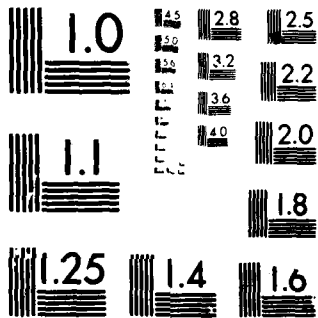END
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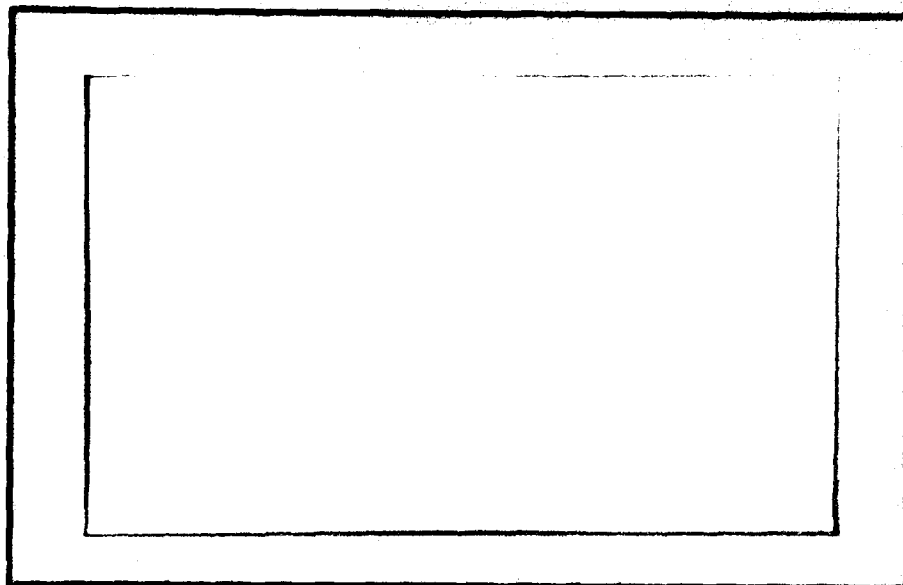
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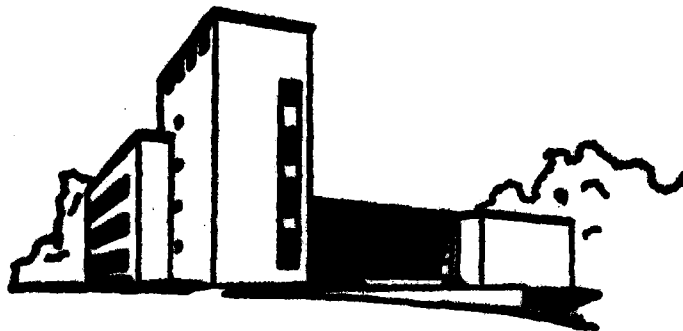
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⑥ THE IMPLICIT COMPLEMENTARITY PROBLEM:

PART II.

by

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THE IMPLICIT COMPLEMENTARITY PROBLEM: PART II

Jong-Shi Pang

ABSTRACT. In Part I of this study, we have defined the implicit complementarity problem and investigated its existence and uniqueness of solution. In the present paper, we establish a convergence theory for a certain iterative algorithm to solve the implicit complementarity problem. We also demonstrate how the algorithm includes as special cases many existing iterative methods for solving a linear complementarity problem.

Key Words.   Implicit complementarity   Successive overrelaxation

                      Iterative method   Convergence theory

1. **Introduction.** Given an n by n matrix A, n-vector b and a mapping m from R^n into itself, the implicit complementarity problem (ICP), denoted by the triple (A,b,m) is to find a vector x in R^n satisfying

$$Ax + b \geq 0, \quad x \geq m(x) \quad \text{and} \quad (Ax + b)^T(x - m(x)) = 0.$$

In a previous paper [11], we have shown how various complementarity problems can be cast as an ICP and studied the existence and uniqueness of solution to the ICP. The fundamental tool employed in our study is a certain implicitly defined mapping F. With this mapping F, we generate a sequence of vectors $\{u^k\}$ iteratively by

$$(1) \qquad u^{k+1} = F(u^k) \qquad k \geq 0$$

where u^o is a given initial vector.

In this paper, we study the convergence of the iterative scheme (1) to a solution of the ICP. Our purpose is twofold. First, we show that many iterative methods for solving the linear complementarity problem can be unified and extended under this general scheme (1). Second we establish a theory for the convergence of the scheme.

Over the past several years, there has been an increasing amount of studies on iterative methods for solving the linear complementarity problem. Several recent references are [1, 2, 3, 4, 5, 7]. In all except two [1, 2] of these previous works, the convergence proofs of the methods rely heavily on the symmetry of matrix involved. Part of the contribution of the present research is that we have provided a general framework for the study of these iterative methods for the linear complementarity problem. More importantly, our method of convergence proof is based on a theory of

contraction mappings and does not rely on matrix symmetry. (This same approach was used by Aganagic and Ahn in their studies.)

The rest of the paper is organized in two sections. In the next section, after briefly reviewing some background materials (including the definition of the mapping F), we show how various iterative methods for the linear complementarity problem can be obtained as special cases of the general scheme (1). In the last section, we develop a convergence theory for the iterative scheme (1) and discuss some of its specializations.

2. **Iterative Methods for the LCP.** Given an n by n matrix A and n-vector b, the linear complementarity problem (LCP), denoted by the pair (A,b) is a special case of the ICP(A,b,m) where m is the zero mapping.

The mapping F used to define the iterative scheme (1) is constructed in the following way. See [11]. Given a P-splitting (B, C) of the matrix A, i.e. $A = B - C$ where B is a P-matrix (i.e. has positive principal minors), for each vector $u \in R^n$, F(u) is the unique solution to the LCP

$$Bx + (b - Cu) \geq 0, \quad x \geq m(u), \quad \text{and} \quad (Bx + b - Cu)^T(x - m(u)) = 0.$$

(This latter problem is not an LCP in the ordinary sense. However, by the obvious translation of variables $y = x - m(u)$, it becomes the LCP(B, b - Cu + Bm(u))). It is easy to see that a vector u* is a fixed point of F if and only if it is a solution of the ICP(A,b,m). For more discussion on this mapping F and its role in the study of the ICP, see [11]. (The P-splitting used to define the mapping F should not be confused with the P-regular splitting of a matrix used frequently in the numerical analysis literature (see Ortega [8] e.g.). For more discussion on various matrix splittings employed in the study of the ICP, see [11]).

Many iterative methods for solving the LCP are of the successive overrelaxation (SOR) type. These methods are based on their counterparts for solving systems of linear equations. The following algorithm, due to Cryer [5] is the modified point SOR method for solving the LCP (M, q) where the matrix M is n by n.

Algorithm I. (Cryer) Let z^0 be an arbitrary nonnegative vector and $\omega*$ a scalar in the interval (0, 2). Generate the sequence $\{z^k\}$ as follows. For $i = 1,\ldots,n$, let

$$\hat{z}_i^{k+1} = - (q_i + \sum_{j < i} m_{ij} z_j^{k+1} + \sum_{j > i} m_{ij} z_j^k)/m_{ii}$$

and

$$z_i^{k+1} = \max\{0, z_i^k + \omega*(\hat{z}_i^{k+1} - z_i^k)\}.$$

In [4], Cottle, Golub and Sacher extended the above point method to a block iterative scheme. More precisely, let the matrix M be partitioned into submatrices M_{ij} $(i,j = 1,\ldots,n)$ where M_{ii} is of order n_i by n_i. The vectors z and q are partitoned accordingly. The algorithm below is the modified block SOR method for solving the LCP (M, q). The point version of the algorithm corresponds to the case where each block size n_i is equal to one.

Algorithm II. (Cottle, Golub and Sacher) Let $z^0 = (z_1^0,\ldots,z_n^0)$ be an arbitrary nonnegative vector and $\omega*$ a scalar in the interval (0, 2). Generate the sequence $\{z^k = (z_1^k,\ldots,z_n^k)\}$ as follows. For $i = 1,\ldots,n$, let $\hat{z}_i^{k+1}$ solve the LCP $(M_{ii}, z_i + \sum_{j < i} M_{ij} z_j^{k+1} + \sum_{j > i} M_{ij} z_j^k)$ and let

$$z_i^{k+1} = z_i^k + \omega_i^{k+1}(\hat{z}_i^{k+1} - z_i^k)$$

where

$$\omega_i^{k+1} = \max\{\omega : \omega \leq \omega^*, \; z_i^k + \omega(\hat{z}_i^{k+1} - z_i^k) \geq 0\}.$$

It was pointed out in [4] that if $\omega^* \in (0, 1]$, then $\omega_i^{k+1} = \omega^*$ for each i and each k; on the other hand, if $\omega^* \in (1, 2)$, then $\omega_i^{k+1} \geq 1$. Moreover, if $\omega^* \in (1, 2)$, then Algorithm I and the point version of Algorithm II are equivalent in the sense that the same sequence of iterates $\{z^k\}$ is generated.

Motivated by several earlier works, including that of Cryer, Mangasarian [7] proposes the following fairly general iterative algorithm for solving the LCP (M, q).

<u>Algorithm III.</u> (Mangasarian) Let z^0 be an arbitrary nonnegative vector, $\lambda \in (0, 1)$ and $\omega^* > 0$. Generate the sequence $\{z^k\}$ as follows. Let

$$z^{k+1} = \lambda \max\{0, z^k - \omega^* E(q + Mz^k + K(z^{k+1} - z^k))\} + (1 - \lambda)z^k$$

where E is a positive definite diagonal matrix.

In its original formulation, the matrices E and K are allowed to vary from one iteration to the next. For our purpose here, they are chosen to be fixed throughout the algorithm. In the reference, Mangasarian commented that in order for the algorithm to be pratical, the matrix K must be either strictly upper or lower triangular. As we shall show later, this restriction on the choice of K can be relaxed considerably.

Recognizing that the LCP can be formulated as a certain fixed point problem, Aganagic [1] proposed the following iterative algorithm for solving the LCP (M, q). Although for the convergence of this algorithm, the initial vector z^0 need not be chosen nonnegative; in order to compare with the above three algorithms, we restrict Aganagic's algorithm to start with a nonnegative vector.

<u>Algorithm IV</u>. (Aganagic) Let z^0 be an arbitrary nonnegative vector, $\lambda \in [0, 1]$ and $\omega^* > 0$. Generate the sequence $\{z^k\}$ as follows. Let

$$z^{k+1} = z^k - \lambda \min\{\omega^*(Mz^k + q), z^k\}.$$

It is easy to see that both Algorithms I and IV are special cases of III. In fact, with $\lambda = 1$, $E = D^{-1}$ and $K = L$ where D and L are respectively the diagonal and strictly lower triangular parts of the matrix M, Algorithm III reduces to I; whereas with $K = 0$ and $E = I$, Algorithm III becomes IV. (Rigorously speaking, the diagonal entries of the matrix M need to be positive in order for Algorithm III to include I as a special case. In Cryer's original paper [5], this requirement is always met because M is assumed to be symmetric positive definite.) In order to see how Algorithms II and III are related, we establish the next two propositions which describe an equivalent way for generating the sequence of iterates.

<u>Proposition 2.1</u>. Suppose that each diagonal block M_{ii} of the matrix M is a P-matrix. Let D, L and U be respectively the block diagonal, the block strictly lower and strictly upper triangular parts of M respectively. Then provided that either each n_i is equal to 1, or the parameter ω^* is in

(0, 1], the sequence $\{z^k\}$ generated by Algorithm II can be obtained in the following way. Given z^k nonnegative, let z^{k+1} be the unique solution to the LCP

(2) $u = \omega^* q + [\omega^* U - (1 - \omega^*)D]z^k + (D + \omega^* L)z \geq 0$

$v = - \max(0, 1 - \omega^*)z^k + z \geq 0, \qquad u^T v = 0.$

<u>Proof</u>. We shall prove the theorem only for the case where each n_i is equal to 1. The other case can be proved in a similar way. First of all, observe that if each n_i is equal to 1, then for each i, $z_i^{k+1} = \max\{0, z_i^k + \omega^*(\hat{z}_i^{k+1} - z_i^k)\}$. Moreover,

$$z_i^k + \omega^*(\hat{z}_i^{k+1} - z_i^k) = \max \{(1 - \omega^*)z_i^k,$$

$$(1 - \omega^*)z_i^k - \omega^*(q_i + \sum_{j < i} m_{ij}z_i^{k+1} + \sum_{j > i} m_{ij}z_j^k)/m_{ii}\}$$

It therefore follows that

$$z_i^{k+1} = \max \{\max(0, (1 - \omega^*)z_i^k),$$

$$(1 - \omega^*)z_i^k - \omega^*(q_i + \sum_{j < i} m_{ij}z_j^{k+1} + \sum_{j > i} m_{ij}z_j^k)/m_{ii}\}$$

or equivalently,

$$u_i = \omega^* q_i + [\omega^* \sum_{j > i} m_{ij}z_j^k - (1 - \omega^*)m_{ii}z_i^k] + (m_{ii}z_i^{k+1} + \omega^* \sum_{j < i} m_{ij}z_j^{k+1}) \geq 0$$

$$v_i = - \max[0, (1 - \omega^*)z_i^k] + z_i^{k+1} \geq 0 \qquad \text{and} \qquad u_i \cdot v_i = 0.$$

Since z^k is nonnegative, $\max[0, (1 - \omega^*)z^k] = [\max(0, 1 - \omega^*)]z^k$. Hence the vector z^{k+1} is a solution to the LCP(2). The reason that it is the unique solution is because the matrix $D + \omega^* L$ is a P-matrix by the assumptions on D and L.   Q.E.D.

Proposition 2.2. The vector z^{k+1} generated by Algorithm III is a solution to the LCP

(3) $u = \lambda\omega*q + (\lambda\omega*(M - K) - D)z^k + (D + \lambda\omega*K)z \geqq 0$

     $v = - (1 - \lambda)z^k + z \geqq 0$, $u^Tv = 0$,

where $D = E^{-1}$.

Proof. By definition,

$$z^{k+1} = \max \{(1 - \lambda)z^k, z^k - \lambda\omega*E(q + (M - K)z^k + Kz^{k+1})\}.$$

Thus, z^{k+1} is a solution to the problem

     $u = \lambda\omega*Eq + (\lambda\omega*E(M - K) - I)z^k + (I + \lambda\omega*EK)z \geqq 0$

     $v = - (1 - \lambda)z^k + z \geqq 0$, $u^Tv = 0$.

Since E is positive diagonal, this latter problem is obviously equivalent to (3).   Q.E.D.

It is important to point out that Proposition 2.2 holds regardless of what the matrix K is. Consequently, a sufficient condition for the sequence $\{z^k\}$ to be well-defined in Algorithm III is that $(D + \lambda\omega*K)$ is a P-matrix. This condition is certainly satisfied if K is strictly triangular and if D is a positive definite diagonal matrix..

From the two Propositions 2.1 and 2.2, it should be obvious that the point version of Algorithm II is a special case of Algorithm III. In fact, if the parameter $\omega*$ (in Algorithm II) is in (0, 1], then by setting in Algorithm III, $\lambda = \omega*$ (of II), $\omega*$ (of III) = 1, and K = L, we obtain Algorithm II. On the other hand, if $\omega*$ (of II) exceeds one, then by setting in Algorithm III, $\lambda = 1$ and K = L, we obtain Algorithm II as well.

For given scalars k_1 and k_2 with $k_1 > 0$ and $k_2 < 1$, the ICP $(k_1 M, k_2 q, m)$ where $m(x) = k_2 x$ for all x, is obviously equivalent to the LCP (M, q). Consider the splitting (B, C) of the matrix $k_1 M$:

(4) $B = D + k_1 K$ and $C = D + k_1 [K - M]$

where D and K are arbitrary matrices such that B is a P-matrix. It is easy to see that by appropriately choosing the scalars k_1 and k_2 and the matrices D and K, the general iterative algorithm defined by (1) for the ICP reduces to Mangasarian's algorithm III (which includes Algorithms I and IV and the point version of Algorithm II) as well as to the general block version (with parameter ω^* not exceeding one) of Algorithm II. It is not clear to the author at this stage whether Algorithm II with ω^* larger than one can also be obtained as a special case of the iterative scheme (1).

3. **A Convergence Theory**. Given an n by n real matrix B, we define its comparison matrix $\bar{B}$ by

$$\bar{B}_{ii} = |B_{ii}| \quad \text{and} \quad \bar{B}_{ij} = -|B_{ij}| \quad \text{for } i \neq j ,$$

(see Varga [13]). The matrix B is said to be an H-matrix if its comparison matrix $\bar{B}$ is a P-matrix (see Ostrowski [9]). We call the splitting (B, C) of the matrix A an H-splitting if B is an H-matrix with positive diagonals. Since any H-matrix with positive diagonal entries is a P-matrix (see Pang [10]), an H-splitting is necessarily a P-splitting. Conversely, a P-splitting (B, C) where B is block triangular with each diagonal block being an H-matrix with positive diagonals, is an H-splitting. In particular, the splitting (4) is an H-splitting if for instance, D is a positive definite diagonal matrix and K is strictly triangular.

In what follows, we fix an arbitrary H-splitting (B, C) of the matrix A and study the iterative scheme defined by (1). For any two vectors u and v, let e(u, v) and n(u, v) be the vectors whose components are the differences $|F(u)_i - F(v)_i|$ and $|m(u)_i - m(v)_i|$ respectively. The result below establishes a basic relationship between these latter two vectors. It is an improvement as well as an extension of the one derived by Ahn [2] for a certain special case.

Proposition 3.1. Let (B, C) be an H-splitting of the matrix A. Then

$$(5) \qquad e(u, v) \leqq \bar{B}^{-1} \max[\bar{D}n(u, v) , |C| |u - v|],$$

where $\bar{D}$ is the diagonal matrix whose diagonal entries are those of B, $|C|$ and $|u - v|$ are respectively, the matrix and vector whose components are the absolute values of those of C and u - v.

Proof. Consider an arbitrary index i and suppose that

$$|F(u)_i - F(v)_i| = F(u)_i - F(v)_i.$$

If $F(u)_i = m(u)_i$, then

$$|F(u)_i - F(v)_i| \leqq m(u)_i - m(v)_i \leqq | m(u)_i - m(v)_i| + \sum_{j \neq i} |B_{ij}| |F(u)_j - F(v)_j|/B_{ii}$$

On the other hand, if $[BF(u) + (b - Cu)]_i = 0$, then by letting C_i to denote the i-th row of the matrix C, we have

$$|F(u)_i - F(v)_i| \leqq \{-[(b - Cu)_i + \sum_{j \neq i} B_{ij}F(u)_j] + [(b - Cv)_i + \sum_{j \neq i} B_{ij}F(v)_j]\}/B_{ii}$$

$$= [C_i(u - v) - \sum_{j \neq i} B_{ij}(F(u) - F(v))_j]/B_{ii}$$

$$\leqq \{|C_i(u - v)| + \sum_{j \neq i} |B_{ij}||F(u)_j - F(v)_j|\}/B_{ii}.$$

Hence, it follows that in either case

$$|F(u)_i - F(v)_i| \leq \max\{|m(u)_i - m(v)_i| \,, \ |C_i(u - v)|/B_{ii}\}$$

$$+ \sum_{j \neq i} |B_{ij}| \ |F(u)_j - F(v)_j|/B_{ii} \,.$$

Consequently, we obtain

$$\bar{B} \ e(u, v) \leq \max\{\bar{D}n(u, v) \,, \ |C| \ |u - v|\} \,.$$

The desired inequality (5) now follows from the fact that $\bar{B}^{-1}$ is nonnegative (see Fiedler and Pták [6], e.g.).

With the proposition above, we state the following principal convergence result for the iterative scheme (1) to solve the ICP (A,b,m).

Theorem 3.2. Let (B, C) be an H-splitting of the matrix A. Suppose that there exists a matrix E such that

(6a) $n(u, v) \leq E|u - v|$ for all u, v

(6b) $\rho(G) < 1$ where $G = \bar{B}^{-1} \max(\bar{D}E \,, \ |C|) \,,$

with ρ denoting the spectral radius. Then for any initial vector u^0, the sequence $\{u^k\}$ generated by (1) converges to a solution of the ICP (A,b,m).

Before proving the theorem, we point out several remarks. First, the matrix E must necessarily be nonnegative. Second, the initial vector u^0 is not required to be nonnegative. Third, the theorem may be considered as giving a set of sufficient conditions for the existence of solution to the ICP.

Proof of Theorem 3.2. (Similar to Ahn [2]) First of all, observe that the matrix G is nonnegative. Conditions (5) and (6a) imply that for $k \geq 1$,

$$|u^{k+1} - u^k| \leqq G|u^k - u^{k-1}|.$$

Since $\rho(G) < 1$, it follows that

(7) $$\lim_{k \to \infty} |u^{k+1} - u^k| = 0.$$

Next, by an inductive argument, we may deduce

$$|u^{k+1} - u^0| \leqq \sum_{i = 0}^{k + 1} G^i |u^1 - u^0| \leqq (I - G)^{-1} |u^1 - u^0|$$

where the last inequality follows from the fact that the matrix G is non-negative and $\rho(G) < 1$ (see Ortega [8, p. 26]). Hence the sequence $\{u^k\}$ is bounded and thus has an accumulation point $u*$.

Let $\{u^{k_i}\}$ be a subsequence converging to $u*$. Then (7) implies that $\{u^{k_i+1}\}$ converges to $u*$ as well. From condition (6a), it follows that the mapping m is continuous. Hence, by passing the limit $k_i \to \infty$ in the conditions

$$Bu^{k_i+1} + (b - Cu^{k_i}) \geqq 0 \; , \; u^{k_i+1} \geqq m(u^{k_i}) \; , \; (u^{k_i+1} - m(u^{k_i}))^T (Bu^{k_i+1} + b - Cu^{k_i}) = 0$$

we deduce that $u*$ is a solution to the ICP (A,b,m). Thus $u*$ is a fixed point of the mapping F. By using Proposition 3.1 again, we obtain

$$|u^{k+1} - u*| \leqq G|u^k - u*| \leqq \ldots \leqq G^k |u^1 - u*|.$$

As $\rho(G) < 1$, it follows that the entire sequence $\{u^k\}$ converges to $u*$. This completes the proof of the theorem.

The argument used in the above proof is a contraction-type reasoning to establish the convergence of sequences. Such argument is used frequently in the study of iterative methods for systems of linear equations (see Ortega [8] e.g.).

Before discussing some special cases of Theorem 3.2, we give necessary
and sufficient conditions for (6a) and (6b) to hold. The next result shows
that condition (6a) is equivalent to the fact that the mapping m is
Lipschitz continuous, i.e. there exists a scalar μ such that

$$(8) \quad \|m(u) - m(v)\| \leq \mu\|u - v\| \quad \text{for all } u, v$$

where the double vertical lines denote a certain norm of R^n.

Proposition 3.3. Condition (6a) is satisfied for some matrix E if and only
if the mapping m is Lipschitz continuous.

Proof. Suppose that m is Lipschitz continuous. With no loss of generality,
we may assume that the norm in (8) is the ∞-norm : $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$.
Then for any u and v, we have

$$\max_{1 \leq i \leq n} |m(u)_i - m(v)_i| \leq \mu \sum_{j=1}^{n} |u_i - v_i|$$

so that

$$|m(u) - m(v)| \leq \mu \tilde{E}|u - v|$$

where $\tilde{E}$ is the matrix of ones. Hence condition (6a) follows. Conversely,
if condition (6a) is satisfied for some matrix E, then

$$\|m(u) - m(v)\|_\infty \leq \max_{1 \leq i \leq n} (\sum_{j=1}^{n} E_{ij})\|u - v\|_\infty .$$

Hence m is Lipschitz continuous. This proves the proposition.

The next proposition gives a necessary and sufficient condition for
(6b) to hold. Recall that a Z-matrix is a square matrix with non-positive
off-diagonal entries and a Minkowski matrix is a Z-matrix which is also

a P-matrix. See Plemmons [12] for various characterizations of a Minkowski matrix. (Minkowski matrices are also called M-matrices or K-matrices in the literature.)

Proposition 3.4. Let $\bar{B}$ and $\bar{E}$ be respectively, a Minkowski matrix and a nonnegative matrix. Then $\rho(\bar{B}^{-1}\bar{E}) < 1$ if and only if the matrix $Q = \bar{B} - \bar{E}$ is a Minkowski matrix.

Proof. This result follows from a characterization of a Minkowski matrix. See Plemmons [12] and Varga [14].

As a consequence of Proposition 3.4, the next result shows that a necessary condition for (6b) to hold is that the matrix A itself is an H-matrix with positive diagonal entires.

Corollary 3.5. Suppose that condition (6b) holds. Then A is an H-matrix with positive diagonals.

Proof. Let $\bar{A}$ be the comparison matrix of A. It is easy to deduce that

$$\bar{A} \geq \bar{B} - \max(\bar{D}E, |C|) = Q.$$

Since $\bar{A}$ is a Z-matrix, the fact that Q is a Minkowski matrix implies that A is an H-matrix (see Fiedler and Pták [6]). Since Q is Minkowski, we have, for each index i,

$$0 < Q_{ii} \leq \bar{B}_{ii} - |C_{ii}| \leq B_{ii} - C_{ii} = A_{ii}$$

because B has positive diagonal entries. This completes the proof of the Corollary.

In the rest of the paper, we derive several specializations of Theorem 3.2. The next result is concerned with the nonlinear complementarity

problem (NLCP)

(9) $y \geq 0$, $g(y) \geq 0$ and $y^T g(y) = 0$.

Corollary 3.6. If by letting $m(y) = y - g(y)$, there exists a matrix E
with $\rho(E) < 1$ such that condition (6a) is satisfied, then for an arbitrary
initial vector u^0, the sequence $\{u^k\}$ defined by

(10) $u^{k+1} = \max\{0, m(u^k)\}$ for $k \geq 0$

converges to a solution of the NLCP(9).

Proof. This follows from Theorem 3.2 by noting that the iteration (10) is
a special case of (1) under the splitting (I, 0) of the identity matrix.

    We mention in the last section that various iterative methods for the
LCP(M, q) can be obtained from the iterative scheme (1) specialized to
the ICP ($k_1 M$, $k_1 q$, m) with $m(x) = k_2 x$ under the splitting (B, C) of the
matrix $k_1 M$ given in (4). The next result is concerned with this
particular ICP.

Corollary 3.7. Let k_1 and k_2 be scalars with $k_1 > 0$ and $k_2 < 1$. Let D
and K be matrices such that the matrix B given in (4) is an H-matrix with
positive diagonals. Suppose that

(11) $\rho(\bar{B}^{-1} \max(|k_2|\bar{D} , |C|)) < 1$

where C is given in (4). Then for any initial vector u^0, the sequence $\{u^k\}$
defined by (1) specialized to the ICP ($k_1 M$, $k_2 q$, m) where $m(x) = k_2 x$,
converges to a solution of the LCP (M, q).

Proof. This corollary is a direct specialization of Theorem 3.2.

Corollary 3.6 extends a result of Ahn [2] which deals mainly with Mangasarian's Algorithm II. Ahn proved the result for strictly triangular K, diagonal D and $k_2 = 0$. The corollary also extends various results of Aganagic [1]. In fact, Aganagic required a somewhat stronger assumption than the one (11) needed in the corollary.

Specializing Corollary 3.7 to the point version of Algorithm II, we obtain the following convergence result for a point SOR method for solving a LCP.

Corollary 3.8. Let M be an H-matrix with positive diagonal entries. Then, provided that $0 < \omega* < 2/[1 + \rho(J_1)]$ where $J_1 = D^{-1}[|L| - |U|]$ and D, L and U are the diagonal, strictly lower and upper triangular parts of M respectively, the sequence $\{z^k\}$ generated by the point version of Algorithm II converges to a solution of the LCP (M, q). If M is symmetric as well, the same conclusion holds if $0 < \omega* < 2$.

Proof. According to Proposition 2.1 and Corollary 3.7, it suffices to verify that

(12) $\rho[(D - \omega*|L|)^{-1}(\omega*|U| + |1 - \omega*|D) < 1.$

Suppose that $0 < \omega* \leq 1$. Then

$$(D - \omega*|L|) - (\omega*|U| + |1 - \omega*|D) = \omega*(D - |L| - |U|) = \omega*\bar{M}$$

where $\bar{M}$ is the comparison matrix of M. Hence by assumption and Proposition 3.4, (12) follows. On the other hand, if $\omega* > 1$, then the left-hand term in (12) is equal to

$$\rho[(D - \omega*|L|)^{-1} ((1 - \omega*) D - \omega*|U|)].$$

By a characterization of H-matrices (see Varga [13]) the latter spectral radius is strictly less than 1 if $\omega^* < 2/[1 + \rho(J_1)]$ where $J_1 = D^{-1}[|L| - |U|]$. The last conclusion of the corollary is obvious.

We conclude this paper by presenting a convergence result for a block Gauss-Sidel method for solving the LCP.

Corollary 3.9. Let M be an H-matrix with positive diagonal entries. Then the sequence $\{z^k\}$ generated by Algorithm II with $\omega^* = 1$ converges to a solution of the LCP (M, q).

Proof. Let D, L and U be as specified in Proposition 2.1. By Proposition 2.1 and Corollary 3.7, it suffices to show that $\rho[(\bar{D} - |L|)^{-1}|U|] < 1$ where $\bar{D}$ is the comparison matrix of D. Since $\bar{D} - |L| - |U| = \tilde{M}$ where $\tilde{M}$ is the comparison matrix of M, the desired conclusion now follows from Proposition 3.4 and the assumption.

REFERENCES

1. Aganagic, M., "Iterative methods for linear complementarity problems,"
    Tech. Report SOL78-10, Systems Optimization Laboratory, Dept. of
    Operations Research, Stanford University (Sept. 1978).

2. Ahn, B.H., "Computation of asymmetric linear complementarity problems
    by iterative methods," manuscript of a paper (July 1979).

3. Cottle, R.W. and M. Goheen, "A special class of large quadratic
    programs," in Nonlinear Programming 3, ed. O.L. Mangasarian, R.R. Meyer
    and S.M. Robinson (Academic Press, Aug. 1978) 361-390.

4. Cottle, R.W., G. Golub and R.S. Sacher, "On the solution of large
    structured linear complementarity problems," Applied Mathematics and
    Optimization 4 (1978) 347-363.

5. Cryer, C.W., "The solution of quadratic programming problems using
    systematic overrelaxation," SIAM Journal on Control 9 (1971) 385-392.

6. Fiedler, M. and J. Pták, "On matrices with nonpositive off-diagonal
    elements and positive principal minors," Czech Journal of Mathematics
    12 (1962) 382-400.

7. Mangasarian, O.L., "Solution of symmetric linear complementarity
    problems by iterative methods," Journal of Optimization Theory and
    Applications 22 (1977) 465-485.

8. Ortega, J.M., Numerical analysis, a second cause (Academic Press 1972).

9. Ostrowski, A.M., "Über die Determinanten mit Überwirgender Hauptdiagonale,"
    Comment Math. Helv. 10 (1937) 69-96.

10. Pang, J.S., "Hidden Z-matrices with positive principal minors," Linear
    Algebra and its Applications 22 (1978) 267-281.

11. Pang, J.S., "The implicit complementarity problem: Part I," MSRR #
    G.S.I.A., Carnegie-Mellon University (April 1980).

12. Plemmons, R.J., "M-matrix characterization I: nonsingular M-matrices,"
    Linear Algebra and its Applications 18 (1977) 175-188.

13. Varga, R.S., "On recurring theorems on diagonal dominance," Linear
    Algebra and its Applications 13 (1976) 1-9.

14. Varga, R.S., Matrix iterative analysis (Prentice-Hall 1962).

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In Part I of this study, we have defined the implicit complementarity problem and investigated its existence and uniqueness of solution. In the present paper, we establish a convergence theory for a certain iterative algorithm to solve the implicit complementarity problem. We also demonstrate how the algorithm includes as special cases many existing iterative methods for solving a linear complementarity problem. established.