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THE PEANO CURVE OF SCHOENBERG IS NOWHERE DIFFERENTIABLE.(U)

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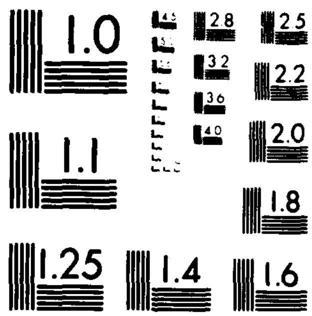

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THE PEANO CURVE OF SCHOENBERG IS NOWHERE DIFFERENTIABLE

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James Alsina
Middlebury College

Technical Summary Report #2043
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ABSTRACT

Let $f(t)$ be defined in $[0,1]$ by

$$f(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \frac{1}{3} \\ 3t-1 & \text{if} \quad \frac{1}{3} \le t \le \frac{2}{3} \\ 1 & \text{if} \quad \frac{2}{3} \le t \le 1 \end{cases}$$

and extended to all real t by requiring that $f(t)$ should be an even function having the period 2 (See Figure 1). The plane arc defined parametrically by the equations

$$x(t) = \sum_{n=0}^{\infty} \frac{f(3^{2n}t)}{2^{n+1}}, \quad y(t) = \sum_{n=0}^{\infty} \frac{f(3^{2n+1}t)}{2^{n+1}}, \quad (0 \le t \le 1),$$

is known to be continuous, and to map the interval $I = \{0 \le x \le 1\}$ onto the entire square $I^2 = \{0 \le x,y \le 1\}$ (See [3]). Here it is shown that this arc is nowhere differentiable, meaning the following: There is no value of t such that both derivatives $x'(t)$ and $y'(t)$ exist and are finite.

AMS(MOS) Subject Classification: 26A24, 54C05

Key Words: Non-differentiability, Peano curves

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SIGNIFICANCE AND EXPLANATION

Well known are examples of area-filling curves, and of continuous functions which are nowhere differentiable. This paper brings together these two pathological properties by showing that the area-filling curve described in [3] lacks, at every point, a finite derivative.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE PEANO CURVE OF SCHOENBERG IS NOWHERE DIFFERENTIABLE

James Alsina
Middlebury College

1. Introduction. It came as quite a surprise to the mathematical world when, in 1875, Weierstrass constructed an everywhere continuous, nowhere differentiable function (see [1]). Equally startling though was the discovery by Giuseppe Peano [2] fifteen years thereafter that the unit interval could be mapped continuously onto the entire unit square I^2.

Well known now are examples of area-filling curves, and of continuous functions which are nowhere differentiable. This paper brings together these two pathological properties by showing that the plane Peano curve of I. J. Schoenberg [3], defined in §3 below, lacks at every point a finite derivative (Theorem 3). An analogous space curve is similarly shown to fill the unit cube I^3 (Theorem 2), and to be nowhere differentiable (Theorem 4).*

2. An identity on the Cantor Set Γ. The foundation of Schoenberg's curve is the continuous function $f(t)$, defined first in $[0,1]$ by

$$(2.1)\qquad f(t) = \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{3} \\ 3t-1, & \text{if } \frac{1}{3} \le t \le \frac{2}{3} \\ 1, & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

We then extend its definition to all real t such that $f(t)$ is an even function of period 2 (See Figure 1 below). Thus

$$f(-t) = f(t), \quad f(t+2) = f(t) \quad \text{for all } t .$$

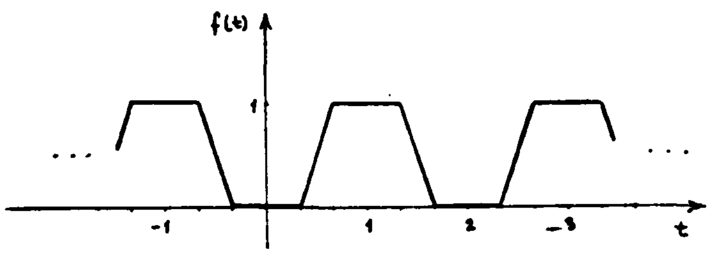

Figure 1

*The author would like to thank Professor Schoenberg for his invaluable suggestions on the preparation of this paper.

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The main property of this function is that it produces the following remarkable identity on Γ.

Lemma 1. If t is an element of Cantor's Set Γ, then

(2.2)
$$t = \sum_{n=0}^{\infty} \frac{2f(3^n t)}{3^{n+1}} .$$

Proof: If indeed $t \in \Gamma$, it can be expressed as

(2.3)
$$t = \sum_{n=0}^{\infty} \frac{a_n}{3^{n+1}} , \qquad (a_n = 0,2)$$

then (2.2) would follow from the relations

(2.4)
$$a_n = 2 \cdot f(3^n t) , \qquad (n = 0,1,2,\ldots).$$

To prove (2.4) observe that (2.3) implies

$$3^n t = 3^n \left(\frac{a_0}{3} + \ldots + \frac{a_{n-1}}{3^n} \right) + \frac{a_n}{3} + \frac{a_{n+1}}{3^2} + \ldots ,$$

whence

(2.5)
$$3^n t = M_n + \frac{a_n}{3} + \frac{a_{n+1}}{3^2} + \ldots , \qquad (M_n \text{ is an even integer}).$$

From the graph of $f(t)$ we conclude the following:

If $a_n = 0$, then $M_n \le 3^n t \le M_n + \frac{2}{3^2} + \frac{2}{3^3} + \ldots = M_n + \frac{1}{3}$

and therefore $f(3^n t) = 0$.

If $a_n = 2$, then $M_n + \frac{2}{3} \le 3^n t \le M_n + \frac{2}{3} + \frac{2}{3^2} + \ldots = M_n + 1$ and so $f(3^n t) = 1$.

This establishes (2.4) and thus the relation (2.2).

3. Schoenberg's curve. This function is defined parametrically by the equations

(3.1)
$$x(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n} t),$$

(3.2)
$$y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n+1} t) , \qquad (0 \le t \le 1).$$

The mapping $t \to (x(t),y(t))$ indeed defines a curve: its continuity follows from the expansions (3.1), (3.2) being not only termwise continuous, but dominated by the series of constants

(3.3)
$$\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = 1 .$$

These conditions insure their uniform convergence, and therefore also the continuity of their sums.

Now if $t \in \Gamma$, hence

(3.4) $$t = \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}} \qquad (a_n = 0,2) ,$$

by (2.4) we may write (3.1) and (3.2) as

(3.5) $$x(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{2n}}{2} , \qquad y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{2n+1}}{2} .$$

We then invert these relationships: let $P = (x(t),y(t))$ be an arbitrarily preassigned point of the square $I^2 = \{0 \le x,y \le 1\}$, and regard (3.5) as the binary expansions of the coordinates of P. This defines a_{2n} and a_{2n+1}, and therefore also the full sequence $\{a_n\}$. With it we define t ($\in \Gamma$) by (3.4), and thus the expressions (3.5), being a consequence of (3.1) and (3.2), show that the point P is on our curve. This proves

<u>Theorem 1.</u> The mapping

$$t \to (x(t), y(t))$$

from I into I^2 defined by (3.1), (3.2), is continuous, and covers the square I^2, even if t is restricted to the Cantor Set Γ.

This result extends naturally to higher dimensions. We discuss only the case of *the space curve*

(3.6) $$X(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n}t) ,$$

(3.7) $$Y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n+1}t) ,$$

(3.8) $$Z(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(x^{3n+2}t) , \qquad (0 \le t \le 1).$$

The continuity of $X(t)$, $Y(t)$, and $Z(t)$, as in the two-dimensional case, is guaranteed by the continuity of each of their terms and by the convergence of the series of constants (3.3). If we define t by (3.4), so $a_n = 0,2$ for $n = 0,1,2,\ldots$, then again (2.4) shows that

(3.9) $$X(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{3n}}{2}, \; Y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{3n+1}}{2} , \; Z(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{3n+2}}{2} .$$

If the right sides are the binary expansions of the coordinates of an arbitrarily chosen point of I^3, then this point of I^3 is reached by our space curve for the value of $t \in \Gamma$ defined by (3.4). Thus we have proven

Theorem 2. The mapping

$$t \to (X(t),\ Y(t),\ Z(t))$$

from I into I^3 defined by (3.6), (3.7), (3.8), is continuous, and fills the cube I^3, even if t is restricted to the Cantor Set Γ.

Theorems 1 and 2 raise an interesting question. Just how does the plane curve, for example, fill the square as t varies from 0 to 1 ? Though by no means may this question be answered completely, we can gain some feeling for the curve's path by viewing it as the point-for-point limit of the sequence of continuous mappings.

(3.10) $$t \to (x_k(t),\ y_k(t)), \qquad (k = 0,1,2,\ldots),$$

where x_k and y_k are the $k^{\underline{th}}$ partial sums of the series (3.1) and (3.2) defining x and y. The graph of this sequence for $k = 0,1,2$ and $0 \leq t \leq 1$ is shown below in Figure 2. (The origin is at the lower left corners, with x_k and y_k on the horizontal and vertical axes, respectively. The dotted lines delineate the boundary of I^2.)

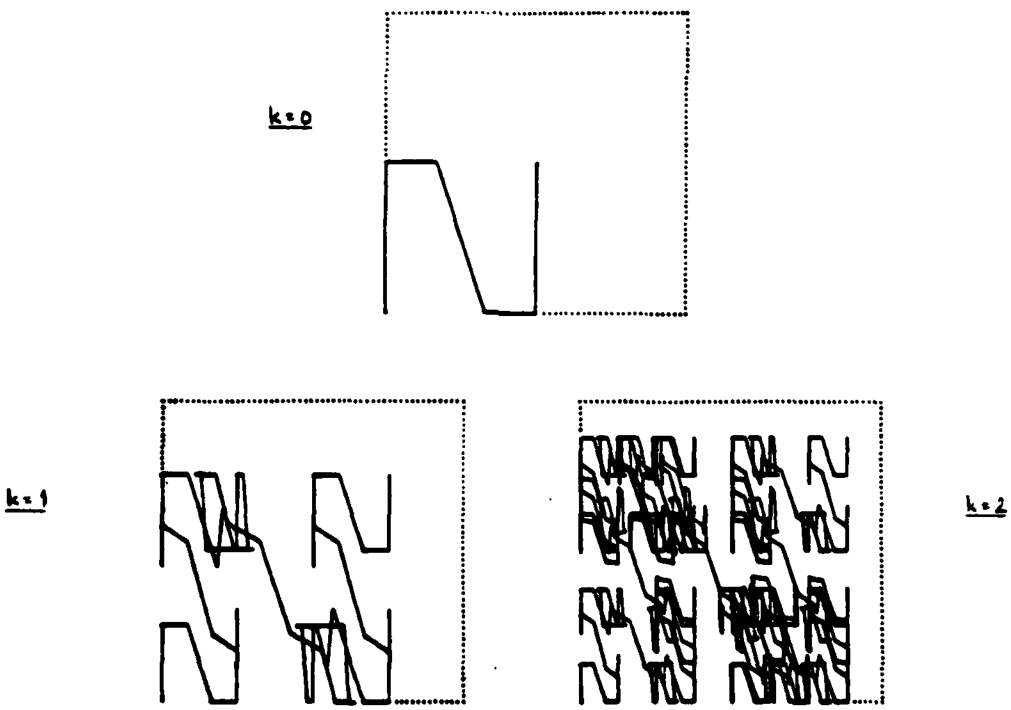

Figure 2. The approximation curves $(x_k(t), y_k(t))$ for $k = 0,1,2$.

Notice in particular in Figure 2 that the curves lack the one-to-one property for $k = 1,2$. This fact, together with the promise for increased complexity in these approximation curves as $k \to \infty$, suggests that the limit curve itself may be many-to-one.

The implication is indeed correct, and not only for the case at hand. If an area-filling curve were one-to-one, it would be a homeomorphism. The unit interval and I^n (for $n \geq 2$), however, are not homeomorphic, since the removal of any interior point disconnects I but not I^n.

The point $(\frac{1}{2},\frac{1}{2})$ of I^2 nicely illustrates this many-to-one property for Schoenberg's curve (3.1), (3.2). Since the number $\frac{1}{2}$ can be expressed in binary form either as .1000... or .0111..., (3.4) and (3.5) imply that $(x(t_0)y(t_0)) = (\frac{1}{2},\frac{1}{2})$ is the image of four distinct elements of the Cantor Set Γ, namely

$$t_0 = \frac{1}{9}, \frac{11}{16}, \frac{25}{36}, \frac{8}{9}.^*$$

In fact, the set of all (x,y) with four pre-images in Γ is dense in the square. Theorem 1 asserted that Γ, a set of Lebesgue measure zero, is sufficiently large to be mapped onto I^2, a set of plane measure 1. It would now seem that Γ has more points than I^2!

In the next section, we explore yet another property of Schoenberg's curve, and prove our main result.

4. The Peano curve of Schoenberg is nowhere differentiable. We say that a plane curve $(x(t),y(t))$ is differentiable at t_0 if both derivatives $x'(t_0)$ and $y'(t_0)$ exist and are finite. Our goal will be to prove

Theorem 3. For no value of t do both functions

(4.1) $$x(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n}t),$$

(4.2) $$y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n+1}t),$$

have finite derivatives $x'(t)$, $y'(t)$.

---

*More precisely, $(\frac{1}{2},\frac{1}{2})$ is a quintuple point of the curve, having its fifth pre-image, $t_0 = \frac{1}{2}$, in $[0,1]\setminus\Gamma$.

Since $f(t)$ is an even function of period two, then so are $x(t)$ and $y(t)$. Thus it suffices to prove Theorem 3 for $t \in I = [0,1]$. The theorem will follow from the proof of two lemmas.

Let t be a fixed number in $[0,1]$, expressed in ternary form by

$$t = \frac{a_0}{3} + \frac{a_1}{3^2} + \dots + \frac{a_n}{3^{n+1}} + \dots, \qquad (a_n = 0,1,2), \tag{4.3}$$

and corresponding to this t, define the following disjoint sets:

$$N_0 = \{n: a_{2n} = 0\},$$
$$N_1 = \{n: a_{2n} = 1\},$$
$$N_2 = \{n: a_{2n} = 2\}.$$

The first of our lemmas is

<u>Lemma 2</u>. $x'(t)$ does not exist finitely if $N_0 \cup N_2$ is an infinite set.

In the proof we make use of several properties of the function $f(t)$:

$$f(t+2) = f(t) \qquad \text{for all } t. \tag{4.4}$$

If M is an integer and $t_1 \in [M, M + \frac{1}{3}]$, $t_2 \in [M + \frac{2}{3}, M + 1]$, then

$$|f(t_1) - f(t_2)| = 1. \tag{4.5}$$

$f(t)$ also satisfies the Lipschitz condition

$$|f(t_1) - f(t_2)| \le 3 \cdot |t_1 - t_2| \qquad \text{for any } t_1, t_2. \tag{4.6}$$

Let us now assume that $m \in N_0 \cup N_2$, hence $a_{2m} = 0$ or $a_{2m} = 2$. For such m, we define the increment

$$\delta_m = \begin{cases} \frac{2}{3} 9^{-m}, & \text{if } a_{2m} = 0, \\ -\frac{2}{3} 9^{-m}, & \text{if } a_{2m} = 2, \end{cases} \tag{4.7}$$

and seek to estimate the corresponding difference quotient

$$\frac{x(t + \delta_m) - x(t)}{\delta_m} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \gamma_{n,m}, \tag{4.8}$$

where

$$\gamma_{n,m} = \frac{f(9^n(t + \delta_m)) - f(9^n t)}{\delta_m}. \tag{4.9}$$

We must distinguish three cases.

(i) n m. By (4.7), $9^n{}_m = \frac{2}{3} 3^{n-m}$, which is an even integer. Thus by (4.4),

we conclude that

(4.10) $\quad {}_{n,m} = 0 \quad$ if $\quad n \quad m$,

regardless of the value of a_{2m} .

(ii) n m . Here we make use of the Lipschitz inequality (4.6) to show that

$$|{}_{n,m}| \leq 3 \cdot \frac{9^n \; {}_m}{{}_m} ,$$

whence

(4.11) $\quad |{}_{n,m}| \leq 3 \cdot 9^n \quad$ for $\quad n \quad m$.

(iii) n = m. By (4.3), we see that

(4.12) $\quad 9^m t = 3^{2m} t = M + \frac{a_{2m}}{3} + \frac{a_{2m+1}}{3^2} + \ldots$, $\quad$ (M is an integer).

Here we must distinguish two subcases:

If $a_{2m} = 0$, and so by (4.7) $9^m{}_m = \frac{2}{3}$, (4.12) implies that $M \leq 9^m t \leq M + \frac{2}{3^2} + \frac{2}{3^3} + \ldots$. Since $\frac{2}{3^2} + \frac{2}{3^3} + \ldots = \frac{1}{3}$, we find that $M \leq 9^m t \leq M + \frac{1}{3}$, and therefore that $M + \frac{2}{3} \leq 9^m t + 9^m{}_m \leq M + 1$.

If $a_{2m} = 2$, then by (4.7) $9^m{}_m = -\frac{2}{3}$. From (4.12), $M + \frac{2}{3} \leq 9^m t \leq M + \frac{2}{3} + \frac{2}{3^2} + \ldots = M + 1$, while $M \leq 9^m t + 9^m{}_m \leq M + \frac{1}{3}$.

In either subcase, we can apply (4.5) to conclude that

(4.13) $\quad |{}_{m,m}| = \frac{1}{|{}_m|} = \frac{3}{2} 9^m$,

regardless of the value of a_{2m} .

The results (4.10), (4.11), and (4.13) hold under the sole assumption

$$m \quad N_0 \quad N_2 .$$

Applying them to the difference quotient

(4.14) $\quad DQ_m = \frac{x(t + {}_m) - x(t)}{{}_m}$,

we find by (4.8) that

$$|DQ_m| = \left|\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \gamma_{n,m}\right|$$

$$= \left|\sum_{n=0}^{m} \frac{1}{2^{n+1}} \gamma_{n,m}\right|$$

$$\geq \frac{1}{2^{m+1}} |\gamma_{m,m}| - \sum_{n=0}^{m-1} \frac{1}{2^{n+1}}|\gamma_{n,m}|$$

$$\geq \frac{1}{2^{m+1}} \cdot \frac{3}{2} 9^m - \sum_{n=0}^{m-1} \frac{1}{2^{n+1}} \cdot 3 \cdot 9^n$$

$$= \frac{3}{4} \left(\frac{9}{2}\right)^m - \frac{3}{7} \left[\left(\frac{9}{2}\right)^m - 1\right].$$

and finally

$$\left|\frac{x(t+\delta_m) - x(t)}{\delta_m}\right| \geq \frac{9}{28} \left(\frac{9}{2}\right)^m + \frac{3}{7} \quad \text{if} \quad m \in N_0 \cup N_2 . \tag{4.15}$$

This establishes Lemma 2 if, in (4.15), we let $m \to \infty$ through the elements of the infinite sequence $N_0 \cup N_2$.

We now turn out attention to the digits of t having odd subscripts, and define the sets

$$N_0' = \{n: a_{2n+1} = 0\}$$
$$N_1' = \{n: a_{2n+1} = 1\}$$
$$N_2' = \{n: a_{2n+1} = 2\} .$$

Now if

$$t = \frac{a_0}{3} + \frac{a_1}{3^2} + \dots + \frac{a_{2n+1}}{3^{2n+2}} + \dots ,$$

then for $\tau = 3t$ we find

$$\tau = a_0 + \frac{a_1}{3} + \dots + \frac{a_{2n+1}}{3^{2n+1}} + \dots .$$

At the same time

$$x(\tau) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n}\tau) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n+1}t) = y(t) .$$

Applying Lemma 2 to $x(t)$ at the point $\tau = 3t$, we see that the digits a_{2n+1} are the digits of τ having <u>even</u> subscripts. We thus obtain

<u>Corollary 1.</u> $y'(t)$ does not exist finitely if $N_0' \cup N_2'$ is an infinite set.

By Lemma 2 and Corollary 1 we can conclude that the only t for which $x'(t)$ and $y'(t)$ might both exist and be finite, is one whose sets $N_0 \cup N_2$ and $N'_0 \cup N'_2$ are finite. This is the case if and only if the digits

(4.16) $\qquad a_n = 1 \quad$ for all sufficiently large n.

On the other hand, to prove the non-differentiability of the mapping $t \to (x(t), y(t))$, it suffices to show that one of the derivatives $x'(t)$, $y'(t)$ fails to exist.

Lemma 3. If t is such that (4.16) holds, then $x'(t)$ does not exist finitely.

The simplest t satisfying (4.16) is the one for which all $a_n = 1$, or

$$t = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} = \tfrac{1}{2}$$

We must, however, treat the general case, where

(4.17) $$t = \frac{a_0}{3} + \frac{a_1}{3^2} + \dots + \frac{a_{2m-1}}{3^{2m}} + \frac{1}{3^{2m+1}} + \frac{1}{3^{2m+2}} + \dots ,$$

with $a_n = 0,1,2$ for $n = 0,1,\dots,2m-1$. To prove the lemma, we proceed as in Lemma 2 by estimating the difference quotient

(4.18) $$\frac{x(t + \delta_m) - x(t)}{\delta_m} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \gamma_{n,m} ,$$

where

(4.19) $$\gamma_{n,m} = \frac{f(9^n(t + \delta_m)) - f(9^n t)}{\delta_m}$$

Here, though, we must abandon our former choice for the increment δ_m in favor of

(4.20) $$\delta_m = \frac{2}{9} 9^{-m} .$$

We will once again examine the quantity $\gamma_{n,m}$ in terms of three cases:

(i) $n > m$. From (4.20), $9^n \delta_m = \frac{2}{9} 9^{n-m}$, which is an even integer. Thus, by property (4.4), the periodicity of $f(t)$, we see that

(4.21) $$\gamma_{n,m} = 0 \quad \text{if} \quad n > m .$$

(ii) $n < m$. In this case, we again use the Lipschitz condition (4.6) to conclude that

(4.22) $$|\gamma_{n,m}| \le 3 \cdot 9^n \quad \text{if} \quad n < m .$$

(iii) $n = m$. By (4.17),

$$9^m t = 3^{2m} t = M + \frac{1}{3} + \frac{1}{3^2} + \dots , \quad (M \text{ is an integer}),$$

whence

(4.23) $$9^m t = M + \tfrac{1}{2}$$

while

(4.24) $$9^m \delta_m = \frac{2}{9} .$$

From the graph of $f(t)$, in Figure 3 below, observe that

(4.25) $$f(N + \tfrac{1}{2}) = f(\tfrac{1}{2}) = \tfrac{1}{2}, \text{ for any integer } N ,$$

and so from (4.23),

(4.26) $$f(9^m t) = \tfrac{1}{2} .$$

The addition of (4.23) and (4.24) gives

$$9^m t + 9^m \delta_m = M + \frac{13}{18} ,$$

and since $\frac{2}{3} < \frac{13}{18} < 1$, Figure 3 shows us that

(4.27) $$f(9^m t + 9^m \delta_m) = \begin{cases} 0 , & \text{if } M \text{ is odd} \\ 1 , & \text{if } M \text{ is even.} \end{cases}$$

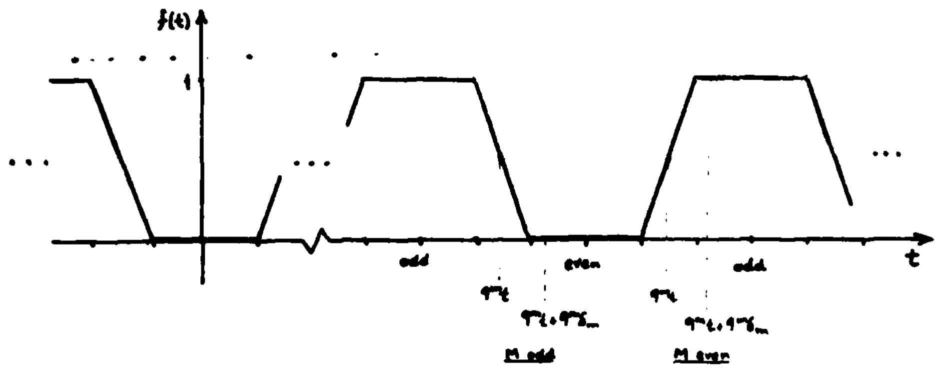

Figure 3

Regardless of the value of M , (4.26) and (4.27) imply that

$$|f(9^m t + 9^m \delta_m) - f(9^m t)| = \tfrac{1}{2},$$

and therefore, by (4.19) and (4.20), that

(4.28) $$|\gamma_{m,m}| = \frac{\frac{1}{2}}{|\delta_m|} = \frac{9}{4} 9^m .$$

Applying the results (4.21), (4.22), and (4.28) to the difference quotient (4.18),

$$DQ_m = \frac{x(t+\delta_m) - x(t)}{\delta_m} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \epsilon_{n,m}$$

$$= \sum_{n=0}^{m} \frac{1}{2^{n+1}} \epsilon_{n,m}$$

$$= \frac{1}{2^{m+1}} \epsilon_{m,m} + \sum_{n=0}^{m-1} \frac{1}{2^{n+1}} \epsilon_{n,m}$$

$$= \frac{1}{2^{m+1}} \cdot \frac{9}{4} 9^m - \sum_{n=0}^{m-1} \frac{1}{2^{n+1}} \cdot 3 \cdot 9^n$$

which yields

(4.29) $$DQ_m \ge \frac{39}{56} \left(\frac{9}{2}\right)^m + \frac{3}{7} .$$

If, in (4.29), we let $m \to \infty$, $DQ_m \to \infty$, hence x is not differentiable at t. This establishes Lemma 3, and therefore also Theorem 3.

While Lemma 3 above is sufficient to prove the nondifferentiability of the mapping

(4.30) $$t \to (x(t), y(t))$$

for t defined by (4.17), $y'(t)$ as well may be shown not to exist for such t. This claim is easily verified by the same argument which produced Corollary 1.

5. The generalization of Theorem 3. Analogous to Schoenberg's plane Peano curve (4.1), (4.2) is the space curve

(5.1) $$X(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n}t)$$

(5.2) $$Y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n+1}t)$$

(5.3) $$Z(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n+2}t), \qquad (0 \le t \le 1),$$

introduced in §3. By way of Theorem 2, we saw that these functions define a Peano curve filling the unit cube I^3. Here, in a similar fashion, we seek to extend Theorem 3 to higher dimensions.

Theorem 4. The Peano curve defined by (5.1), (5.2), (5.3) above is nowhere differentiable.

The technique of proof used for Theorem 3 will apply nicely; again we shall have two lemmas and a corollary.

Indeed, with t defined by

$$t = \frac{a_0}{3} + \frac{a_1}{3^2} + \dots + \frac{a_n}{3^{n+1}} + \dots , \qquad (a_n = 0,1,2),$$

we define the corresponding sets of integers

$$M_0 = \{n: a_{3n} = 0\}, \quad M_1 = \{n: a_{3n} = 1\}, \quad M_2 = \{n: a_{3n} = 2\},$$

and state

Lemma 4. The derivative $X'(t)$ does not exist finitely if $M_0 \cup M_2$ is an infinite set.

For $m \in M_0 \cup M_2$, we define the increment

$$\delta_m = \begin{cases} \frac{2}{3}\, 3^{-3m}, & \text{if} \quad a_{3m} = 0, \\ -\frac{2}{3}\, 3^{-3m}, & \text{if} \quad a_{3m} = 2, \end{cases}$$

and investigate the difference quotient

$$DQ_m = \frac{X(t+\delta_m) - X(t)}{\delta_m} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \gamma_{n,m}$$

where

$$\gamma_{n,m} = \frac{f(3^{3n}(t+\delta_m)) - f(3^{3n}t)}{\delta_m}.$$

Proceeding as in the proof of Lemma 2, we find that

$$|DQ_m| \geq \frac{3}{4}\left(\frac{27}{2}\right)^m - \frac{3}{25}\left(\left(\frac{27}{2}\right)^m - 1\right),$$

which proves Lemma 4, if we let $m \to \infty$ through the elements of $M_0 \cup M_2$.

Using the identities $Y(t) = X(3t)$, $Z(t) = X(3^2t)$, we obtain the following:

Corollary 2. (i) If the sets $M_0' = \{n: a_{3n+1} = 0\}$, $M_2' = \{n: a_{3n+1} = 2\}$ are such that $M_0' \cup M_2'$ is an infinite set, then $Y'(t)$ does not exist finitely. (ii) If the sets $M_0'' = \{n: a_{3n+2} = 0\}$, $M_2'' = \{n: a_{3n+2} = 2\}$ are such that $M_0'' \cup M_2''$ is an infinite set, then $Z'(t)$ does not exist finitely.

The only t for which all the derivatives $X'(t)$, $Y'(t)$, $Z'(t)$ might still exist is one whose sets

$$M_0 \cup M_2,\ M_0' \cup M_2',\ M_0'' \cup M_2''$$

are all finite. This condition is true if and only if

(5.4) $\qquad a_n = 1$ for all sufficiently large n .

We now state our final

Lemma 5. Suppose t satisfies (5.4). Then none of the derivatives $X'(t)$, $Y'(t)$, $Z'(t)$ exists and is finite.

The proof of the claim for $X'(t)$ follows from the choice of

$$\delta_m = \frac{2}{9} 3^{-3m} ,$$

and those for $Y'(t)$ and $Z'(t)$ from arguments similar to the proof of Corollary 1 in §4.

6. A final remark. In its nowhere differentiability, Schoenberg's plane curve provides an interesting contrast to the Peano curve from which it is derived, that of H. Lebesgue (see [3]).

Under Lebesgue's mapping $L(t)$, each (x_0,y_0) of I^2 , expressed as

$$x_0 = \frac{\alpha_0}{2} + \frac{\alpha_2}{2^2} + \frac{\alpha_4}{2^3} + \cdots$$

$$y_0 = \frac{\alpha_1}{2} + \frac{\alpha_3}{2^2} + \frac{\alpha_5}{2^3} + \cdots , \qquad (\alpha_i = 0,1),$$

is the image of a point t_0 in Cantor's Set Γ of the form

$$t_0 = \frac{2\alpha_0}{3} + \frac{2\alpha_1}{3^2} + \frac{2\alpha_2}{3^3} + \cdots .$$

This correspondence we now recognize as a restatement of the relations (3.5). As such, $L(t)$ coincides with Schoenberg's curve on Γ , and thus must lack a finite derivative there.

Lebesgue then extends the domain of $L(t)$ to all of $[0,1]$ by means of linear interpolation over each of the open intervals which comprise the complement of Γ . Defined in this manner, $L(t)$ must indeed be differentiable on $[0,1]\backslash\Gamma$, and hence constitutes an example of a Peano curve which, unlike Schoenberg's, is differentiable almost everywhere.

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Let $f(t)$ be defined in $[0,1]$ by

$$f(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \frac{1}{3} \\ 3t-1 & \text{if} \quad \frac{1}{3} \le t \le \frac{2}{3} \\ 1 & \text{if} \quad \frac{2}{3} \le t \le 1 \end{cases}$$

and extended to all real t by requiring that $f(t)$ should be an even

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function having the period 2 (See Figure 1). The plane arc defined parametrically by the equations

$$x(t) = \sum_{n=0}^{\infty} \frac{f(3^{2n}t)}{2^{n+1}}, \quad y(t) = \sum_{n=0}^{\infty} \frac{f(3^{2n+1}t)}{2^{n+1}}, \quad (0 \leq t \leq 1),$$

is known to be continuous, and to map the interval $I = \{0 \leq x \leq 1\}$ onto the entire square $I^2 = \{0 \leq x,y \leq 1\}$ (See [3]). Here it is shown that this arc is nowhere differentiable, meaning the following: There is no value of t such that both derivatives $x'(t)$ and $y'(t)$ exist and are finite.