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Estimation of Non-Centrality Parameter of a Chi-Square Distribution

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Abstract

The non-central chi-square distribution arises in various statistical analyses. The estimation of the non-centrality parameter of the distribution is of importance in some problems. In this paper it is shown that the maximum likelihood estimator is inadmissible with respect to the squared error loss function. It is trivially minimax since all estimators have unbounded maximum risk. A class of estimators is given which are admissible and minimax for a modified loss function.

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1. Introduction. The non-central chi-square distribution arises in various statistical analyses, such as, the analysis of variance and Pearson's chi-square test for goodness of fit. A discussion of various applications of the distribution is given in Johnson & Kotz ([3], §28.9). For an example in electrical engineering, Spruill (1979) has shown that the measurement of electrical power in a circuit is related to the estimation of the non-centrality parameter of a chi-square distribution.

Let X be distributed according to the non-central chi-square distribution with p degrees of freedom and non-centrality parameter equal to λ. It is known that $X-p$ is a uniformly minimum variance unbiased estimator (UMVUE) of λ. Perlman and Rasmussen (1975) have shown that a class of estimators, given by

$$(1.1) \qquad \delta(X) = \frac{b}{X}, \qquad p \geq 5,\ 0 < b < 4(p-r)$$

has uniformly smaller mean squared error (MSE) than the UMVUE. Neff and Strawderman (1976) have extended the class to the family of estimators, given by

$$(1.2) \qquad \delta(X) = X-p+\frac{b}{X^a}, \qquad 0 < a < \frac{p}{4}$$

and

$$(1.3) \qquad \delta(X) = X-p+\frac{b}{X+c}, \qquad p \geq 5$$

where b and c are positive numbers.

It is interesting to observe that the UMVUE has smaller MSE than the estimator $X+p$ which is a Bayes estimator with respect to an improper prior distribution (see deWaal (1974)) and Perlman and Rasmussen (1975).

On the other hand, the UMVUE is itself inadmissible, as it is dominated by $(X-p)^+$, the positive part of X-p, given by

$$(1.4) \qquad (X-p)^+ = \begin{cases} 0 , & X \leq p \\ X-p , & X > p. \end{cases}$$

Here and throughout the following, admissibility is tacitly defined with respect to the squared error (SE) loss function. Let $c \geq 0$. We shall show that $(X-c)^+$ is inadmissible for $c < p$ and that there are no two values of $c \geq p$ for which one estimator dominates the other.

Comparing the MSE of the estimators given by (1.2) and (1.3) with that of $(X-p)^+$ near the origin, it is seen that none of them dominates $(X-p)^+$. It is not known whether any of these estimators is admissible. On the other hand, we shall show that all estimators have unbounded maximum risk (MSE). Therefore, all estimators are trivially minimax with respect to the squared error loss. However, if the loss function is changed to $(SE)/(\lambda+\varepsilon) = L_\varepsilon$, say, where ε is a positive number, then there exists a class of estimators with bounded maximum risk. We have derived a class of proper Bayes estimaters which are shown to be mimimax with respect to L_ε, and also admissible.

Meyer (1967) has considered the maximum likelihood estimator (MLE) for $p = 2$. We show that the MLE is inadmissible for $p \geq 1$ but mimimax with respect to L_ε. Also, we consider the derivation of the MLE in the general case when the sample size is larger than 1.

The question of admissibility of the estimator $(X-p)^+$ is of special significance. We have not been able to establish its admissibility (inadmissibility). However, we conjecture that the estimator is admissible.

2. Admissible minimax estimators. The following results pertain to the non-central gamma distribution whose density function is given by

$$(2.1) \qquad f_\lambda(x) = x^{p-1} e^{-\lambda-x} \sum_{r=0}^{\infty} \frac{(\lambda x)^r}{\Gamma(p+r)r!}$$

$$= e^{-\lambda-x} \left(\frac{x}{\lambda}\right)^{(p-1)/2} I_{p-1}(2\sqrt{\lambda x}), \quad x > 0$$

where $I_p(x)$ denotes the modified Bessel function with parameter p. Let X be distributed according to the distribution (2.1) then 2X is distributed according to the chi-square distribution with 2p degrees of freedom and non-centrality parameter 2λ.

We consider an a'priori distribution for λ which is a mixture of gamma distributions with ν degrees of freedom and scale factor c, and $\theta = (1+c)^{-1}$ being distributed according to a beta distribution, given by the density function

$$h(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\beta(\alpha,\beta)}, \quad \theta > 0.$$

Given c, the conditional density function of λ is given by

$$(2.2) \qquad g_{c,\nu}(\lambda) = \frac{c^\nu \lambda^{\nu-1} e^{-c\lambda}}{\Gamma(\nu)}, \quad \lambda > 0.$$

Let

$$\Phi(a,b;x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \cdots$$

denote the confluent hypergeometric function, and let

$$F(a,b;c,d;x) = \sum_{r=0}^{\infty} \frac{\Gamma(a+r)\Gamma(b+r)}{\Gamma(c+r)\Gamma(d+r)} \frac{x^r}{r!}$$

denote the generalized hypergeometric function. The posterior mean of λ is given by

$$(2.3) \qquad \hat{\lambda} = (\nu\int_0^1 \theta(1-\theta)^{\nu}\phi(\nu+1,p;\theta x)h(\theta)d\theta)/(\int_0^1 (1-\theta)^{\nu}\phi(\nu,p;\theta x)h(\theta)d\theta)$$

$$= F(\nu+1,\alpha+1;p,\alpha+\beta+\nu+1;X) \;/\; F(\nu,\alpha;p,\alpha+\beta+\nu;X).$$

The posterior mean given by (2.3) is a Bayes estimator of λ with respect to the squared error loss and the given prior distribution. The Bayes estimator is admissible since the prior distribution assigns positive probability measure to every open interval.

From (2.3) and the formula for the asymptotic expansion of the generalized hypergeometric function (see e.g., Wright [8]) the value of $\hat{\lambda}$ for large X is given by

$$(2.4) \qquad \hat{\lambda} = X(1-\frac{p+2\beta}{X} + O(X^{-2})) .$$

From (2.1) it is seen that the distribution of X is stochastically increasing in λ, and its moments are given by $EX=p+\lambda$, $var(x)=p+2\lambda$

$$E(X^{-r}) = \frac{\Gamma(p-r)}{\Gamma(p)} e^{-\lambda}\phi(p-r,p;\lambda)$$

$$= \lambda^{-r} + O(\lambda^{-r-1}) \quad \text{for large values of } \lambda .$$

Therefore,

$$(2.5) \qquad MSE\hat{\lambda} = p+4\beta^2+2\lambda+O(\lambda^{-1/2}).$$

Consider a gamma prior for λ , given by the density function (2.2) with c being a fixed positive number. The posterior mean with respect to this distribution is given by

$$(2.6) \qquad \hat{\delta}_{\nu} = \frac{\nu\Phi(\nu+1,p;X/(1+c))}{(1+c)\Phi(\nu,p;X/(1+c))}$$

$$= \frac{p}{1+c} + \frac{X}{(1+c)^2} \quad \text{for } \nu = p \ .$$

From (2.6) we have

$$(2.7) \qquad \text{MSE } \hat{\delta}_p = (1+c)^{-4}[p+2\lambda+(2+c)^2(p-c\lambda)^2]$$

and

$$(2.8) \qquad \int_0^{\infty} (\text{MSE}\hat{\delta}_p) g_{c,p}(\lambda)d\lambda = (1+c)^{-4}[p+\frac{2p}{c} + p(2+c)^2]$$

$$= \rho(c), \quad \text{say.}$$

Since $\hat{\delta}_p$ is a Bayes estimator and $\rho(c) \to \infty$ as $c \to 0$, it follows that all estimators have unbounded maximum risk, and are therefore trivially minimax with respect to the squared error loss. Hence, $\hat{\lambda}$ and $\hat{\delta}_{\nu}$ are minimax and admissible.

As shown above, all estimators have unbounded maximum risk with respect to the squared error loss. Therefore, we consider the loss function L_{ε} and a new prior distribution for λ, given by the density function $k(\lambda+\varepsilon)g_{c,p}(\lambda)$, where k is a normalizing factor, equal to $c/(p+c\varepsilon)$. Clearly, $\hat{\delta}_p$, given by (2.6), is Bayes with respect to L_{ε} and the new prior distribution, with respect to which the average loss is equal to $c\rho(c)/(p+c\varepsilon)$. Since

$$\frac{c\rho(c)}{\rho+c} \to 2 \quad \text{as } c \to 0$$

we conclude that any estimator whose maximum risk is bounded above by 2 is minimax with respect to L_ε. From (2.5) it is seen that the maximum risk of $\hat{\lambda}$ is bounded above by 2 for sufficiently large values of ε. Therefore, $\hat{\lambda}$ is minimax with respect to L_ε for sufficiently large values of ε. We summarize the preceding results in the following theorem.

Theorem 2.1. The estimator $\hat{\lambda}$ given by (2.3) is admissible and minimax with respect to L_ε for sufficiently large values of ε.

We consider now the class of estimators $(x-c)^+$, $c \geq 0$. Let A_ρ denote the subclass for which $c \geq \rho$. We find, as shown below, that $(x-c)^+$ is inadmissible for $c < \rho$ and that A_ρ is irreducible in the sense that there are no two values of c for which one estimator dominates the other. Let $M_c(\lambda)$ denote the mean squared error of $(X-c)^+$, and let prime denote its derivative with respect to c. We have

$$(2.9) \qquad M_c(\lambda) = \rho+2\lambda + (c-\rho)^2 + \int_0^c \left(\lambda^2-(x-c-\lambda)^2\right) f_\lambda(x)\,dx$$
$$> M_\rho(\lambda) \text{ for } c < \rho.$$

$$(2.10) \qquad \tfrac{1}{2}M_c'(\lambda) = c-\rho + \int_0^c (x-c-\lambda) f_\lambda(x)\,dx$$

$$= \int_c^\infty (c+\lambda-x)\, f_\lambda(x)\,dx$$

$$\tfrac{1}{2}M_c''(\lambda) = -\lambda f_\lambda(c) + \int_c^\infty f_\lambda(x)\,dx$$

$$(2.11) \qquad \tfrac{1}{2}M_c'''(\lambda) = \lambda^2 f_\lambda(c)\left[\frac{\lambda-1}{\lambda^2} - \left(I_{\rho-2}(2\sqrt{c\lambda})/\sqrt{c\lambda}\; I_{\rho-1}(2\sqrt{c\lambda})\right)\right].$$

The inequality in (2.9) shows that $(X-c)^+$ is inadmisible for $c < p$. From (2.10) it is seen that $M_c'(\lambda)$ is negative for $c \leq p$ and that it tends to 0 as $c \to \infty$. The quantity inside the square bracket on the right hand side of (2.11) is negative for $\lambda \leq 1$ and is increasing in c (by lemma A2 of the Appendix), tending to $(\lambda-1)/\lambda^2$ as $c \to \infty$. Therefore $M_c'(\lambda)$ is concave in c for $\lambda \leq 1$ and for $\lambda > 1$ it is first concave then convex as c varies from 0 to ∞. It follows that for each λ there is a unique value of $c = c_\lambda (>p)$, say, for which $M_c(\lambda)$ is minimized (given by $M_c'(\lambda)=0$) as c varies from 0 to ∞, where $c_\lambda = \infty$ for $\lambda \leq 1$. Moreover, for each $c > p$ there exists a value of $\lambda > 1$, equal to λ_c, say, such that $M_c'(\lambda_c) = 0$, since $M_c'(0) < 0$ and $M_c'(\infty) = 2(c-p) > 0$. Thus there exists a λ_c such that $M_c(\lambda_c) = \min_{\nu>p} M_\nu(\lambda_c)$.

Let $c' > c > p$. From the preceding results we have that the inequality $M_c(\lambda) > M_{c'}$ holds for $\lambda \leq 1$ and that $M_c(\lambda_c) < M_{c'}(\lambda_c)$. Therefore neither of the estimators $(X-c)^+$ and $(X-c')^+$ dominates the other.

Let $h_c(\lambda) = M_p(\lambda) - M_c(\lambda)$. It is easy to show that $h_c(\infty)<0$ and $h_c(0)>0$. Therefore between the estimators $(X-c)^+$ and $(X-p)^+$ neither dominates the other for any $c > p$.

From (2.9) it is seen that the maximum risk of $(X-c)^+$ with respect to L_ε is bounded above by 2 for sufficiently large values of ε. Therefore $(X-c)^+$ is minimax with respect to L_ε. We summarize the preceding results in the following theorem.

Theorem 2.2. The estimator $(X-c)^+$ is minimax for $c \geq 0$ with respect to L_ε for sufficiently large values of ε. It is inadmissible for $c < p$. The class of estimators A_p is irreducible.

3. Maximum likelihood estimator. Some properties of the Bessel function are given in the Appendix. The given result will be used below to show the inadmissibility and the mimimax property of the MLE. Equating to 0 the derivative with respect to λ of the density function given by (2.1), we get

$$(3.1) \qquad 1 = \frac{x}{\sqrt{\lambda x}} I_p(2\sqrt{\lambda x}) / I_{p-1}(2\sqrt{\lambda x}).$$

Let λ^* denote the MLE. The quantity on the right hand side of (3.1) is monotone decreasing in λ by Lemma A2 of Appendix, and tends to x/p as $\lambda \to 0$. Therefore, $\lambda^* = 0$ for $x \le p$, and for $x > p$ it is uniquely given as a solution of (3.1).

From (3.1), we have

$$\begin{aligned}(3.2) \qquad \lambda &= \sqrt{\lambda x}\, I_p(2\sqrt{\lambda x}) \,/\, I_{p-1}(2\sqrt{\lambda x}) \\ &= \sqrt{\lambda x}\, I_p(2\sqrt{\lambda x}) \,/\, \left(\frac{p}{\sqrt{\lambda x}} I_p(2\sqrt{\lambda x}) + I_{p+1}(2\sqrt{\lambda x})\right) \\ &\ge \sqrt{\lambda x}\, I_{p-1}(2\sqrt{\lambda x}) \,/\, \left(\frac{p}{\sqrt{\lambda x}} I_{p-1}(2\sqrt{\lambda x}) + I_p(2\sqrt{\lambda x})\right) \\ &= \sqrt{\lambda x} \,/\, \left(\frac{p}{\sqrt{\lambda x}} + \sqrt{\frac{\lambda}{x}}\right).\end{aligned}$$

The above inequality follows from Lemma A3. From Lemma A1 and the first equality in (3.2) it follows that $\lambda^* \le x$, and from the last equality it follows that $\lambda^* \ge x-p$. Thus we have the interesting result

$$(3.3) \qquad x - p < \lambda^* \le x \quad \text{for } x > p.$$

Let $Z = 2(\lambda^* x)^{\frac{1}{2}}$ and write λ for λ^*. Then

$$\lambda + x\frac{d\lambda}{dx} = \frac{Z}{2}\frac{dZ}{dx} \tag{3.4}$$

and from (3.1)

$$Z\, I_{p-1}(Z) = 2x\, I_p(Z)$$

or

$$\sum_{r=0}^{\infty} \frac{(Z^2/4)^r}{\Gamma(p+r)r!} = x \sum_{r=0}^{\infty} \frac{(Z^2/4)^r}{r!\Gamma(p+r+1)} \tag{3.5}$$

Differentiating both sides of (3.5) with respect to x, we get after simplification

$$\frac{Z}{2}\frac{dz}{dx} = \frac{\lambda}{\lambda - x + p} \quad . \tag{3.6}$$

Hence from (3.4) and (3.6) we get

$$x\frac{d\lambda}{dx} = \lambda\left(\frac{1}{\lambda - x + p} - 1\right) . \tag{3.7}$$

Differentiating (3.7) with respect to x we get

$$x\frac{d^2\lambda}{dx^2} = -\left(2 + \frac{x-p}{(\lambda - x + p)^2}\right)\frac{d\lambda}{dx} + \frac{\lambda}{(\lambda - x + p)^2} \quad . \tag{3.8}$$

Consider the behavior of λ as a function of x. From the second equality in (3.2) we have

$$\frac{\sqrt{\lambda x}\; I_{p+1}(2\sqrt{\lambda x})}{I_p(2\sqrt{\lambda x})} = (x-p).$$

The above relation shows that $\lambda \to 0$ and $\frac{\lambda}{x-p} \to \frac{p+1}{p}$ as $x \to p + 0$. Then from (3.7) we have that $\frac{d\lambda}{dx} \to \frac{p+1}{p}$, as $x \to p+0$. Let x_0 denote the smallest value of $x > p$ for which $\frac{d\lambda}{dx} = 0$. Then $\frac{d^2\lambda}{dx^2} \leq 0$ for $x = x_0$. On the other hand, from (3.8) it is seen that $\frac{d^2\lambda}{dx^2} > 0$ at $x = x_0$. Therefore, $\frac{d\lambda}{dx} > 0$ for all $x > p$. Hence, from (3.7) we have that $\lambda^* < x-p+1$.

Let x^0 denote the smallest value of $x > p$ for which $\frac{d\lambda}{dx} = 1$. From (3.9) the value of $x = x^0$ is given by

$$(3.9) \qquad x = \lambda\left(\frac{1}{\lambda-x+p} - 1\right)$$

or

$$(3.10) \qquad 2\lambda = 1-p + ((2x-p)^2 - 2p+1)^{\frac{1}{2}} .$$

Therefore

$$(3.11) \qquad 1-2(\lambda-x+p) = 2x-p - ((2x-p)^2 - 2p+1)^{\frac{1}{2}}$$

$$> 0 \quad \text{for } p > \tfrac{1}{2}.$$

Putting $\frac{d\lambda}{dx} = 1$ in (3.8) and using (3.8) we get, after simplification

$$(3.12) \qquad x(\lambda-x+p)^2 \frac{d^2\lambda}{dx^2} = (\lambda-x+p)(1-2\lambda+2x-2p)$$

$$> 0 \text{ by (3.3) and (3.11)} .$$

Since $\frac{d\lambda}{dx} \to \frac{p+1}{p} > 1$ as $x \to p+0$, it follows that $\frac{d^2\lambda}{dx^2} \leq 0$ at $x = x^0$, contrary to (3.12). Therefore $\frac{d\lambda}{dx} > 1$ for all $x > p$. Hence, $\lambda^* - (x-p)^+$ is increasing in x. From this result it follows that λ^* is inadmissible, as shown below.

Let $p > \frac{1}{2}$. We have

$$(3.13) \qquad \text{MSE } \lambda^* - \text{MSE } (x-p)^+ = E(\lambda^*-(x-p)^+)(\lambda^*+(x-p)^+-2\lambda)$$

$$\geq E(\lambda^*-(x-p)^+)\, E(\lambda^*+(x-p)^+-2\lambda)$$

$$> 0 .$$

The first inequality in (3.12) follows from the fact that each of the quantities $\lambda^* - (x-p)^+$ and $\lambda^* + (x-p)^+ - 2\lambda$ is increasing in x. Te second inequality follows from (3.3) and from

$$E(\lambda^*+(x-p)^+-2\lambda) > 2E(x-p-\lambda) = 0.$$

Thus, the MLE is inadmissible, being dominated by the estimator $(x-p)^+$.

From the asymptotic expansion of the Bessel function we can show that for large x

$$\lambda^* = x-p + \tfrac{1}{2} + O(\tfrac{1}{x}).$$

Hence, for large values of λ

$$\text{MSE } \lambda^* = 2\lambda + O(1).$$

Therefore, λ^* is minimax with respect to L_ε for sufficiently large values of ε. We summarize the preceding results in the following Theorem.

Theorem 3.1. The maximum likelihood estimator λ^* is uniquely determined by (3.1) for $x > p$ and is equal to 0 for $x \leq p$. Moreover, $X-p < \lambda^* < X-p+1$. The MLE is inadmissible, being dominated by $(X-p)^+$, and is minimax with respect to L_ε for sufficiently large values of ε.

Theorem 3.1 pertains to the MLE for a sample of size n=1. We consider now the case $n > 1$. Let $x_1,\ldots,x_n$ be the sample values, and let λ_n^* denote the MLE. The likelihood equation for the MLE is given by

$$(3.14) \qquad n = \sum_{i=1}^{n} \frac{x_i}{\sqrt{\lambda x_i}} \left(I_p(2\sqrt{\lambda x_i}) \;/\; I_{p-1}(2\sqrt{\lambda x_i}) \right) .$$

Each term of the summation on the right hand side of (3.14) is monotone decreasing in λ by Lemma A 2 and the maximum value of the sum corresponding to $\lambda \to 0$ is equal to $n\bar{x}/p$, where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ denotes the sample mean. Therefore, $\lambda_n^* = 0$ for $\bar{x} \leq p$ and for $\bar{x} > p$ the value of λ_n^* is uniquely given as a solution of (3.14).

From (3.14) and Lemma A 1 we have

$$n \leq \sum_{i=1}^{n} \sqrt{x_i/\lambda}$$

Therefore

$$(3.15) \qquad \lambda_n^* \leq \left(\frac{1}{n} \sum_{i=1}^{n} \sqrt{x_i} \right)^2 .$$

For a lower bound on the value of λ_n^*, we have from (3.14)

$$n\sqrt{\lambda} = \sum_{i=1}^{n} \sqrt{x_i}\; I_p\left(2\sqrt{\lambda x_i}\right) / I_{p-1}\left(2\sqrt{\lambda x_i}\right)$$

$$= \sum_{i=1}^{n} \sqrt{x_i} \left(\frac{p}{\sqrt{\lambda x_i}} + \frac{I_{p+1}(2\sqrt{\lambda x_i})}{I_p(2\sqrt{\lambda x_i})} \right)^{-1}$$

$$\geq \left(\sum_{i=1}^{n} \sqrt{x_i} \right) \left(\frac{np}{\sqrt{\lambda}} + \sum_{i=1}^{n} \sqrt{x_i}\; I_{p+1}(2\sqrt{\lambda x_i}) / I_p(2\sqrt{\lambda x_i}) \right)^{-1} .$$

by Jensen's inequality

$$\geq \left(\sum_{i=1}^{n} \sqrt{x_i} \right)^2 \left(\frac{np}{\sqrt{\lambda}} + \sum_{i=1}^{n} \sqrt{x_i}\; I_p(2\sqrt{\lambda x_i}) / I_{p-1}(2\sqrt{\lambda x_i}) \right)^{-1}$$

by Lemma A 3

$$= \left(\sum_{i=1}^{n} \sqrt{x_i}\right)^2 \left(\frac{np}{\sqrt{\lambda}} + n\sqrt{\lambda}\right)^{-1}.$$

Therefore,

$$\lambda_n^* \geq \left(\frac{1}{n}\sum_{i=1}^{n} \sqrt{x_i}\right)^2 - p. \tag{3.16}$$

The inequalities (3.15) giving the upper and lower bounds on the values of λ_n^* are useful for the computation of the MLE.

Appendix

Let $Q_p(x) = I_p(x)/I_{p-1}(x)$ denote the ratio of two Bessel functions.

Lemma A 1. $Q_p(x) \leq 1$ for $p \geq 1$ and $x > 0$

Proof: From the series expansion of the modified Bessel function, the inequality $Q_p(x) \leq 1$ reduces to

$$(1) \qquad \sum_{r=0}^{\infty} \frac{(x^2/4)^r}{r!\Gamma(p+r)} \geq (\frac{x}{2}) \sum_{r=0}^{\infty} \frac{(x^2/4)^r}{r!\Gamma(p+r+1)}.$$

Denoting the left hand side of (1) by $\sum_{r=1}^{\infty} a_r$, where a_r denotes the (r+1)th term of the series, we have

$$\sum_{r=0}^{\infty} a_r = \frac{1}{2} a_0 + \frac{1}{2} (a_0+a_1) + \frac{1}{2} (a_1+a_2) + \ldots$$

$$> \frac{1}{2} \sum_{r=0}^{\infty} (a_r+a_{r+1})$$

$$\geq \sum_{r=0}^{\infty} (a_r a_{r+1})^{\frac{1}{2}}$$

$$= \sum_{r=0}^{\infty} \left[\frac{(x^2/4)^{2r+1}}{r!(r+1)!\Gamma(p+r)\Gamma(p+r+1)}\right]^{\frac{1}{2}}$$

$$\geq \frac{x}{2} \sum_{r=0}^{\infty} \frac{(x^2/4)^r}{r!\Gamma(p+r+1)} \qquad \text{for } p \geq 1.$$

Lemma A 2. $xQ_p(x)$ is increasing in x and $Q_p(x)/x$ is decreasing in x.

Proof: We have

$$xQ_p(x) = \left[(\frac{x^2}{2}) \sum_{r=0}^{\infty} \frac{(x^2/4)^r}{r!\Gamma(p+r+1)}\right] / \left[\sum_{r=0}^{\infty} \frac{(x^2/4)^r}{r!\Gamma(p+r)}\right]$$

$$= 2E(J)$$

where J denotes a discrete rancom variable whose distribution is given by

$$P\{J=r\} = \frac{(x^2/4)^r}{r!\Gamma(p+r)}\left(\sum_{m=0}^{\infty} \frac{(x^2/4)^m}{m!\Gamma(p+r)}\right)^{-1} , \qquad r=0,1,\ldots,$$

Since the distribution of J has monotone likelihood ratio property in x, it follows that E(J)is increasing in x. That $Q_p(x)/x$ is decreasing in x follows from the preceding relation and the recurrence relation

$$I_{p-1}(x) - I_{p+1}(x) = \frac{2p}{x} I_p(x) .$$

Lemma A 3. $Q_p(x) > Q_{p+1}(x)$ for all p and $x > 0$.

Proof: We have

$$(2) \qquad [(x^{-p}I_p(x)/(x^{-p+1}I_{p-1}(x)) = Q_p(x)/x.$$

The derivative with respect to x of the quantity on the left hand side of (2) is negative, since $Q_p(x)/x$ is decreasing in x by Lemma A 2 Therefore

$$x^{-p}I_p(x)\,\frac{d}{dx}(x^{-p+1}I_{p-1}(x)) > x^{-p+1}I_{p-1}(x)\,\frac{d}{dx}(x^{-p}I_p(x)).$$

or

$$I_p(x)I_p(x) > I_{p-1}(x)I_{p+1}(x)$$

by Formula 9.6.28 of Abramowitz and Stegun [1]. Hence

$Q_p(x) > Q_{p+1}(x)$.

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