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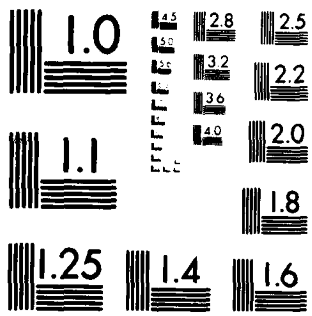

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A NONLINEAR VOLTERRA EQUATION IN VARIABLE DOMAIN

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Hedy Attouch† and Alain Damlamian‡

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ABSTRACT

We prove the existence and uniqueness of a strong solution of a nonlinear heat flow equation governing the temperature in a homogeneous material with memory when the temperature outside of a smooth time dependent domain $\Omega(t)$ is prescribed.

AMS(MOS) Subject Classifications: 35K05, 35K60, 35KXX
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Key Words: Nonlinear partial differential equations, Heat equation, Volterra equation, parabolic equations, convexity and subdifferential.

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SIGNIFICANCE AND EXPLANATION

We consider nonlinear heat flow in a homogeneous material in which the internal energy and the heat flux depend on the history of temperature and the history of the gradient of temperature respectively. We prescribe at each time t the temperature outside of a domain $\Omega(t)$ in x space, and we desire to find the temperature inside $\Omega(t)$. For example, in one space dimension imagine a homogeneous bar of a material with memory such that the temperature is maintained at zero for each time t outside the interval $[\alpha(t),\beta(t)]$ for $t \in [0,T]$, where $T > 0$, the functions α,β prescribe how the endpoints of the interval are changing with time. The problem in this case is to find the temperature at a point (x,t) where $0 < t \leq T$ and $\alpha(t) < x < \beta(t)$. Assuming that $\Omega(t)$ depends smoothly on t, we use methods of nonlinear analysis to prove the existence and uniqueness of a regular solution to the above problem.

A NONLINEAR VOLTERRA EQUATION IN VARIABLE DOMAIN

Hedy Attouch† and Alain Damlamian‡

Consider a domain Ω in $\mathbb{R}^n$ consisting of a homogeneous material of "memory" type. We consider nonlinear heat flow in Ω, prescribing the temperature outside of a subdomain $\Omega(t)$ of Ω for each time t in $[0,T]$, (smooth dependence of $\Omega(t)$ with respect to t is assumed). Using the techniques of Crandall-Nohel [1] for abstract Volterra equations, and of Attouch-Damlamian [1], Kenmochi [1] for evolution equations with time-dependent operators, we prove existence and uniqueness of the strong solution for the above problem.

I. The physical problem.

Let Ω be a bounded domain in $\mathbb{R}^n$, and for each t in $[0,T]$ let $\Omega(t)$ be a strict subdomain of Ω. Let us denote by $u = u(t,x)$ the temperature at time t and position x. For each t, u is a prescribed constant on $\Omega \setminus \Omega(t)$; for simplicity we shall assume that this constant is independent of t and we normalize it to zero. The history of u is prescribed for $t \leq 0$. The equation satisfied by u is obtained as usual, from heat balance between the internal energy E and the heat flow Q which in this case are assumed to be given by the following functionals of the temperature and the gradient of temperature respectively:

$$E(t,x) = u(t,x) + \int_{-\infty}^{t} \beta(t-x)u(s,x)ds,$$

$$Q(t,x) = -\sigma(\nabla u(t,x)) + \int_{-\infty}^{t} \gamma(t-s)\sigma(\nabla u(s,x))dx,$$

where β and γ are smooth real valued functions and $t \geq 0$, $x \in \Omega$. For simplicity we assume here that the history of the temperature u is zero for $t < 0$; if not this would alter the forcing term G in (1) below.

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We assume σ to be the derivative of a convex continuous coercive function J on $\mathbb{R}^n$. (here we mainly follow Ph. Clément, R. C. Mac Camy and J. Nohel [1, Sec. 2.[illegible]] to which the reader is referred for more details).

The following physical hypotheses are made:

$$
\text{(H)} \qquad \begin{cases} 1 + \int_0^{+\infty} \beta(t)dt > 0 \\ 1 - \int_0^{\infty} \gamma(t)dt > 0 \\ 1 + \mathrm{Re}\hat{\beta}(i\eta) > 0 \quad \text{for all} \quad \eta \in \mathbb{R}. \end{cases}
$$

The heat balance equation can be transformed from the form

$$
\frac{\partial}{\partial t}[u(t,x) + (\beta * u)(t,x)] - \mathrm{div}[\sigma(\nabla u(t,x)) - \gamma * \sigma(\nabla u)(t,x)] = h(t,x)
$$

to the nonlinear Volterra equation:

$$
(1) \qquad u + \beta * u = c * \mathrm{div}\sigma(\nabla u) + G
$$

where $*$ denotes convolution with respect to time on $[0,+\infty]$

$$
((a*b)(t) = \int_0^t a(t-s)b(s)ds \;,
$$

$$
c(t) = 1 - \int_0^t \gamma(s)ds
$$

$$
G(t,x) = u_0(x) + \int_0^t h(s,x)ds
$$

, with the prescribed "boundary" condition:

$$
(\text{B.C}) \qquad u(t,x) = 0 \quad \text{for} \quad x \quad \text{in} \quad \Omega \backslash \Omega(t) \;,
$$

and $u(0,x) = u_0(x)$, a given function. Defining ρ as the resolvent kernel of $\rho + \beta + \beta * \rho = 0$, where by assumptions (H) one easily shows that $\rho \in L^1(0,+\infty)$, (1) is seen to be equivalent to the Volterra equation

$$
(2) \qquad u + b * Au = F \quad \text{on} \quad [0,T] \;, \quad \text{where}
$$

$$
b = c + \rho * c
$$

$$
F = G + \rho * G \quad \text{and}
$$

$$
Au = -\mathrm{div}\,\sigma(\nabla u) \quad \text{for } u \text{ satisfying (B.C).}
$$

Using Attouch-Damlamian [1], one can see that for each $t \in [0,T]$, A together with the "boundary condition" at time t is the subdifferential of a lower semicontinuous, convex, proper function φ^t on $L^2(\Omega)$, namely

$$\varphi^t(u) = \begin{cases} \int_\Omega J(\nabla u(x))dx & u \in H^1(\Omega) \quad u|_{\Omega\setminus\Omega(t)} = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

(J is a "primitive" of σ). Therefore (2) is a special case of the following abstract Volterra equation in the Hilbert space $H = L^2(\Omega)$:

(3) $\qquad u + b * \partial\varphi^t(u) \ni F \qquad t \in [0,T]$.

In the following paragraph we prove an existence and uniqueness theorem for (3) in an abstract setting; we return to the original problem in the last section.

II. The Abstract theorem

(2.1) Theorem 1

Let H be a real Hilbert space and $(\varphi^t)_{t\in[0,T]}$ be a family of lower-semi-continuous proper convex functions from H into $]-\infty,+\infty]$ satisfying the following time dependence condition (where k is a positive constant):

$$(K) \quad \begin{cases} \forall\, 0 \le s \le t \le T\,, \quad \forall\, z \in D(\varphi^s) \;\; \exists\, \tilde{z} \in D(\varphi^t) \text{ such that:} \\ |z-\tilde{z}|_H \le k|t-s| \; (|\varphi^s(z)|^{\frac{1}{2}} + |z| + 1) \\ \varphi^t(\tilde{z}) \le \varphi^s(z) + k|t-s|(1+|z| + \varphi^s(z)) \quad . \end{cases}$$

Assume that b is Lipschitz continuous on $[0,T]$, with $b' \in BV(0,T;\mathbb{R})$, $b(0) = 1$. Let F be in $W^{1,2}(0,T;H)$ with $F(0) \in D(\varphi^0)$. Then, there exists a unique solution u of

(2.2) $\qquad u + b * \partial\varphi^t(u) \ni F$, belonging to $W^{1,2}(0,T;H)$ with $t \mapsto \varphi^t(u(t))$ in $L^\infty(0,T)$.

Proof of Theorem 1

We use the same transformation of the problem as Crandall-Nohel [1] to reduce (2.2) to (2.3):

$$(2.3) \quad \begin{cases} \dfrac{du}{dt} + \partial\varphi^t(u(t)) \ni [G(u)](t) \\ u(0) = F(0) \end{cases}$$

with $G(u)(t) = F'(t) + (r * F')(t) + b'(0)u(t) + r(t)F(0) - (u * r')(t)$ where $r \in BV(0,T;\mathbb{R})$ is defined by

$$r + b' * r = -b' .$$

Here also, one can see that G maps $C([0,T];H)$ into $L^2(0,T;H)$ and satisfies:

(2.4) $\forall u,v \in C([0,T];H)\ \forall t \in [0,T]\ \|G(u)-G(v)\|_{L^1(0,t;H)} \leq \int_0^t \gamma(s)\|u-v\|_{L^\infty(0,s;H)} ds$

where $\gamma(s) = |r(0)| + \mathrm{var}_{[0,s]}(r)$. (cf. Crandall-Nohel [1], Theorem 4.) On the other hand from Kenmochi [1], Yamada [1] and Attouch-Damlamian [1], assumption K implies the existence and uniqueness - for all f in $L^2(0,T;H)$ and u_0 in $D(\varphi^0)$ - of the solution u for

$$(2.5) \quad \begin{cases} \frac{du}{dt} + \partial\varphi^t(u(t)) & f \\ u(0) = u_0 . \end{cases}$$

Such u is in $W^{1,2}(0,T:H)$ and the function $t \mapsto \varphi^t(u(t))$ is bounded. Furthermore, the mapping $S:(f,u_0) \mapsto u$ satisfies:

$$\forall\ u = S(f,u_0)\ , \quad v = S(g,v_0)$$

$$(2.6) \quad \|u-v\|_{L^\infty(0,t;H)} \leq |u_0-v_0|_H + \|f-g\|_{L^1(0,t;H)}$$

Combining (2.4) and (2.6) with a fixed point theorem, one concludes the proof of Theorem (2.1).

<u>Remark</u>

1) Making use of the regularizing effect in (2.5), one can weaken some hypotheses of Theorem (2.1), namely $F(0) \in \overline{D(\varphi^0)}$, $F' \in L^1(0,T;H)$, $\sqrt{t}\ F' \in L^2(0,T;H)$. The conclusion is weakened to $u \in C([0,T];H)$, $\sqrt{t}\ \frac{du}{dt} \in L^2(0,T;H)$, $t \mapsto t\varphi^t(u(t))$ is bounded.

2) Using results of Attouch-Damlamian [1] one can slightly generalize the time dependence condition (K) assuming that

$$\begin{cases} \forall\ 0 \leq s \leq t \leq T \quad \forall z \in D(\varphi^s)\ \exists\ \tilde{z} \in D(\varphi^t) \text{ such that} \\ |z-\tilde{z}|_H \leq |a(t)-a(s)|(|\varphi^s(z)|^{\frac{1}{2}} + |z| + 1) \\ \varphi^t(\tilde{z}) \leq \varphi^s(z) + (e(t)-e(s))(1 + |z| + \varphi^s(z)) \end{cases}$$

where a is in $W^{1,2}(0,T;\mathbb{R})$ and e is an increasing function.

III. The Physical Problem.

We now give sufficient conditions on $\Omega(t)$ so that the physical problem falls into the framework of Theorem 2.1, i.e. for conditions (K) to hold.

(3.1) Proposition

Assume that there exists a Lipschitz continuous mapping θ from $[0,T]$ into the open set (in $C^1(\Omega)$) of C^1-diffeomorphisms of Ω such that for all t, $\Omega(t) = \theta(t)\Omega(0)$. Assume further that the convex function J has a quadratic behaviour at infinity, i.e. there exist c_1, c_2, c_3 belonging to $\mathbb{R}^{+*}$ such that:

(3.2) $$\forall\, r \in \mathbb{R}^n \quad c_1|r|^2 - c_2 \leq J(r) \leq c_3(|r|^2 + 1) .$$

Then (K) is satisfied.

Proof of Proposition (3.1)

We give it for convenience (for more details, see Damlamian-Kenmochi [1]). Let us denote $\bar{\theta} = \theta^{-1}$; for z_0 in $H_0^1(\Omega(0))$, put $z(t,y) = z_0(\bar{\theta}(t,y))$. Clearly $z(t) \in H_0(\Omega(t))$,

$$\frac{\partial z}{\partial t}(t,y) = \sum_i \frac{\partial z_0}{\partial x_i}(\bar{\theta}(t,y)) \cdot \frac{\partial \bar{\theta}_i}{\partial t}(t,y) .$$

$$\left\| \frac{\partial z}{\partial t}(t) \right\|_{L^2(\Omega)} \leq M \left\| \nabla z_0 \right\|_{L^2(\Omega)}$$

(where we use the Lipschitz property of the map $t \to \bar{\theta}(t)$; hence,

$$\forall\, t \in [0,T] \quad \|z(t) - z_0\|_{L^2(\Omega)} \leq Mt\, \|\nabla z_0\|_{L^2(\Omega)} ,$$

from which we get

$$\|z(t) - z_0\|_{L^2(\Omega)} \leq \frac{Mt}{c_1^{\frac{1}{2}}} \left[(c_2|\Omega|)^{\frac{1}{2}} + |\phi^0(z_0)|^{\frac{1}{2}} \right]$$

A similar computation yields

$$\|z(t) - z(s)\|_{L^2(\Omega)} \leq c_1^{-\frac{1}{2}} M|t-s| \left[(c_2|\Omega|) + |\phi^s(z(s)|^{\frac{1}{2}} \right] .$$

For the second condition in (K) we denote by $\alpha(t,y)$ the Jacobian matrix of $\bar{\theta}(t,y)$ (with respect to y) and by $\alpha(t,x) = \alpha(t,\theta(t,x))$, $\beta(t,x) = [\det \alpha(t,x)]^{-1}$.

Then,

$$\phi^t(z(t)) - \phi^0(z_0) = \int_\Omega \{J(\alpha(t,x)\nabla z_0(x))\beta(t,x) - J(\nabla z_0(x))\}dx$$
$$= \int_\Omega J(\alpha\nabla z_0)(\beta-1)dx + \int_\Omega J(\alpha\nabla z_0) - J(\nabla z_0)dx .$$

Since $\|\alpha(t,x) - I\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{R}^N))} \leq Kt$

$$|\beta(t,\cdot) - 1|_{L^\infty(\Omega)} \leq Kt$$

$$|J| \leq M(J+1)$$

$$|\partial J(r)| \leq C(|r|+1) \quad \text{(the two last inequalities follow from 3.2)} ,$$

one deduces

$$\phi^t(z(t)) - \phi^0(z_0) \leq MKt \int_\Omega (J(\alpha\nabla z_0)+1)dx + \int_\Omega \|\alpha-1\| \, |\nabla z_0(x)|\cdot|\partial J(\alpha\nabla z_0)|dx.$$

Since

$$\int_\Omega J(\alpha\nabla z_0)dx \leq \int_\Omega J(\nabla z_0)dx + \int_\Omega |\partial J(\alpha\nabla z_0)|\cdot\|\alpha-1\|\cdot|\nabla z_0|dx$$

$$\phi^t(z(t)) - \phi^0(z_0) \leq MKt[\int_\Omega (J(\nabla z_0)+1)dx + Kt\int_\Omega |\partial J(\alpha\nabla z_0)|\cdot|\nabla z_0|dx]$$
$$+ Kt\int_\Omega |\nabla z_0(x)|\cdot|\partial J(\alpha\nabla z_0)|dx$$
$$\leq MKt[\phi^0(z_0) + K'\int_\Omega |\partial J(\alpha\nabla z_0)|\cdot|\nabla z_0|dx + K''].$$

But $|\partial J(\alpha\nabla z_0)|\cdot|\nabla z_0| \leq C(|\alpha\nabla z_0|+1)(|\nabla z_0|$

$$\leq C'(J(\nabla z_0)+1) \quad \text{so} ,$$

$\phi^t(z(t)) - \phi^0(z_0) \leq \lambda t[\phi^0(z_0)+1]$; as above this extends to

$\forall\, 0 \leq s \leq t \leq T \quad \phi^t(z(t)) - \phi^s(z(s)) \leq \lambda(t-s)[\phi^s(z(s)+1]$.

Remark

The previous results can be extended to the case where

$$\forall\, r \in \mathbb{R}^n \quad C_1|r|^p - C_2 \leq J(r) \leq C_3(|r|^p+1) \quad \text{for some } p \geq 2 .$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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