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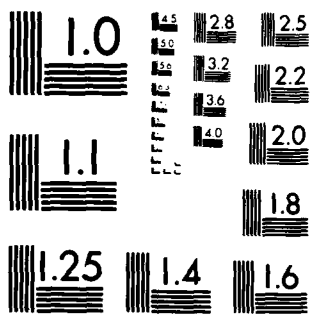

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COINCIDENT BIFURCATION OF EQUILIBRIUM AND PERIODIC SOLUTIONS OF EVOLUTION EQUATIONS

Michael Shearer

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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COINCIDENT BIFURCATION OF EQUILIBRIUM AND PERIODIC SOLUTIONS OF EVOLUTION EQUATIONS

Michael Shearer†

Technical Summary Report #2027
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ABSTRACT

Bifurcation of equilibrium and periodic solutions of nonlinear evolution equations is considered in the neighbourhood of an equilibrium solution for which the corresponding linear problem admits both non-zero equilibrium and non-constant periodic solutions. These solutions of the linear problem are related to those of the nonlinear equation by deriving bifurcation equations possessing a simple symmetry property. This results in a simplification of the bifurcation analysis, illustrated by a discussion of two important special cases exhibiting secondary bifurcation of periodic solutions.

AMS (MOS) Subject Classifications: 34C20, 34C25, 34G20, 47H17, 58F14, 58F35

Key Words: Time periodic bifurcation, Evolution equations, Projection method, Bifurcation and symmetry

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†Department of Mathematics, Duke University, Durham, North Carolina, 27706.

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SIGNIFICANCE AND EXPLANATION

Suppose there is a known equilibirum solution $u = u_0$ of the equation

$$\frac{du}{dt} + F(\xi,u) = 0, \qquad u = u(t) \in X, \ \xi \in \mathbb{R} \tag{1}$$

for some value $\xi = \xi_0$ of a control parameter ξ, where F is a differentiable nonlinear mapping into a finite or infinite dimensional Banach space X. The problem discussed in this paper is that of characterizing equilibrium and t-periodic solutions of (1) near u_0, when ξ is allowed to vary near ξ_0. This is a bifurcation problem when the corresponding linear equation

$$\frac{dv}{dt} + Lv = 0 \qquad v = v(t) \in X, \ v \neq 0 \tag{2}$$

possesses equilibrium or periodic solutions. Here, $Lv = D_u F(\xi_0,u_0)v$ is the linear part of an expansion of $F(\xi_0,u_0 + v)$ around $v = 0$.

This problem is of importance in chemical reactions, for which (1) will represent the reaction and diffusion dynamics of the variable concentration and temperature u, and ξ may for example measure a controlled concentration of reactant, or a diffusion constant, etc.

Assuming that L satisfies assumptions guaranteeing in particular that (2) admits both equilibrium and periodic solutions, the bifurcation problem is reduced to that of solving a pair of real equations

$$f_0(\xi,\alpha,\beta) = 0, \ f_1(\xi,\alpha,\beta) = 0 \tag{3}$$

known as the bifurcation equations. The functions f_0, f_1, and the real variables α,β are so chosen that the first equation is even in β, while the second is odd in β. This simple symmetry property is shown to greatly simplify the study of the bifurcation problem.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

COINCIDENT BIFURCATION OF EQUILIBRIUM AND
PERIODIC SOLUTIONS OF EVOLUTION EQUATIONS

Michael Shearer[†]

1. INTRODUCTION.

Let Ω be a neighbourhood of zero in $\mathbb{R}^{m+1} \times \mathbb{R}^n$, with $m \geq 0$, $n \geq 3$, and let $F \in C^p(\Omega, \mathbb{R}^n)$, with $p \geq 3$, satisfy

(H1) $$F(0,0) = 0$$

For $\xi \in \mathbb{R}^{m+1}$ near zero, we consider equilibrium and periodic solutions u near $u = 0$, of the equation

$$\frac{du}{dt} + F(\xi,u) = 0 \,, \tag{1.1}$$

under the following assumption concerning the spectrum $\sigma(L) \subset \mathbb{C}$ of the linear operator $L = F_u(0,0)$.

(H2)
- (a) i is an algebraically simple eigenvalue of L
- (b) 0 is an algebraically simple eigenvalue of $L : \mathbb{R}^n \to \mathbb{R}^n$
- (c) $ni \notin \sigma(L)$ for $n = 2,3,\ldots$

In particular, (H2) implies that the linear equation

$$\frac{du}{dt} + Lu = 0 \tag{1.2}$$

possesses non-zero equilibrium solutions, and periodic solutions with least period 2π. To show how these solutions of (1.2) generate solutions of the nonlinear equation (1.1), we derive a pair of bifurcation equations in section two. The bifurcation equations possess a simple symmetry, corresponding to the invariance of (1.1) under the translation of t by π.

In section three, we replace $\mathbb{R}^n$ in (1.1) by a real Banach space X, and let $F : \mathbb{R}^{m+1} \times X \to X$ satisfy (H1), together with appropriate regularity conditions, and a spectral assumption corresponding to (H2) (see (A1), (A2)). We adopt the quite

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[†] Department of Mathematics, Duke University, Durham, North Carolina, 27706.

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general setting of Crandall and Rabinowitz [2], which involves the study of an integrated form of equation (1.1). The derivation of the bifurcation equations is slightly complicated by this device, the comparison with the straightforward analysis of section two being of some interest. In this context, the Hilbert space approach of Joseph and Sattinger [8], and Kielhofer [10, 11] should be mentioned, in which Hopf bifurcation for equation (1.1) (with $\mathbb{R}^n$ replaced by a Hilbert space) is studied by working directly with (1.1).

In section four, we briefly discuss the bifurcation equations, appealing to the bifurcation theory of Golubitsky and Schaeffer [5], together with the results in [13]. In particular, the secondary bifurcation of periodic solutions observed by Keener [9] and Langford [12], is explained in terms of the symmetry in the bifurcation equations. Other approaches to the bifurcation problem for (1.1) under (H1), (H2), include those of Cronin [3], using degree theory, and of Hoyle [7], involving the center manifold theory.

Throughout, we refer to the papers of Crandall and Rabinowitz [1], [2], for preliminary results, generalizing these to the present context without proof, where appropriate. The fundamental difference between the situation considered here and that of Hopf bifurcation, in the sense of [1], [2], is that equation (1.2) is here assumed to possess non-zero equilibrium solutions, whereas for Hopf bifurcation, such solutions of (1.2) are excluded. A consequence of this is that it is worthwhile considering perturbed bifurcation problems, for which purpose we allow $m \geq 1$, and write $\xi = (\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^m$, where λ is the real bifurcation parameter, and $\mu \in \mathbb{R}^m$ parametrizes possible perturbations. This is not the case for Hopf bifurcation however, which would be qualitatively the same for $\mu = 0$ and small $\mu \neq 0$, thus rendering the generality of $m \geq 1$ superfluous.

Notation. Subscripts are used to denote partial (Fréchet) derivatives. The null space and range of a linear operator A are denoted by $N(A)$, $R(A)$ respectively. If f is p times continuously Fréchet differentiable on a set U, with values in a set V, we write $f \in C^p(U,V)$, or say $f : U \to V$ is of class C^p.

2. THE FINITE DIMENSIONAL CASE

In this section, we derive bifurcation equations for the finite dimensional problem defined in the introduction. Specifically, for integers $m \geq 0$, $n \geq 3$, $p \geq 3$, we consider the equation

$$\frac{du}{dt} + F(\xi,u) = 0 \qquad (\xi,u) \in \Omega \tag{2.1}$$

where $F \in C^p(\Omega,\mathbb{R}^n)$ for some neighbourhood $\Omega \subset \mathbb{R}^{m+1} \times \mathbb{R}^n$ of zero, and F is assumed to satisfy (H1), (H2).

Let $\tau = \rho^{-1}t$. Then $2\pi\rho$-periodic solutions of (2.1) correspond to 2π-periodic solutions of the equation

$$u' + \rho F(\xi,u) = 0 \tag{2.2}$$

where a prime denotes $\frac{d}{d\tau}$. Note that the parameter ρ has to be determined as part of the solution of (2.2). Let $C_{2\pi}(\mathbb{R},\mathbb{R}^n)$, $C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)$ denote respectively the Banach spaces of continuous, and continuously differentiable, 2π-periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$, with norms

$$\|u\|_{2\pi} = \max\{|u(\tau)| : \tau \in [0,2\pi]\}$$

$$\|u\|_{2\pi,1} = \|u\|_{2\pi} + \|u'\|_{2\pi}$$

Set

$$\mathcal{F}(\rho,\xi,u) = u' + \rho F(\xi,u)$$

Then there is a neighbourhood U of zero in $\mathbb{R}^{m+1}$, and a neighbourhood W of zero in $C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)$ such that $\mathcal{F} : \mathbb{R} \times U \times W \to C_{2\pi}(\mathbb{R},\mathbb{R}^n)$ is p times continuously differentiable.

For u,v in $L^2([0,2\pi],\mathbb{C}^n)$, define

$$(u,v) = \int_0^{2\pi} (u(\tau),\overline{v(\tau)})_{\mathbb{C}^n} d\tau$$

where $(\cdot,\cdot)_{\mathbb{C}^n}$ denotes the usual scaler product in $\mathbb{C}^n$. Letting L^* denote the adjoint of L, condition (H2) implies that there exist vectors φ_0,ψ_0 in $\mathbb{R}^n\backslash\{0\}$ and a,b in $\mathbb{C}^n\backslash\{0\}$ such that

$$L\varphi_0 = 0,\ L^*\psi_0^* = 0,\ La = ia,\ L^*b = -ib$$

$$(\varphi_0,\psi_0)_{\mathbb{C}^n} = (2\pi)^{-1},\ (a,b)_{\mathbb{C}^n} = \pi^{-1}$$

Now set

$$\varphi_1 = \mathrm{Re}(e^{-i\tau}a),\quad \varphi_2 = \varphi_1' = \mathrm{Im}(e^{-i\tau}a)$$

$$\psi_1 = \mathrm{Re}(e^{i\tau}\bar{b}),\quad \psi_2 = \psi_1' = -\mathrm{Im}(e^{i\tau}\bar{b})$$

Then $(\varphi_i,\psi_j) = \delta_{ij}$ $i,j = 0,1,2$ and the following lemma characterizes $N\left(\frac{d}{d\tau} + L\right)$ and $R\left(\frac{d}{d\tau} + L\right)$ (for details of the proof, see [1]).

__Lemma 2.1__. __Suppose__ $L : \mathbb{R}^n \to \mathbb{R}^n$ __satisfies__ (H2), __and let__ φ_k,ψ_k, $k = 0,1,2$ __be defined as above__. __Then__ $\{\varphi_0,\varphi_1,\varphi_2\}$ __is a basis for__ $N\left(\frac{d}{d\tau} + L\right)$ __in__ $C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)$, __and__

$$R\left(\frac{d}{d\tau} + L\right) = \{f \in C_{2\pi}(\mathbb{R},\mathbb{R}^n) : (f,\psi_k) = 0,\ k=0,1,2\}\ .$$

For $\theta \in \mathbb{R}$, define a bounded linear operator $S_\theta : C_{2\pi}(\mathbb{R},\mathbb{R}^n) \to C_{2\pi}(\mathbb{R},\mathbb{R}^n)$ by $(S_\theta w)(\tau) = w(\tau + \theta)$, $\tau \in \mathbb{R}$, and note that S_θ also maps $C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)$ into itself. The invariance of equation (2.2) under translations of τ may be expressed by the property.

$$F(\rho,\xi,S_\theta u) = S_\theta F(\rho,\xi,u) \qquad (\rho,\xi,u) \in \mathbb{R} \times U \times W \tag{2.3}$$

for each $\theta \in \mathbb{R}$. The operator $S = S_\pi$ is of particular importance here, as $S \circ S = I$, the identity operator, and

$$S\varphi_0 = \varphi_0,\ S\psi_0 = \psi_0;\ S\varphi_j = -\varphi_j,\ S\psi_j = -\psi_j\ (j = 1,2) \tag{2.4}$$

Moreover, S is self adjoint:

$$(S\varphi,\psi) = (\varphi,S\psi) \quad \text{for } \varphi,\psi \text{ in } C_{2\pi}(\mathbb{R},\mathbb{R}^n) \tag{2.5}$$

The following subspaces are invariant under S:

$$C^1_{\pm} = (I \pm S)C^1_{2\pi}(\mathbb{R},\mathbb{R}^n),\ C_{\pm} = (I \pm S)C_{2\pi}(\mathbb{R},\mathbb{R}^n)\ ,$$

and

$$C^1_{2\pi}(\mathbb{R},\mathbb{R}^n) = C^1_+ \oplus C^1_-;\ C_{2\pi}(\mathbb{R},\mathbb{R}^n) = C_+ \oplus C_-\ .$$

By (2.3), F maps $\mathbb{R} \times W \times (U \cap C^1_+)$ into C_+, and $F_v(1,\cdot,0) = \frac{d}{d\tau} + L$ maps [illegible] into $C_\pm$, respectively. Let $V_+ = \{v \in C^1_+ : (v,\psi_0) = 0\}$. Then V_+ is a closed linear subspace of C^1_+ complementary to span $\{\varphi_0\}$. Similarly, set $V_- = \{v \in C^1_- : (v,\psi_i) = 0,\ i = 1,2\}$. Then V_- is complementary to span $\{\varphi_1,\varphi_2\}$ in C^1_-, and $V = V_+ \oplus V_-$ is complementary to $N\left(\frac{d}{d\tau} + L\right)$ in $C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)$. Define a projection $P : C_{2\pi}(\mathbb{R},\mathbb{R}^n) \to R\left(\frac{d}{d\tau} + L\right)$ by

$$Pu = u - \sum_{k=0}^{2} (u,\psi_k)\varphi_k$$

Clearly,

$$PS_\theta = S_\theta P \tag{2.6}$$

for all $\theta \in [0,2\pi)$.

By the implicit function theorem, the equation

$$PF(\rho,\xi,\alpha\varphi_0 + \beta\varphi_1 + v) = 0 \tag{2.7}$$

has a unique solution $v = \hat{v}(\rho,\xi,\alpha,\beta)$ in an open ball $B \subset V$ around 0, for (ρ,ξ,α,β) in a neighbourhood $D \subset \mathbb{R} \times \mathbb{R}^{m+1} \times \mathbb{R} \times \mathbb{R}$ of $(1,0,0,0)$. The function $\hat{v} : D \to B$ is of class C^p, and (H1) implies $\hat{v}(\rho,0,0,0) = 0$ identically. Operating on (2.7) with S, and using (2.3), (2.6),

$$PF(\rho,\xi,S(\alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,\beta)) = 0 \qquad (\rho,\xi,\alpha,\beta) \in D$$

which implies that D may be taken to be symmetric about $\beta = 0$, and

$$S\hat{v}(\rho,\xi,\alpha,\beta) = \hat{v}(\rho,\xi,\alpha,-\beta) \tag{2.8}$$

It follows that $\hat{v}(\rho,\xi,\alpha,0) \in V_+$, but we require the following stronger result.

__Lemma 2.2__. __$\hat{v}(\rho,\xi,\alpha,0)$ is a constant function of__ τ, __for each__ (ρ,ξ,α), __and is independent of__ $\rho \neq 0$.

__Proof__. Set $\beta = 0$ in (2.7) and consider only constant v (i.e. $v \in \mathbb{R}^n \cap V$). Then the left hand side of (2.7) lies in $\mathbb{R}^n$, and P projects onto $R(L)$. This leaves the equation

$$\rho P_0 F(\xi,\alpha\varphi_0 + v) = 0 \qquad v \in \mathbb{R}^n \cap V \tag{2.9}$$

where $P_0 : \mathbb{R}^m \to R(L) : w \mapsto w - (w,\psi_0)\varphi_0$. Dividing (2.9) by $\rho \neq 0$, and applying the implicit function theorem in a neighbourhood of $(\xi,\alpha) = (0,0)$, $v = 0$, yields a unique solution $v = \bar{v}(\xi,\alpha)$ of (2.9). But $v = \bar{v}(\xi,\alpha)$ is a solution of (2.8) when $\beta = 0$. Therefore, $\bar{v}(\xi,\alpha) = \hat{v}(\rho,\xi,\alpha,0)$ is independent of τ and $\rho \neq 0$.

Since V is invariant under S_θ $(\theta \in \mathbb{R})$, if $(\rho,\xi,u) \in \mathbb{R} \times U \times W$ is a solution of (2.2) near $(1,0,0)$, then $u = S_\theta(\alpha\varphi_0 + \beta\varphi_1 + v)$, for $(\alpha,\beta,v) \in \mathbb{R}^2 \times V$ and $\theta \in [0,2\pi)$. But then $v = \hat{v}(\rho,\xi,\alpha,\beta)$, and $(\rho,\xi,\alpha,\beta) \in D$ must satisfy

$$(F(\rho,\xi,\alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,\beta)),\psi_k) = 0 \qquad k = 0,1,2 \tag{2.10}$$

Conversely, each solution $(\rho,\xi,\alpha,\beta) \in D$ of (2.10) generates a family $\{u_\theta = S_\theta(\alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,\beta)) : \theta \in [0,2\pi)\}$ of solutions of (2.2), the elements u_θ of which, differ only in phase θ.

Let $g_k(\rho,\xi,\alpha,\beta)$ denote the left hand side of (2.10), $k = 0,1,2$. Then (2.3)-(2.5), (2.8) imply

$$g_0(\rho,\xi,\alpha,-\beta) = g_0(\rho,\xi,\alpha,\beta) \tag{2.11}$$

$$g_k(\rho,\xi,\alpha,-\beta) = -g_k(\rho,\xi,\alpha,\beta) \qquad k = 1,2 \tag{2.12}$$

The next step in deriving the bifurcation equations is to eliminate ρ from (2.10), by solving the equation

$$g_2(\rho,\xi,\alpha,\beta) = 0 \tag{2.13}$$

for ρ near 1 in terms of (ξ,α,β) near $(0,0,0)$. First differentiate the identity

$$PF(\rho,\xi,\alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,\beta)) = 0 \tag{2.14}$$

with respect to β at $(\xi,\alpha,\beta) = (0,0,0)$, $\rho \neq 0$:

$$PF_u(\rho,0,0)(\varphi_1 + \frac{\partial\hat{v}}{\partial\beta}(\rho,0,0,0)) = 0$$

But $PF_u(\rho,0,0)\varphi_1 = P(\varphi_1' + \rho L\varphi_1) = (\rho - 1)PL\varphi_1 = (1 - \rho)P\varphi_2 = 0$, and the restriction of $F_u(\rho,0,0)$ to V is one-to-one for all ρ near 1. Therefore

$$\frac{\partial\hat{v}}{\partial\beta}(\rho,0,0,0) = 0 \text{ for all } \rho \text{ near } 1\text{ ,}$$

so that

$$\frac{\partial\hat{v}}{\partial\beta}(1,0,0,0) = 0 = \frac{\partial^2\hat{v}}{\partial\rho\,\partial\beta}(1,0,0,0) \tag{2.15}$$

Now differentiate the identity

$$\frac{\partial g_2}{\partial \beta} = (F_u(\rho,\xi,\alpha\varphi_0 + \beta\varphi_1 + v)(\varphi_1 + \frac{\partial v}{\partial \beta}), \psi_2) \tag{2.16}$$

with respect to ρ at $\rho = 1$, $(\xi,\alpha,\beta) = (0,0,0)$, using (2.15):

$$\frac{\partial^2 g_2}{\partial\rho\,\partial\beta}(1,0,0,0) = (L\varphi_1, \psi_2) = -1 \tag{2.17}$$

A similar argument gives

$$\left.\begin{aligned} \frac{\partial^2 g_0}{\partial\rho\,\partial\alpha}(1,0,0,0) &= (L\varphi_0, \psi_0) = 0 \\ \frac{\partial^2 g_1}{\partial\rho\,\partial\beta}(1,0,0,0) &= (L\varphi_1, \psi_1) = 0 \end{aligned}\right\} \tag{2.18}$$

which will be used in section four. Define a function $h : D \to \mathbb{R}$ of class C^{p-1}, by

$$h(\rho,\xi,\alpha,\beta) = \beta^{-1} g_2(\rho,\xi,\alpha,\beta) \qquad (\beta \neq 0)$$

$$h(\rho,\xi,\alpha,0) = \frac{\partial g_2}{\partial \beta}(\rho,\xi,\alpha,0)$$

Then $h(1,0,0,0) = 0$ (by (2.15), (2.16)), and $\frac{\partial h}{\partial \rho}(1,0,0,0) = -1$, by (2.17). The implicit function theorem therefore implies that there exist positive numbers η, ε, and a function $\hat{\rho} : B_\eta \to \mathbb{R}$ (B_η denoting the ball in $\mathbb{R}^{m+1} \times \mathbb{R} \times \mathbb{R}$ with center zero and radius η) of class C^{p-1}, such that $\hat{\rho}(0,0,0) = 1$ and $\rho = \hat{\rho}(\xi,\alpha,\beta)$ is the unique solution of

$$h(\rho,\xi,\alpha,\beta) = 0 \qquad |\rho - 1| < \varepsilon,\ (\xi,\alpha,\beta) \in B_\eta .$$

By lemma 2.2, $\hat{v}(\rho,\xi,\alpha,0)$ is independent of ρ and τ. Therefore $g_0(\rho,\xi,\alpha,0)$ is linear in ρ, and $g_1(\rho,\xi,\alpha,0) = 0$ identically. Consequently, setting $\beta = 0$ in (2.10) is equivalent to seeking equilibrium solutions of equation (2.2), for which ρ is undetermined. In this sense, $\rho = \hat{\rho}(\xi,\alpha,\beta)$ describes all the solutions of (2.13) of interest. Since $h(\rho,\xi,\alpha,\beta)$ is even in β (by (2.12)), we have

$$\hat{\rho}(\xi,\alpha,-\beta) = \hat{\rho}(\xi,\alpha,\beta) \tag{2.19}$$

Substituting $\rho = \hat{\rho}$ into $g_k = 0$, $k = 0,1$, we obtain the <u>bifurcation equations</u>:

$$f_k(\xi,\alpha,\beta) = 0 \qquad k = 0,1 \tag{2.20}$$

where $f_k(\xi,\alpha,\beta) = g_k(\hat{\rho}(\xi,\alpha,\beta),\xi,\alpha,\beta)$ satisfy

$$f_k(\xi,\alpha,-\beta) = (-1)^k f_k(\xi,\alpha,\beta) \qquad k = 0,1$$

In particular, if (ξ,α,β) is a solution of (2.20), then so is $(\xi,\alpha,-\beta)$, but t[illegible] solutions of (2.20) correspond to solutions of (2.2) differing only by a phase of [illegible].

The mapping $(f_0,f_1) : B_\eta \to \mathbb{R}^2$ is of class C^{p-1}, and of class C^∞ away from $\beta = 0$. From (H1),

$$f_k(0,0,0) = 0 \qquad k = 0,1 \tag{2.22}$$

and (H2) implies

$$\frac{\partial f_k}{\partial \alpha}(0,0,0) = \frac{\partial f_k}{\partial \beta}(0,0,0) = 0 \qquad k = 0,1 \tag{2.23}$$

To summarize, (2.21)-(2.23) represent the basic properties of the bifurcation equations (2.20) for equation (2.1), assuming (H1) and (H2).

3. THE INFINITE DIMENSIONAL CASE.

In this section, we show how the Lyapunov-Schmidt method of section two, for finite dimensional problems may be carried over to infinite dimensions. The main result of this section, proposition 3.2, states that the bifurcation equation may be written in the form (2.20), and satisfy properties (2.21)-(2.23). The demonstration of these properties differs significantly from that in section two, however, and provides an interesting generalization of the usual Lyapunov-Schmidt procedure. As for the finite dimensional case, the setting and preliminary theory derive largely from the papers of Crandall and Rabinowitz [1,2].

Let X be a real Banach space with norm $\|\cdot\|$, and let $X_c = X + iX$ be the complexification of X. We use the same symbol A to denote the extension to X_c of a linear operator A in X; $\sigma(A)$ denotes the (complex) spectrum of A.

Let L be a densely defined linear operator on A, satisfying

A1): (i) $-L$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on X.

(ii) $T(t)$ is a holomorphic semigroup on X_c.

(iii) $(\lambda I - L)^{-1}$ is compact for all λ in the resolvent set of L.

(iv) i is an algebraically simple eigenvalue of L.

(v) 0 is an algebraically simple eigenvalue of $L : X \to X$

(vi) $ni \notin \sigma(L)$, $n = 2,3,\ldots$

If $\eta > -\mathrm{Re}\,\lambda$ for all $\lambda \in \sigma(L)$, then the fractional powers $(L + \eta I)^{\alpha}$ are defined for $\alpha \geq 0$, and have domains $D((L + \eta I)^{\alpha})$ dense in X. Let X_{α} denote the Banach space $D((L + \eta I)^{\alpha})$, with norm defined by

$$\|x\|_{\alpha} = \|(L + \eta I)^{\alpha} x\| \quad \text{for} \quad x \in X_{\alpha}$$

We consider an equation of the form

$$\frac{du}{dt} + Lu + f(\xi,u) = 0 \quad (\xi,u) \in \mathbb{R}^{m+1} \times X \tag{3.1}$$

where $m \geq 0$, and f satisfies

(A2): For some $\alpha \in [0,1)$, $p \geq 3$, there is a neighbourhood Ω of $(0,0)$ in $\mathbb{R}^{m+1} \times X_\alpha$ such that $f \in C^p(\Omega,X)$. In addition, $f(0,0) = 0$ and $f_u(0,0)X = \{0\}$.

Hypotheses (A1), (A2) correspond respectively to (HL), (Hf) in [2]. The values of α and p will henceforth be considered fixed by condition (A2).

Setting $\tau = \rho^{-1}t$, we see that $2\pi\rho$-periodic solutions of (3.1) correspond to 2π-periodic solutions of the equation

$$u' + \rho(Lu + f(\xi,u)) = 0 \tag{3.2}$$

where a prime denotes $\frac{d}{d\tau}$.

The following lemma, relating solutions of (3.2) to those of an integrated form of the equation, is proved in [4, 6].

Lemma 3.1. *Suppose* (A1), (A2) *hold, let* $r > 0$, *and let* $u \in C([0,r],X_\alpha)$. *The following statements are then equivalent.*

(i) $u' \in C((0,r],X)$, $u((0,r]) \subset D(L)$, *and* (3.2) *is satisfied on* $(0,r)$.

(ii) $u(\tau) - T(\rho\tau)u(0) + \rho \int_0^\tau T(\rho(\tau - s))f(\xi,u(s))ds = 0$, *for* $0 \leq \tau \leq r$.

We say u is a solution of (3.2) if $u \in C([0,r],X_\alpha)$ and (ii) of lemma 3.1 is satisfied. Let $C_{2\pi}(\mathbb{R},X_\alpha)$ be the Banach space of 2π-periodic functions from $\mathbb{R}$ to X_α, and let $C_0([0,2\pi],X_\alpha)$ be the Banach space of continuous functions $h : [0,2\pi] \to X_\alpha$ such that $h(0) = 0$. If w is a function from $\mathbb{R}$ to X, and $\theta \geq 0$, define $(S_\theta w)(\tau) = w(\tau + \theta)$.

Proposition 3.2. *Let* (A1), (A2) *hold. Then there exist neighbourhoods* U *of* 0 *in* $C_{2\pi}(\mathbb{R},X_\alpha)$ *and* W *of* $(0,0)$ *in* $\mathbb{R}^{m+1} \times \mathbb{R}^2$, *together with functions* $(f_0,f_1) : W \to \mathbb{R}^2$, $\hat{\rho} : W \to \mathbb{R}$, $\hat{u} : W \to U$, *each of class* C^{p-1} *and satisfying*

(i) *If* $(\xi,\eta) \in W$ *and* $f_k(\xi,\eta) = 0$, $k = 0,1$, *then* $u = \hat{u}(\xi,\eta)$ *is a solution of* (3.2) *for this value of* $\xi \in \mathbb{R}^{m+1}$, *with* $\rho = \hat{\rho}(\xi,\eta)$.

(ii) $\hat{u}(0) = 0$, $\hat{\rho}(0) = 1$, $f_k(0) = 0$, $k = 0,1$ *and the* 2×2 *matrix* $D_\eta(f_0,f_1)(0,\eta)|_{\eta=(0,0)}$ *has all entries zero.*

(iii) $\hat{u}(\xi,\alpha,-\beta) = S_\pi\hat{u}(\xi,\alpha,\beta)$, $\hat{\rho}(\xi,\alpha,-\beta) = \hat{\rho}(\xi,\alpha,\beta)$, $f_k(\xi,\alpha,-\beta) = (-1)^k f_k(\xi,\alpha,\beta)$, $k = 0,1$, *for all* $(\xi,\alpha,\beta) \in W$.

(iv) There exists $\varepsilon > 0$ such that if $(\rho,\xi,u) \in \mathbb{R} \times \mathbb{R}^{m+1} \times U$, $|\rho - 1| < \varepsilon$, $\|\xi\| < \varepsilon$, and (ρ,ξ,u) is a solution of (3.2), then for some $(\alpha,\beta) \in \mathbb{R}^2$ such that $(\xi,\alpha,\beta) \in W$, $f_k(\xi,\alpha,\beta) = 0$, $k = 0,1$, $u = S_\theta \hat{u}(\xi,\alpha,\beta)$ for some $\theta \in [0,2\pi)$, and either $\beta = 0$ (if $u(\tau)$ is independent of $\tau \in \mathbb{R}$), or $\rho = \hat{\rho}(\xi,\alpha,\beta)$.

(v) $\hat{u}(\xi,\alpha,0)$ is a constant function for all $(\xi,\alpha,0) \in W$.

The proof of proposition 3.2 occupies the rest of this section. Define

$$F(\rho,\xi,u)(\tau) = u(\tau) - T(\rho\tau)u(0) + \rho \int_0^\tau T(\rho(\tau - s))f(\xi,u(s))ds \tag{3.3}$$

F is to be regarded as a mapping of that subset of $\mathbb{R} \times \mathbb{R}^{m+1} \times C_{2\pi}(\mathbb{R},X_\alpha)$ for which (3.3) makes sense into $C_0([0,2\pi],X_\alpha)$.

Lemma 3.3. Suppose (A1), (A2) are satisfied. Then F is p times continuously differentiable from its domain into $C_0([0,2\pi],X_\alpha)$ and $F(\rho,0,0) = 0$ for $\rho \in (0,\infty)$. Moreover, for $v \in C_{2\pi}(\mathbb{R},X_\alpha)$ and $\rho > 0$,

$$(F_u(\rho,0,0)v)(\tau) = v(\tau) - T(\rho\tau)v(0) \quad \tau \geq 0 \tag{3.4}$$

Lemma 3.3 is proved in [2].

Let $A : C_{2\pi}(\mathbb{R},X_\alpha) \to C_0([0,2\pi],X_\alpha)$ denote the linear operator $F_u(1,0,0)$. By (3.4), $(Au)(\tau) = u(\tau) - T(\tau)u(0)$. The following characterization of $N(A)$, $R(A)$ is a straightforward generalization of lemma 1.13 of [2], to the present situation.

Lemma 3.4. Let (A1) hold. Then

(i) $N(I + L^2) \oplus N(L) = N(I - T(2\pi))$ and $N(I + L^{*2}) \oplus N(L^*) = N(I - T(2\pi)^*)$.

(ii) There exists $x_1 \in N(I - T(2\pi))$ such that $x_1, x_2 = Lx_1$ span $N(I + L^2)$ and

$$T(\tau)x_k = (\cos\tau)x_k + (\sin\tau)Lx_k, \quad k = 1,2, \ \tau \geq 0 \tag{3.5}$$

If x_0 spans $N(L)$, then $T(\tau)x_0 = x_0$, $\tau \geq 0$.

(iii) There exists $x_1^* \in N(I - T(2\pi)^*)$ such that $x_1^*, x_2^* = Lx_1^*$ span $N(I + L^{*2})$. If x_0^* spans $N(L^*)$, then $h \in R(A)$ if and only if $(h(2\pi),x_k^*) = 0$, $k = 0,1,2$. That is, $h \in R(A)$ if and only if $h(2\pi) \in R(I - T(2\pi))$.

(iv) x_0^*, x_1^* may be chosen in (iii) so that

$$(x_i,x_j^*) = \delta_{ij} \quad i,j = 0,1,2$$

(v) For $\tau \geq 0$, set $\varphi_k(\tau) = T(\tau)x_k$, $(k = 0,1,2)$.

Then $\{\varphi_0,\varphi_1,\varphi_2\}$ is a basis for $N(A)$.

In lemma 3.4, $(\cdot,\cdot)$ denotes the pairing between X and its dual X^*; [illegible] of an operator $M : X \to X$ is denoted by M^*, and I is the identity operator in [illegible] X and X^*.

We define a family $\{P_\rho : \rho \in (0,2)\}$ of projections on $C_0([0,2\pi],X_\alpha)$ as follows. For $\rho \in (0,2)$ and $\tau \geq 0$, let $E_\rho(\tau)$ be the linear mapping in $N(I - T(2\pi))$ defined by

$$E_\rho(\tau)x_0 = (\tau/2\pi)x_0, \quad E_\rho(\tau)x_k = [T(\tau) - T(\rho\tau)]x_k, \quad k = 1,2$$

and let $M_\rho(\tau)$ be the linear mapping in $N(I - T(2\pi))$ defined by

$$M_\rho(\tau) = [E_\rho(2\pi)]^{-1}E_\rho(\tau), \text{ if } \rho \neq 1 \text{ and } M_1(\tau)x_k = (\tau/2\pi)T(\tau)x_k, \quad k = 0,1,2 .$$

Then, for $w \in C_0([0,2\pi],X_\alpha)$, $\rho \in (0,2)$, set

$$(P_\rho w)(\tau) = w(\tau) - \sum_{k=0}^{2} (w(2\pi),x_k^*)M_\rho(\tau)x_k$$

Lemma 3.5. Let (A1) hold. Then

(a) For each $\rho \in (0,2)$, P_ρ is a projection of $C_0([0,2\pi],X_\alpha)$ onto $R(A)$.

(b) The mapping $(\rho,w) \to P_\rho w$ from $(0,2) \times C_0([0,2\pi],X_\alpha)$ to $R(A)$ is analytic.

Proof. Part (a) follows immediately from lemma 3.4 (iii). To prove (b), note that (3.5) implies that $P_\rho w$ is analytic in (ρ,w), except possibly at $\rho = 1$. It is easily shown that $[E_\rho(2\pi)]^{-1}E_\rho(\tau)x_k$ $(k = 1,2)$ involves singular terms only of the form $(\sin(\rho - 1)\tau/\sin(\rho - 1)\pi)x_j$ $(j \in \{1,2\})$, each of which has a removable singularity at $\rho = 1$. Hence result.

In the standard Lyapunov-Schmidt procedure, it would be natural to use the single projection P_1 of $C_0([0,2\pi],X_\alpha)$ onto $R(A)$. However, the family $\{P_\rho : \rho \in (0,2)\}$ of such projections is important here, and lemma 3.5 shows that the equation $F = 0$ may be replaced by the system

$$P_\rho F(\rho,\xi,u) = 0 \tag{3.6}$$

$$(I - P_\rho)F(\rho,\xi,u) = 0 \tag{3.7}$$

The reason for doing this is that we wish to preserve in this procedure, the symmetry induced by the invariance of equation (3.2) under translations of τ. This invariance is expressed in terms of F by the identity

$$F(\rho,\xi,S_\theta u)(\tau) = [S_\theta F(\rho,\xi,u)](\tau) - T(\rho\tau)F(\rho,\xi,u)(\theta) \qquad (3.8)$$

which holds for all $\theta \geq 0$, $\tau \geq 0$, and (ρ,ξ,u) in the domain of F.

Lemma 3.6. Let (A1), (A2) hold, and suppose $P_\rho F(\rho,\xi,u) = 0$. Then

(i) $F(\rho,\xi,u)(\tau) = \sum_{k=0}^{2} (F(\rho,\xi,u)(2\pi),x_k^*)M_\rho(\tau)x_k$, for all $\tau \geq 0$.

(ii) $F(\rho,\xi,S_\theta u) = T(\theta)F(\rho,\xi,u)$, for all $\theta \geq 0$.

(iii) $P_\rho F(\rho,\xi,S_\theta u) = 0$, for all $\theta \geq 0$.

Proof. Suppose $P_\rho F(\rho,\xi,u) = 0$. Then

$$F(\rho,\xi,u)(\tau) = \sum_{k=0}^{2} a_k M_\rho(\tau)x_k \qquad (3.9)$$

for $\tau \in [0,2\pi]$, where $a_k = (F(\rho,\xi,u)(2\pi),x_k^*)$, $k = 0,1,2$. We require the identity

$$M_\rho(\tau + \theta) = T(\theta)M_\rho(\tau) + T(\rho\tau)M_\rho(\theta) \qquad (3.10)$$

which holds for all $\rho \in (0,2)$, $\tau \geq 0$, $\theta \geq 0$.

Suppose (3.9) holds for all $\tau \in [0,2\pi n]$, for some integer $n \geq 1$. Let $\tau \in [0,2\pi n]$, and set $\theta = 2\pi$ in (3.8).

$$\begin{aligned} F(\rho,\xi,u)(\tau + 2\pi) &= F(\rho,\xi,S_{2\pi}u)(\tau) + T(\rho\tau)F(\rho,\xi,u)(2\pi) \\ &= F(\rho,\xi,u)(\tau) + T(\rho\tau)\sum_{k=0}^{2} a_k x_k \\ &= \sum_{k=0}^{2} a_k[M_\rho(\tau) + T(\rho\tau)M_\rho(2\pi)]x_k \\ &= \sum_{k=0}^{2} a_k M_\rho(\tau + 2\pi)x_k, \quad \text{by (3.10)} . \end{aligned}$$

Thus, (3.9) holds for all $\tau \in [0,2\pi(n + 1)]$, and (i) is proved by induction.

From (3.8) and part (i), we have

$$F(\rho,\xi,S_\theta u)(\tau) = \sum_{k=0}^{2} a_k[M_\rho(\tau+\theta) - T(\rho\tau)M_\rho(\theta)]x_k$$

$$= \sum_{k=0}^{2} a_k T(\theta) M_\rho(\tau) x_k = T(\theta)F(\rho,\xi,u)(\tau), \qquad \text{by (3.10)}$$

which proves (ii). Moreover

$$P_\rho T(\theta)F(\rho,\xi,u)(\tau) = P_\rho \sum_{k=0}^{2} a_k M_\rho(\tau)T(\theta)x_k$$

$$= M_\rho(\tau)\left\{\sum_{k=0}^{2} a_k T(\theta)x_k - \sum_{j=0}^{2}\sum_{k=0}^{2} a_k (T(\theta)x_k, x_j^*)x_j\right\} = 0 .$$

This proves (iii).

Let V be the closed linear subspace of $C_{2\pi}(\mathbb{R},X_\alpha)$ defined by

$$V = \left\{v \in C_{2\pi}(\mathbb{R},X_\alpha) : (\int_0^{2\pi} T(2\pi - s)v(s)ds, x_k^*) = 0,\ k = 0,1,2\right\}$$

Then $V \oplus N(A) = C_{2\pi}(\mathbb{R},X_\alpha)$. Writing $u = \varphi + v$, $(\varphi,v) \in N(A) \times V$, in (3.6), we solve (3.6) for v as a function of φ,ρ,ξ by the implicit function theorem. Set $G(\rho,\xi,\alpha,\beta,v) = P_\rho F(\rho,\xi,\alpha\varphi_0 + \beta\varphi_1 + v)$. Then G is of class C^p from a neighbourhood of $(1,0,0,0,0)$ in $(0,2) \times \mathbb{R}^{m+1} \times \mathbb{R} \times \mathbb{R} \times V$, into $R(A)$. Moreover, $G(1,0,0,0,0) = 0$ and $G_v(1,0,0,0,0) = A : V \to R(A)$ is a linear homeomorphism. By the implicit function theorem, there exists a neighbourhood D of $(1,0,0,0)$ in $\mathbb{R}^{m+4}$, and a neighbourhood B of zero in V, together with a function $\hat{v} : D \to B$ of class C^p, such that $\hat{v}(1,0,0,0) = 0$, and

$$P_\rho F(\rho,\xi,\alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,\beta)) = 0 \quad \text{for} \quad (\rho,\xi,\alpha,\beta) \in D \tag{3.11}$$

Moreover, for each $(\rho,\xi,\alpha,\beta) \in D$, $v = \hat{v}(\rho,\xi,\alpha,\beta)$ is the unique solution in B of $G = 0$. This together with lemma 3.6 (ii), and (3.11), implies

$$P_\rho F(\rho,\xi,\alpha\varphi_0 + \beta S_\theta\varphi_1 + S_\theta\hat{v}(\rho,\xi,\alpha,\beta)) = 0 \tag{3.12}$$

identically, so that

$$S_\theta\hat{v}(\rho,\xi,\alpha,0) = \hat{v}(\rho,\xi,\alpha,0) \quad \text{for all} \quad \theta \geq 0 \tag{3.13}$$

and

$$S_-\hat{v}(\rho,\xi,\alpha,\beta) = \hat{v}(\rho,\xi,\alpha,-\beta) \tag{3.14}$$

In particular, (3.13) means that $\hat{v}(\rho,\xi,\alpha,0)(\tau)$ is a constant (i.e. independent of τ). Now substitute $u = \alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,\beta)$ into (3.7), which becomes, by lemma 3.4 (iii).

$$(F(\rho,\xi,\alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,\beta))(2\pi),x_k^*) = 0, \quad k = 0,1,2 \tag{3.15}$$

Let $g_k(\rho,\xi,\alpha,\beta)$ denote the left hand side of (3.15), $k = 0,1,2$. As in section two, we wish to show that g_0,g_1,g_2 satisfy the relations (2.11), (2.12). Noting that

$$T(\pi)^*x_0^* = x_0^*, \quad T(\pi)^*x_k^* = -x_k^*, \quad k = 1,2 ,$$

(2.11), (2.12) follow from the identity

$$\begin{aligned} g_k(\rho,\xi,\alpha,-\beta) &= (F(\rho,\xi,\alpha\varphi_0 - \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,-\beta))(2\pi),x_k^*) \\ &= (T(\pi)F(\rho,\xi,\alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,\beta))(2\pi),x_k^*) \\ &= (F(\rho,\xi,\alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\rho,\xi,\alpha,\beta))(2\pi),T(\pi)^*x_k^*) . \end{aligned}$$

In order that ρ be undetermined for equilibrium solutions of (3.2), it is enough to show that $\hat{v}(\rho,\xi,\alpha,0)$ is independent of ρ whenever $F(\rho,\xi,\alpha\varphi_0 + \hat{v}(\rho,\xi,\alpha,0)) = 0$. We remark that in the finite dimensional case, it was possible to show that the corresponding $\hat{v}(\rho,\xi,\alpha,0)$ is independent of ρ, without qualification.

__Lemma 3.7__. __Let__ (A1), (A2) __hold__. __If__ $(\rho,\xi,\alpha,0) \in D$ __satisfies__ $F(\rho,\xi,\alpha\varphi_0 + \hat{v}(\rho,\xi,\alpha,0)) = 0$, __then__ $\hat{v}(\rho,\xi,\alpha,0)(\tau)$ __is independent of__ $\tau \in \mathbb{R}$, __and__ ρ.

__Proof__. Suppose $u \in X_\alpha$ is a constant satisfying $F(\rho,\xi,u) = 0$ for some (ρ,ξ), $\rho \neq 0$. Then

$$u = T(\rho\tau)u - \rho\int_0^\tau T(\rho(\tau - s))f(\xi,u)ds = g(\rho)(\tau)$$

say, and $g(\rho)(\tau)$ is consequently independent of $\tau \geq 0$. But

$$g(\rho + \bar{\rho})(\tau) = g(\rho)((\rho + \bar{\rho})\tau/\rho) \quad \text{for all } \bar{\rho} > -\rho$$

which implies that $g(\rho)$ is independent of $\rho > 0$. Therefore

$$F(\bar{\rho},\xi,u)(\tau) = u - g(\bar{\rho})(\tau) = u - g(\rho)(\tau) = 0$$

for all $\bar{\rho} > 0$, $\tau \geq 0$.

As remarked earlier, (3.13) implies that $\hat{v}(\rho,\xi,\alpha,0)(\tau)$ is independent [illegible]

for each (ρ,ξ,α). Suppose $(\rho,\xi,\alpha,0) \in D$ satisfies $F(\rho,\xi,\alpha\varphi_0 + \hat{v}($[illegible]$)) = $ [illegible].

Then, since $\varphi_0(\tau)$ is also independent of $\tau \in \mathbb{R}$, the above discussion implies

$$F(\bar{\rho},\xi,\alpha\varphi_0 + \hat{v}(\rho,\xi,\alpha,0)) = 0 \quad \text{for all} \quad \bar{\rho} > 0 .$$

In particular,

$$P_{\bar{\rho}}F(\bar{\rho},\xi,\alpha\varphi_0 + \hat{v}(\rho,\xi,\alpha,0)) = 0 \quad \text{for all} \quad \bar{\rho} > 0$$

which implies that $\hat{v}(\rho,\xi,\alpha,0) = \hat{v}(\bar{\rho},\xi,\alpha,0)$ whenever $\rho,\bar{\rho}$ are near 1. Hence result.

<u>Lemma 3.8</u>. <u>Let</u> (A1), (A2) <u>hold</u>. <u>Then there exists</u> $\epsilon > 0$ <u>such that for</u> $|\rho - 1|$ [illegible],

$$\frac{\partial\hat{v}}{\partial\beta}(\rho,0,0,0) = 0 .$$

<u>Proof</u>. Differentiating (3.11) with respect to β at $\xi = 0$, $\alpha = \beta = 0$, and setting $\hat{v}_\beta(\tau) = \frac{\partial\hat{v}}{\partial\beta}(\rho,0,0,0)(\tau)$, $\psi(\tau) = \varphi_1(\tau) - T(\rho\tau)\varphi_1(0)$, we have

$$0 = P_\rho F_u(\rho,0,0)(\varphi_1 + \hat{v}_\beta) = P_\rho\psi + P_\rho F_u(\rho,0,0)\hat{v}_\beta = P_\rho F_u(\rho,0,0)\hat{v}_\beta \tag{3.16}$$

Now the restriction of $A = P_1F_u(1,0,0)$ to V is one-to-one. Therefore, since $\rho \to P_\rho F_u(\rho,0,0)$ is continuous, the restriction of $P_\rho F_u(\rho,0,0)$ to V is one-to-one for all ρ near 1, say $|\rho - 1| < \epsilon$. Since $\hat{v}_\beta \in V$, (3.16) proves the result.

In particular, it follows from lemma 3.8 that

$$\frac{\partial g_2}{\partial\beta}(\rho,0,0,0) = (\varphi_1(2\pi) - T(2\pi\rho)\varphi_1(0),x_2^*) = -\sin 2\pi\rho \tag{3.17}$$

so that

$$\frac{\partial g_2}{\partial\beta}(1,0,0,0) = 0, \quad \frac{\partial^2 g_2}{\partial\rho\,\partial\beta}(1,0,0,0) = -2\pi \tag{3.18}$$

As in section two, define a function $h : D \to \mathbb{R}$ of class C^{p-1} by

$$h(\rho,\xi,\alpha,\beta) = \beta^{-1}g_2(\rho,\xi,\alpha,\beta) \quad \text{if} \quad \beta \neq 0$$

$$h(\rho,\xi,\alpha,0) = \frac{\partial g_2}{\partial\beta}(\rho,\xi,\alpha,0)$$

From (3.18) and the implicit function theorem, it follows that there exist positive numbers ϵ,η and a function $\hat{\rho} : B_\eta \to \mathbb{R}$ of class C^{p-1} (and of class C^p in the region $\beta \neq 0$) such that, writing $x = (\xi,\alpha,\beta) \in B_\eta$,

$\hat{\rho}(0) = 1$, $h(\hat{\rho}(x),x) = 0$ for all $x \in B_\eta$, and if [illegible] $- 1$ [illegible], $x \in B_\eta$ satisfy $h(\rho,x) = 0$, then $\rho = \hat{\rho}(x)$.

Since $h(\rho,\xi,\alpha,-\beta) = h(\rho,\xi,\alpha,\beta)$, $\hat{\rho}$ satisfies

$$\hat{\rho}(\xi,\alpha,-\beta) = \hat{\rho}(\xi,\alpha,\beta) \quad \text{for all} \quad (\xi,\alpha,\beta) \in B_\eta \tag{3.19}$$

Setting $f_k(\xi,\alpha,\beta) = g_k(\hat{\rho}(\xi,\alpha,\beta),\xi,\alpha,\beta)$, $k = 0,1$, we have the <u>bifurcation equations</u>

$$f_0(\xi,\alpha,\beta) = 0, \; f_1(\xi,\alpha,\beta) = 0, \; (\xi,\alpha,\beta) \in B_\eta \tag{3.20}$$

where $(f_0,f_1) : B_\eta \to \mathbb{R}^2$ is of class C^{p-1}, and of class C^p away from [illegible] $= 0$. Moreover, (3.19) and the symmetry properties (2.11), (2.12) imply that

$$f_k(\xi,\alpha,-\beta) = (-1)^k f_k(\xi,\alpha,\beta), \; k = 0,1 \tag{3.21}$$

identically in B_η.

Since V is invariant under $T(\theta)$, $\theta \in [0,2\pi]$, and $S_\theta = T(\theta)$ on $N(A)$, the argument of section two may be repeated here, to show that solutions $(\xi,\alpha,\beta) \in B_{[illegible]}$ of (3.20) correspond to small norm equilibrium and 2π-periodic solutions of equation (3.2), with $\rho = \hat{\rho}(\xi,\alpha,\beta)$ (see proposition 3.2 (iv)). Setting $\hat{u}(\xi,\alpha,\beta) = \alpha\varphi_0 + \beta\varphi_1 + \hat{v}(\hat{\rho}(\xi,\alpha,\beta),\xi,\alpha,\beta)$, and choosing U and W appropriately, we have proved proposition 3.2.

4. BIFURCATION.

To discuss bifurcation of equilibrium and periodic solutions of (2.1) or (3.1), we set $\xi = (\lambda,\mu) \in \mathbb{R} \times \mathbb{R}^m$, where λ is to be considered a real bifurcation parameter, and $\mu \in \mathbb{R}^m$ parameterizes possible perturbations. The bifurcation problem is to describe the local structure of equilibrium and periodic solutions of (2.1) or (3.1) in (λ,u) - space (i.e. near $(\lambda,u) = (0,0)$) for each fixed μ near zero. For periodic solutions with period near 2π, we have shown that under the conditions (H1), (H2) or (A1), (A2), this is equivalent (in the sense specified by proposition 3.2) to the bifurcation problem for the bifurcation equations

$$f_0(\lambda,\mu,\alpha,\beta) = 0, \quad f_1(\lambda,\mu,\alpha,\beta) = 0, \quad (\lambda,\mu,\alpha,\beta) \in B_\eta \tag{4.1}$$

where f_0 f_1 are real valued functions of class C^{p-1}, satisfying (2.21)-(2.23).

The natural setting for a discussion of bifurcation for equations (4.1), is the singularity theory developed by Golubitsky and Schaeffer [5], at least when $p = \infty$. It is not our purpose here to attempt a general analysis of (4.1), however, but simply to emphasize two important cases (for which the possible bifurcation structure is well documented in [5]), and to remark on how these relate to the work of Keener [9] and Langford [12], on secondary bifurcation of periodic solutions.

We shall need to specify conditions on derivatives of $F : \mathbb{R}^{m+1} \times \mathbb{R}^n \to \mathbb{R}^n$ (when considering (2.1)), and corresponding conditions on $f : \mathbb{R}^{m+1} \times X_\alpha \to X$ (when considering (3.1)). To save duplication, we state these only for F; the corresponding conditions on f are readily obtained.

We consider the following two situations

I: $\mu = (\delta,\nu) \in \mathbb{R}^2$, and $F(\lambda,\delta,0;0) = 0$ for all (λ,δ) near $(0,0)$.

II: $\mu \in \mathbb{R}$, and $F(\lambda,\mu;-u) = -F(\lambda,\mu;u)$ identically.

<u>Case I</u>. We observe that $\hat{v}(\rho,\lambda,\delta,0,0,0) = 0$ identically, so that $f_k(\lambda,\delta,0,0,0) = 0$, $k = 0,1$, identically. Together with (2.21)-(2.23), this implies that (f_0,f_1) $(\lambda,\delta,\nu,\alpha,\beta)$ has a Taylor expansion about zero in $\mathbb{R}^5$, of the form

$$\left.\begin{aligned} f_0 &\equiv (a_0\lambda + b_0\delta)\alpha + q\alpha^2 + r\beta^2 + c\nu + R_0(\lambda,\delta,\nu,\alpha,\beta) \\ f_1 &\equiv (a_1\lambda + b_1\delta)\beta + s\alpha\beta + \beta R_1(\lambda,\delta,\nu,\alpha,\beta) \end{aligned}\right\} \qquad (4.2)$$

where $(R_0, \beta R_1)$ is of class C^{p-1}, and contains terms of higher order than those written explicitly in (4.2). Set

$$L_1 = F^0_{u\lambda},\ L_2 = F^0_{u\delta},\ Q = \frac{1}{2}F^0_{uu},\ \omega = F^0_{\nu}\ ,$$

a superscript 0 indicating that each of these derivatives is evaluated at (0,0,0;0). The coefficients in (4.2) are given by

$$\begin{aligned} a_0 &= 2\pi(L_1\varphi_0,\psi_0)_{\mathbb{C}^n}; & a_1 &= \pi\mathrm{Re}\,(L_1 a,b)_{\mathbb{C}^n} \\ b_0 &= 2\pi(L_2\varphi_0,\psi_0)_{\mathbb{C}^n}; & b_1 &= \pi\mathrm{Re}\,(L_2 a,b)_{\mathbb{C}^n} \\ q &= 2\pi(Q(\varphi_0,\varphi_0),\psi_0)_{\mathbb{C}^n}; & s &= 2(Q(\varphi_0,\varphi_1),\psi_1) \\ r &= (Q(\varphi_1,\varphi_1),\psi_0); & c &= 2\pi(\omega,\psi_0)_{\mathbb{C}^n} \end{aligned}$$

Note that we have here used (2.18), (2.19) to ensure that $\hat{\rho}(\lambda,\mu,\nu,\alpha,\beta)$ does not contribute to these coefficients.

If we assume the non-degeneracy condition

$$a_0b_1 \neq a_1b_0,\ a_0q \neq a_1p,\ qrs \neq 0,\ c \neq 0 \qquad (4.3)$$

then bifurcation for equations (4.1) is qualitatively described by the following truncated form of those equations.

$$\left.\begin{aligned} (a_0\lambda + b_0\delta)\alpha + q\alpha^2 + r\beta^2 + c\nu &= 0 \\ (a_1\lambda + b_1\delta)\beta + s\alpha\beta &= 0 \end{aligned}\right\} \qquad (4.4)$$

This result is given precise meaning for $p = \infty$ in [5]. For any $p \geq 4$, and $\nu = 0$, the correspondence between solutions of (4.4) and those of (4.1) is established in [13], and may easily be generalized to $\nu \neq 0$.

In particular, when $\nu = 0$, (4.4), and consequently (4.1), admits exactly one secondary bifurcation for all $\delta \neq 0$ near 0 [13]. Secondary branches for (4.1) have $\beta \neq 0$ away from the bifurcation point, which lies on a primary branch with $\beta = 0$.

This corresponds to secondary bifurcation of periodic solutions of (1.1) from a [illegible] branch of equilibrium solutions, which agrees with the observations of Keener and Langford [12]. For $\nu \neq 0$ however, the (primary) bifurcation of equilibrium solutions is destroyed, while bifurcations of periodic solutions from curves of equilibrium solutions are preserved (see [5] for a complete set of bifurcation diagrams).

Case II. We here consider one parameter perturbations, $\mu \in \mathbb{R}$, when $F(\ ,\ ,u)$ is [illegible] with respect to u. Recalling that $S : C_{2\pi}(\mathbb{R},\mathbb{R}^n) \to C_{2\pi}(\mathbb{R},\mathbb{R}^n)$ is the operator which translates τ by π, $(Su)(\tau) = u(\tau + \pi)$, set $R = -S$. Then $G = \{I,R,S,-I\}$ is an Abelian group of linear operators that commutes with equation (2.2) in the sense that

$$GF(\rho,\lambda,\mu,u) = F(\rho,\lambda,\mu,Gu) \tag{4.5}$$

For all $G \in G$ and $(\rho,\lambda,\mu,u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times C^1_{2\pi}(\mathbb{R},\mathbb{R}^n)$ near $(1,0,0,0)$. Moreover,

$$R\varphi_0 = -\varphi_0,\ R\varphi_k = \varphi_k,\ k = 1,2 \tag{4.6}$$

and

$$R\psi_0 = -\psi_0,\ R\psi_k = \psi_k,\ k = 1,2 \tag{4.7}$$

The relations (4.5)-(4.7) imply that the bifurcation equations (4.1) are odd with respect to (α,β). As in case I, we expand (f_0,f_1) as a Taylor series:

$$\left.\begin{aligned} f_0 &= (a_0\lambda + b_0\mu)\alpha + p\alpha^3 + r\alpha\beta^2 + \alpha R_0(\lambda,\mu,\alpha,\beta) \\ f_1 &= (a_1\lambda + b_1\mu)\alpha + q\alpha^2\beta + s\beta^3 + \beta R_1(\lambda,\mu,\alpha,\beta) \end{aligned}\right\} \tag{4.8}$$

where $(\alpha R_0,\beta R_1)$ represents the remaining terms, each $R_k(\lambda,\mu,\alpha,\beta)$, $k = 0,1$, is even in (α,β). The coefficients a_k,b_k, $k = 0,1$, are defined as for case I, with μ replacing δ, and p,q,r,s depend upon $C = \frac{1}{6}F^0_{uuu}$ as follows

$$p = 2\pi(C(\varphi_0,\varphi_0,\varphi_0),\psi_0)_{\mathbb{C}^n}; \quad q = 3(C(\varphi_0,\varphi_0,\varphi_1),\psi_1)$$

$$r = 3\ (C(\varphi_0,\varphi_1,\varphi_1),\psi_0); \quad s = (C(\varphi_1,\varphi_1,\varphi_1),\psi_1)$$

The corresponding non-degeneracy condition for this case is

$$a_0b_1 \neq a_1b_0,\ a_0q \neq a_1p,\ a_0s \neq a_1r,\ ps \neq rq \tag{4.9}$$

Assuming (4.9), bifurcation for equations (4.1) is qualitatively described by the truncated form

$$(a_0[illegible] + b[illegible])[illegible] + [illegible]^2 + r[illegible]^2 = [illegible]$$

$$(a_1[illegible] + b_1[illegible])[illegible] + q[illegible]^2 + s[illegible]^2 = [illegible]$$

The correspondence between solutions of (4.10) and those of (4.1) i[illegible]

In discussing bifurcation for (4.10) in relation to bifurcation for [illegible], it should be recalled that the analysis is valid only locally, so that all [illegible] are to be considered with this restriction.

Letting $\alpha = 0$, $\beta = 0$ in turn in (4.10) gives primary branches [illegible] of respectively non-constant periodic, and equilibrium solutions of (2.1) or [illegible] for each μ (near zero). These primary branches bifurcate from $u = [illegible]$ at values of [illegible] given respectively by $\lambda = -b_1\mu/a_1 + o(|\mu|)$ and $\lambda = -b_0\mu/a_0 + o(|\mu|)$. Next, divide the first equation in (4.10) by α, and the second by β:

$$\left.\begin{aligned} a_0\lambda + b_0\mu + p\alpha^2 + r\beta^2 &= 0 \\ a_1\lambda + b_1\mu + q\alpha^2 + s\beta^2 &= 0 \end{aligned}\right\} \qquad (4.11)$$

Solutions of (4.11) lie on secondary branches of solutions of equations (4.1), corresponding to secondary branches of solutions of (2.1) or (3.1). The broad structure of the secondary bifurcation may be described in terms of the coefficients in (4.11), by distinguishing between the following two cases.

(A): If $(a_1r - a_0s)(a_1p - a_0q) < 0$, then there are exactly two secondary bifurcation points $(\lambda^*(\mu), \pm u^*(\mu)) \in D_\mu$ for each μ satisfying

$$(a_1r - a_0s)(a_1b_0 - a_0b_1)\mu > 0 \qquad (4.12)$$

When the inequality (4.12) is reversed, there is exactly one secondary bifurcation point $(\lambda^*(\mu), u^*(\mu)) \in C_\mu$. There are no other secondary bifurcation points for (2.1), in the local sense of the analysis. The secondary branches consist of non-constant periodic solutions of (2.1), and have values of λ satisfying $(\lambda - \lambda^*(\mu))(a_1r - a_0s) > 0$.

(B): If $(a_1 r - a_0 s)(a_1 p - a_0 q)$ [illegible] 0, then there are exactly three secondary bifurcation points $(\lambda_1^*(\mu), u_1^*(\mu)) \in C_\mu$, $(\lambda_2^*(\mu), \pm u_2^*(\mu)) \in D_\mu$, for each μ satisfying (4.12). $(\lambda_1^*(\mu), u_1^*(\mu))$ is connected to each of $(\lambda_2^*(\mu), \pm u_2^*(\mu))$ by secondary branches of non-constant periodic solutions, with values of λ lying between $\lambda_1^*(\mu)$ and $\lambda_2^*(\mu)$. There are no secondary bifurcation points for other values of μ.

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Bifurcation of equilibrium and periodic solutions of nonlinear evolution equations is considered in the neighbourhood of an equilibrium solution for which the corresponding linear problem admits both non-zero equilibrium and non-constant periodic solutions. These solutions of the linear problem are related to those of the nonlinear equation by deriving bifurcation equations possessing a simple symmetry property. This results in a simplification of the bifurcation analysis, illustrated by a discussion of two important special cases exhibiting secondary bifurcation of periodic solutions.

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