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THE SOLUTIONS TO A SMOOTH PDE CAN BE DENSE IN C(I).(U)
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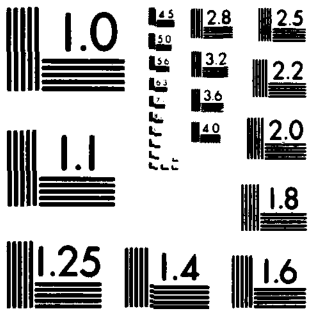

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THE SOLUTIONS TO A SMOOTH PDE CAN BE DENSE IN C[I]

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

The solutions to a partial differential equation that arises in the physical sciences are expected to be restricted in nature, since they are intended to represent some physically significant behavior. In particular, they can approximate the general continuous function on a compact set K only if K is "thin", e.g. nowhere dense in R^n. The purpose of this note is to show that this does not hold for <u>all</u> smooth partial differential equations. Specifically, for any $n \geqq 2$ there exist partial differential equations of polynomial type on R^n whose C^∞ solutions are uniformly dense in the space $C[I]$ of all continuous functions on the n-cube I.

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SIGNIFICANCE AND EXPLANATION

The solutions to a partial differential equation that arises in the physical sciences are expected to be restricted in nature, since they are intended to represent some physically significant behavior. In particular, they can approximate the general continuous function on a compact set K only if K is "thin", e.g. nowhere dense in R^n. The purpose of this note is to show that this does not hold for all smooth partial differential equations. Specifically, for any $n \geq 2$ there exist partial differential equations of polynomial type on R^n whose C^∞ solutions are uniformly dense in the space $C[I]$ of all continuous functions on the n-cube I.

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THE SOLUTIONS TO A SMOOTH PDE CAN BE DENSE IN $C[I]$

R. C. Buck

The solutions to a partial differential equation that arises in the physical sciences are expected to be restricted in nature, since they are intended to represent some physically significant behavior. In particular, they can approximate the general continuous function on a compact set K only if K is "thin", e.g. nowhere dense in R^n. The purpose of this note is to show that this does not hold for all smooth partial differential equations. Specifically, for any $n \geq 2$ there exist partial differential equations of polynomial type on R^n whose C^∞ solutions are uniformly dense in the space $C[I]$ of all continuous functions on the n-cube I.

Here, I is the set of all $x = (x_1, x_2, \ldots, x_n) \in R^n$ with $0 \leq x_i \leq 1$. The vector exponent $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ has integer components $\alpha_i \geq 0$, and $|\alpha| = \sum \alpha_i$; the general multinomial, denoted by

$$x^\alpha = \prod_1^n x_i^{\alpha_i}$$

is said to have (total) degree $|\alpha|$.

We also use this notation for partial derivatives, so that

$$\left(\frac{\partial}{\partial x}\right)^\alpha F = \prod_1^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i} F$$

is one of the derivatives of F of order $|\alpha|$.

We prove the following result:

Main Theorem. *If $n \geq 2$ there are integers $k = k(n)$ and $\Delta = \Delta(n)$ and a partial differential equation on R^n of order not greater than k, whose C^∞ solutions $F(x)$ are uniformly dense in $C[I]$, the space of all continuous real valued functions on the n-cube I. The form of the differential equation is*

(1) $$Q(w_1, w_2, \ldots, w_N) = 0$$

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where the w_i are a listing of all the partial derivatives of F of order $|\alpha| \leq k$, and where Q is a nontrivial polynomial of total degree at most .

The construction of such a "universal" PDE which we will outline depends upon a combination of familiar facts, of which the central one is the Kolmogorov solution of Hilbert's 13th Problem (See [3]). This asserts that every continuous function F on I can be represented there in the form

$$F = \sum_{1}^{2n+1} f_j \circ g_j \tag{2}$$

where the f_j are continuous functions of one variable, and the g_j are sums of such functions:

$$g_j(x) = {}_j\phi_1(x_1) + {}_j\phi_2(x_2) + \dots + {}_j\phi_n(x_n) \quad . \tag{3}$$

The various proofs of this surprising fact depend on the Baire category theorem, and are not constructive. It is known however that one cannot in general hope for all the component functions f_j and ${}_j\phi_i$ to be differentiable, even if F itself is reasonably smooth (See again [3]).

We introduce the special term smooth Kolmogorov function for a function F defined on I that is represented there in the format (2) and (3), where each of the component functions is of class C^∞. As observed above, these form a proper subset of $C[I]$.

Lemma 1. The class of smooth Kolmogorov functions are uniformly dense in $C[I]$.

This depends on the elementary observation that if $g_n : A \to B$ and $f_n : B \to C$, are continuous mappings between compact metric spaces, and if $\{f_n\}$ and $\{g_n\}$ converge uniformly on their domains to f and g, respectively, then $f_n \circ g_n$ converges to $f \circ g$, uniformly on A.

The proof of the main theorem will be completed by showing that a PDE, of the form (1), can be constructed whose solutions include all the smooth Kolmogorov functions. The argument uses two elementary results dealing with polynomials.

Lemma 2. The collection of polynomials in r indeterminates of total degree at most d forms a vector space $V(r,d)$ of dimension $\binom{r+d}{r}$.

The second result is the standard theorem on algebraic dependence.

Lemma 3. (i) If $N > m$ and $P_1(u), P_2(u), \ldots, P_N(u)$ is a collection of polynomials in the indeterminates $u = (u_1, u_2, \ldots, u_m)$, then there is a non-trivial polynomial Q such that

(4) $$Q(P_1, P_2, \ldots, P_N) = 0 \quad .$$

(ii) If $\deg(P_i) \leqq \delta$ for each i, then

(5) $$\deg(Q) \leqq N\,\delta^{m/N-m)} \quad .$$

Proof:

Set $w_i = P_i(u_1, u_2, \ldots, u_m)$, for $i = 1, 2, \ldots, N$. The goal is to eliminate the indeterminates u_i, obtaining (4). Choosing an as yet unspecified Δ, form all the multinomials w^α for $|\alpha| \leqq \Delta$. By Lemma 2, we will have

(6) $$s(\Delta) = \binom{\Delta + N}{N}$$

of these. But,

(7) $$w^\alpha = \prod_{i=1}^{N} P_i(u_1, u_2, \ldots, u_m)^{\alpha_i}$$

which is a polynomial in the u_i of total degree at most $\sum \delta\alpha_i = \delta|\alpha| \leqq \delta\Delta$, belonging to the vector space $V(m, \delta\Delta)$. By Lemma 2, the dimension of this space is

(8) $$t(\Delta) = \binom{\delta\Delta + m}{m}$$

Thus, if $s(\Delta) > t(\Delta)$, the polynomials will be linearly dependent, and there will exist scalars c_α, not all zero, such that $0 = \sum_\alpha c_\alpha w^\alpha = \sum_\alpha c_\alpha \prod_1^N w_i^{\alpha_i}$ $Q(w_1, \ldots, w_N)$, a polynomial of degree at most Δ such that (4) holds. To obtain

$s(\Delta) > t(\Delta)$, we observe that (6) shows that $s(\Delta) = O(\Delta^N)$, while (8) gives $t(\Delta) = O(\Delta^m)$; since $N > m$, $s(\Delta) > t(\Delta)$ when Δ is sufficiently large.

The explicit estimate given in (5) is easily obtained, since

$$s(\Delta) > \left(\frac{\Delta}{N} \right)^{N-m} \frac{1}{\sigma^m} t(\Delta)$$

The proof of the main theorem is now completed by returning to (2) and differentiating it repeatedly, up to partial derivatives of total order at most k, thereby obtaining a large number of equations of the form

$$\left(\frac{\partial}{\partial x} \right)^{\alpha} F = F_{\alpha} = P_{\alpha}(u_1, u_2, \ldots , u_m) \tag{9}$$

where the u_i are the functions arising from differentiating the component functions entering into the Kolmogorov format representation of F, and P_α is a polynomial.

Consideration of a related case may clarify this (See [1]). Suppose that

$$G(x_1,x_2) = f(\phi(x_1) + \psi(x_2)) = (f \circ g)(x) \tag{10}$$

where f, ϕ and ψ are in C^∞. By differentiation, we obtain

$$\begin{aligned} G_{1,0} &= u_1 u_2 \\ G_{0,1} &= u_1 u_3 \\ G_{2,0} &= u_1 u_5 + u_4 (u_2)^2 \\ G_{1,1} &= u_4 u_2 u_3 \end{aligned}$$

etc

where

$$\begin{aligned} u_1 &= f' \circ g \;, & u_2 &= \phi' \;, & u_3 &= \psi' \\ u_4 &= f'' \circ g \;, & u_5 &= \phi'' \;, & u_6 &= \psi'' \end{aligned}$$

etc.

In this example, it may be seen that the u_i can be eliminated, resulting in a polynomial relation among the G_α, namely:

$$(G_{10})^2 G_{01} G_{12} - G_{10}(G_{01})^2 G_{21} - (G_{10})^2 G_{11} G_{02} + (G_{01})^2 G_{11} G_{20} = 0$$

This is a partial differential equation of order 3 and degree 4 satisfied by all the functions G that can be represented in the format (10) with smooth component functions f, ϕ, and ψ. Note that this PDE does not depend upon f, ϕ, or ψ, but only upon the specific format displayed in (10).

The procedure in the Kolmogorov case is essentially the same. By differentiating (2) up to order k, we produce N equations of the form

$$F_\alpha = P_\alpha(u_1, u_2, \ldots, u_m) \quad , \qquad 1 \le |\alpha| \le k$$

where the u_i are the functions $D^p \, {}_j\phi_i$ and $(D^p f_j) \circ g_j$ for $p = 1, 2, \ldots, k$, $j = 1, 2, \ldots, 2n+1$, $i = 1, 2, \ldots, n$, and D is d/dt. Thus, $m = k(2n+1)(n+1)$. By Lemma 2, the number of equations obtained, N, is the dimension of $V(n,k)$ minus 1, since we omit the 0th order derivative. Hence,

$$N = \binom{k+n}{n} - 1$$

Moreover, each of the polynomials P_α has degree at most $k + 1$, and integral coefficients. Since $N = O(k^n)$ while $n \ge 2$ and $m = O(k)$, we see that $N > m$ when k is sufficiently large. Thus, by Lemma 3, there is a polynomial Q with rational coefficients and of degree less than the integer Δ given in (5) such that

$$Q(\ \ldots\ , \ \left(\frac{\partial}{\partial x}\right)^\alpha F, \ \ldots\ ) = 0$$

which depends only upon the integer n and the format given in (2) and (3) and not upon the specific functions f_j and ${}_j\phi_i$ that appear there. Thus, this partial differential equation will have as solutions all the smooth Kolmogorov functions of n variables on I. The main theorem follows.

Explicit values of k and Δ can be found for any $n \geq 2$. However, it is not likely that anyone will wish to exhibit Q itself, since the order and degree for low dimensions n are so large. Interest in this theorem lies in the fact that such "universal" differential equations exist, and that they are themselves smooth (of polynomial type) and that it is the C^∞ solutions of these equations that have the approximation property.

For the record, we record that if $n = 2$, $k = 28$, $N = 434$, $m = 420$; if $n = 10$, $k = 4$, $N = 1000$, $m = 924$; that if $n = 32$, $k = 3$, $N = 6544$, $m = 6435$. All of the upper bound estimates for Δ are very large.

There is no reason to believe that "universal" differential equations of much lower order, degree, and dimension may not exist. However, the present line of argument does not yield any PDE of order 2. It would be of interest to show that no second order partial differential equation of polynomial type in n variables can have the approximation property. It is very plausible that this is true for polynomial equations in one variable, of any order.

The restriction to equations of polynomial type may be an essential feature of the example given in this paper. In subsequent discussions, Professors Crandall and Turner, [2], have shown that it is relatively easy to construct ODEs of the form $Q(u',u") = 0$ with solution sets that are uniformly dense on intervals, and with Q of class C^∞. These are constructed by careful selection of the desired solution sets S in advance, and then construction of the function Q so that it vanishes on a countable family of curves γ_u where

$$\gamma_u(t) = (u'(t),u''(t)), \quad \text{for} \quad t \in I \quad \text{and} \quad u \in S_0 \subset S \ .$$

This of course forces S_0 to be a subset of the solutions of $Q(u',u") = 0$. In particular, such a construction of S_0 is possible for any C^∞ function Q with compact support.

REFERENCES

[1] R. C. Buck, Approximate complexity and functional representation, J. Math. Anal. and Applic. v. 70 (1979), pp. 280 - 298.

[2] M. G. Crandall and R. E. L. Turner, private communication.

[3] G. G. Lorentz, The 13th problem of Hilbert, in "Mathematical Developments Arising from Hilbert's Problems," Proc. Symp. Pure Math., v. 28, Amer. Math. Soc., Providence, R.I. 1976.

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ABSTRACT (Continued)

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