}_i \right] [PF(j) - \bar{\theta}_j \, PF(j-1)], & \text{if } n < c' \\ \\ \sum_{j=1}^{n-1} \frac{p_n}{\bar{\phi}_n} e^{-\lambda(t_n - t_j)} \left[\prod_{i=j+1}^{n-1} \bar{\theta}_i \right] [PF(j) - \bar{\theta}_j \, PF(j-1)], & \text{if } n > c' \end{cases} \tag{B-8b}$$

$$R(n) = \begin{cases} \sum_{j=1}^{n-1} \frac{\bar{p}_j}{\bar{\phi}_n} \left[\prod_{i=j+1}^{n-1} \bar{\theta}_j \right] [PF(j) - \bar{\theta}_j \, PF(j-1)], & \text{if } n \le c' \\ \\ \sum_{j=1}^{n-1} \frac{\bar{p}_n}{\bar{\phi}_n} \left[\prod_{i=j+1}^{n-1} \bar{\theta}_i \right] [PF(j) - \bar{\theta}_j \, PF(j-1)], & \text{if } n > c', \end{cases} \tag{B-8c}$$

and

$$S(n) = \prod_{i=1}^{n-1} \bar{\theta}_i . \tag{B-8d}$$

Difference equations may be written for the quantities in equations (B-8). Equation (B-8a) gives a difference equation for PF(n). Clearly, $\phi_{n+1} = \max\{\phi_n, p_{n+1}\}$ and

$$S(n+1) = \bar{\theta}_n \, S(n). \tag{B-9}$$

If $n < c'$, so $p_n \le p_{n+1}$, then

$$Q(n+1) = e^{-\lambda(t_{n+1} - t_n)} \left\{ \bar{\theta}_n \, Q(n) + \frac{p_n}{\bar{\phi}_n} [PF(n) - \bar{\theta}_n \, PF(n-1)] \right\}, \tag{B-10a}$$

$$R(n+1) = \bar{\theta}_n \, R(n) + \frac{p_n}{\bar{\phi}_n} [PF(n) - \bar{\theta}_n \, PF(n-1)]. \tag{B-10b}$$

If $n > c'$, then $p_n \geq p_{n+1}$, and

$$Q(n+1) = e^{-\lambda(t_{n+1}-t_n)} \left\{ \frac{p_{n+1}}{p_n} \bar{\theta}_n Q(n) + \frac{p_{n+1}}{\bar{\phi}_n} [PF(n) - \bar{\theta}_n PF(n-1)] \right\}. \quad \text{(B-11a)}$$

$$R(n+1) = \frac{\bar{p}_{n+1}}{\bar{p}_n} \bar{\theta}_n R(n) + \frac{\bar{p}_{n+1}}{\bar{\phi}_n} [PF(n) - \bar{\theta}_n PF(n-1)]. \quad \text{(B-11b)}$$

Note in this last case that $\phi_n = \phi_{n+1}$.

<u>The SCREEN Algorithms for Cumulative Detection Probability</u>

The SCREEN cdp algorithm essentially involves using equations (B-10) when $p_n \leq p_{n+1}$ and equations (B-11) when $p_n \geq p_{n+1}$, regardless of whether the snapshot probabilities are unimodal. (Note that when $p_n = p_{n+1}$, equations (B-10) and (B-11) give the same result.) Such an algorithm may be described as follows. Initialize by $PF(0) = 1$, $S(1) = 1$, $R(0) = Q(0) = 0$, $\phi_0 = 1$. Then the iteration is described by:

$$\left.\begin{aligned}
\bar{\phi}_n &= \min\{\bar{\phi}_{n-1}, \bar{p}_n\} \\
\psi_{n+1} &= \min\{p_n, p_{n+1}\} \\
Q(n+1) &= \psi_{n+1}\, e^{-\lambda(t_{n+1}-t_n)} \left\{ \frac{\bar{\theta}_n}{p_n} Q(n) + \frac{1}{\bar{\phi}_n} [PF(n) - \bar{\theta}_n PF(n-1)] \right\} \\
R(n+1) &= \bar{\psi}_{n+1} \left\{ \frac{\bar{\theta}_n}{\bar{p}_n} R(n) + \frac{1}{\bar{\phi}_n} [PF(n) - \bar{\theta}_n PF(n-1)] \right\} \\
S(n+1) &= \bar{\theta}_n S(n) \\
PF(n) &= \bar{\theta}_n PF(n-1) + \theta_n p_n S(n) + \theta_n \bar{\phi}_n [R(n) + Q(n)].
\end{aligned}\right\} \quad \text{(B-12)}$$

The actual SCREEN cdp algorithm, described by equations (II-13) in the main text, is a slight recasting of equations (B-12) done with machine implementation in mind. The modifications indicated there for use with nonunimodal snapshot probabilities are intended to reduce discrepancies in the value of PF(n) which arise from applying a "unimodal"-based algorithm in a nonunimodal setting.

The algorithm of (B-12) gives only an approximation to PF(n), even in the unimodal case. Discrepancies arise because the algorithm essentially ignores the different forms that Q(n) and R(n) take when $n > c'$ vice when $n \leq c'$ (cf. equations (B-8b) and (B-8c)). The remainder of this appendix will examine these discrepancies more closely. For notational simplicity in what follows, denote c' by c. Recall then that $p_c = \max_i p_i$.

The first discrepancy arises in the transition from step c to step c+1 and we will examine it first. Let $Q(c+1)_t$ be the true value of Q(c+1), given by setting n = c+1 in the second of equations (A-8b). Let $Q(c+1)_e$ be the value of Q(c+1) obtained from Q(c) by applying the algorithm of equations (B-12), i.e., by "naively" applying equation (B-11a). Define $R(c+1)_t$ and $R(c+1)_e$ similarly, and let $PF(c+1)_t$ and $PF(c+1)_e$ be obtained from equation (B-8a). Then $PF(c+1)_t$ is the true value of PF(c+1), and $PF(c+1)_e$ is the value which would be returned by the algorithm of equations (B-12). It is these quantities we wish to compare. Now,

$$Q(c+1)_e - Q(c+1)_t = \sum_{j=1}^{c-1} \left[\frac{p_{c+1}}{p_c}\frac{p_j}{\bar{\phi}_j} - \frac{p_{c+1}}{\bar{\phi}_{c+1}}\right] e^{-\lambda(t_{c+1}-t_j)}\left[\prod_{i=j+1}^{c} \bar{\theta}_i\right][PF(j) - \bar{\theta}_j PF(j-1)] \quad \text{(B-13)}$$

and

$$R(c+1)_e - R(c+1)_t = \sum_{j=1}^{c-1} \left[\frac{\bar{p}_{c+1}}{\bar{p}_c}\frac{\bar{p}_j}{\bar{\phi}_j} - \frac{\bar{p}_{c+1}}{\bar{\phi}_{c+1}}\right] \left[\prod_{i=j+1}^{c} \bar{\theta}_i\right] [PF(j) - \bar{\theta}_j PF(j-1)]. \quad \text{(B-14)}$$

Combining equations (B-8a), (B-13), and (B-14) obtains

$$PF(c+1)_e - PF(c+1)_t = \theta_{c+1} \sum_{j=1}^{c-1} \left[\left(\frac{\bar{\phi}_{c+1}}{p_c} \cdot \frac{p_j}{\bar{\phi}_j} - 1\right) p_{c+1} e^{-\lambda(t_{c+1}-t_j)} + \left(\frac{\bar{\phi}_{c+1}}{\bar{p}_c}\frac{\bar{p}_j}{\bar{\phi}_j} - 1\right)\bar{p}_{c+1}\right]\left[\prod_{i=j+1}^{c} \bar{\theta}_i\right][PF(j) - \bar{\theta}_j PF(j-1)]. \quad \text{(B-15)}$$

Now, $\phi_{c+1} = p_c$, and $\phi_j = p_j$ for $j = 1, \ldots, c$; so equation (B-15) reduces to

$$PF(c+1)_e - PF(c+1)_t = \theta_{c+1} \sum_{j=1}^{c-1} \left[\frac{\bar{p}_c}{p_c} \frac{p_j}{\bar{p}_j} - 1 \right] p_{c+1} e^{-\lambda (t_{c+1} - t_j)} \left[\prod_{i=j+1}^{c} \bar{\theta}_i \right] (PF(j) - \bar{\theta}_j PF(j-1)). \tag{B-16}$$

For all j, $p_j \leq p_c$, and hence

$$1 \geq \frac{\bar{p}_c}{p_c} \frac{p_j}{\bar{p}_j} .$$

It then follows from equation (B-16) that $PF(c+1)_e < PF(c+1)_t$. Thus, the algorithm of equations (B-12) gives an optimistic view of cdp at time c+1, even in the unimodal case.

Note that for all $j = 1, \ldots, c$,

$$-1 \leq \frac{\bar{p}_c}{p_c} \frac{p_j}{\bar{p}_j} - 1 \leq 0,$$

with the extremes occurring at $p_j = 0$ and $p_j = p_c$, respectively. Thus, the terms for which the quantity

$$\frac{\bar{p}_c}{p_c} \frac{p_j}{\bar{p}_j} - 1$$

is the largest, also contain the largest exponential terms

$$e^{-\lambda (t_{c+1} - t_j)} \left[\prod_{i=j+1}^{c} \bar{\theta}_i \right] .$$

This "exponential damping" keeps the discrepancy within tolerable limits. Moreover, since $\phi_n = p_c$ if $n \geq c$, one may show by retracing the foregoing development that

$$PF(n+1)_e - PF(n+1)_t = \frac{\theta_{n+1}}{\theta_{c+1}} e^{-\lambda(t_{n+1}-t_{c+1})} \left[\prod_{i=c+1}^{n} \bar{\theta}_i \right] (PF(c+1)_e - PF(c+1)_t),$$

so that further exponential damping--and hence an improved approximation to $PF(c+1)_t$--takes place as the algorithm progresses.

The foregoing discussion pertains only to the unimodal case. However, as mentioned in Chapter II, studies have shown the SCREEN cdp algorithm to give good approximation even in the unimodal case. See reference [j].

Remark. The iterative algorithm for PF(n) given above makes it appear that six quantities must be carried from step n-1 to step n: ϕ_{n-1}, p_{n-1}, PF(n-1), Q(n), R(n), and S(n). However, this need not be the case, as PF(n-1) is recoverable from the other quantities. To see this, consider the formula for R(n) given by equation (B-8c). Note that if $n \leq c'$, then $p_j = \phi_j$ for $j = 1, \ldots, n-1$. Thus, if

$$c_n = \begin{cases} 1 & \text{if } n \leq c' \\ \dfrac{\bar{p}_n}{\bar{\phi}_n} & \text{if } n \geq c' \end{cases},$$

then, using equation (B-8d), equation (B-8c) may be written as

$$R(n) = c(n) \sum_{j=1}^{n-1} \frac{S(n)}{S(j+1)} [PF(j) - \bar{\theta}_j PF(j-1)]$$

$$= c(n)\, S(n) \sum_{j=1}^{n-1} \left[\frac{PF(j)}{S(j+1)} - \frac{PF(j-1)}{S(j)} \right]$$

$$= c(n)\, S(n) \left(\frac{PF(n-1)}{S(n)} - 1 \right),$$

whence

$$PF(n-1) = \frac{R(n)}{c(n)} + S(n) \qquad \text{(B-17)}$$

follows.

APPENDIX C

SUMMARY OF DETECTION ALGORITHMS IN SCREEN

by David P. Kierstead

This appendix summarizes how the detection algorithms are implemented in SCREEN. Detailed program operations are discussed in reference [a].

Snapshot Detection Probabilities

The snapshot detection probabilities (for individual sensors) are obtained from equation (II-7):

$$p(t) = \int_{-\infty}^{10 \log_{10} E[\mathscr{D}(t)]} n(y; RD, \sigma^2)dy$$

$$= \int_{-\infty}^{[10 \log_{10} E[\mathscr{D}(t)] - RD]/\sigma} n(y; 0, 1)dy. \qquad \text{(C-1)}$$

According to equation (II-6),

$$E[\mathscr{D}(t)] = \frac{\int_{-\infty}^{t} E[I_S(s)]dW(s)}{\int_{-\infty}^{t} E[I_N(S)]dW(s)},$$

where

$$dW(s) = \frac{1}{\omega} \exp\left[- \frac{(t-s)}{\omega}\right] ds.$$

In the SCREEN model, it is assumed that I_N remains constant over short time intervals. This assumption, together with the exponential weight function W(s), makes it possible to compute:

$$\int_{-\infty}^{t} E[I_N(s)]dW(s) \approx \int_{-\infty}^{t} E[I_N(t)]dW(s)$$

$$= E[I_N(t)].$$

Thus,

$$E[\mathcal{D}(t)] = \frac{J_S(t)}{E[I_n(t)]} ,$$

where

$$J_S(t) = \int_{-\infty}^{t} E[I_S(s)]dW(s),$$

and 10 $\log_{10}$ $E[I_N(t)]$ and 10 $\log_{10}$ $E[I_S(t)]$ are the terms BN and SL-PL (resp., BLM* and SL+TS-PL) of the passive (resp., active)sonar equation. $J_S(t)$ may be computed via the iteration of text equation (II-9a).

In order to implement equation (C-1), the quantity

$$U(t) = [10 \log_{10} E[\mathcal{D}(t)]-RD]/\sigma$$

must be computed. This computation is performed by the function SENPRB according to the following algorithm.

Recall the sonar equation

$$SE = SL-PL-BN-RD.$$

(This is the form for a passive sensor. In the case of an active sensor, SL and BN must be replaced by SL+TS and BLM, respectively, throughout.)

In order to compute $U(t+\delta)$, we set:

---

* BLM is a modified form of the BL in Appendix A. The modification makes the active sonar equation appear as

$$FOM = SL+TS - BLM-RD.$$

$$H_W = 10 \log_{10}(e^{-\delta/\omega}),$$

$$H_N = 10 \log_{10}(1-e^{-\delta/\omega}),$$

$$H(t) = 10 \log_{10} J_S(t) - BN-RD-H_N.$$

Then

$$H(t+\delta) = 10 \log_{10}\left[\frac{J_S(t+\delta)}{10^{BN/10} \cdot 10^{RD/10} \cdot 10^{HN/10}}\right]$$

$$= 10 \log_{10}\left[\frac{J_S(t) \cdot 10^{H_W/10} + 10^{HN/10} \cdot 10^{SL-PL/10}}{10^{BN/10} \cdot 10^{RD/10} \cdot 10^{HN/10}}\right]$$

$$= [H(t) + H_W] \oplus [SL-PL-BN-RD]$$

$$= [H(t) + H_W] \oplus [SE];$$

where $\oplus$ is the power sum operation. This gives an iteration for H(t) (called HISTRY in SCREEN) from which U(t) may be computed according to the formula

$$U(t) = 10 \log_{10}(E[\mathcal{D}(t)]) - RD$$

$$= 10 \log_{10}(J(t)) - BN - RD$$

$$= H(t) + H_N.$$

CDPs

The computation of cdp for a single sensor is guided by the subroutine PIMAP and function SCRINF, with most of the computations done by SENPRB and CUMPRB. In order to obtain meaningful results from the iteration of equation (II-13), it is necessary to discretize time to intervals (t_n, t_{n+1}) which are much finer than the program time steps. These intervals are determined in PIMAP according to the criterion:

No interval (t_n, t_{n+1}) may be long enough that the distance between the target and any sensor decreases by more than 1 nm during the interval.

Once the time intervals have been chosen, PIMAP calls SCRINF at successive subincrements. SCRINF calls SENPRB to obtain the snapshot PD p_n. If $p_n \neq 0$, then CUMPRB is called to implement equation (II-13) (with a slight modification). In the special case that $p_n = 0$, the iteration of equation (II-13) does not work. In this case, only QQQ must be changed (to be ready for the next iteration). The adjustment is: $QQQ = \exp[-\lambda(t_n - t_{n-1})] \cdot QQQ$. Equation (II-13) assumes unimodality but gives a good approximation to CDP even in the nonunimodal case if the following change is made:

In the event that a peak value of p_j has been passed and $p_n > p_{n-1}$ (so unimodality is violated and the p_i's are increasing), ψ_n must be set equal to p_n and $\exp[-\lambda(t_n - t_{n-1})]$ must be replaced by $\exp[-\lambda(t_n - t_{n-2})]$. *

Group Correlation

Once the snapshot PDs have been computed for each sensor in a group, they are combined (as described in the second section of Chapter II) to produce a snapshot PD for the group. These PDs may be called forth using PDSTEP.

There are two alternative methods for computing CDP for the group:

(1) the group PDs could be used directly, or

(2) CDP could be computed for each sensor in the group, and the results combined (using the correlation coefficient) to yield the group CDP.

The second alternative is used in SCREEN.

* This is an empirical correction. For a discussion of its effect, see reference [j].

APPENDIX D

CORRELATION MATRICES

This appendix presents a discussion of correlation matrices pertinent to their use in this report and, in particular, Chapter III.

Let Y be a random variable with covariance matrix Var(Y). There is an orthogonal matrix O such that

$$\mathrm{Var}(Y) = O \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \sigma_n^2 \end{bmatrix} O^T,$$

where σ_i^2 are the principal variances, and O^T is the transpose of O. Then the "standard deviation" matrix of Y is defined by

$$\sigma(Y) = O \begin{bmatrix} \sigma_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \sigma_n \end{bmatrix} O^T.$$

If $Y_1 = \begin{bmatrix} y_{11} \\ \cdot \\ \cdot \\ y_{1m} \end{bmatrix}$ and $Y_2 \begin{bmatrix} y_{21} \\ \cdot \\ \cdot \\ y_{2n} \end{bmatrix}$ are vector-valued random variables with covariance matrix

$$\mathrm{Cov}(Y_1, Y_2) = \begin{bmatrix} & \vdots & \\ \cdots & \mathrm{Cov}(y_{1i}, y_{2j}) & \cdots \\ & \vdots & \end{bmatrix}_{\substack{i = 1, \ldots, m \\ j = 1, \ldots, n}},$$

then the correlation matrix $\rho(Y_1, Y_2)$ between Y_1 and Y_2 is defined by

$$\rho(Y_1, Y_2) = \sigma(Y_1)^{-1}\mathrm{Cov}(Y_1, Y_2)\sigma(Y_2)^{-1}. \tag{D-1}$$

Note that $\rho(Y_1, Y_2)$ need not be symmetric even if $m = n$. If $Y_1 = Y_2$, then $\rho(Y_1, Y_2)$ is the identity matrix.

Next, consider a sequence of vector random variables

$$X_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

such that x_i and y_i, which may also be vector-valued, are uncorrelated:

$$\mathrm{Var}X_i = \begin{bmatrix} \mathrm{Var}(x_i) & 0 \\ 0 & \mathrm{Var}(y_i) \end{bmatrix}, \quad \mathrm{Cov}(x_i, y_i) = 0.$$

(The following discussion will pertain to X_i with two uncorrelated components. The generalization to an arbitrary number of uncorrelated components is straightforward.) Suppose componentwise correlations are specified:

$$\rho(x_i, x_j) = \rho^1_{ij}, \ \rho(y_i, y_j) = \rho^2_{ij}, \qquad i, j = 1, \ldots .$$

Then

$$\mathrm{Cov}(X_i, X_j) = \begin{bmatrix} \mathrm{Cov}(x_i, x_j) & \mathrm{Cov}(x_i, y_j) \\ \mathrm{Cov}(y_i, x_j) & \mathrm{Cov}(y_i, y_j) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma(x_i)\rho^1_{ij}\sigma(x_j) & 0 \\ 0 & \sigma(y_i)\rho^2_{ij}\sigma(y_j) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma(x_i) & 0 \\ 0 & \sigma(y_i) \end{bmatrix} \begin{bmatrix} \rho^1_{ij} & 0 \\ 0 & \rho^2_{ij} \end{bmatrix} \begin{bmatrix} \sigma(x_j) & 0 \\ 0 & \sigma(y_j) \end{bmatrix} . \qquad \text{(D-2)}$$

Note also that since

$$\sigma(X_i) = \begin{bmatrix} \sigma(x_i) & 0 \\ 0 & \sigma(y_i) \end{bmatrix},$$

it follows from equations (D-1) and (D-2) that

$$\rho(X_i, X_j) = \begin{bmatrix} \rho^1_{ij} & 0 \\ 0 & \rho^2_{ij} \end{bmatrix}.$$

Now, let O_i be a sequence of orthogonal matrices, and let $Y_i = O_i X_i$. Then $\mathrm{Var}(Y_i) = O_i \mathrm{Var}(X_i) O_i^T$, so

$$\sigma(Y_i) = O_i \begin{bmatrix} \sigma(x_i) & 0 \\ 0 & \sigma(y_i) \end{bmatrix} O_i^T.$$

The correlation between Y_i and Y_j is given by

$$\rho(Y_i, Y_j) = \sigma(Y_i)^{-1} \mathrm{Cov}(Y_i, Y_j) \sigma(Y_j)^{-1}$$

$$= O_i \begin{bmatrix} \sigma(x_i)^{-1} & 0 \\ 0 & \sigma(y_i)^{-1} \end{bmatrix} O_i^{-1} \cdot O_i \mathrm{Cov}(X_i, X_j) O_j^T \cdot O_j \begin{bmatrix} \sigma(x_j)^{-1} & 0 \\ 0 & \sigma(y_j)^{-1} \end{bmatrix} O_j^{-1}$$

$$= O_i \begin{bmatrix} \sigma(x_i)^{-1} & 0 \\ 0 & \sigma(y_i)^{-1} \end{bmatrix} \mathrm{Cov}(X_i, X_j) \begin{bmatrix} \sigma(x_j)^{-1} & 0 \\ 0 & \sigma(y_j)^{-1} \end{bmatrix} O_j^{-1}$$

$$= O_i \begin{bmatrix} \rho_{ij}^1 & 0 \\ 0 & \rho_{ij}^2 \end{bmatrix} O_j^T \ . \qquad \text{(D-3)}$$

Equation (D-3) follows from equation (D-2) and the fact that the matrices O_i are orthogonal.

If $\rho_{ij}^m \rho_{jk}^m = \rho_{ik}^m$ for $m = 1, 2$ and $i \le j \le k$ then $\rho(Y_i, Y_j)\rho(Y_j, Y_k) = \rho(Y_i, Y_k)$, for $i \le j \le k$. Conversely, consider the following observation, due to Dr. F. P. Engel. If $\rho_{ij} : 1 \le i \le j$ is a collection of matrices satisfying

(i) ρ_{ii} is the identity for all i;

(ii) $\rho_{ij}\rho_{jk} = \rho_{ik}$ for $i \le j \le k$;

then there is a sequence of matrices A_n, $n = 1, 2, \ldots$, such that

$$\rho_{ij} = \prod_{k=i+1}^{j} A_k . \qquad \text{(D-4)}$$

Indeed, $A_k = \rho_{k,k+1}$. (For example, in Appendix E, typically $A_k = \rho^{t_{k+1}-t_k}$ where $t_1 \leq t_2 \leq \ldots$ is a sequence of sampling times.) Moreover, if

(iii) $\rho_{ij} = \rho_{i+k,\, j+k}$ for all i, j, k,

then clearly there is a matrix A such that

$$\rho_{ij} = A^{j-i}.$$

APPENDIX E

THE KALMAN ALGORITHM IN SCREEN

This appendix shows how the correlated-observation algorithm of Chapter III given by equations (III-13) applies to sequences of observations of the types in reference [g]. Observations of this type are also addressed in the SCREEN model. The "SPA" will be addressed first, and given the most extensive treatment. It will be shown next that bearing/range and bearing observations' information adjustment schemes are special cases of the scheme derived for a SPA. In fact, all observations will be seen to have essentially the form of a SPA.

In each example below, the state vector X is

$$\begin{bmatrix} x_0 \\ y_0 \\ s_x \\ s_y \end{bmatrix},$$

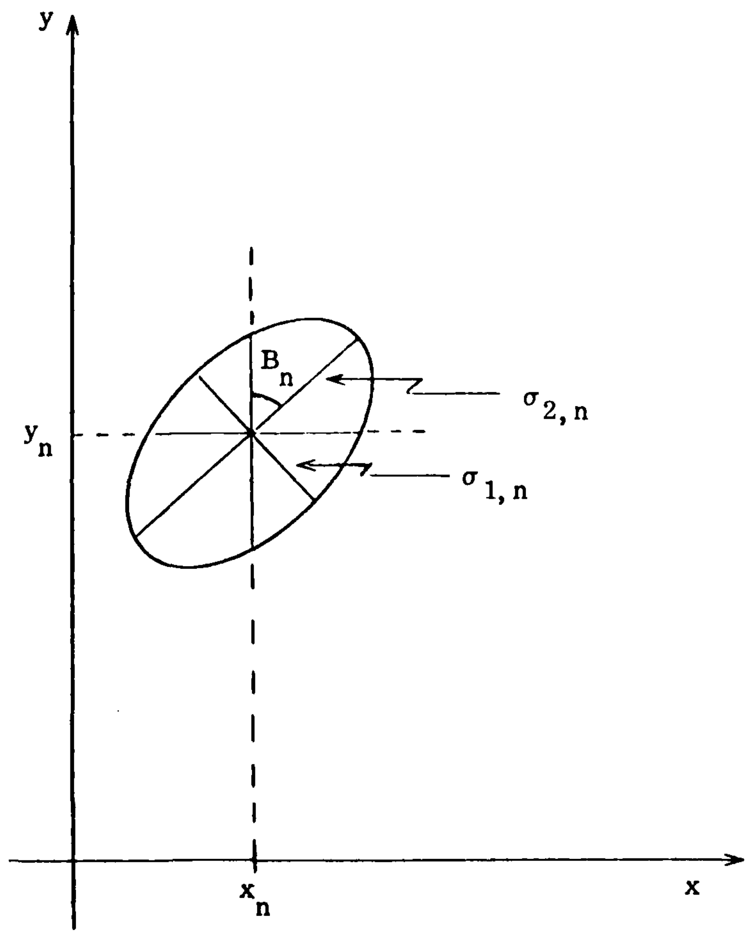
where (x_0, y_0) is the target position at a reference time t_0, and (s_x, s_y) is the (constant) velocity of the target. Such targets are the focus of reference [g] and the basis of SCREEN localization performance measures. The actual target motion model in SCREEN is more complex (see Chapter IV), but localization measures within that context have not been developed as this report goes to press.

SPA

A SPA (the acronym usually stands for "search probability area") is a position probability ellipse, centered at a point (x_n, y_n), which has variances $(\sigma_{1,n})^2$ and $(\sigma_{2,n})^2$ along the axes, and a specified orientation B_n, which will be taken to be the bearing of the $\sigma_{2,n}$ axis. See Figure E-1. (The subscript n is an observation index in the sense of equation (III-1). The equation (III-1) quantities corresponding to the SPA at time t_n just described are

FIGURE E-1

GEOMETRY OF A SPA



$$z_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$l_n = \begin{bmatrix} 1 & 0 & t_n - t_0 & 0 \\ 0 & 1 & 0 & t_n - t_0 \end{bmatrix};$$

The "standard deviation" matrix for ϵ_n is given by

$$\sigma_n = A(B_n)^T \begin{bmatrix} \sigma_{1,n} & 0 \\ 0 & \sigma_{2,n} \end{bmatrix} A(B_n)$$

where, for an angle B

$$A(B) = \begin{bmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{bmatrix}.$$

If successive principal axes are correlated in time--with respective correlation constants ρ_1 and ρ_2--then the correlation between ϵ_i and ϵ_j for $i \leq j$ is

$$\rho_{ij} = A(B_i)^T \begin{bmatrix} \rho_1^{|t_j - t_i|} & 0 \\ 0 & \rho_2^{|t_j - t_i|} \end{bmatrix} A(B_j).$$

These correlation matrices satisfy the conditions (III-8). (See Appendix D for a discussion of such correlation matrices.)

Using the orthogonality of the matrices $A(B_n)$ and $A(B_{n+1})$, the invocation of equations (III-14) and (III-15) becomes straightforward. In order to express the results more clearly, define the auxiliary quantities $U_{n,n+1}$ and $V_{n,n+1}$ by

$$U_{n,n+1} = \begin{bmatrix} \left(1-\rho_1^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} \sigma_{1,n+1}^{-1} & 0 \\ 0 & \left(1-\rho_2^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} \sigma_{2,n+1}^{-1} \end{bmatrix}, \tag{E-1a}$$

$$V_{n,n+1} = \begin{bmatrix} \left(1-\rho_1^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} \rho_1^{(t_{n+1}-t_n)} \sigma_{1,n}^{-1} & 0 \\ 0 & \left(1-\rho_2^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} \rho_2^{(t_{n+1}-t_n)} \sigma_{2,n}^{-1} \end{bmatrix} \tag{E-1b}$$

Then the equation (III-15) quantities are given by

$$L_{n,n+1} = A(B_{n+1})^T \left[U_{n,n+1}A(B_{n+1})-V_{n,n+1}A(B_n) \;\middle|\; t_{n+1}U_{n,n+1}A(B_{n+1})-t_nV_{n,n+1}A(B_n) \right], \tag{E-2a}$$

$$Z_{n,n+1} = A(B_{n+1})^T \left[U_{n,n+1}A(B_{n+1})z_{n+1}-V_{n,n+1}A(B_n)z_n \right]. \tag{E-2b}$$

The equations (III-16) may then be used to construct the necessary information update quantities.

Note that the data which must be saved from time step n (corresponding to time t_n) are the quantities:

$$\sigma_{1,n},\ \sigma_{2,n},\ B_n,\ x_n,\ y_n.$$

Conceivably t_n and the correlation constants ρ_1 and ρ_2 also "count" toward the storage requirement. In either case, this storage requirement is approximately equal--in terms of number of elements--to the storage of $\sigma_n^{-1} z_n$ and $\sigma_n^{-1} l_n$, the requirement for whose storage in the general case was indicated toward the end of the second section of Chapter III. Furthermore, the computation of the "square root" matrix R_{n+1} is contained in the development of the equations (E-2).

Bearing/Range Observations (Active Directional Sonar)

At time t_n a range R_n and a bearing B_n is taken from a sensor position (u_n, v_n). The uncertainty ϵ_{R_n} in the range is assumed to be stochastically independent of the bearing uncertainty ϵ_{B_n}. It is furthermore assumed that each of the uncertainties is correlated in time; i.e., there are constants ρ_R, ρ_B such that

$$\rho(\epsilon_{R_i}, \epsilon_{R_j}) = \rho_R^{|t_j - t_i|},$$

$$\rho(\epsilon_{B_i}, \epsilon_{B_j}) = \rho_B^{|t_j - t_i|}.$$

The equation (III-1) quantities corresponding to this case are:

$$z_n = \begin{bmatrix} R_n \sin B_n + u_n \\ R_n \cos B_n + v_n \end{bmatrix},$$

$$l_n = \begin{bmatrix} 1 & 0 & t_n - t_o & 0 \\ 0 & 1 & 0 & t_n - t_o \end{bmatrix}.$$

The error term ϵ_n is a zero-mean bivariate normal random variable with covariance

$$\sigma_n^2 = A(B_n)^T \begin{bmatrix} R_n^2 \sigma_{B_n}^2 & \\ & \sigma_{R_n}^2 \end{bmatrix} A(B_n).$$

The line of range uncertainty coincides with the line of sight, whereas the bearing uncertainty, scaled by the range, translates to an uncertainty across the line of sight. It follows from the discussion given in Appendix D that the correlation between ϵ_i and ϵ_j for $t_i \leq t_j$ is given by

$$\rho_{i,j} = A(B_i)^T \begin{bmatrix} \rho_B^{t_j - t_i} & 0 \\ 0 & \rho_R^{t_j - t_i} \end{bmatrix} A(B_j).$$

Clearly, this satisfies the conditions (III-8).

Note that the bearing/range observation essentially has been assigned a "SPA" as in Figure B-2. Thus, the equations (E-1) and (E-2) may be used to construct the equation (III-15) quantities for a bearing/range observation, if one has made the substitutions:

$$\sigma_{1,n} = R_n \sigma_{B_n}, \tag{E-3a}$$

$$\sigma_{2,n} = \sigma_{R_n} \tag{E-3b}$$

(where σ_{B_n} and σ_{R_n} are the standard deviations in ϵ_{B_n} and ϵ_{R_n}, respectively),

$$x_n = u_n + R_n \sin B_n, \tag{E-3c}$$

$$y_n = v_n + R_n \cos B_n, \tag{E-3d}$$

$$\rho_1 = \rho_B, \tag{E-3e}$$

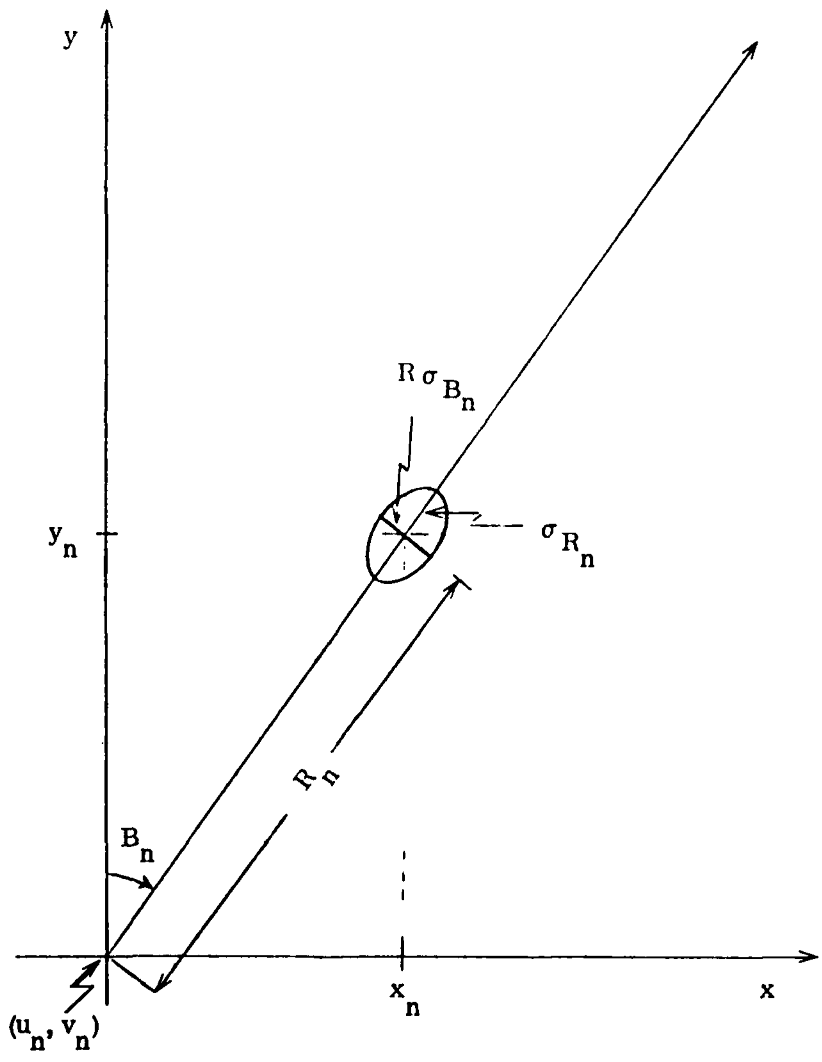
$$\rho_2 = \rho_R. \tag{E-3f}$$

Analogous storage requirements between observations apply. The equations (E-1) and (E-2) after the substitutions (E-3) have been made appear as equations (E-8).

FIGURE E-2

BEARING/RANGE OBSERVATION

Note: σ_{B_n} and σ_{R_n} are the standard deviations in the errors ϵ_{B_n} and ϵ_{R_n}, respectively.



Bearing Observations (Passive Directional Sonar)

Suppose a sequence of bearings B_n is taken from a fixed position (u, v). Assume the errors ϵ_{B_n} in the bearings are correlated in time: that is, there is some constant ρ, $0 \leq \rho \leq 1$ such that

$$\rho(\epsilon_{B_i}, \epsilon_{B_j}) = \rho^{|t_j - t_i|}.$$

Let (x_n, y_n) denote the position of the target at time t_n. Then following reference [g], write

$$\epsilon_{B_n} \approx \frac{\sin B_n}{R} (y_n - v) - \frac{\cos B_n}{R} (x_n - u), \tag{E-4}$$

where R is a nominal range from the sensor to the target, applicable during the time interval involved. The choice of R and its effect on a Kalman filter solution are discussed in reference [g]. For the present, it will suffice to remark that this effect is not significant. Rewriting (E-4) as

$$u \cos B_n - v \sin B_n = x_n \cos B_n - y_n \sin B_n + R\, \epsilon_{B_n}$$

and noting that

$$x_n = x_o + (t_n - t_o) s_x$$

$$y_n = y_o + (t_n - t_o) s_y,$$

the equation (III-1) quantities for the bearing observation at time t_n are seen to be

$$z_n = u \cos B_n - v \sin B_n,$$

$$l_n = \left[\cos B_n, -\sin B_n, (t_n - t_o) \cos B_n, -(t_n - t_o) \sin B_n\right],$$

$$\epsilon_n = R\, \epsilon_{B_n}.$$

Thus, $\sigma_n = R\,\sigma_{B_n}$, where σ_{B_n} is the standard deviation in the bearing error ϵ_{B_n}. Clearly, $\rho_{n,n+1} = \rho^{t_{n+1}-t_n}$ satisfies the conditions (III-8).

Now, straightforward computations using equations (III-14) and (III-15) yield the equation (III-16) quantities needed to construct the information update quantities (III-16). They are:

$$L_{n,n+1} = (1-\rho^{2(t_{n+1}-t_n)})^{-\frac{1}{2}} R^{-1} [F_{n,n+1} \vdots G_{n,n+1}], \tag{E-5a}$$

$$Z_{n,n+1} = (1-\rho^{2(t_{n+1}-t_n)})^{-\frac{1}{2}} R^{-1} F_{n,n+1} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{E-5b}$$

where $F_{n,n+1}$ is the 2-vector

$$F_{n,n+1} = \Big[\sigma_{B_{n+1}}^{-1} \cos B_{n+1} - \rho^{(t_{n+1}-t_n)} \sigma_{B_n}^{-1} \cos B_n,\ \sigma_{B_{n+1}}^{-1} \sin B_{n+1} + \rho^{(t_{n+1}-t_n)} \sigma_{B_n}^{-1} \sin B_n\Big], \tag{E-6a}$$

and $G_{n,n+1}$ is the 2-vector

$$G_{n,n+1} = \Big[t_{n+1}\sigma_{B_{n+1}}^{-1} \cos B_{n+1} - t_n \rho^{(t_{n+1}-t_n)} \sigma_{B_n}^{-1} \cos B_n,\ -t_{n+1}\sigma_{B_{n+1}}^{-1} \sin B_{n+1} + t_n \rho^{(t_{n+1}-t_n)} \sigma_{B_n}^{-1} \sin B_n\Big]. \tag{E-6b}$$

It is not clear from the foregoing development that the bearing observation is a special case of a "SPA." However, a bearing measurement may be viewed as a bearing/range measurement in which the range uncertainty is infinitely large. Then equations (E-1), (E-2), and (E-3) may be used to construct the equation (III-15) quantities for a bearing observation if one makes the substitutions $R_n = R$, $u_n = u$, $v_n = v$, $\rho_B = \rho$, $\sigma_{R_n}^{-1} = 0$. (Obviously, ρ_R is irrelevant.) The resulting equation (III-15) quantities are those of equations (E-5) premultiplied by

$$\begin{bmatrix} \cos B_{n+1} \\ -\sin B_{n+1} \end{bmatrix}. \tag{E-7}$$

Note that, although the equation (III-15) quantities under the two approaches differ by the matrix (E-7), the information update quantities, computed in accordance with equations (III-16), will be the same under each approach. This development is outlined in the next section.

Bearing Observations as "SPAs"

The purpose of this section is to outline the derivation of the information update quantities for a bearing observation in the event that the latter is viewed as a bearing/range observation with infinite range uncertainty, and to reconcile the derivation with the earlier development which culminated in equations (E-5). It is the development outlined below which is reflected in Table E-1.

It will be helpful to first present, for a frame of reference, the equation (III-15) quantities for a bearing/range observation. These are obtained by making the substitutions (E-3) in equations (E-1) and (E-2):

$$U_{n,n+1} = \begin{bmatrix} \left(1-\rho_B^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} \left(R_{n+1}\,\sigma_{B_{n+1}}\right)^{-1} & 0 \\ 0 & \left(1-\rho_R^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} \rho_R^{(t_{n+1}-t_n)} \sigma_{R_n}^{-1} \end{bmatrix}, \quad \text{(E-8a)}$$

$$V_{n,n+1} = \begin{bmatrix} \left(1-\rho_B^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} \rho_B^{(t_{n+1}-t_n)} \left(R_n \sigma_{B_n}\right)^{-1} & 0 \\ 0 & \left(1-\rho_R^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} \rho_R^{(t_{n+1}-t_n)} \sigma_{R_n}^{-1} \end{bmatrix}, \quad \text{(E-8b)}$$

(E-8c)

$$L_{n,n+1} = A(B_{n+1})^T \left[U_{n,n+1}A(B_{n+1}) - V_{n,n+1}A(B_n) \vdots t_{n+1}U_{n,n+1}A(B_{n+1}) - t_n V_{n,n+1}A(B_n) \right],$$

(E-8d)

$$Z_{n,n+1} = A(B_{n+1})^T \left\{ U_{n,n+1}A(B_{n+1}) \begin{bmatrix} u_{n+1}+R_{n+1}\sin B_{n+1} \\ v_{n+1}+R_{n+1}\cos B_{n+1} \end{bmatrix} - V_{n,n+1}A(B_n) \begin{bmatrix} u_n+R_n \sin B_n \\ v_n+R_n \cos B_n \end{bmatrix} \right\},$$

where, of course

$$A(B_n) = \begin{bmatrix} \cos B_n & -\sin B_n \\ \sin B_n & \cos B_n \end{bmatrix}.$$

Now, a bearing observation realizes the following substitutions: $R_n = R$, $u_n = u$, $v_n = v$, $\rho_B = \rho$, $\sigma_{R_n}^{-1} = 0$; ρ_R is irrelevant. With these substitutions, it is straightforward to show that

$$A(B_{n+1})^T U_{n,n+1} A(B_{n+1}) =$$

$$\left(1-\rho^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} (R\sigma_{B_{n+1}})^{-1} \begin{bmatrix} \cos B_{n+1} \\ -\sin B_{n+1} \end{bmatrix} \left[\cos B_{n+1}, -\sin B_{n+1}\right]$$

and

$$A(B_{n+1})^T V_{n,n+1} A(B_n) =$$

$$\left(1-\rho^{2(t_{n+1}-t_n)}\right)^{-\frac{1}{2}} \rho^{t_{n+1}-t_n} (R\sigma_n)^{-1} \begin{bmatrix} \cos B_{n+1} \\ -\sin B_{n+1} \end{bmatrix} \left[\cos B_n, -\sin B_n\right].$$

Using these expressions, it is then also straightforward to show that

$$L_{n,n+1} = 1-\rho^{2(t_{n+1}-t_n)} {}^{-\frac{1}{2}} R^{-1} \begin{bmatrix} \cos B_{n+1} \\ -\sin B_{n+1} \end{bmatrix} [F_{n,n+1} \vdots G_{n,n+1}], \qquad \text{(E-9a)}$$

$$Z_{n,n+1} = 1-\rho^{2(t_{n+1}-t_n)} {}^{-\frac{1}{2}} R^{-1} \begin{bmatrix} \cos B_{n+1} \\ -\sin B_{n+1} \end{bmatrix} F_{n,n+1} \begin{bmatrix} u \\ v \end{bmatrix} . \qquad \text{(E-9b)}$$

where $F_{n,n+1}$ and $G_{n,n+1}$ are as defined by equations (E-6) above. Evidently, these expressions are those of equations (E-5) premultiplied by

$$\begin{bmatrix} \cos B_{n+1} \\ -\sin B_{n+1} \end{bmatrix} .$$

However, since

$$\begin{bmatrix} \cos B_{n+1}, & -\sin B_{n+1} \end{bmatrix} \begin{bmatrix} \cos B_{n+1} \\ -\sin B_{n+1} \end{bmatrix} = 1,$$

either the expressions (E-5) or the expressions (E-9) may be inserted into equations (III-16); the same information update quantities will result.

Inversion of the Information Matrix

As mentioned in Chapter III, the information matrices encountered in SCREEN are 4 x 4 matrices which partition into two-dimensional submatrices corresponding to the position and speed components of the state vector. Thus, we can write:

$$\mathcal{I} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} ,$$

where A corresponds to location information, C corresponds to velocity information, and B corresponds to the correlation between position and velocity.

When the information matrix $\mathscr{I}$ is singular, it may still be possible to solve for target position. This will occur, for example, if there is a bearing crossfix or an active sonar contact at a single time. In such a case the target's position is known, but its velocity components are impossible to determine without further measurements. It is desirable to perform a partial inversion of $\mathscr{I}$ in those cases in order to obtain a position covariance. Similar remarks apply to the "expected" information matrix.

Partition the inverse of $\mathscr{I}$ as follows:

$$P = \mathscr{I}^{-1} = \begin{pmatrix} D & E \\ E^T & F \end{pmatrix}.$$

Inversion of a partitioned matrix is discussed by Proposition G-1. Table E-1 gives a FORTRAN function which implements the inversion. The quantity SINVER corresponds to L^2, where L is the quantity of equation (III-20). The function has three possible conditions for termination:

(1) Normal Return. In this case, the matrix $\hat{\mathscr{I}}$ is nonsingular. The inverse of the matrix is returned and the function SINVER is set to the trace of the position covariance matrix corresponding to the "best SPA time," which is the time at which Trace Var $\begin{bmatrix} x_t \\ y_t \end{bmatrix}$ is minimized (see reference [g]).

(2) Partial Inversion: Only Position Covariance Matrix is Nonsingular. In this case, the submatrix A^{-1} (the position covariance matrix) is returned and the function SINVER is set to the negative of the trace of A^{-1}.

(3) No Inverse Performed. The position covariance matrix is singular. In this case, the value SINVER = 0 is returned.

The test for singularity consists in checking the magnitude of the determinant of the appropriate matrix. The singularity test fails (i.e., the matrix is considered noninvertible) if the determinant cannot be determined to at least two significant digits. In case single precision variables are carried to about 7 digits ($2^{-23} \sim 10^{-7}$), the determinant involves the product of two such variables, hence its accuracy is about 2^{-22} or 2×10^{-7}. Requiring two digits of accuracy in the difference translates to less than 2^{-18} or 2×10^{-6}.

TABLE E-1

```
C
C
      FUNCTION SINVER(A,B,C,D,E,F)
C     COMPUTES THE INVERSE OF A 4X4 MATRIX PARTITIONED INTO 2X2 SUBMATRICES:
C
C    INVERSE OF (A B) = (D E) WHERE A = (A(1) A(2))
C               (B C) = (E F)              (A(2) A(3))    ETC.
C
C     THE SUBMATRICES ARE SYMMETRIC.
C
C      THE FORMULA USED IS:
C             INVERSE[A        B] = [INV(A) 0] + [-S]*INV(T)*[-TRAN(S) I]
C                   [TRAN(B) C]   [0      0]   [I ]
C          WHERE:
C                S = INV(A)*B
C                T = C - TRAN(B)*INV(A)*B
C
C
C              -- IF MATRIX IS SINGULAR, SINVER=0 IS RETURNED.
C              -- IF VEL COV IS SINGULAR, MINUS TRACE IS RETURNED
C
      DIMENSION A(3),B(3),C(3),D(3),E(4),F(3)
C      TRACE 200
      CCON=2.**(-13)         /* REAL*4 ROUNDOFF ABT. 2**-23*VALUE
      SINVER=0
      DETI=A(1)*A(3)-A(2)*A(2)
      IF(DETI.LE.CCON*SIZE(A)) RETURN  /* POSITION SUBMATRIX IS SINGULAR-- RETURN
      QETI=1/DETI                                                          ZERO
      D(1)=A(3)*QETI    /* D() IS INVERSE OF A() AT THIS POINT
      D(2)=-A(2)*QETI
      D(3)=A(1)*QETI
      SINVER=-D(1)-D(3)    /* MINUS TRACE OF POSITION SUBMATRIX
```

TABLE E-1 (Continued)

```
C
C  A,B,C  SYMMETRIC DO NOT IMPLY S IS SYMMETRIC
C
         S1=D(1)*B(1)+D(2)*B(2)
         S2=D(1)*B(2)+D(2)*B(3)
         S3=D(2)*B(1)+D(3)*B(2)
         S4=D(2)*B(2)+D(3)*B(3)
            E(1)=C(1)-B(1)*S1-B(2)*S3
            E(2)=C(2)-B(2)*S1-B(3)*S3        /*B
            E(3)=C(3)-B(2)*S2-B(3)*S4

C    CHECK IF MAG.OF E() APPROXIMATES ROUNDOFF ERROR IN C() (REAL*4)
C
      SIZEE=SIZE(E)

           IF(SIZEE.LE.CCON*SIZE(C))RETURN   /* RETURN NEGATIVE POS'N SUBMATRIX TRACE

C    CHECK FOR SMALL DETERMINANT

      DET2=E(1)*E(3)-E(2)*E(2)
            IF(DET2.LE.CCON*SIZEE) RETURN  /* RETURNS NEGATIVE POS'N SUBMATRIX TRACE
            DET2=1/DET2
            TEMP=E(1)
      E(1)=E(3)*DET2
      E(2)=-E(2)*DET2
      E(3)=TEMP*DET2
C  A,B,C  SYMMETRIC DO NOT IMPLYTHAT E IS SYMMETRIC
         E(1)=-S1*E(1)-S2*E(2)
         E(2)=-S1*E(2)-S2*E(3)
         E(3)=-S3*E(1)-S1*E(2)
         E(4)=-S3*E(2)-S4*E(3)
            D(1)=D(1)-E(1)*S1-E(2)*S2
            D(2)=D(2)-E(1)*S3-E(2)*S4
            D(3)=D(3)-E(3)*S3-E(4)*S4

C   SINVER IS THE TRACE OF THE POSITION COV MX AT THE BEST SPA TIME

      SINVER=D(1)+D(3)-(E(1)+E(4))*(E(1)+E(4))/(E(1)+E(3))
200   RETURN
      END
C
```

APPENDIX F

SCREEN LOCALIZATION ROUTINES

by David P. Kierstead

This appendix summarizes the localization algorithms as they appear in SCREEN. These computations are performed by the same subroutines that compute cdps and snapshot PDs, with user inputs controlling whether the results are given as output.

As the development of Appendix E shows, all types of observations considered in SCREEN may be cast in the form of SPAs. This simply requires that appropriate quantities be constructed to take the place of $\sigma_{1,n}^{-1}$, $\sigma_{2,n}^{-1}$, ρ_1, ρ_2, and z_n in equations (E-1, 2). In all cases, B_n is the bearing (of the major axis in the case of a SPA, of the observation in the other two cases). Table F-1 gives the necessary substitutions.

Cumulative Localization

Subroutine PIMAP guides the whole computation. First, PIMAP breaks the program time steps into small subintervals (see Appendix K), then SCRINF is called at each subinterval.

When the function SCRINF is called in PIMAP, all of the sensors are dealt with in turn and the following operations are performed for each:

(1) the quantities in Table F-1 are computed,

(2) a quantity SAVAIL is computed (this is the probability that the given sensor detects the target, given the target's position), and

TABLE F-1

VARIABLES DETERMINED BY OBSERVATIONS

FORTRAN Variable	SPA	Bearing/Range	Bearing
1/SIGMA1	$\sigma_{1,n}^{-1}$	$(R_n \sigma_{B_n})^{-1}$	$(R_n \sigma_{B_n})^{-1}$
1/SIGMA2	$\sigma_{2,n}^{-1}$	$\sigma_{R_n}^{-1}$	0
RHO1	$\rho_1^{(t_{n+1}-t_n)}$	$\rho_B^{(t_{n+1}-t_n)}$	$\rho_B^{(t_{n+1}-t_n)}$
RHO2	$\rho_2^{(t_{n+1}-t_n)}$	$\rho_R^{(t_{n+1}-t_n)}$	(not required)
Z1, Z2*	z_n	$\begin{bmatrix} u_n+R_n \sin B_n \\ v_n+R_n \cos B_n \end{bmatrix}$	$\begin{bmatrix} u_n+R_n \sin B_n \\ v_n+R_n \cos B_n \end{bmatrix}$

* These quantities are only used in computing $\mathscr{X}$ and $\mathscr{R}$. SCREEN does not currently compute either, and thus does not deal with Z1, Z2.

(3) subroutine INFADD is called to:

(a) compute the information from the sensor (given a detection) via equations (E-1, 2)*,

(b) multiply the result of (a) by SAVAIL to yield "expected information" from the sensor, and

(c) add the result of (b) to the accumulating "expected information" matrix.

When the information matrix has been adjusted for the entire time span of the problem, PIMAPS calls SINVER to compute the localization measure. (See Appendix E.) Finally, this quantity is multiplied by the cdp for the entire screen to yield expected localization, given detection. (Equivalently, the "expected information" matrix is divided by cdp and then inverted to compute the localization measure.)

Snapshot Localization

Subroutine PDSTEP guides the computation of snapshot localization. At the desired time step, PDSTEP calls the function PRSTEP which performs essentially the operations of SCRINF, INFADD, and SINVER. That is, PRSTEP evaluates the snapshot analogues of equations (E-1, 2) for each sensor and accumulates the results into an "expected localization" matrix. The localization measure is computed and is then multiplied by the screen's snapshot PD.

The differences between equations (E-1, 2) and their snapshot analogues stem from the fact that there are no previous time steps to be considered. Thus, $\rho_1 = \rho_2 = 0$, so V is the zero matrix. Further, there can be no velocity information from snapshot observations, so only the upper left 2 x 2 block of $\hat{\mathcal{J}}$ need be computed. The associated localization measure is the trace of the inverse of this block (see Appendix E).

---

* There is a slight deviation from equations (E-1, 2). Recall that values of the previous observation from the sensor in question were required in order to compute the incremental information from that sensor. SCREEN assumes that the previous observation is the same as the current one. This is reasonable because the time increments are very small.

APPENDIX G

MATRIX ALGEBRA

Various matrix algebra computations and conventions which are needed for the analysis and description of the SCREEN target motion model are the subject of this appendix.

The first part is devoted to a careful description of the notation conventions used in describing the operations on the DIOU process. In what follows, let $u = \{u_1, \ldots, u_k\}$ and $v = \{v_1, \ldots, v_l\}$ be (not necessarily distinct) sets of indices drawn from $\{1, \ldots, \tau\}$*. When such a set consists of a single index j, then the index j will be used instead of $\{j\}$. If M is a matrix, then the transpose of M will be denoted M^T.

Let x be a (2τ)-vector written as

$$\begin{bmatrix} x_1 \\ \vdots \\ x_\tau \end{bmatrix},$$

where each x_i is a 2-vector. Then x(u) shall be the (2k)-dimensional subvector defined by

$$x(u) = \begin{bmatrix} x_{u_1} \\ \vdots \\ x_{u_k} \end{bmatrix}. \qquad \text{(G-1)}$$

* It is clear that the indexing may start with any number, with corresponding changes in vector and matrix dimensions required. In the SCREEN target motion model, e.g., the indexing begins with 0 (see equations (IV-3) and (IV-4)).

Next, suppose A is a $(2\tau) \times (2\tau)$ matrix partitioned as

$$A = \begin{bmatrix} a_{11} & \cdots & A_{1\tau} \\ \vdots & & \vdots \\ a_{\tau 1} & \cdots & a_{\tau\tau} \end{bmatrix},$$

where each a_{ij} is a 2×2 matrix. Then A_{uu} shall be the $(2k) \times (2k)$ matrix defined by

$$A_{uu} = \begin{bmatrix} a_{u_1 u_1} & \cdots & a_{u_1 u_k} \\ \vdots & & \vdots \\ a_{u_k u_1} & \cdots & a_{u_k u_k} \end{bmatrix}; \qquad \text{(G-2)}$$

and A(u) will denote the $(2\tau) \times (2k)$ matrix consisting of those columns of A which contain elements of A_{uu}:

$$A_u = \begin{bmatrix} a_{1,u_1} & \cdots & a_{1,u_k} \\ \vdots & & \vdots \\ a_{\tau,u_1} & \cdots & a_{\tau,u_k} \end{bmatrix}. \qquad \text{(G-3)}$$

For example, if $\tau = 3$, $u = \{1, 3\}$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then

$$A(u) = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix},$$

and

$$A_{uu} = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} .$$

Recall that each a_{ij} is a 2 x 2 matrix. Note that one can also view A(u) as being obtained from A by deleting all columns not corresponding to indices in u. Then A_{uu} may be obtained from A(u) be deleting those rows of A(u) not corresponding to indices in u.

Now let I_u be the (2k) x (2τ) matrix which is the identity matrix when restricted to the index set u and is the zero matrix elsewhere. Examples follow:

(a) $\tau = 3$, $u = \{2\}$; I_u is the 2 x 6 matrix given by

$$I_u = \left[\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] .$$

(b) $\tau = 3$, $u = \{1, 2\}$; I_u is the 4 x 6 matrix given by

$$I_u = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \underbrace{0 \quad 0}_{u_1} & & \underbrace{0 \quad 1}_{u_2} & & 0 & 0 \end{array}\right] .$$

(c) $\tau = 4$, $u = \{1, 3\}$; I_u is the 4 x 8 matrix given by

$$I_u = \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \underbrace{0 \quad 0}_{u_1} & & 0 & 0 & \underbrace{0 \quad 1}_{u_2} & & 0 & 0 \end{array}\right] .$$

Let A be a $(2\tau) \times (2\tau)$ matrix, and suppose D is a (2k) x (2k) matrix which is "associated" with the index set u, in the same sense as the information adjustment quantities in the text pertain to a set of indices (i.e., sampling times). (See the discussion of equations (IV-10) and (IV-11).) Then the operation (+) is defined by

$$A \; (+) \; D = A + I_u^T D I_u, \tag{G-4}$$

the idea being that the entries of D are to be added to those entries of A corresponding to the index set u. For example, if $\tau = 3$, $u = \{1, 3\}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(each a_{ij} being a 2 x 2 matrix), and

$$D = \begin{bmatrix} f & g \\ h & k \end{bmatrix},$$

where f, g, h, and k are 2 x 2 matrices, then

$$A \; (+) \; D = \begin{bmatrix} a_{11}+f & a_{12} & a_{13}+g \\ a_{21} & a_{22} & a_{23} \\ a_{31}+h & a_{32} & a_{33}+k \end{bmatrix}.$$

The operation (+) can also be extended to vectors. If x is a 2τ-dimensional vector, and d is a (2k)-dimensional vector which is associated with the index set u, then by definition

$$x \; (+) \; d = x + I_u^T d. \tag{G-5}$$

For example, if $\tau = 4$, $u = \{1, 3, 4\}$,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad d = \begin{bmatrix} \xi \\ \eta \\ \theta \end{bmatrix} \quad ,$$

then

$$x \ (+) \ d = \begin{bmatrix} x_1 + \xi \\ x_2 \\ x_3 + \eta \\ x_4 + \theta \end{bmatrix} .$$

Finally, if A is a $(2\tau) \times (2\tau)$ matrix and $u = \{u_1, \ldots, u_k\}$ and $v = \{v_1, \ldots, v_l\}$ (not necessarily distinct set of indices drawn from $\{1, \ldots, \tau\}$), then A_{uv} is the $(2k \times 2l)$ matrix defined by

$$A_{uv} = \begin{bmatrix} a_{u_1 v_1} & \cdots & a_{u_1 v_l} \\ \vdots & & \vdots \\ a_{u_k u_1} & \cdots & a_{u_k v_l} \end{bmatrix} . \qquad \text{(G-6)}$$

For example, if $\tau = 3$, $u = \{1, 3\}$, $v = \{3\}$, then

$$A_{uv} = \begin{bmatrix} a_{13} \\ \hline a_{33} \end{bmatrix}$$

(recall that each a_{ij} is a 2×2 matrix), whereas

$$A_{vu} = \left[a_{31} \ \vdots \ a_{33} \right] .$$

Generally, for the matrices encountered in this report, $(A_{uv}) = A^T_{vu}$. If $u = v$, then A_{uv} is just A_{uu} as defined by equation (G-2).

The remainder of this appendix contains various propositions in matrix algebra which are used in the analysis underlying the target motion model. Throughout the sequel, I will denote the identity matrix, and $\hat{I}$ will denote the matrix $[I, 0]$. The dimensions will be determined by the context.

Lemma G-1. If $x, y \in \mathbb{R}^n$ and C is an n x n symmetric invertible matrix, then

$$x^T Cx - (y^T x + x^T y) = (x - C^{-1}y)^T C(x - C^{-1}y) - y^T C^{-1} y.$$

Proof. Expand and simplify the right-hand side of the formula. QED.

Lemma G-2. $\begin{bmatrix} A & B \\ 0 & T \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ B & T \end{bmatrix}$ are invertible if and only if A and T are, and in this case

$$\begin{bmatrix} A & B \\ 0 & T \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BT^{-1} \\ 0 & T^{-1} \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ B & T \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -T^{-1}BA^{-1} & T^{-1} \end{bmatrix}.$$

Proof. Note that the determinant of $\begin{bmatrix} A & B \\ 0 & T \end{bmatrix}$ is (det A) (det T). The inversion formulas are easily verified. QED.

Proposition G-1. Let

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

be an invertible symmetric matrix where A and C are symmetric (and invertible) matrices not necessarily of the same dimensions. If

$$S = A^{-1}B$$

$$T = C - B^T S = C - B^T A^{-1} B$$

then

$$\det M = (\det A)\,(\det T)$$

and

$$M^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1}[-S^T \quad I].$$

Proof. It is clear that

$$\begin{bmatrix} I & 0 \\ -S^T & I \end{bmatrix} M = \begin{bmatrix} A & B \\ 0 & T \end{bmatrix},$$

so the determinant assertion obtains immediately. It also follows that T is invertible. Next, from Lemma G-2,

$$M^{-1} = \begin{bmatrix} A^{-1} & -ST^{-1} \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -S^T & I \end{bmatrix} = \begin{bmatrix} A^{-1}+ST^{-1}S^T & -ST^{-1} \\ -T^{-1}S^T & T^{-1} \end{bmatrix},$$

from which follows the last statement of the proposition. QED.

Proposition G-2. Let Σ and D be symmetric matrices such that:

(i) Σ is invertible.

(ii) The dimensions of Σ are at least those of D.

(iii) $(\Sigma^{-1} + \hat{I}^T D \hat{I})$ is invertible.

Let $\Sigma_{11} = \hat{I} \Sigma \hat{I}^T$ and $\begin{bmatrix} \Sigma_{11} \\ \Sigma_{21} \end{bmatrix} = \Sigma \hat{I}^T$. Then:

(a) $(I + \Sigma_{11} D)$ is invertible;

(b) $(\Sigma^{-1} + \hat{I}^T D \hat{I})^{-1} = \Sigma - \Sigma \hat{I}^T G \hat{I} \Sigma$, where

$$G = D(I + \Sigma_{11} D)^{-1};$$

(c) $\det(\Sigma^{-1} + \hat{I}^T D\hat{I})^{-1} = \det(I + \Sigma_{11} D)^{-1} \det \Sigma$;

(d) D is invertible if and only if G is, and in this case

$$G = (D^{-1} + \Sigma_{11})^{-1};$$

(e) if Σ_{11} is invertible, then so is $\Sigma_{11}^{-1} + D = \Phi^{-1}$, and then

$$G = \Sigma_{11}^{-1}(\Sigma_{11} - \Phi)\Sigma_{11}^{-1}.$$

Proof. We have

$$(\Sigma^{-1} + \hat{I}^T D\hat{I})^{-1} = \left(I + \Sigma \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^{-1} \Sigma$$

$$= \begin{bmatrix} I + \Sigma_{11} D & 0 \\ \Sigma_{21} D & I \end{bmatrix}^{-1} \Sigma.$$

Lemma G-2 now yields statements (a) and (c), and also

$$(\Sigma^{-1} + I^T DI)^{-1} = \begin{bmatrix} (I + \Sigma_{11} D)^{-1} & 0 \\ -\Sigma_{21} D (I + \Sigma_{11} D)^{-1} & I \end{bmatrix} \Sigma;$$

since

$$I = (I + \Sigma_{11} D)^{-1} + \Sigma_{11} D (I + \Sigma_{11} D)^{-1},$$

$$(\Sigma^{-1} + \hat{I}^T D\hat{I})^{-1} = \begin{bmatrix} I - \Sigma_{11} D (I + \Sigma_{11} D)^{-1} & 0 \\ -\Sigma_{21} D (I + \Sigma_{11} D)^{-1} & I \end{bmatrix} \Sigma$$

$$= \left\{ I - \begin{bmatrix} \Sigma_{11} G & 0 \\ \Sigma_{21} G & 0 \end{bmatrix} \right\} \Sigma$$

$$= \left\{ I - \begin{bmatrix} \Sigma_{11} \\ \Sigma_{21} \end{bmatrix} G\hat{I} \right\} \Sigma,$$

which gives statement (b). Statement (d) is obvious. Finally, if Σ_{11} is invertible, then

$$(I+\Sigma_{11}D) = \Sigma_{11}(\Sigma_{11}^{-1} + D),$$

which, by statement (a), may be written as $\Sigma_{11}\Phi^{-1}$; furthermore,

$$\begin{aligned} G &= D(I+\Sigma_{11}D)^{-1} \\ &= (\Phi^{-1}-\Sigma_{11}^{-1})\,(\Sigma_{11}^{-1}+D)^{-1}\,\Sigma_{11}^{-1} \\ &= (\Phi^{-1}-\Sigma_{11}^{-1})\,\Phi\Sigma_{11}^{-1} \\ &= (I-\Sigma_{11}^{-1}\,\Phi)\,\Sigma_{11}^{-1} \\ &= \Sigma_{11}^{-1}\,(\Sigma_{11}-\Phi)\Sigma_{11}^{-1}. \quad \text{QED.} \end{aligned}$$

Corollary. Let Σ and D be symmetric matrices whose dimensions are k x k and j x j, respectively, with $j \leq k$. Let Σ be invertible and let A be a j x k matrix of rank j. If $\Sigma^{-1} + A^TDA$ is invertible, then

$$(\Sigma^{-1}+A^TDA)^{-1} = \Sigma-\Sigma A^TD(I+A\Sigma A^TD)^{-1}A\Sigma,$$

$$\det(\Sigma^{-1}+A^TDA)^{-1} = \det(I+A\Sigma A^TD)^{-1}\,\det\Sigma.$$

Proof. Since A has maximal rank, $A = \hat{I}P$ for some invertible k x k matrix P. Thus,

$$\begin{aligned} (\Sigma^{-1}+A^TDA)^{-1} &= (\Sigma^{-1}+P^T\hat{I}^T\,D\hat{I}P)^{-1} \\ &= P^{-1}[(P\Sigma P^T)^{-1} + \hat{I}^T\,D\hat{I}]^{-1}(P^T)^{-1}; \end{aligned}$$

so Proposition G-2 implies that

$$P(\Sigma^{-1}+A^TDA)^{-1}P^T = P\Sigma P^T - P\Sigma P^T\hat{I}^TD(I+\hat{I}P\Sigma P^T\hat{I}^TD)^{-1}\hat{I}P\Sigma P^T$$

and

$$\det(P[\Sigma^{-1}+A^TDA]^{-1}P^T = \det(I+\hat{I}P\Sigma P^T\hat{I}^TD)\det(P\Sigma P^T),$$

from which the statements of the corollary follow easily. QED.

A particular example of the corollary is a proposition due to S. S. Brown and R. V. Kohn (reference [o]). Suppose $D = s^{-1}$ and

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} ,$$

where Σ_{22} is a symmetric matrix which has the same dimensions as s. Then the corollary, with $A = [0 \;\; I \;\; 0]$, implies that

$$\left\{\Sigma^{-1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & s^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}^{-1} = \Sigma - \Sigma A^T s^{-1}(I+\Sigma_{22}s^{-1})^{-1}A\Sigma$$

$$= \Sigma - \begin{bmatrix} \Sigma_{12} \\ \Sigma_{22} \\ \Sigma_{32} \end{bmatrix} (s+\Sigma_{22})^{-1} [\Sigma_{21}\,\Sigma_{22}\,\Sigma_{23}].$$

The following proposition is sort of a converse to Proposition G-2, and is due to Dr. L. K. Arnold.

Proposition G-3. Let

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

with Σ_{11} invertible, and let $H = \Sigma - \Sigma \hat{I}^T G \hat{I} \Sigma$. We then have the following:

(a) If, for some invertible matrix Φ, $D = \Phi^{-1} - \Sigma_{11}^{-1}$ and

$$G = \Sigma_{11}^{-1} (\Sigma_{11} - \Phi) \Sigma_{11}^{-1},$$

then H is invertible and $H^{-1} = \Sigma^{-1} + \hat{I}^T D \hat{I}$.

(b) The matrix D is invertible if and only if G is, and in this case, $G = (D^{-1} + \Sigma_{11})^{-1}$.

Proof. (a) If Σ_{11} and Φ are both invertible, then so is $\Sigma_{11} \Phi^{-1} = I + \Sigma_{11} D$. Since

$$\Sigma^{-1} + \hat{I}^T D \hat{I} = \Sigma^{-1} \begin{bmatrix} I + \Sigma_{11} D & 0 \\ \Sigma_{21} & I \end{bmatrix},$$

Lemma G-2 shows that the inverse of $\Sigma^{-1} + \hat{I}^T D \hat{I}$ exists; Proposition G-2 says it is equal to H.

(b) This follows from part (d) of Proposition G-2. QED.

The corollary to Proposition G-2 extended statements (b) and (c) of that proposition. Statements (d) and (e) may also be extended, with Σ_{11} replaced by $A \Sigma A^T$. Using these extensions, analogous versions of Proposition G-3 may be proved as well. For example, if Σ is partitioned as in the Brown-Kohn proposition discussed above, Σ_{22} is invertible, and

$$H = \Sigma - \begin{bmatrix} \Sigma_{12} \\ \Sigma_{22} \\ \Sigma_{32} \end{bmatrix} \Sigma_{22}^{-1} (\Sigma_{22} - \Phi) \Sigma_{22}^{-1} [\Sigma_{21}\ \Sigma_{22}\ \Sigma_{23}]$$

for some invertible Φ, then, if $D = \Phi^{-1} - \Sigma_{22}^{-1}$, H is invertible and

$$H^{-1} = \Sigma^{-1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the text and subsequent appendices, such extensions of Propositions G-2 and G-3 will generally be invoked without explicitly stating the particular form being used. In those cases, A will usually be an appropriate permutation matrix.

APPENDIX H

GAUSSIAN ANALYSIS

This appendix collects some useful facts and manipulations concerning multivariate Gaussian densities which underlie the analysis of the target diffusion model. Two sources for basic facts about multivariate Gaussian distributions are references [n], [p], and [r].

For x, m ϵ $\mathbb{R}^N$ and C an n x n symmetric positive definite matrix, define

$$n(x;m,C) = (2\pi)^{-\frac{1}{2}} \left|C\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x-m)^T C^{-1}(x-m)\right\} \tag{H-1}$$

$$= \exp\left\{-\frac{1}{2}\left[(x-m)^T C^{-1}(x-m)+K\right]\right\},\ K = \log\left|C\right| + N\log 2\pi,$$

where $|C|$ = det C is the determinant of C. Thus, n($\cdot$;m, C) is the multivariate Gaussian density function on $\mathbb{R}^N$ with mean m and covariance matrix C. That (H-1) describes a probability density function follows from

$$\int_{\mathbb{R}^N} \exp\{-\frac{1}{2}(x-m)^T C^{-1}(x-m)\}\, dx = (2\pi)^{\frac{N}{2}} \left|C\right|^{\frac{1}{2}} = e^K. \tag{H-2}$$

When it is desired to emphasize the dimension of the distribution, the function in (H-1) will be denoted by n(x : $\mathbb{R}^N$;m, C).

<u>Proposition H-1.</u> Suppose a random variable X has the density function

$$n(x:\mathbb{R}^k;m,C).$$

If F is a j x k matrix of rank j, for $j \leq k$, then $Y = FX$ has the density function

$$n(y : \mathbb{R}^j ; Fm, FCF^T).$$

The proof of Proposition H-1 is essentially a change of variables in equaion (H-1). The typical use of Proposition H-1 is illustrated by Proposition H-3 below, and discussed further in the remark following the proof of that proposition.

Proposition H-2. Let a Gaussian random vector Y be partitioned as

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

and let its mean vector μ and covariance matrix V be partitioned as

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \text{and } V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix},$$

with respect to this partition. Then

$$p(Y_1=y \mid Y_2=z) = n(y;\mu_1+v_{12}v_{22}^{-1}(z-\mu_2),\ v_{11}+v_{12}^T v_{22}^{-1} v_{21}).$$

Proof. See reference [r], pg. 63.

Proposition H-3. Let a Gaussian random vector Y be partitioned as

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix},$$

with corresponding partitions for its mean μ and covariance matrix V given by

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}, \quad V = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}.$$

Then

$$p\left(\begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, Y_2 = z\right) = n\left(\begin{bmatrix} x \\ y \end{bmatrix}; m(z), Q\right),$$

where

$$m(z) = \begin{bmatrix} \mu_1 \\ \mu_3 \end{bmatrix} + \begin{bmatrix} v_{12} \\ v_{32} \end{bmatrix} v_{22}^{-1} (z - \mu_2),$$

$$Q = \begin{bmatrix} v_{11} & v_{13} \\ v_{31} & v_{33} \end{bmatrix} + \begin{bmatrix} v_{12} \\ v_{32} \end{bmatrix} v_{22}^{-1} [v_{21} \quad v_{23}].$$

<u>Proof.</u> Let X be defined by $X_1 = Y_1$, $X_2 = Y_3$, $X_3 = Y_2$; thus, X=PY where P is the permutation matrix

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}.$$

Then X has mean

$$P\mu = \begin{bmatrix} \mu_1 \\ \mu_3 \\ \cdots \\ \mu_2 \end{bmatrix}$$

and covariance matrix

$$PVP^T = \left[\begin{array}{cc|c} v_{11} & v_{13} & v_{12} \\ v_{31} & v_{33} & v_{32} \\ \hline v_{21} & v_{23} & v_{22} \end{array}\right],$$

by Proposition H-1. Now,

$$p\left(\begin{bmatrix} Y^1 \\ Y^3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \mid Y^2 = z\right) = p\left(\begin{bmatrix} X^1 \\ X^2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \mid X^3 = z\right);$$

therefore, applying Proposition H-2 to the random vector

$$X = \begin{bmatrix} X^1 \\ X^2 \\ \cdots \\ X^3 \end{bmatrix}$$

yields the present result. QED.

Remark. Note that other results along the lines of Proposition H-3--e.g., the distribution of Y_2 given Y_1--may be derived from Proposition H-2 using Proposition H-1. (Note further that P need not necessarily be a permutation matrix.) These "change of variables" adjustments will be true of other results as well; for example, applying such an adjustment to Proposition H-5 below yields the equations (I-1). However, since the spirit of these adjustments has been made incarnate in the proof of Proposition H-3 and by the Brown-Kohn example in Appendix G, they will generally not be made as explicit as they were in those instances.

Proposition H-4. Let $X \in \mathbb{R}^j$ and $Y \in \mathbb{R}^k$ be (vector-valued) random variables jointly distributed such that the conditional density of X given Y has the form

$$p(X = x \mid Y = y) = n(x; a+Ay, \Sigma_1),$$

where $a \in \mathbf{R}^j$, A is a j x k matrix, and Σ_1 is independent of y; and such that the marginal density for Y is

$$p(Y=y) = n(y;\mu,\Sigma_2).$$

Then:

(i) the distribution of (X, Y) has the density function

$$n\left(\begin{bmatrix} x \\ y \end{bmatrix}; \begin{bmatrix} z+A\mu \\ \mu \end{bmatrix}, \begin{bmatrix} \Sigma_1+A\Sigma_2A^T & A\Sigma_2 \\ \Sigma_2A^T & \Sigma_2 \end{bmatrix}\right),$$

and

(ii) the conditional distribution of Y given X is Gaussian, with moments given by

$$E(Y|X) = [\Sigma_2^{-1}+A^T\Sigma_1^{-1}A]^{-1}[\Sigma_2^{-1}\mu+A^T\Sigma_1^{-1}(X-a)],$$

$$Var(Y|X) = [\Sigma_2^{-1}+A^T\Sigma_1^{-1}A]^{-1}.$$

<u>Proof</u>. If f(x, y) denotes the joint density for X and Y, then

$$f(x,y) = p(X=x|Y=y)\,p(Y=y), \tag{H-3}$$

so f(x, y) has the form

$$f(x,y) = (2\pi)^{-\left(\frac{j+k}{2}\right)} |\Sigma_1|^{-\frac{1}{2}} |\Sigma_2|^{-\frac{1}{2}} \exp(-\tfrac{1}{2}Q(x,y)),$$

where

$$Q(x,y) = (x-a-Ay)^T\Sigma_1^{-1}(x-a-Ay) + (y-\mu)^T\Sigma_2^{-1}(y-\mu).$$

This can be written as

$$Q(\underline{x}) = \underline{x}^T\Sigma\underline{x} - (\underline{x}^T\xi + \xi^T\underline{x}) + a^T\Sigma_1^{-1}a + \mu^T\Sigma_2^{-1}\mu, \qquad \text{(H-4)}$$

where

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\xi^{T} = (a^T\Sigma_1^{-1}, \mu^T\Sigma_2^{-1} - a^T\Sigma_1^{-1}A). \qquad \text{(H-5)}$$

$$\Sigma = \begin{bmatrix} \Sigma_1^{-1} & -\Sigma_1^{-1}A \\ -A^T\Sigma_1^{-1} & A^T\Sigma_1^{-1}A+\Sigma_2^{-1} \end{bmatrix}. \qquad \text{(H-6)}$$

Note that Σ is positive definite, since both Σ_1 and Σ_2 are. Applying Lemma G-1 to the first two terms on the right-hand side of (H-4) gives

$$Q(\underline{x}) = (\underline{x}-\Sigma^{-1}\xi)^T\,\Sigma(\underline{x}-\Sigma^{-1}\xi) - [\xi^T\Sigma^{-1}\xi - a^T\Sigma_1^{-1}a - \mu^T\Sigma_2^{-1}\mu].$$

Now, f(x, y) has the form

$$f(x,y) = K\exp\left(-\frac{1}{2}(\underline{x}-\Sigma^{-1}\xi)^T \quad (\underline{x}-\Sigma^{-1}\xi)\right), \qquad \text{(H-7)}$$

where K is a positive constant. Since f(x, y) integrates to one over $\mathbb{R}^{j+k}$, (H-2) implies that $K^2 = [(2\pi)^{j+k}|\Sigma^{-1}|]$.

Now, from (H-6), it is apparent that

$$\begin{bmatrix} I & 0 \\ A^T & I \end{bmatrix} \Sigma = \begin{bmatrix} \Sigma_1^{-1} & -\Sigma_1^{-1}A \\ 0 & \Sigma_2^{-1} \end{bmatrix}.$$

so

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_1^{-1} & -\Sigma_1^{-1}A \\ 0 & \Sigma_2^{-1} \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ A^T & I \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_1 & A\Sigma_2 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ A^T & I \end{bmatrix},$$

by Lemma G-2. Now it follows that Σ^{-1}, the covariance matrix of (X, Y), is given by

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_1 + A\Sigma_2 A^T & A\Sigma_2 \\ \Sigma_2 A^T & \Sigma_2 \end{bmatrix}. \qquad \text{(H-8)}$$

From this and (H-5) the mean of (X, Y), which equals $\Sigma^{-1}\xi$ according to (H-7), may be explicity computed to be

$$\begin{bmatrix} a+A\mu \\ \mu \end{bmatrix}.$$

This proves (i).

To prove (ii), note that by Proposition H-2,

$$\begin{aligned} Var(Y \mid X) &= Var(Y) - Cov(Y,X)\,(Var\,X)^{-1}\,Cov(X,Y) \\ E(Y \mid X) &= E(Y) + Cov(Y,X)\,(Var\,X)^{-1}\,(X - E(X)). \end{aligned} \qquad \text{(H-9)}$$

Substituting from the matrix of (H-8) and noting that $E(X) = a+A\mu$ changes (H-9) into

$$\mathrm{Var}(Y \mid X) = \Sigma_2 - \Sigma_2 A^T (A\Sigma_2 A^T + \Sigma_1)^{-1} A\Sigma_2, \tag{H-10}$$

$$E(Y \mid X) = \mu + \Sigma_2 A^T (A\Sigma_2 A^T + \Sigma_1^{-1}) (X - a - A\mu). \tag{H-11}$$

The results (b) and (d) of Proposition G-2 suggest multiplying the right-hand side of (H-10) by $\Sigma_2^{-1} + A^T\Sigma_1^{-1}A$. Doing this and simplifying give the identity matrix; therefore,

$$\mathrm{Var}(Y \mid X) = [\Sigma_2^{-1} + A^T\Sigma_1^{-1}A]^{-1}. \tag{H-12}$$

Furthermore,

$$E(Y \mid X) = [\Sigma_2^{-1} + A^T\Sigma_1^{-1}A]^{-1} [\Sigma_2^{-1}\mu + A^T\Sigma_1^{-1}(X-a)], \tag{H-13}$$

because if the equality between (H-12) and (H-10) is used on the right-hand side of (H-13) and if the resulting product is expanded and simplified, then (H-11) obtains. The formulas (H-12) and (H-13) prove (ii). QED.

Proposition (H-4) essentially describes Bayesian updating with Gaussian random variables. The final topic of this appendix is the constraining of a marginal distribution.

To begin, consider a random vector Y--not necessarily Gaussian--which is partitioned as:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}.$$

In the following manner, Y may be constrained so that Y_2 has a particular density f. Given t with the same dimension as Y_2, the conditional distribution of (Y_1, Y_3) given that $Y_2 = t$ has some distribution function $F([\text{-}\cdot\text{-}];t)$. Let

$$G\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \int_{-\infty}^{y} F\left(\begin{bmatrix} x \\ z \end{bmatrix} ; t\right) f(t)dt, \tag{H-14}$$

where the integral is the appropriate multi-fold integration. A random variable with distribution function G is said to be obtained from Y by <u>constraining</u> Y_2 to have density f.

In case the conditional distribution function F has a density p(· ;t), then (H-14) implies that G has a density function g defined by

$$g(x, y, z) = p(x, z;t)\ f(t). \tag{H-15}$$

Note that (H-14) and (H-15) obtain if f is the marginal density function of Y_2 determined by the distribution of Y.

It is worth noting that the foregoing construction and that of Proposition H-4 are pretty much the same: both a conditional and a marginal distribution are specified, and the joint distribution derived. (Compare equations (H-3) and (H-15).) This similarity is illustrated in the proof of the following proposition.

Proposition H-5. Suppose $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$ is a Gaussian random vector with mean $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$ and covariance matrix

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}.$$

Let Y* be the random variable obtained from Y by constraining Y_2 to have a Gaussian distribution with mean α and covariance matrix C. Then Y* has a Gaussian distribution with mean

$$\beta + \begin{bmatrix} B_{12} \\ B_{22} \\ B_{32} \end{bmatrix} B_{22}^{-1} (\alpha - \beta_2)$$

and covariance matrix

$$B - \begin{bmatrix} B_{12} \\ B_{22} \\ B_{32} \end{bmatrix} B_{22}^{-1} (B_{22} - C)\ B_{22}^{-1} [B_{21}\ \ B_{22}\ \ B_{32}].$$

Proof. From Proposition H-3, the conditional distribution of $\begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix}$ given that $Y_2 = z$ has density

$$n\left(\begin{pmatrix} x \\ y \end{pmatrix} ; a+Az,\ Q\right), \tag{H-16}$$

where

$$A = \begin{bmatrix} B_{12} \\ B_{32} \end{bmatrix} B_{22}^{-1},$$

$$a = \begin{bmatrix} \beta_1 \\ \beta_3 \end{bmatrix} - A\beta_2$$

$$Q = \begin{bmatrix} B_{11} & B_{13} \\ B_{31} & B_{33} \end{bmatrix} - A[B_{21} \quad B_{23}].$$

Proposition (H-4) and equation (H-15) imply that $\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$ has a Gaussian distribution with mean

$$\begin{bmatrix} a + A\alpha \\ \alpha \end{bmatrix} \tag{H-17}$$

and covariance matrix

$$\begin{bmatrix} Q + ACA^T & AC \\ CA^T & C \end{bmatrix}. \tag{H-18}$$

Now,

$$a+A\alpha = \begin{bmatrix}\beta_1\\ \beta_3\end{bmatrix} - A\beta_2 + A\alpha = \begin{bmatrix}\beta_1\\ \beta_3\end{bmatrix} + \begin{bmatrix}B_{12}\\ B_{32}\end{bmatrix} B_{22}^{-1}(\alpha-\beta_2),$$

so the mean vector (H-17) can be rewritten as

$$\begin{bmatrix}\beta_1\\ \beta_3\\ \beta_2\end{bmatrix} - \begin{bmatrix}B_{12}\\ B_{32}\\ B_{22}\end{bmatrix} B_{22}^{-1}(\alpha-\beta_2). \qquad \text{(H-19)}$$

(Note, in particular, that the third component is equal to α.) Next,

$$AC = \begin{bmatrix}B_{12}\\ \\ B_{32}\end{bmatrix} B_{22}^{-1} C, \qquad \text{(H-20)}$$

$$CA^T = C\, B_{22}^{-1}[B_{21} \quad B_{23}]; \qquad \text{(H-21)}$$

and

$$ACA^T = \begin{bmatrix}B_{12}\\ B_{32}\end{bmatrix} B_{22}^{-1} C B_{22}^{-1} [B_{21} \quad B_{23}];$$

$$Q+ACA^T = \begin{bmatrix}B_{11} & B_{13}\\ B_{31} & B_{33}\end{bmatrix} - \begin{bmatrix}B_{12}\\ B_{32}\end{bmatrix} B_{22}^{-1}[B_{21} \quad B_{23}] + \begin{bmatrix}B_{12}\\ B_{32}\end{bmatrix} B_{22}^{-1} C B_{22}^{-1}[B_{21} \quad B_{23}]$$

$$= \begin{bmatrix}B_{11} & B_{13}\\ B_{31} & B_{33}\end{bmatrix} - \begin{bmatrix}B_{12}\\ B_{32}\end{bmatrix} (I-B_{22}^{-1}C)\, B_{22}^{-1}[B_{21} \quad B_{23}]$$

$$= \begin{bmatrix}B_{11} & B_{13}\\ B_{31} & B_{33}\end{bmatrix} - \begin{bmatrix}B_{12}\\ B_{32}\end{bmatrix} B_{22}^{-1}(B_{22}-C)\, B_{22}^{-1}[B_{21} \quad B_{23}].$$

If

$$\hat{B}_{ij} = B_{ij} - B_{12}B_{22}^{-1}(B_{22}-C)B_{22}^{-1}B_{2j}, \tag{H-22}$$

then the last equation says that

$$Q + ACA^T = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{13} \\ \hat{B}_{31} & \hat{B}_{33} \end{bmatrix};$$

moreover, (H-20) and (H-21) become

$$AC = \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{32} \end{bmatrix}, \quad CA^T = [\hat{B}_{21} \quad \hat{B}_{23}]$$

and $C = \hat{B}_{22}$. Now the covariance matrix (H-11) for $\begin{bmatrix} Y_1 \\ Y_3 \\ Y_2 \end{bmatrix}$ becomes

$$\begin{bmatrix} \hat{B}_{11} & \hat{B}_{13} & \hat{B}_{12} \\ \hat{B}_{31} & \hat{B}_{33} & \hat{B}_{32} \\ \hat{B}_{21} & \hat{B}_{23} & \hat{B}_{22} \end{bmatrix}$$

with $\hat{B}_{ij}$ defined by (H-22). To finish the proof, apply the permutation matrix

$$P = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$$

according to Proposition (H-1) to the mean vector (H-19) and covariance matrix (H-23). QED.

APPENDIX I

OPERATIONS ON THE DIOU

This appendix discusses in further detail the formulas which accomplish the marginal constraint and contact data incorporation operations. The time step increase operation is addressed in Appendix J. The application of a marginal constraint will be presented first, followed by the incorporation of contact data. From this discussion derives the methodology of equations (IV-10)-(IV-12) in the text. The appendix closes by discussing the removal of a constraint and the application (or removal) of a constraint in the presence of other constraints.

Marginal constraint. Let $u = \{u_1, \ldots, u_k\}$ be a set of indices and let $z(u)$ be the subvector

$$\begin{bmatrix} z_{u_1} \\ \vdots \\ z_{u_k} \end{bmatrix}$$

of the random vector Z_τ. It is desired to constrain, in the sense of Appendix H, the UPD for Z_τ in such a way that the marginal distribution for $z(u)$ has specified mean ϕ_n and specified covariance matrix Φ_u. Let the covariance domain representation of the UPD be $(B_\tau, \beta_\tau, K_\tau^o)$ and that of the CPD be $(C_\tau, \sigma_\tau, K_\tau)$. Then Proposition (H-5) implies that

$$\gamma_\tau = \beta_\tau + B_\tau(u)(B_{uu})^{-1}(\phi_u - \beta_\tau(u)), \tag{I-1a}$$

$$C_\tau = B_\tau - B_\tau(u)(B_{uu})^{-1}(B_{uu} - \Phi_u)(B_{uu})^{-1}B_\tau(u)^T, \tag{I-1b}$$

where $\beta_\tau(u)$ is the subvector of β_τ corresponding to u,

$$B_{uu} = (B_\tau)_{uu} = [b_{u_i, u_j}] \quad i, j = 1, \ldots, k$$

and $B_\tau(u)$ is the column matrix of B_τ containing B_{uu}. (See equations (G-1) through (G-3) in Appendix G.)

Now let $(\mathscr{I}_\tau^o, \mathscr{X}_\tau^o)$ and $(\mathscr{I}_\tau, \mathscr{X}_\tau)$ be the information domain representations of the UPD and CDP, respectively. If $D = \Phi_u^{-1} - (B_{uu})^{-1}$, then Proposition (G-3) says that

$$C_\tau^{-1} = B_\tau^{-1} \ (+) \ D,$$

i.e.,

$$\mathscr{I}_\tau = \mathscr{I}_\tau^o \ (+) \ (\Phi_u^{-1} - (B_{uu})^{-1}), \tag{I-2}$$

where the (+) operation is defined by equation (G-4). Furthermore, by Proposition (G-2(c)),

$$\det C_\tau = \det(I + B_{uu}(\Phi_u^{-1} - B_u^{-1} - B_{uu}^{-1}))^{-1} \det B_\tau$$

$$= \det(B_{uu})^{-1} \det \Phi_u \det B_\tau ,$$

whence

$$K_\tau = K_\tau^o + \log \det \Phi_u - \log \det B_{uu}. \tag{I-3}$$

Now,

$$\mathscr{X}_\tau = C_\tau^{-1} \gamma_\tau = [B_\tau^{-1} \ (+) \ (\Phi_u^{-1} - B_{uu}^{-1})] \ [\beta_\tau + B_\tau(u) B_{uu}^{-1} (\phi_u - \beta_\tau(u)]$$

$$= B_\tau^{-1} \beta_\tau \ (+) \ [B_{uu}^{-1}(\phi_u - \beta_\tau(u)) + (\Phi_u^{-1} - B_{uu}^{-1}) \beta_\tau(u) + (\Phi_u^{-1} - B_{uu}^{-1})(\phi_u - \beta_\tau(u))]$$

which, upon simplification, yields

$$\mathscr{X}^{\cdot}_{\tau} = \mathscr{X}^{o}_{\tau} (+) [\Phi_u^{-1}\phi_u - B_{uu}^{-1}\beta_\tau (u)]. \tag{I-4}$$

Other operations involving marginal constraints will be discussed later in this appendix.

Contact Data; Basic Methodology

For the remainder of this appendix, C_τ, γ_τ, etc. will denote parameters for the modified distribution before an operation is performed, and $\tilde{C}_\tau, \tilde{\mathscr{X}}^{\cdot}_\tau$, etc. will denote the corresponding parameters after performing the operation.

It is argued in the text of Chapter IV that the incorporation of a contact datum results in equations of the following form for the information matrices and information vectors:

$$\tilde{\mathscr{I}}_\tau = \mathscr{I}_\tau (+) D \tag{I-5}$$

$$\tilde{\mathscr{X}}^{\cdot}_\tau = \mathscr{X}^{\cdot}_\tau (+) d \tag{I-6}$$

where D and d take forms depending on the type of contact datum involved. D is, moreover, a symmetric matrix. By examining equations (I-5) and (I-6), it is apparent that, in order to remove from processing a contact datum already incorporated, the negatives of the corresponding D and d should be used. The (+) sign has the same meaning as described by equation (I-4) and (I-5), with $u = \{j\}$, where t_j is the time of the contact.

Now, Proposition G-2b implies that

$$\tilde{C}_\tau = C_\tau - C_\tau(t)\, D(I+C_{tt}D)^{-1} C_\tau(t)^T \tag{I-7}$$

and

$$\det \tilde{C}_\tau = \det(I+C_{tt}D)^{-1} \det C_\tau .$$

Since det $\tilde{C}_\tau$ and det C_τ are positive, so is $\det(I+C_{tt}D)$, and hence

$$\log\det\tilde{C}_\tau = \log\det(I+C_{tt}D)^{-1} + \log\det C_\tau ,$$

from which

$$\tilde{K}_\tau = K_\tau - \log\det(I+C_{tt}D) \qquad \text{(I-8)}$$

follows.

Next, using (I-6) and (I-7),

$$\tilde{\gamma}_\tau = \tilde{C}_\tau \tilde{I}_\tau = \gamma_\tau + C_\tau(t)d - C_\tau(t)\,D(I+C_{tt}D)^{-1}C_\tau(t)\,C_\tau^{-1}\gamma_\tau$$

$$- C_\tau(t)\,D(I+C_{tt}D)^{-1}\,C_{tt}d$$

$$= \gamma_\tau + C_\tau(t)\,[I-D(I+C_{tt}D)^{-1}C_{tt}]d - C_\tau(t)\,D(I+C_{tt}D)^{-1}\,\gamma_\tau(t). \qquad \text{(I-9)}$$

Since C_{tt} is invertible,

$$I - D(I+C_{tt}D)^{-1}C_{tt} = I - D(C_{tt}^{-1}+D)^{-1}$$

$$= C_{tt}^{-1}(C_{tt}^{-1}+D)^{-1}$$

$$= (I+DC_{tt})^{-1}.$$

Substituting this into (I-9) gives

$$\tilde{\gamma}_\tau = \gamma_\tau + C_\tau(t)\,(I+DC_{tt})^{-1}d - C_\tau(t)\,D(I+C_{tt}D)^{-1}\,\gamma_\tau(t). \qquad \text{(I-10)}$$

The following is also worth noting for independent interest. If $D = \Delta^{-1}$ and $d = \Delta^{-1}\delta$, for a positive definite symmetric matrix Δ, then

$$(I+C_{tt}D)^{-1} = \Delta(\Delta+C_{tt})^{-1}$$

and

$$(I+DC_{tt})^{-1} = (\Delta+C_{tt})^{-1}\Delta,$$

so equations (I-7), (I-8), and (I-10) become

$$\tilde{C}_T = C_T - C_T(t)(\Delta+C_{tt})^{-1} C_T(t)^T, \tag{I-11a}$$

$$K_T = K_T - \log\det(\Delta+C_{tt}) + \log\det\Delta, \tag{I-11b}$$

$$\gamma_T = \gamma_T + C_T(t)(\Delta+C_{tt})^{-1}(\delta-\gamma_T(t)). \tag{I-11c}$$

Of the operations under consideration, the only one which, in general, fits the criteria necessary for (I-11) is the incorporation of a "SPA." Hence, the equations (I-5) through (I-8) and (I-10) are recommended for use.

Now, the equations (I-5), (I-6), (I-7), (I-8), and (I-10) form the basic methodology described by equations (IV-10), (IV-11), and (IV-12) in the text. The remainder of this appendix will be used to show how the various constraint operations fit into this methodological framework.

Constraint Operations and the Basic Methodology

Let ϕ_u and Φ_u be the mean and covariance matrix corresponding to a marginal constraint as discussed above, and suppose (D_i, d_i) $(i=1,\ldots,n)$ are information adjustment quantities corresponding to various contact data as per equations (I-5) and (I-6). Suppose all of these are to be incorporated into the UPD. Then (I-2), (I-4), (I-5), and (I-6) combine to give the following equations for the resulting information matrix and information vector:

$$\tilde{\mathscr{I}}_T = \mathscr{I}_T^o \,(+)\, (\Phi_u^{-1} - B_{uu}^{-1}) \,(+)\, D_1 \,(+) \ldots (+)\, D_n \tag{I-12}$$

$$\tilde{\mathscr{X}}_T = \mathscr{X}_T^{\cdot} \,(+)\, (\Phi_u^{-1}\phi_u - B_{uu}^{-1}\beta_T(u)) \,(+)\, d_1 \,(+) \ldots (+)\, d_n. \tag{I-13}$$

In (I-12) each quantity to the right of a (+) sign is to be incorporated into a $(2\tau) \times (2\tau)$ matrix (either $\mathcal{I}_\tau^o$ or an already modified $\mathcal{I}_\tau^o$), according to the definition of (+). A similar remark applies to equation (I-13). However, it is evident that this "addition" is commutative among the quantities to the right of the first (+) sign. Therefore, if the constraint is to be applied after all of the contact data are incorporated, then the following equations obtain:

$$\tilde{\mathcal{I}}_\tau = \mathcal{I}_\tau \; (+) \; (\Phi_u^{-1} - B_{uu}^{-1}), \tag{I-14}$$

$$\tilde{\mathcal{X}}_\tau^{\cdot} = \mathcal{X}_\tau^{\cdot} \; (+) \; (\Phi_u^{-1}\phi_u - B_{uu}^{-1}\beta_\tau(u)). \tag{I-15}$$

Equations (I-14) and (I-15) give rise to the assignments described by equations (I-14) of the text. Now the equations (I-7), (I-8), and (I-10), which are equations (IV-12) of the text, may be applied. Thus, a marginal constraint may be applied even after the incorporation of contact data. Note that if this methodology is applied in this case when the prior distribution is the UPD (i.e., no contact data is present), then the equations (I-1) and (I-3) ultimately obtain.

By subjecting equations (I-14) and (I-15) to the same examination that was applied to equations (I-5) and (I-6), it is apparent that a marginal constraint may be removed by replacing the corresponding D and d by their negatives and applying the same methodology. Hence, the assignments given by equations (IV-15) of the text are justified.

As mentioned in the text, the foregoing remarks apply only when no other constraints are involved. Suppose now that $u = \{u_1, \ldots, u_k\}$ and $v = \{v_1, \ldots, v_l\}$ are (disjoint) index sets, and for notational simplicity suppose that $u_i < v_j$ for all i and j. Let $u' = \{u_1, \ldots, u_k, v_1, \ldots, v_l\}$. Assume further that the constrained distributions for z(u) and z(v) are uncorrelated, with parameters (ϕ_u, Φ_u) and (ϕ_v, Φ_v), respectively. (This will be the case in practice.) Thus z(u') will have the distribution with parameters

$$\phi_{u'} = \begin{bmatrix} \phi_u \\ \\ \phi_v \end{bmatrix}, \qquad \Phi_{u'} = \begin{bmatrix} \Phi_u & 0 \\ \\ 0 & \Phi_v \end{bmatrix}.$$

The idea is to be able to apply the constraint on z(v) after the constraint on z(u) and several contact data have been incorporated.

Now,

$$\Phi_{u'} = \begin{bmatrix} \Phi_u & 0 \\ 0 & \Phi_v \end{bmatrix};$$

furthermore,

$$B_{u'u'} = \begin{bmatrix} B_{uu} & B_{uv} \\ B_{vu} & B_{vv} \end{bmatrix}$$

where, e.g.,

$$B_{uv} = [\mathrm{cov}(z_{u_i}, z_{v_j})] \quad i = 1, \ldots, k;\ j = 1, \ldots, l.$$

Let $\tilde{S} = B_{uu}^{-1} B_{uv}$, $\tilde{T} = B_{vv} - B_{vu} B_{uu}^{-1} B_{uv}$. According to Proposition G-1.

$$B_{u'u'}^{-1} = \begin{bmatrix} B_{uu}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\tilde{S} \\ I \end{bmatrix} \tilde{T}^{-1} [-\tilde{S}^T \quad I].$$

With all constraints and contact data incorporated, the information matrix is given by

$$\tilde{\mathscr{I}}_\tau = \mathscr{I}_\tau^o \,(+)\, (\Phi_{u'}^{-1} - B_{u'u'}^{-1}) \,(+)\, D_1 \,(+) \ldots (+)\, D_n$$

$$= \mathscr{I}_\tau^o \,(+) \left(\begin{bmatrix} \Phi_u^{-1} & 0 \\ 0 & \Phi_v^{-1} \end{bmatrix} - \begin{bmatrix} B_{uu}^{-1} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -\tilde{S} \\ I \end{bmatrix} \tilde{T}^{-1} [-\tilde{S}^T \quad I] \right) (+)\, D_1 \,(+) \ldots (+)\, D_n$$

$$= \mathscr{I}_\tau^o \,(+)\, (\Phi_u^{-1} - B_{uu}^{-1}) \,(+)\, D_1 \,(+) \ldots (+)\, D_n \,(+) \begin{bmatrix} -\tilde{S} \\ I \end{bmatrix} (-\tilde{T}^{-1}) [-\tilde{S}^T \quad I] \,(+)\, \Phi_v^{-1};$$

so

$$\tilde{\mathcal{I}}_{\tau} = \mathcal{I}_{\tau} \; (+) \begin{bmatrix} -\tilde{S} \\ I \end{bmatrix} (-\tilde{T}^{-1}) \; [-\tilde{S}^{T} \quad I] \; (+) \; \Phi_{v}^{-1}. \tag{I-16}$$

If

$$\tilde{L} = \begin{bmatrix} -\tilde{S} \\ I \end{bmatrix} (-\tilde{T}^{-1}) \; [-\tilde{S}^{T} \quad I],$$

then the assignment given by equation (IV-16a) of the text obtains. Next,

$$B_{u'u'}^{-1} \beta_{\tau}(u') = \begin{bmatrix} B_{uu}^{-1} \quad \beta_{\tau}(u) \\ \\ 0 \end{bmatrix} + \begin{bmatrix} -\tilde{S} \\ I \end{bmatrix} \tilde{T}^{-1} \; [-\tilde{S}^{T} \quad I],$$

so

$$\begin{aligned} \mathcal{X}_{\tau}^{\cdot} &= \mathcal{X}_{\tau}^{o} \; (+) \; (\Phi_{u'}^{-1} \; \phi_{u'} - B_{u'u'} \beta_{\tau}(u')) \; (+) \; d_1 \; (+) \ldots (+) \; d_n \\ &= \mathcal{X}_{\tau}^{\cdot} \; (+) \; \Phi_{v}^{-1} \; \phi_{v}^{-1} \quad {}_{v} \; (+) \; L \beta_{\tau+1}(u') \end{aligned} \tag{I-17}$$

may be derived in a manner similar to the derivation of (I-16), justifying the assignment given by the text's equation (IV-16b).

A straightforward examination of the derivations of equations (I-16) and (I-17) reveals the following. Suppose the constraints corresponding to u and v have been incorporated into the modified distribution, and it is desired to remove the constraint corresponding to v. Then the methodology of equations (IV-10)-(IV-13) of the text (equations (I-5)-(I-8) and (I-10) above)may be applied, taking for D and d the negatives of the quantities given by the text equations.

Equations (I-16) and (I-17) may be extended to the case where $u_i > v_j$ for some i and j using a permutation argument. See the remark following Proposition (H-3).

APPENDIX J

TIME STEP INCREASE

This appendix considers the third operation on the DIOU, that of increasing the time interval over which the process is studied from $[0, t_\tau)$ to $[0, t_{\tau+1})$. The other two operations, corresponding to marginal constraints and contact data, are discussed in Appendix I. The time step increase (or "updating") operation will first be applied to the UPD, and then those results will in turn be applied to the modified distribution. Initially, it will be assumed that $\tau > 1$; the cases $\tau = 0$ and $\tau = 1$ will be addressed at the end.

Updating the UPD

According to the relations (IV-6) in the text,

$$B_{\tau+1} = \begin{bmatrix} B_\tau & H_\tau \\ H_\tau^T & b_{\tau+1,\tau+1} \end{bmatrix}, \tag{J-1}$$

where

$$H_\tau = \begin{bmatrix} b_{1,\tau+1} \\ \vdots \\ b_{\tau,\tau+1} \end{bmatrix},$$

and $b_{\tau+1,\tau+1}$ are constructed by the recursion of equations (IV-6). Let $S = B_\tau^{-1} H_\tau$ and $T = b_{\tau+1,\tau+1} - H_\tau^T S$. Then, according to Proposition G-1, T is invertible.

$$B_{\tau+1}^{-1} = \begin{bmatrix} B_\tau^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1}[-S^T \quad I], \tag{J-2}$$

where I is the 2 x 2 identity, and

$$\det B_{\tau+1} = (\det B_\tau)(\det T).$$

Since det B_τ and det $B_{\tau+1}$ are positive, so is det T, and equation (IV-27) of the text obtains:

$$K_{\tau+1}^{o} = K_\tau^{o} + 2 \log 2\pi + \log \det T. \tag{J-3}$$

The relations involving equation (J-2) simplify, given the specific form of B_τ contained in the relations (IV-6). Let e_τ be as in those relations, from which follows

$$b_{i,\tau+1} = b_{i,\tau} + e_\tau (b_{i,\tau} - b_{i,\tau-1}) \tag{J-4}$$

for $0 \leq i \leq \tau$. Now, if $\tau > 1$, use (J-1) to write

$$B_\tau = \left[\begin{array}{cc|c} B_{\tau-2} & H_{\tau-2} & \\ \hline H_{\tau-2}^T & b_{\tau-1,\tau-1} & H_{\tau-1} \\ \hline \multicolumn{2}{c|}{H_{\tau-1}^T} & b_{\tau,\tau} \end{array}\right]. \tag{J-5}$$

Then (J-4) and (J-5) imply

$$H_\tau = \begin{bmatrix} H_{\tau-1} \\ \cdots\cdots \\ b_{\tau,\tau} \end{bmatrix} + e_\tau \left\{ \begin{bmatrix} H_{\tau-1} \\ \cdots\cdots \\ b_{\tau,\tau} \end{bmatrix} - \begin{bmatrix} H_{\tau-2} \\ K_{\tau-2} \\ b_{\tau,\tau-1} \end{bmatrix} \right\};$$

Note that vectors on the right of this equation can be identified with columns of B_τ. Since $B_\tau^{-1}B_\tau$ equals the identity matrix, this in turn implies that

$$S = B_\tau^{-1}H_\tau = \begin{bmatrix} 0 \\ -e_\tau I \\ (1+e_\tau)I \end{bmatrix}, \qquad \text{(J-6)}$$

in which I is the 2 x 2 identity matrix. Furthermore,

$$T = b_{\tau+1,\tau+1} - H_\tau^T S = b_{\tau+1,\tau+1} + e_\tau b_{\tau+1,\tau-1} - (1+e_\tau)b_{\tau+1,\tau}. \qquad \text{(J-7)}$$

Equations (J-6) and (J-7) are the text equations (IV-26), except that (IV-26c) is derived in Note 3 at the end of the chapter. Note that not only computing T^{-1} is a computationally simple task, since T is only a 2 x 2 matrix, but obtaining T is easier than its original definition might suggest. Indeed, equations (J-6) and (J-7) show that both S and T take on simple forms, dependent on process parameters at $\tau-1$, τ, and $\tau+1$. This lies at the heart of the simplifications.

Next, let L be the $(2\tau+2) \times (2\tau+2)$ matrix defined by

$$L = \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1} [-S^T \quad I], \qquad \text{(J-8)}$$

so that (J-2) may be written as

$$B_{\tau+1} = \left[\begin{array}{c|c} B_\tau^{-1} & 0 \\ \hline 0 & 0 \end{array}\right] + L, \qquad \text{(J-9)}$$

which is text equation (IV-28a). Note that since S takes the form given by equation (J-6), (J-8) can be rewritten

$$L = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & Q \end{array}\right], \qquad \text{(J-10a)}$$

where Q is the 6 x 6 matrix

$$Q = \begin{bmatrix} e_\tau I \\ -(1+e_\tau)I \\ I \end{bmatrix} T^{-1}[e_\tau I, -(1+e_\tau)I, I]. \quad \text{(J-10b)}$$

The reader should beware that the matrix partitions in (J-9) and (J-10a) are not (nor are they intended to be) compatible.

Now, the equations (IV-5) of the text immediately give

$$\beta_{\tau+1} = \begin{bmatrix} \beta_\tau \\ \text{-----------} \\ \beta_\tau(\tau)+\delta_\tau \nu_\tau \end{bmatrix}; \quad \text{(J-11)}$$

Let, e.g., $\beta_{\tau-1}(\tau-1)$ denote the $(\tau-1)^{th}$ 2-dimensional subvector of β_τ (see discussion of equation (G-1)). Then (J-10) and (J-11) combine to give

$$L\beta_{\tau+1} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \beta_{\tau+1} = \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1}(\beta_{\tau+1}(\tau+1)-(1+e_\tau)\,\beta_{\tau+1}(\tau)+e_\tau \beta_\tau(\tau-1))$$

$$= \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1}(\beta_{\tau+1}(\tau+1)-\beta_{\tau+1}(\tau)-e_\tau(\beta_{\tau+1}(\tau)-\beta_\tau(\tau-1)) \quad \text{(J-12)}$$

$$= \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1}[\delta_\tau \nu_\tau - e_\tau \delta_{\tau-1} \nu_{\tau-1}].$$

In deriving (J-12), use was made of the fact, apparent from (J-11), that $\beta_\tau(i) = \beta_{\tau+1}(i)$ if $i \leq \tau$. To continue,

$$\mathscr{X}^{Q}_{\tau+1} = B^{-1}_{\tau+1}\beta_{\tau+1} = \left\{\begin{bmatrix} B^{-1}_{\tau} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}\beta_{\tau+1}$$

$$= \begin{bmatrix} B^{-1}_{\tau}\beta_{\tau} \\ \hline 0 \end{bmatrix} + L\beta_{\tau+1}, \qquad \text{(J-13)}$$

which gives text equation (IV-28b).

Updating the Modified Distribution

The foregoing results will be used to update the modified distribution. It is assumed that the latter involves only constraints and contact data pertaining to times preceding t_{τ}. Let D denote the information adjustment quantity as in text equation (IV-10) (see also equation (I-5)) corresponding to such a constraint or contact datum; conceptually, D should stand for the sum such as is added to $\mathscr{I}^{Q}_{\tau}$ in equation (I-12). Then

$$C^{-1}_{\tau+1} = B^{-1}_{\tau+1} (+) D$$

$$= \left(\begin{bmatrix} B^{-1}_{\tau} & 0 \\ 0 & 0 \end{bmatrix} + L\right) (+) D$$

$$= \left(\begin{bmatrix} B^{-1}_{\tau} & 0 \\ 0 & 0 \end{bmatrix} (+) D\right) + L;$$

using the fact that the constraint, etc., correspond to times less than t_{τ}, this gives

$$C^{-1}_{\tau+1} = \begin{bmatrix} C^{-1}_{\tau} & 0 \\ 0 & 0 \end{bmatrix} + L, \qquad \text{(J-14)}$$

which is equation (IV-30a) of the text. Now, let S and T be as in (J-2). Then examining Proposition G-1 reveals that

$$C_{\tau+1} = \begin{bmatrix} C_\tau & C_\tau S \\ (C_\tau S)^T & T+S^T C_\tau S \end{bmatrix}. \qquad \text{(J-15)}$$

which is text equation (IV-29a). Since S takes the form (J-6), equation (J-15) can be written in another form. Let V_1 and V_2 be the 2τ x 2 matrices which form the last four columns of C_τ, and let w_{11}, w_{12}, w_{22} form the lower right-hand 4 x 4 block of C_τ:

$$C_\tau = [\ldots | V_1 | V_2] = \left[\begin{array}{c|cc} & & \\ \hline & w_{11} & w_{12} \\ & w_{12}^T & w_{22} \end{array}\right].$$

Then

$$C_{\tau+1} = \begin{bmatrix} C_\tau & V \\ V^T & T+W \end{bmatrix}, \qquad \text{(J-16)}$$

where

$$V = C_\tau S = (1+e_\tau)\, V_2 - e_\tau\, V_1,$$

$$W = S^T C_\tau S = (1+e_\tau)^2 w_{22} - e_\tau [(1+e_\tau)(w_{12}+w_{12}^T) - e_\tau w_{11}],$$

thus continuing the theme of involving only times $\tau-1$, τ, and $\tau+1$ in the updating process.

Proposition G-1 also implies that

$$\det C_{\tau+1} = (\det C_\tau)(\det T),$$

whence text equation (IV-31) follows:

$$K_{\tau+1} = K_{\tau} + 2 \log 2\pi + \log \det T. \tag{J-17}$$

Now, if d is the information adjustment quantity of text equation (IV-11) (or equation (I-6)), or a "sum" such as is added to $\mathscr{X}_\tau^0$ in equation (I-13), then

$$\mathscr{X}_{\tau+1} = B_{\tau+1}^{-1}\beta_{\tau+1} \,(+)\, d$$

$$= \left(\begin{bmatrix} \mathscr{X}_\tau^0 \\ 0 \end{bmatrix} + L\beta_{\tau+1}\right) (+)\, d$$

$$= \left(\begin{bmatrix} \mathscr{X}_\tau^0 \\ 0 \end{bmatrix} (+)\, d\right) + L\beta_{\tau+1};$$

analogously to (J-14), this gives

$$\mathscr{X}_{\tau+1} = \begin{bmatrix} \mathscr{X}_\tau \\ 0 \end{bmatrix} + L\beta_{\tau+1} \tag{J-18}$$

which together with (J-12), yields text equation (IV-30b).

Next, equations (J-15) and (J-18) combine as follows to give equation (IV-29b) of the text.

$$\begin{aligned} \gamma_{\tau+1} &= C_{\tau+1}\,\mathscr{X}_{\tau+1} \\ &= \begin{bmatrix} C_\tau & C_\tau S \\ (C_\tau S)^T & T+S^T C_\tau S \end{bmatrix} \left\{\begin{bmatrix} \mathscr{X}_\tau \\ 0 \end{bmatrix} + L\beta_{\tau+1}\right\} \\ &= \begin{bmatrix} C_\tau \mathscr{X}_\tau \\ S^T \gamma_\tau \end{bmatrix} + C_{\tau+1} L\beta_{\tau+1} \\ &= \begin{bmatrix} I \\ S^T \end{bmatrix} \gamma_\tau + C_{\tau+1} L\beta_{\tau+1}. \end{aligned} \tag{J-19}$$

From (J-12) and (J-15) comes

$$C_{\tau+1} L \beta_{\tau+1} = \begin{bmatrix} C_\tau & C_\tau S \\ S^T C_\tau & T+S^T C_\tau S \end{bmatrix} \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1}(\delta_\tau \nu_\tau - e_\tau \delta_{\tau-1} \nu_{\tau-1})$$

$$= \begin{bmatrix} 0 \\ T \end{bmatrix} T^{-1}(\delta_\tau \nu_\tau - e_\tau \delta_{\tau-1} \nu_{\tau-1}) \tag{J-20}$$

$$= \begin{bmatrix} 0 \\ \hline \delta_\tau \nu_\tau - e_\tau \delta_{\tau-1} \nu_{\tau-1} \end{bmatrix}.$$

Since $S^T \gamma_\tau = (1+e_\tau)\, \gamma_\tau(\tau) - e_\tau \gamma_\tau(\tau-1)$, equations (J-19) and (J-20) combine to give the desired result:

$$\gamma_{\tau+1} = \begin{bmatrix} \gamma_\tau \\ \hline (1+e_\tau)\gamma_\tau(\tau) + \delta_\tau \nu_\tau - e_\tau(\gamma_\tau(\tau-1) + \delta_{\tau-1}\nu_{\tau-1}) \end{bmatrix}. \tag{J-21}$$

The following identity was used in the course of developing equation (J-12):

$$\beta_{\tau+1}(\tau+1) - (1+e_\tau)\beta_\tau(\tau) + e_\tau \beta_\tau(\tau-1) = \delta_\tau \nu_\tau - e_\tau \delta_{\tau-1} \nu_{\tau-1}. \tag{J-22}$$

Equations (J-21) and (J-22) give

$$\gamma_{\tau+1} = \begin{bmatrix} \gamma_\tau \\ \hline \beta_\tau(\tau) + \delta_\tau \nu_n + (1+e_\tau)(\gamma_\tau(\tau) - \beta_\tau(\tau)) - e_\tau(\gamma_\tau(\tau-1) - \beta_\tau(\tau-1)) \end{bmatrix}. \tag{J-23}$$

Equation (J-23) and a straightforward examination of equations (J-14), (J-15), (J-17), and (J-18) show that if the modified distribution coincides with the UPD, then the formulas just derived to update the modified distribution reduce to those derived initially to update the UPD.

Special Cases

The development in this appendix to this point has assumed that $\tau > 1$. The remainder will consider the special cases $\tau = 0$ and $\tau = 1$. Specifically, the necessary changes in the foregoing development will be examined.

To begin, suppose $\tau = 0$. Then

$$B_{\tau+1} = B_1 = \begin{bmatrix} B_0 & H_0 \\ H_0^T & b_{11} \end{bmatrix},$$

where $H_0 = b_{01}$; $B_0 = b_{01}$, of course, equations (J-2) and (J-3) still hold, but (J-4) is replaced by

$$b_{01} = b_{00} + h_{01} = b_{00},$$

since $h_{0j} = 0$ for all j. Therefore,

$$S = B_\tau^{-1} H_\tau = b_{00}^{-1} b_{01} \tag{J-24}$$

replaces (J-6), and

$$T = K_\tau - H_\tau^T S = b_{11} - b_{01}^T = b_{11} - b_{00}$$

replaces (J-7). Now equations (J-8) and (J-9) remain valid. However, (J-8) and (J-24) imply that

$$L = \begin{bmatrix} -I \\ I \end{bmatrix} T^{-1} [-I \quad I]$$

(so Q is a 4 x 4 matrix); this and the still valid (J-11) give

$$L\beta_1 = \begin{bmatrix} -S \\ I \end{bmatrix} T^{-1} (\delta_0 \nu_0),$$

a different form of (J-12). Equation (J-13) remains valid.

Turning to the updating of the modified distribution, only a couple of changes are necessary. Firstly, in order for equation (J-16) to remain valid, the equation preceding it must be rewritten as

$$C_0 = V_2 = w_{22},$$

so that

$$V = C_0 S = C_0$$

and

$$W = S^T C_0 S = C_0$$

follow. The only other change is that equation (J-23) should be replaced by

$$\gamma_1 = \gamma_{\tau+1} = \left[\frac{\gamma_0}{\beta + (\gamma_0(0) - \beta_0(0))} \right] .$$

Note that since it is unlikely that a constraint will be imposed before considering the first time step (such an action would be merely rechoosing an initial distribution), $\gamma_1 = \beta_1$ is most likely.

The case $\tau = 1$ requires even fewer modifications. Then equation (J-5) still holds, since $H_0^T = b_{01}^T = b_{10}$. Then

$$S = \begin{bmatrix} -e_\tau I \\ (1+e_\tau)I \end{bmatrix}; \qquad \text{(J-25)}$$

T is still given by equation (J-7). The only other changes to note are that

$$C_1 = [V_1 \quad V_2] = \left[\begin{array}{c|c} w_{11} & w_{12} \\ \hline w_{12}^T & w_{22} \end{array}\right]$$

in order for equation (J-16) to remain valid, and that, since S is given by (J-25), L = Q.

APPENDIX K

SUMMARY OF ALGORITHMS

by David P. Kierstead

This appendix gives a brief description of how the text equations are actually implemented by SCREEN. The quantities which must be produced are the mean and covariance matrix of the target's distribution (as predicted from the motion assumptions and positive contact information).

In some portions of SCREEN it is desirable to know $E[(z_1^T, \ldots, z_\tau^T)^T \mid z_0]$, which may be computed according to the formula

$$E\left(\begin{bmatrix} z_1 \\ \vdots \\ z_\tau \end{bmatrix} \middle| \; z_0\right) = E\left(\begin{bmatrix} z_1 \\ \vdots \\ z_\tau \end{bmatrix}\right) + \text{Cov}\left(\begin{bmatrix} z_1 \\ \vdots \\ z_\tau \end{bmatrix}, z_0\right) (\text{Var}(z_0))^{-1} (z_0 - E(z_0)).$$

It is clear that this is a linear equation in z_0, so what are actually computed are the coefficients of this equation. These are stored as SMEANO and SMEAN1. Thus,

$$E\left(\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \middle| \; z_0\right) = \text{SMEAN0} + \text{SMEAN1} \cdot z_0.$$

In the following summaries, which appear as Tables K-1 through K-4, the notation of the text and Appendix J is preserved.

TABLE K-1

Variable	Identity	Program Variable (at time (τ))
δ	length of time step	DELTAT
γ_t	mean target speed for times t to t+1	TARS records $\delta\nu_0, \ldots, \delta\nu_{\tau-1}$;SPD is $\delta\nu_\tau$
Γ_t	diffusion covariance matrix for times t to t+1	TDIFF records $\delta^2\Gamma_0, \ldots, \delta^2\Gamma_{\tau-1}$; SPDCOV is $\delta^2\Gamma_\tau$
μ_t	mean number of course changes per hour for times t to t+1	TARM(2) holds μ_τ
γ_t	mean target location at time t	TMEAN
C_τ	covariance matrix of target positions for times 0 through τ	SCOV0 is cols. for time τ-1 SCOV1 is cols. for time τ PSPACV is block diag.
$E[(z_1^T, \ldots, z_\tau^T)^T \mid z_0]$	mean position, conditioned on the starting position z_0	$E[(z_1^T, \ldots, z_\tau^T)^T \mid z_0]$=SMEAN0 + SMEAN1$\cdot z_0$

TABLE K-2

TIME STEP UPDATE, $\tau = 0$

(Initialization and Update from Times 0 to 1)

Step	Calculations	Program Variables
Initialization	Input:	
	δ	DELTAT
	μ_0	TARM(2)
	$\delta \nu_0$	SPD
	$\delta^2 \Gamma_0$	SPDCOV
	$\sigma_0 = \beta_0$	TMEAN, SPAPOS, BETN
	C_0	SPACOV, PSPACV
1	$e_1 = \exp[-\delta \mu_0]$	CSCALE
2	$\beta_1 = \left[\frac{\beta_0}{\beta_0 + \delta \nu_0}\right]$	BET NP1 = $\beta_1(1)$
3	$\gamma_1 \sigma_1 = \left[\frac{\gamma_0}{\beta_1(1)}\right]$	TMEAN
4	vacuous	
5	$C_1 = \begin{bmatrix} C_0 & C_0 \\ C_0 & C_0 + \delta^2 \Gamma_0 \end{bmatrix}$	SCOV0 = $C_1(0)$ SCOV1 = $C_1(1)$
6	SMEAN0 = $\gamma_1(1) - \gamma_0$ SMEAN1 = I	

TABLE K-2 (Continued)

Step	Calculations	Program Variables
7	Reset auxiliary quantities to be ready for next time update:	
	β_0	BETNM1
	$\beta_1(1)$	BETN
	$\delta^2\Gamma_0$	GAMNM1
	W_{11}	W11MAT
	W_{12}	W12MAT
	W_{22}	W22MAT
	$\tau = \tau + 1$	TARM(1) = 1

TABLE K-3

TIME STEP UPDATE, $\tau > 0$

(Update from Times τ to $\tau+1$)

Step	Calculations	Program Variables
	Input: μ_τ, ν_τ, Γ_τ	
1	$e_{\tau+1} = \exp[-\delta\mu_\tau]$	CSCALE
2	$\beta_{\tau+1} = \left[\frac{\beta_\tau}{\beta_\tau(\tau)+\delta\nu_\tau}\right]$	BETNP1 = $\beta_{\tau+1}(\tau+1)$
3	$\gamma_{\tau+1} = \left[\frac{\gamma_\tau}{\beta_{\tau+1}(\tau+1)+(1+e_n)(\gamma_\tau(\tau)-\beta_\tau(\tau))-e_n(\gamma_\tau(\tau-1)-\beta_\tau(\tau-1))}\right]$	TMEAN
4	$V = (1+e_\tau)\, C_\tau(\tau)-e_\tau\, C_\tau(\tau-1)$	VMAT
	$W = (1+e_\tau)^2 W_{22}-e_\tau[(1+e_\tau)(W_{12}+W_{12}^T)-e_\tau W_{11}]$	WMAT
	$T = \delta^2\Gamma_\tau - e_\tau^2\delta^2\Gamma_{\tau-1}$	TMAT
5	$C_{\tau+1}(\tau) = \left[\frac{C_\tau(\tau)}{(1+e_\tau)W_{22}-e_\tau W_{12}}\right]$	SCOV0
	$C_{\tau+1}(\tau+1) = \left[\frac{V}{T+W}\right]$	SCOV1
6	$\text{SMEAN0} = \text{SMEAN0} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline \gamma_{\tau+1}(\tau+1)-\text{Cov}(z_{\tau+1})z_0)\text{Var}(z_0)^{-1}E(z_0) \end{bmatrix}$	
	$\text{SMEAN1} = \text{SMEAN1} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline \text{Cov}(z_{\tau+1})z_0)\text{Var}(z_0)^{-1} \end{bmatrix}$	

TABLE K-3 (Continued)

Step	Calculations	Program Variables
7	Reset auxiliary quantities to be ready for next time update:	
	$\beta_{\tau}(\tau)$	BETNM1
	$\beta_{\tau+1}(\tau+1)$	BETN
	$\delta^2\Gamma_{\tau}$	GAMNM1
	W_{11}	W11MAT
	W_{12}	W12MAT
	W_{22}	W22MAT
	$\tau = \tau+1$	TARM(1)

TABLE K-4

INCORPORATION OF INFORMATION

Step	Calculation	Program Variables
1	Input:	
	D	ADDMAT
	d	ADDVEC
	t	ITSTP2
2	Retrieve $(C(t))^T$	COVRO
3	Compute $D(I+C_{tt}D)^{-1}$	PROD
4	$\tilde{C} = C-C(t)[D(I+C_{tt}D)^{-1}]C(t)^T$	SCOV0 SCOV1 PSPACV
5	Compute $(I+DC_{tt})^{-1}$	TEMINV
6	$\tilde{\gamma} = \gamma+C(t)(I+DC_{tt})^{-1}d-C(t)[D(I+C_{tt}D)^{-1}]\gamma(t)$	TMEAN

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