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TRANSIENT ACOUSTIC WAVE PROPAGATION IN AN EPSTEIN DUCT.(U)
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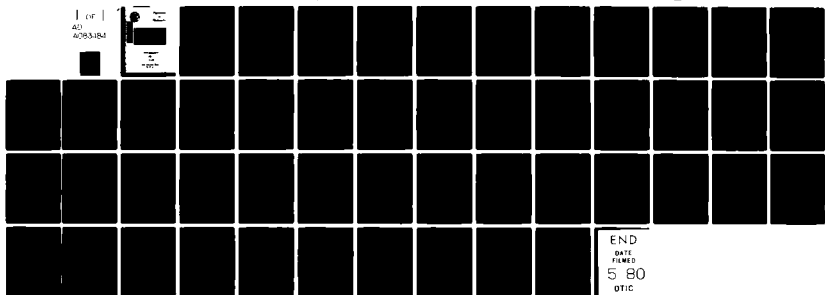
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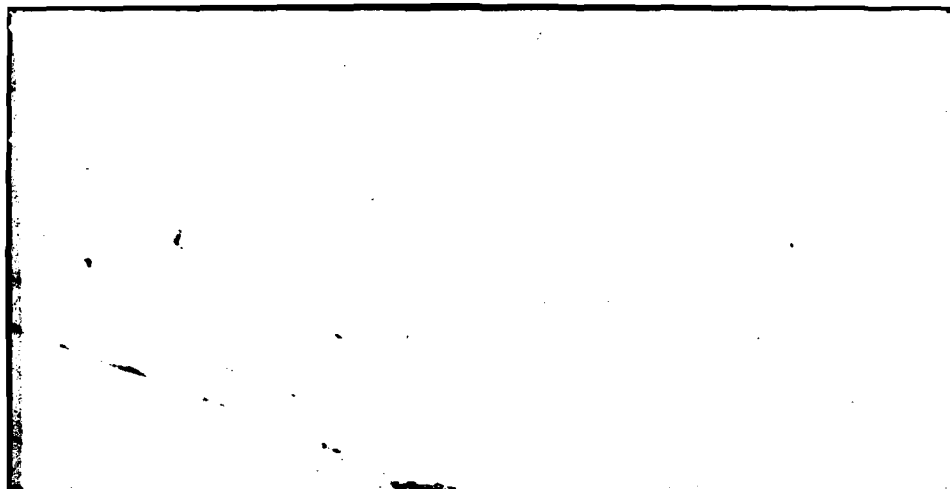
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TRANSIENT ACOUSTIC WAVE PROPAGATION
IN AN EPSTEIN DUCT

C. H. Wilcox

Technical Summary Report #36

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Abstract.

Transient acoustic wave propagation is analyzed for the case of an unlimited plane-stratified fluid having constant density and sound speed $c(y)$ at depth y given by the Epstein profile

$$c^{-2}(y) = K \operatorname{sech}^2 (y/H) + L \tanh (y/H) + M$$

The acoustic potential is a solution of the wave equation

$$D_t^2 u - c^2(y) (D_x^2 u + D_z^2 u + D_y^2 u) = f(t, x, y)$$

where $x = (x_1, x_2)$ are horizontal coordinates and $f(t, x, y)$ characterizes the wave sources. The principal results of the analysis show that u is the sum of a free component, which behaves like a diverging spherical wave for large t , and a guided component which is approximately localized in a region $|y - y_0| \leq h$ and propagates outward in horizontal planes like a diverging cylindrical wave.

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§1. Introduction.

This paper presents an analysis of transient acoustic wave propagation in a stationary unlimited plane stratified fluid with constant density and sound speed $c(y)$ at depth y defined by the Epstein profile

$$c(y) = [K \operatorname{sech}^2 (y/H) + L \tanh (y/H) + M]^{-1/2} \quad (1.1)$$

where H , K , L and M are constants such that $c(y) > 0$. The acoustic field is characterized by a real-valued potential $u(t, x, y)$ that satisfies the wave equation [8]

$$D_t^2 u - c^2(y)(D_1^2 u + D_2^2 u + D_y^2 u) = f(t, x, y) \quad (1.2)$$

where t is a time coordinate, $x = (x_1, x_2)$ are Cartesian coordinates in a horizontal plane, $f(t, x, y)$ is a function that characterizes the wave sources and $D_j = \partial/\partial x_j$, $D_t = \partial/\partial t$, $D_y = \partial/\partial y$.

The sound speed profiles (1.1) were introduced by P. S. Epstein [2] who discovered that (1.2) with this choice of $c(y)$ can be integrated by means of hypergeometric functions. This fact is of interest in theoretical acoustics because the Epstein profile provides an example of the physically interesting phenomenon of an acoustic duct when $c(y)$ has a minimum. This case is characterized by the parameter values [3,4]

$$H > 0, M > L \geq 0, K > L/2 \quad (1.3)$$

For these values the limits $c(\pm\infty) = (M \pm L)^{-1/2}$ are finite and $c(y)$ has a unique minimum $c(y_0) < c(\infty) \leq c(-\infty)$ at $y_0 = H \tanh^{-1}(L/2K)$.

The integration of equation (1.2) below is based on the spectral theory of the Epstein operator A in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3, c^{-2}(y)dx dy)$ as developed in [3,4] and the notation and results of that paper are used. Equation (1.2) is interpreted as the equation

$$D_t^2 u + Au = f(t, \cdot), \quad t \in \mathbb{R}, \quad (1.4)$$

for a function $t \rightarrow u(t, \cdot) \in \mathcal{H}$. The wave sources are assumed to act during the time interval $[0, T]$, so that $\text{supp } f \subset [0, T]$. The corresponding acoustic wave is the solution of (1.4) that satisfies the initial condition

$$u(t, \cdot) = 0 \text{ for all } t < 0 \quad (1.5)$$

The solution is given by Duhamel's integral

$$u(t, \cdot) = \int_0^t \{A^{-1/2} \sin(t - \tau) A^{1/2}\} f(\tau, \cdot) d\tau, \quad t \geq 0 \quad (1.6)$$

Indeed, if $f \in C([0, T], \mathcal{H})$ then (1.6) is the unique "solution with finite energy" of [9], while if $f \in C([0, T], D(A^{1/2}))$ then (1.6) is the "strict solution with finite energy." In addition, if $f \in C([0, T], D(A^{-1/2}))$ then

$$u(t, x, y) = \text{Re} \{v(t, x, y)\} \quad (1.7)$$

where $v(t, \cdot)$ is the complex-valued potential defined by

$$v(t, \cdot) = i \exp \{-it A^{1/2}\} A^{-1/2} \int_0^t \exp \{i\tau A^{1/2}\} f(\tau, \cdot) d\tau$$

In particular,

$$v(t, \cdot) = \exp \{-it A^{1/2}\} h, \quad t \geq T \quad (1.8)$$

where

$$h = i A^{-1/2} \int_0^T \exp \{it A^{1/2}\} f(\tau, \cdot) d\tau \quad (1.9)$$

The initial-value problem

$$D_t^2 u + Au = 0 \text{ for all } t > 0 \quad (1.10)$$

$$u(0) = f, \quad D_t u(0) = g \quad (1.11)$$

can be treated by the same formalism. Indeed, if $f \in D(A^{1/2})$ and $g \in D(A^{-1/2})$ then the solution of (1.10), (1.11) is given by (1.7), (1.8) where $h = f + i A^{-1/2} g \in D(A^{1/2})$ (cf. [11, Ch.3]).

The integral

$$E(u, K, t) = \int_K \{ (D_1 u)^2 + (D_2 u)^2 + (D_y u)^2 + c^{-2}(y) (D_t u)^2 \} dx dy \quad (1.12)$$

may be interpreted as the energy of the acoustic field u in the set $K \subset \mathbb{R}^3$ at time t . Moreover, A is the selfadjoint operator associated with the Dirichlet integral

$$A(u, v) = \sum_{j=1}^2 (D_j u, D_j v)_{L_2(\mathbb{R}^3)} + (D_y u, D_y v)_{L_2(\mathbb{R}^3)}$$

in the sense of T. Kato's theory of sesquilinear forms [5]. Indeed, if the domain of A is $D(A) = L_2^1(\mathbb{R}^3)$, the first Sobolev space, then A is precisely the associated operator of Kato's theory. It follows from Kato's second representation theorem [5, p. 331] that

$$D(A^{1/2}) = L_2^1(\mathbb{R}^3) \quad (1.13)$$

and

$$\|A^{1/2} u\|_{\mathcal{H}}^2 = A(u, u) = \sum_{j=1}^2 \|D_j u\|_{L_2(R^3)}^2 + \|D_y u\|_{L_2(R^3)}^2 \quad (1.14)$$

Hence the total energy satisfies

$$E(u, R^3, t) = \|A^{1/2} u\|_{\mathcal{H}}^2 + \|D_t u\|_{\mathcal{H}}^2 \quad (1.15)$$

If $h \in D(A^{1/2})$ and u is defined by (1.7), (1.8) then a simple calculation shows that

$$E(u, R^3, t) = \|A^{1/2} h\|_{\mathcal{H}}^2 \text{ for all } t \geq T \quad (1.16)$$

The analysis of the structure of the acoustic potential (1.7), (1.8) presented below is based on the eigenfunction expansion of [4]. The orthogonal projections in \mathcal{H}

$$P_f = P_+ + P_- + P_0 \quad (1.17)$$

and

$$P_g = \sum_{k=1}^{\infty} P_k \quad (1.18)$$

defined by the eigenfunction expansion provide a decomposition

$$u(t, \cdot) = u_f(t, \cdot) + u_g(t, \cdot)$$

into orthogonal partial waves

$$u_f(t, \cdot) = P_f u(t, \cdot), \quad u_g(t, \cdot) = P_g u(t, \cdot)$$

The first, called the free component, will be shown to behave for large times like a diverging spherical wave in a homogeneous fluid. The

second, called the guided component, will be shown to be approximately localized near the plane $y = y_0$ and to propagate outward in horizontal planes like a diverging cylindrical wave. This second component shows the profound effect of an acoustic duct on transient acoustic waves. It is absent when $c(y)$ has no minimum.

Transient wave propagation in the analogous but simpler case of the Pekeris duct ($c(y)$ piece-wise constant) was analyzed by Wilcox in [12] and [13]. This paper is a sequel to the report [10] of Wilcox and the article [4] of Guillot and Wilcox. In [10] the special case of a symmetric Epstein profile ($L = 0$) was analyzed without detailed proofs. [4] presented a complete spectral analysis of the Epstein operator. Here the results of [4] are used to treat the general Epstein duct and to supply the proofs that were omitted in [10]. Some of the results of this paper were announced in [1].

The remainder of this paper is organized as follows. In §2 the eigenfunction expansion of [4] is reformulated to provide a convenient starting point for the analysis of the free component u_f . The behavior of $u_f(t, \cdot)$ for $t \rightarrow \infty$ is calculated in §3. The justification of these calculations is technically more difficult than for the Pekeris profile treated in [13] and proofs of the results are presented here for the first time. §4 presents a calculation of the asymptotic behavior for $t \rightarrow \infty$ of the guided component $u_g(t, \cdot)$. §5 presents applications of the results of §§3 and 4 to calculating asymptotic distributions of energy for large times. The proofs of the results in §§4 and 5 are essentially the same as those for the Pekeris profile, given in [13], and are therefore omitted. A version of the method of stationary phase for oscillatory integrals containing parameters is given in an Appendix.

§2. Eigenfunction Expansions for the Epstein Operator.

The generalized eigenfunctions $\psi_{\pm}(x, y, p, \lambda)$ and $\psi_0(x, y, p, \lambda)$ were defined in [4] as multiples of certain hypergeometric functions. The normalizing constants $a_{\pm}(p, \lambda)$ and $a_0(p, \lambda)$ are determined only up to factors of modulus 1. In [4] these factors were chosen to make $a_{\pm}(p, \lambda)$ and $a_0(p, \lambda)$ real and positive [4, pp. 92-93]. Here it will be convenient to renormalize to make the constants $c_{\pm}(p, \lambda)$ and $c_0(p, \lambda)$ of the asymptotic forms [4, (1.7), (1.8), (1.11)] real and positive. Calculation of $c_{\pm}(p, \lambda)$ and $c_0(p, \lambda)$ when $a_{\pm}(p, \lambda)$ and $a_0(p, \lambda)$ are defined as in [4] shows that the quantities $(4\pi q_{\pm})^{1/2} c_{\pm}(p, \lambda) = \exp \{i \Psi_{\pm}(p, \lambda)\}$ and $(4\pi q_{+})^{1/2} c_0(p, \lambda) = \exp \{i \Psi_0(p, \lambda)\}$ have modulus 1. Hence, if ψ_{\pm} and ψ_0 are renormalized by replacing $a_{\pm}(p, \lambda)$ and $a_0(p, \lambda)$ by $a_{\pm}(p, \lambda) \exp \{-i \Psi_{\pm}(p, \lambda)\}$ and $a_0(p, \lambda) \exp \{-i \Psi_0(p, \lambda)\}$ then

$$(4\pi q_{\pm})^{1/2} c_{\pm}(p, \lambda) = 1, \quad (4\pi q_{+})^{1/2} c_0(p, \lambda) = 1 \quad (2.1)$$

Explicit expressions for $\exp \{i \Psi_{\pm}(p, \lambda)\}$ and $\exp \{i \Psi_0(p, \lambda)\}$ as quotients of products of Γ -functions are easily obtained but will not be needed here. The normalization (2.1) is employed in the remainder of this paper.

The generalized eigenfunctions ψ_{\pm} and ψ_0 , renormalized as above, satisfy

$$\psi_{\pm}(y, p, \lambda) = (4\pi q_{\pm})^{-1/2} T_{\pm}(y, p, \lambda) \exp \{\mp i q_{\pm} y\} \quad (2.2)$$

$$\psi_0(y, p, \lambda) = (4\pi q_{+})^{-1/2} T_0(y, p, \lambda) \exp \{\tilde{\beta} y/H\} \quad (2.3)$$

where

$$T_+(y, p, \lambda) = (4\pi q_+)^{1/2} a_+(p, \lambda) \exp \{-i\Psi_+(p, \lambda) + iq_- y\} \phi_3(y, p, \lambda)$$

$$T_-(y, p, \lambda) = (4\pi q_-)^{1/2} a_-(p, \lambda) \exp \{-i\Psi_-(p, \lambda) - iq_+ y\} \phi_4(y, p, \lambda)$$

$$T_0(y, p, \lambda) = (4\pi q_+)^{1/2} a_0(p, \lambda) \exp \{-i\Psi_0(p, \lambda) - \tilde{\beta}y/H\} \phi_1(y, p, \lambda)$$

and

$$\lim_{y \rightarrow \mp\infty} T_{\pm}(y, p, \lambda) = T_{\pm}(p, \lambda) \quad (2.4)$$

$$\lim_{y \rightarrow -\infty} T_0(y, p, \lambda) = T_0(p, \lambda) \quad (2.5)$$

Similarly, well-known identities for the hypergeometric functions imply [3]

$$\psi_{\pm}(y, p, \lambda) = (4\pi q_{\pm})^{-1/2} \{I_{\pm}(y, p, \lambda) \exp(\mp iq_{\pm} y) + R_{\pm}(y, p, \lambda) \exp(\pm iq_{\pm} y)\} \quad (2.6)$$

$$\psi_0(y, p, \lambda) = (4\pi q_+)^{-1/2} \{I_0(y, p, \lambda) \exp(-iq_+ y) + R_0(y, p, \lambda) \exp(iq_+ y)\} \quad (2.7)$$

where

$$I_+(y, p, \lambda) = (2/H)^{1/2} \exp(iq_+ y) \phi_2(y, p, \lambda)$$

$$I_-(y, p, \lambda) = (2/H)^{1/2} \exp(-iq_- y) \phi_1(y, p, \lambda)$$

$$I_0(y, p, \lambda) = (2/H)^{1/2} \exp(iq_+ y) \phi_2(y, p, \lambda)$$

$$R_+(y, p, \lambda) = (2/H)^{1/2} \exp(-iq_+ y) R_+(p, \lambda) \phi_4(y, p, \lambda)$$

$$R_-(y, p, \lambda) = (2/H)^{1/2} \exp(iq_- y) R_-(p, \lambda) \phi_3(y, p, \lambda)$$

$$R_0(y, p, \lambda) = (2/H)^{1/2} \exp(-iq_+ y) R_0(p, \lambda) \phi_4(y, p, \lambda)$$

and

$$\lim_{y \rightarrow \pm\infty} I_{\pm}(y, p, \lambda) = 1 \quad (2.8)$$

$$\lim_{y \rightarrow \pm\infty} I_0(y, p, \lambda) = 1 \quad (2.9)$$

$$\lim_{y \rightarrow \pm\infty} R_{\pm}(y, p, \lambda) = R_{\pm}(p, \lambda) \quad (2.10)$$

$$\lim_{y \rightarrow \pm\infty} R_0(y, p, \lambda) = R_0(p, \lambda) \quad (2.11)$$

The easily calculated expressions for $R_{\pm}(p, \lambda)$, $R_0(p, \lambda)$, $T_{\pm}(p, \lambda)$ and $T_0(p, \lambda)$ as quotients of products of Γ -functions were given in [3] and will not be needed here. They imply that

$$q_{\pm} |R_{\pm}|^2 + q_{\mp} |T_{\pm}|^2 = q_{\pm}, \quad |R_0| = 1 \quad (2.12)$$

The construction of $u_f(t, \cdot) = P_f u(t, \cdot)$ will be simplified by representing $P_f = P_+ + P_- + P_0$ by means of a single family of generalized eigenfunctions, rather than the three families ψ_+ , ψ_- and ψ_0 . This may be motivated by noting that ψ_+ , ψ_- and ψ_0 , collectively, represent the response of the Epstein fluid to the incident plane waves $\exp \{i(p \cdot x - q y)\}$, $(p, q) \in \mathbb{R}^3$. To see this consider the mappings

$$(p, q) = X_+(p, \lambda) = (p, q_+(p, \lambda)), \quad (p, \lambda) \in \Omega$$

$$(p, q) = X_0(p, \lambda) = (p, q_+(p, \lambda)), \quad (p, \lambda) \in \Omega_0$$

$$(p, q) = X_-(p, \lambda) = (p, -q_-(p, \lambda)), \quad (p, \lambda) \in \Omega$$

X_+ is an analytic transformation of Ω onto the cone

$$C_+ = \{(p, q): q > a|p|\}$$

where

$$a = ((c(-\infty)/c(\infty))^2 - 1)^{1/2} \geq 0$$

Similarly, X_0 is an analytic transformation of Ω_0 onto the cone

$$C_0 = \{(p, q): 0 < q < a|p|\}$$

and X_- is an analytic transformation of Ω onto the cone

$$C_- = \{(p, q): q < 0\}$$

Thus, the asymptotic forms of ψ_{\pm} and ψ_0 for $y \rightarrow \pm\infty$ [4, (1.7), (1.8), (1.11)] show that $\psi_+(x, y, p, \lambda)$ with $(p, \lambda) \in \Omega$ is the response of the medium to a plane wave $\exp \{i(p \cdot x - q y)\}$ with $(p, q) \in C_+$, $\psi_0(x, y, p, \lambda)$ is the response to a plane wave with $(p, q) \in C_0$ and $\psi_-(x, y, p, \lambda)$ is the response to a plane wave with $(p, q) \in C_-$. Note that

$$R^3 = C_+ \cup C_0 \cup C_- \cup N$$

where N is a Lebesgue null set.

The interpretation of ψ_+ , ψ_- and ψ_0 given above suggests defining the composite eigenfunction

$$\phi_+(x, y, p, q) = (2\pi)^{-1} \exp(ip \cdot x) \phi_+(y, p, q), \quad (p, q) \in C_+ \cup C_0 \cup C_- \quad (2.13)$$

by

$$\phi_+(y, p, q) = \begin{cases} (2q)^{1/2} c(\infty) \psi_+(y, p, \lambda), & (p, \lambda) = X_+^{-1}(p, q), \quad (p, q) \in C_+ \\ (2q)^{1/2} c(\infty) \psi_0(y, p, \lambda), & (p, \lambda) = X_0^{-1}(p, q), \quad (p, q) \in C_0 \\ (2|q|)^{1/2} c(-\infty) \psi_-(y, p, \lambda), & (p, \lambda) = X_-^{-1}(p, q), \quad (p, q) \in C_- \end{cases} \quad (2.14)$$

The terms $(2q)^{1/2} c(\infty)$ and $(2|q|)^{1/2} c(-\infty)$ are normalizing factors. Note that $2q c^2(\infty)$ is the Jacobian of X_+^{-1} and X_0^{-1} and $2q c^2(-\infty)$ is the Jacobian of X_-^{-1} .

The eigenfunction expansion of [4] will be reformulated in terms of ϕ_+ . To this end let $h \in \mathcal{H}$ and note that $h_f = P_f h = P_+ h + P_- h + P_0 h$ has the representation

$$\begin{aligned} h_f(x, y) = & \int_{\Omega} \psi_+(x, y, p, \lambda) \tilde{h}_+(p, \lambda) dp d\lambda \\ & + \int_{\Omega_0} \psi_0(x, y, p, \lambda) \tilde{h}_0(p, \lambda) dp d\lambda \\ & + \int_{\Omega} \psi_-(x, y, p, \lambda) \tilde{h}_-(p, \lambda) dp d\lambda \end{aligned}$$

where the integrals converge in \mathcal{H} . The \mathcal{H} -lim notation will be suppressed for brevity. Changing the variables in the three integrals by means of X_+ , X_0 and X_- , respectively, gives

$$\begin{aligned} h_f(x, y) = & \int_{C_+} \phi_+(x, y, p, q) \tilde{h}_+(p, \lambda) c(\infty) (2q)^{1/2} dp dq \\ & + \int_{C_0} \phi_+(x, y, p, q) \tilde{h}_0(p, \lambda) c(\infty) (2q)^{1/2} dp dq \\ & + \int_{C_-} \phi_+(x, y, p, q) \tilde{h}_-(p, \lambda) c(-\infty) (2|q|)^{1/2} dp dq \\ = & \int_{R^3} \phi_+(x, y, p, q) \hat{h}_+(p, q) dp dq \end{aligned}$$

where

$$\lambda = \lambda(p, q) = \begin{cases} c^2(\infty)(|p|^2 + q^2), & q > 0 \\ c^2(-\infty)(|p|^2 + q^2), & q < 0 \end{cases} \quad (2.15)$$

and

$$\hat{h}_+(p, q) = \begin{cases} (2q)^{1/2} c(\infty) \tilde{h}_+(p, \lambda(p, q)), & (p, q) \in C_+ \\ (2q)^{1/2} c(\infty) \tilde{h}_0(p, \lambda(p, q)), & (p, q) \in C_0 \\ (2|q|)^{1/2} c(-\infty) \tilde{h}_-(p, \lambda(p, q)), & (p, q) \in C_- \end{cases} \quad (2.16)$$

It is easy to verify by considering the three cones C_+ , C_0 and C_- separately that

$$\hat{h}_+(p, q) = \int_{\mathbb{R}^3} \phi_+(x, y, p, q)^* h(x, y) c^{-2}(y) dx dy$$

where the integral converges in $L_2(\mathbb{R}^3)$. Moreover, it can be shown by direct calculation, using the Parseval formula of [4], that

$$\|h_f\|_{\mathcal{H}} = \|\hat{h}_+\|_{L_2(\mathbb{R}^3)}$$

These considerations and the eigenfunction expansion theory of [4, §5] imply

Theorem 2.1. For all $h \in \mathcal{H}$ the strong limit

$$\hat{h}_+(p, q) = L_2(\mathbb{R}^3)\text{-}\lim_{M \rightarrow \infty} \int_{|x|^2 + y^2 \leq M^2} \phi_+(x, y, p, q)^* h(x, y) c^{-2}(y) dx dy \quad (2.17)$$

exists. Moreover, the mapping $\Omega_+ : \mathcal{H} \rightarrow L_2(\mathbb{R}^3)$ defined by $\Omega_+ h = \hat{h}_+$ is a partial isometry such that

$$\Omega_+ \Omega_+^* = 1 \text{ and } \Omega_+^* \Omega_+ = P_f$$

The adjoint mapping $h_f = \Omega_+^* \hat{h}_+$ is given by

$$h_f(x, y) = \mathcal{H}\text{-}\lim_{M \rightarrow \infty} \int_{|p|^2 + q^2 \leq M^2} \phi_+(x, y, p, q) \hat{h}_+(p, q) dp dq \quad (2.18)$$

Finally, Ω_+ is a spectral mapping for A in the sense that for all $h \in D(A)$ one has

$$(\Omega_+ A h)(p, q) = \lambda(p, q) \Omega_+ h(p, q) \quad (2.19)$$

where $\lambda(p, q)$ is defined by (2.15).

Note that these results are simply a reformation of the results of [4, §5] and not a new theorem.

It is important for the calculations of §3 to recognize that another family of generalized eigenfunctions of A is defined by

$$\phi_-(x, y, p, q) = \phi_+(x, y, -p, q)^*, \quad (p, q) \in \mathbb{R}^3 - N$$

It is clear that $A\phi_- = \lambda(p, q)\phi_-$ and

$$\phi_-(x, y, p, q) = (2\pi)^{-1} \exp(ip \cdot x) \phi_-(y, p, q)$$

Moreover,

$$\phi_-(y, p, q) = \phi_+(y, p, q)^* \quad (2.20)$$

because $\psi_{\pm}(y, p, \lambda)$, $\psi_0(y, p, \lambda)$ and therefore $\phi_{\pm}(y, p, q)$, depend on p through $|p|$ alone. The asymptotic behavior of ϕ_+ and ϕ_- for $y \rightarrow \infty$ may be seen from (2.2), (2.6), (2.7), (2.14) and (2.20). It is given by

$$\phi_+(y, p, q) \sim (2\pi)^{-1/2} \begin{cases} c(\infty) \{ \exp(-iqy) + R_+(p, \lambda) \exp(iqy) \}, & (p, q) \in C_+ \\ c(\infty) \{ \exp(-iqy) + R_0(p, \lambda) \exp(iqy) \}, & (p, q) \in C_0 \\ c(-\infty) T_-(p, \lambda) \exp(iq_+ y), & (p, q) \in C_- \end{cases}$$

and

$$\phi_-(y, p, q) \sim (2\pi)^{-1/2} \begin{cases} c(\infty) \{ \exp(iqy) + R_+(p, \lambda)^* \exp(-iqy) \}, & (p, q) \in C_+ \\ c(\infty) \{ \exp(iqy) + R_0(p, \lambda)^* \exp(-iqy) \}, & (p, q) \in C_0 \\ c(-\infty) T_-(p, \lambda)^* \exp(-iq_+ y), & (p, q) \in C_- \end{cases}$$

These relations clearly imply that $\phi_-(y, p, \lambda)$ is not simply a multiple of $\phi_+(y, p, \lambda)$. By contrast, the guided mode eigenfunctions have the symmetry property

$$\psi_k(y, p) = \psi_k(y, -p)^*, \quad k = 1, 2, \dots \quad (2.21)$$

because they are real-valued and depend on p only through $|p|$.

The family $\phi_-(x, y, p, q)$, $(p, q) \in R^3 - N$, is a second family of generalized eigenfunctions for A that spans the reducing subspace $\mathcal{H}_f = P_f \mathcal{H}$. In fact, the following exact counterpart of Theorem 2.1 holds.

Theorem 2.2. For all $h \in \mathcal{H}$ the strong limit

$$\hat{h}_-(p, q) = L_2(R^3)\text{-}\lim_{M \rightarrow \infty} \int_{|x|^2 + y^2 \leq M^2} \phi_-(x, y, p, q)^* h(x, y) c^{-2}(y) dx dy$$

exists. Moreover, the mapping $\Omega_-: \mathcal{H} \rightarrow L_2(R^3)$ defined by $\Omega_- h = \hat{h}_-$ is a partial isometry such that

$$\Omega_- \Omega_-^* = 1 \text{ and } \Omega_-^* \Omega_- = P_f$$

The adjoint mapping $h_f = \Omega_-^* \hat{h}_-$ is given by

$$h_f(x,y) = \mathcal{H}\text{-}\lim_{M \rightarrow \infty} \int_{|p|^2 + q^2 \leq M^2} \phi_-(x,y,p,q) \hat{h}_-(p,q) \, dp dq$$

Finally, Ω_- is a spectral mapping for A in the sense that for all $h \in D(A)$ one has

$$(\Omega_- A h)(p,q) = \lambda(p,q) \Omega_- h(p,q) \quad (2.22)$$

Theorem 2.2 is a direct corollary of Theorem 2.1. This follows from the observations that $f(p,q) \rightarrow f(-p,q)$ defines a unitary transformation in $L_2(\mathbb{R}^3)$ while $f \rightarrow f^*$ defines a unitary transformation in both \mathcal{H} and $L_2(\mathbb{R}^3)$.

§3. Transient Free Waves.

The eigenfunction expansion of Theorem 2.2 is used in this section to calculate the asymptotic behavior for $t \rightarrow \infty$ of the free component $u_f(t, \cdot) = P_f u(t, \cdot)$. The principal result is that in each of the half-spaces R_+^3 and R_-^3 , where $R_\pm^3 = \{(x, y): \pm y > 0\}$, $u_f(t, \cdot)$ is asymptotically equal to a wave function for a homogeneous fluid with propagation speed $c(\infty)$ and $c(-\infty)$, respectively. It is this behavior that motivates the term "free component" for $u_f(t, \cdot)$.

It will be assumed that the total acoustic potential u satisfies $u(t, \cdot) = \text{Re}\{v(t, \cdot)\}$ where $v(t, \cdot) = \exp\{-itA^{1/2}\}h$ and $h \in D(A^{1/2})$ (see §1). The corresponding partial waves $u_k(t, \cdot) = P_k u(t, \cdot)$ with $k \geq 1$ satisfy $u_k(t, \cdot) = \text{Re}\{\exp(-itA^{1/2})P_k h\}$. This follows from relations (2.21) which imply that $P_k(h^*) = (P_k h)^*$. It follows by addition that $u_g(t, \cdot) = \text{Re}\{\exp(-itA^{1/2})P_g h\} = \text{Re}\{v_g(t, \cdot)\}$ and hence

$$u_f(t, \cdot) = \text{Re}\{v_f(t, \cdot)\} \quad (3.1)$$

where $v_f(t, \cdot) = \exp(-itA^{1/2})P_f h = \exp(-itA^{1/2})h_f$. The starting point for calculating the asymptotic behavior of $u_f(t, \cdot)$ will be the eigenfunction expansions of §2. Theorems 2.1 and 2.2 imply the representations

$$v_f(t, x, y) = \int_{R^3} \phi_\pm(x, y, p, q) \exp\{-it\lambda^{1/2}(p, q)\} \hat{h}_\pm(p, q) dp dq \quad (3.2)$$

convergent in \mathcal{K} . The \mathcal{K} -lim notation will be suppressed.

Equation (3.2) gives two representations of v_f corresponding to the two families ϕ_+ and ϕ_- . The calculations below are based on the ϕ_- -representation which has been found to yield the simplest form of the

asymptotic wave function. It will be convenient to introduce the characteristic functions χ_+ , χ_0 , and χ_- of the cones C_+ , C_0 and C_- in (p,q) -space and to decompose \hat{h}_- as

$$\hat{h}_-(p,q) = a(p,q) + b(p,q) + c(p,q) \quad (3.3)$$

where $a = \chi_+ \hat{h}_-$, $b = \chi_0 \hat{h}_-$ and $c = \chi_- \hat{h}_-$. The corresponding decomposition of v_f is

$$v_f = v_a + v_b + v_c \quad (3.4)$$

where

$$\left. \begin{aligned} v_a &= \exp(-itA^{1/2}) \Omega_-^* a \\ v_b &= \exp(-itA^{1/2}) \Omega_-^* b \\ v_c &= \exp(-itA^{1/2}) \Omega_-^* c \end{aligned} \right\} \quad (3.5)$$

The behavior for $t \rightarrow \infty$ of these three functions will be analyzed separately.

Behavior of v_a . The partial wave v_a has the representation

$$v_a(t,x,y) = \int_{C_+} \phi_-(x,y,p,q) \exp(-it\omega_+(p,q)) a(p,q) dp dq \quad (3.6)$$

where

$$\omega_{\pm}(p,q) = c(\pm\infty) \sqrt{|p|^2 + q^2}$$

To discover the behavior of $v_a(t,x,y)$ for $(x,y) \in R_+^3$, $t \rightarrow \infty$ the representation (2.6) for $\phi_-(y,p,q) = (2q)^{1/2} c(\infty) \psi_+(y,p,\lambda)^*$ on C_+ is substituted into (3.6). The result is

$$v_a(t, x, y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp \{i(x \cdot p + yq - t\omega_+(p, q))\} I_+(y, p, \lambda)^* a(p, q) dp dq \\ + \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp \{i(x \cdot p - yq - t\omega_+(p, q))\} R_+(y, p, \lambda)^* a(p, q) dp dq$$

It is natural to expect that in R_+^3 the partial wave $v_a(t, x, y)$ will propagate as $t \rightarrow \infty$ into regions where y is large and hence $I_+(y, p, \lambda)$ and $R_+(y, p, \lambda)$ are near their limiting values. Thus one is led to conjecture that

$$v_a(t, \cdot) \sim v_a^0(t, \cdot) + v_a^1(t, \cdot) \text{ in } L_2(R_+^3), \quad t \rightarrow \infty \quad (3.7)$$

where

$$v_a^0(t, x, y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp \{i(x \cdot p + yq - t\omega_+(p, q))\} a(p, q) dp dq$$

and

$$v_a^1(t, x, y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp \{i(x \cdot p - yq - t\omega_+(p, q))\} R_+(p, \lambda)^* a(p, q) dp dq \\ = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{-C_+} \exp \{i(x \cdot p + yq - t\omega_+(p, q))\} R_+(p, \lambda)^* a(p, -q) dp dq$$

where $-C_+ = \{(p, q) : (p, -q) \in C_+\} = \{(p, q) : q < -a|p|\}$. This conjecture is proved below. Note that

$$\left. \begin{aligned} v_a^0(t, \cdot) &= \exp(-it c(\infty) A_0^{1/2}) h_a \\ v_a^1(t, \cdot) &= \exp(-it c(\infty) A_0^{1/2}) h_a^1 \end{aligned} \right\} \quad (3.8)$$

where A_0 is the selfadjoint realization in $L_2(R^3)$ of $-\Delta = -(D_1^2 + D_2^2 + D_y^2)$

and h_a and h_a^1 are the functions in $L_2(R^3)$ whose Fourier transforms are

$$\left. \begin{aligned} \hat{h}_a(p, q) &= c(\infty) a(p, q) = c(\infty) \chi_+(p, q) \hat{h}_-(p, q) \\ \hat{h}_a^1(p, q) &= c(\infty) R_+(p, \lambda)^* a(p, -q) = c(\infty) R_+(p, \lambda)^* (1 - \chi_+(p, q)) \hat{h}_-(p, -q) \end{aligned} \right\} (3.9)$$

Both are in $L_2(R^3)$ because $\hat{h}_- \in L_2(R^3)$ and $R_+(p, \lambda)$ is bounded, by (2.12). Moreover, $\text{supp } \hat{h}_a^1 \subset -C_+$ and hence the theory of asymptotic wave functions for d'Alembert's equation [11, Ch. 2] implies that $v_a^1(t, \cdot) \sim 0$ in $L_2(R_+^3)$ when $t \rightarrow \infty$. Combining this with (3.7) gives

$$v_a(t, \cdot) \sim v_a^0(t, \cdot) \text{ in } L_2(R_+^3), \quad t \rightarrow \infty \quad (3.10)$$

Now consider the behavior of $v_a(t, x, y)$ for $(x, y) \in R_-^3$, $t \rightarrow \infty$.

Substituting the representation (2.2) for $\phi_-(y, p, \lambda) = (2q)^{1/2} c(\infty) \psi_+(y, p, \lambda)^*$ into (3.6) gives

$$v_a(t, x, y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp \{i(x \cdot p + yq_- - t\omega_+(p, q))\} T_+(y, p, \lambda)^* a(p, q) dp dq$$

which suggests the conjecture that

$$v_a(t, \cdot) \sim v_a^2(t, \cdot) \text{ in } L_2(R_-^3), \quad t \rightarrow \infty \quad (3.11)$$

where

$$v_a^2(t, x, y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp \{i(x \cdot p + yq_- - t\omega_+(p, q))\} T_+(p, \lambda)^* a(p, q) dp dq$$

Now the mapping $(p, q) \rightarrow (p, q') = X'(p, q) = (p, q_-(p, \lambda(p, q)))$ has range $X'(C_+) = R_+^3$, Jacobian $\partial(p, q)/\partial(p, q') = c^2(-\infty)q'/c^2(\infty)q$ and satisfies $\omega_+(p, q) = \omega_-(p, q')$. Thus

$$v_a^2(t, x, y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{R_+^3} \exp \{i(x \cdot p + yq' - t\omega_-(p, q'))\} \times \\ \times T_+(p, \omega_-^2(p, q'))^* a(p, q) (c^2(-\infty)q' / c^2(\infty)q) dpdq'$$

which may be written

$$v_a^2(t, \cdot) = \exp(-it c(-\infty) A_0^{1/2}) h_a^2$$

where $h_a^2 \in L_2(R^3)$ has Fourier transform

$$\hat{h}_a^2(p, q') = c(\infty) T_+(p, \omega_-^2(p, q'))^* a(p, q(p, q')) c^2(-\infty)q' / c^2(\infty)q$$

Since $\text{supp } \hat{h}_a^2 \subset R_+^3$ the results of [11, Ch. 2] imply $v_a^2(t, \cdot) \sim 0$ in $L_2(R_-^3)$ when $t \rightarrow \infty$. Combining this with (3.11) gives

$$v_a(t, \cdot) \sim 0 \text{ in } L_2(R_-^3), t \rightarrow \infty \quad (3.12)$$

Analogous conjectures concerning $v_b(t, \cdot)$ and $v_c(t, \cdot)$ will now be formulated. Only the main steps of the calculations will be given since the method is the same as for $v_a(t, \cdot)$.

Behavior of v_b . Combining (3.5) and (2.7) gives

$$v_b(t, x, y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_0} \exp \{i(x \cdot p + yq - t\omega_+(p, q))\} I_0(y, p, \lambda)^* b(p, q) dpdq \\ + \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_0} \exp \{i(x \cdot p - yq - t\omega_+(p, q))\} R_0(y, p, \lambda)^* b(p, q) dpdq$$

which suggests the conjecture

$$v_b(t, \cdot) \sim v_b^0(t, \cdot) + v_b^1(t, \cdot) \text{ in } L_2(R_+^3), t \rightarrow \infty \quad (3.13)$$

where

$$\left. \begin{aligned} v_b^0(t, \cdot) &= \exp(-it c(\infty) A_0^{1/2}) h_b \\ v_b^1(t, \cdot) &= \exp(-it c(\infty) A_0^{1/2}) h_b^1 \end{aligned} \right\} \quad (3.14)$$

and h_b and h_b^1 are the functions in $L_2(R^3)$ whose Fourier transforms are

$$\left. \begin{aligned} \hat{h}_b(p, q) &= c(\infty) b(p, q) = c(\infty) \chi_0(p, q) \hat{h}_-(p, q) \\ \hat{h}_b^1(p, q) &= c(\infty) R_0(p, \lambda)^* b(p, -q) = c(\infty) R_0(p, \lambda)^* (1 - \chi_0(p, q)) \hat{h}_-(p, -q) \end{aligned} \right\} \quad (3.15)$$

Since $\text{supp } \hat{h}_b^1 \subset -C_0 \subset R_-^3$, $v_b^1(t, \cdot) \sim 0$ in $L_3(R_+^3)$ when $t \rightarrow \infty$, by [11, Ch. 2]. Combining this with (3.13) gives

$$v_b(t, \cdot) \sim v_b^0(t, \cdot) \text{ in } L_2(R_+^3), \quad t \rightarrow \infty \quad (3.16)$$

Similarly (3.5) and (2.3) imply the representation

$$\begin{aligned} v_b(t, x, y) &= \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_0} \exp\{i(x \cdot p - t\omega_+(p, q))\} T_0(y, p, \lambda)^* \exp(\tilde{\beta}y/H) b(p, q) dp dq \end{aligned}$$

and since

$$\lim_{y \rightarrow -\infty} T_0(y, p, \lambda) \exp(\tilde{\beta}y/H) = 0$$

one expects that

$$v_b(t, \cdot) \sim 0 \text{ in } L_2(R_-^3), \quad t \rightarrow \infty \quad (3.17)$$

Behavior of v_c . Combining (3.5) and (2.2) gives

$$v_c(t, x, y)$$

$$= \frac{c(-\infty)}{(2\pi)^{3/2}} \int_{C_-} \exp \{i(x \cdot p - yq_+ - t\omega_-(p, q))\} T_-(y, p, \lambda)^* c(p, q) dp dq$$

which suggests that

$$v_c(t, \cdot) \sim v_c^1(t, \cdot) \text{ in } L_2(R_+^3), t \rightarrow \infty \quad (3.18)$$

where

$$v_c^1(t, x, y) = \frac{c(-\infty)}{(2\pi)^{3/2}} \int_{C_-} \exp \{i(x \cdot p - yq_+ - t\omega_-(p, q))\} T_-(p, \lambda)^* c(p, q) dp dq$$

The mapping $(p, q) \rightarrow (p, q') = X''(p, q) = (p, -q_+(p, \lambda(p, q)))$ has range $X''(C_-) = -C_+$, Jacobian $\partial(p, q)/\partial(p, q') = c^2(\infty)q'/c^2(-\infty)q$ and satisfies $\omega_-(p, q) = \omega_+(p, q')$. Thus

$$v_c^1(t, \cdot) = \exp(-it c(\infty) A_0^{1/2}) h_c^1$$

where h_c^1 has Fourier transform

$$\hat{h}_c^1(p, q') = c(-\infty) T_-(p, \omega_+^2(p, q'))^* c(p, q(p, q')) c^2(\infty)q'/c^2(-\infty)q$$

Moreover, $\text{supp } \hat{h}_c^1 \subset -C_+ \subset R_-^3$ and hence one expects $v_c^1(t, \cdot) \sim 0$ in $L_2(R_+^3)$, $t \rightarrow \infty$. Combining this with (3.18) gives

$$v_c(t, \cdot) \sim 0 \text{ in } L_2(R_+^3), t \rightarrow \infty \quad (3.19)$$

Finally, combining (3.5) and (2.2) gives

$$\begin{aligned} v_c(t, x, y) &= \frac{c(-\infty)}{(2\pi)^{3/2}} \int_{C_-} \exp \{i(x \cdot p + yq - t\omega_-(p, q))\} I_-(y, p, \lambda)^* c(p, q) dp dq \\ &\quad + \frac{c(-\infty)}{(2\pi)^{3/2}} \int_{C_-} \exp \{i(x \cdot p - yq - t\omega_-(p, q))\} R_-(y, p, \lambda)^* c(p, q) dp dq \end{aligned}$$

which suggests that

$$v_c(t, \cdot) \sim v_c^0(t, \cdot) + v_c^1(t, \cdot) \text{ in } L_2(R_-^3), t \rightarrow \infty \quad (3.20)$$

where

$$\left. \begin{aligned} v_c^0(t, \cdot) &= \exp(-it c(-\infty) A_0^{1/2}) h_c \\ v_c^1(t, \cdot) &= \exp(-it c(-\infty) A_0^{1/2}) h_c^1 \end{aligned} \right\} \quad (3.21)$$

and h_c and h_c^1 are the functions in $L_2(R^3)$ whose Fourier transforms are

$$\left. \begin{aligned} \hat{h}_c(p, q) &= c(-\infty) c(p, q) = c(-\infty) \chi_-(p, q) \hat{h}_-(p, q) \\ \hat{h}_c^1(p, q) &= c(-\infty) R_-(p, \lambda)^* c(p, -q) = c(-\infty) R_-(p, \lambda)^* (1 - \chi_-(p, q)) \hat{h}_-(p, -q) \end{aligned} \right\} \quad (3.22)$$

Since $\text{supp } \hat{h}_c^1 \subset -C_- \subset R_+^3$, $v_c^1(t, \cdot) \sim 0$ in $L_2(R_-^3)$ when $t \rightarrow \infty$, by [11, Ch. 2]. Combining this with (3.20) gives

$$v_c(t, \cdot) \sim v_c^0(t, \cdot) \text{ in } L_2(R_-^3), t \rightarrow \infty \quad (3.23)$$

The asymptotic behavior of $v_f(t, \cdot)$ for $t \rightarrow \infty$ is obtained from the three cases analyzed above by superposition, equation (3.4). Thus, equations (3.10), (3.12), (3.16), (3.17), (3.19) and (3.23) imply

$$v_f(t, \cdot) \sim \left\{ \begin{aligned} &v_a^0(t, \cdot) + v_b^0(t, \cdot) \text{ in } L_2(R_+^3) \\ &v_c^0(t, \cdot) \text{ in } L_2(R_-^3) \end{aligned} \right\} t \rightarrow \infty$$

Combining this with the definition of v_a^0 , v_b^0 and v_c^0 , equations (3.8), (3.9), (3.14), (3.15), (3.21) and (3.22) suggests

Theorem 3.1. For all $h \in \mathcal{K}$ let $v_f^0(t, \cdot)$ be defined by

$$v_f^0(t, x, y) = \left\langle \begin{array}{ll} \exp(-it c(\infty) A_0^{1/2}) h^+(x, y), & (x, y) \in R_+^3 \\ \exp(-it c(-\infty) A_0^{1/2}) h^-(x, y), & (x, y) \in R_-^3 \end{array} \right\rangle \quad (3.24)$$

where h^+ and h^- are the functions in $L_2(R^3)$ whose Fourier transforms are given by

$$\hat{h}^+(p, q) = \left\langle \begin{array}{ll} c(\infty) \hat{h}_-(p, q), & (p, q) \in R_+^3 \\ 0, & (p, q) \in R_-^3 \end{array} \right\rangle \quad (3.25)$$

and

$$\hat{h}^-(p, q) = \left\langle \begin{array}{ll} 0, & (p, q) \in R_+^3 \\ c(-\infty) \hat{h}_-(p, q), & (p, q) \in R_-^3 \end{array} \right\rangle \quad (3.26)$$

Then

$$\lim_{t \rightarrow \infty} \|v_f(t, \cdot) - v_f^0(t, \cdot)\|_{\mathcal{H}} = 0 \quad (3.27)$$

Proof of Theorem 3.1. The decomposition (3.3) will be used for the proof. Moreover, for brevity, only the asymptotic equality (3.7) for $v_a(t, \cdot)$ will be proved. The five remaining cases, namely (3.11), (3.13), (3.17), (3.18) and (3.20), can be proved by the method used for v_a .

As a first step, (3.7) will be proved for the special case where $a(p, q) \in C_0^\infty(C_+)$. The general case will then be proved by using the fact that $C_0^\infty(C_+)$ is dense in $L_2(C_+)$.

When $a(p, q) \in C_0^\infty(C_+)$ the integrals defining v_a , v_a^0 and v_a^1 converge point-wise, as well as in \mathcal{H} , and one has

$$v_a(t, x, y) - v_a^0(t, x, y) - v_a^1(t, x, y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{R^2} \exp(ix \cdot p) w(t, y, p) dp \quad (3.28)$$

where

$$w(t, y, p) = w^0(t, y, p) + w^1(t, y, p) \quad (3.29)$$

$$w^0(t, y, p) = \int_{a|p|}^{\infty} \exp\{i(yq - t\omega_+(p, q))\} (I_+(y, p, \lambda) - 1)^* a(p, q) dq \quad (3.30)$$

and

$$\begin{aligned} w^1(t, y, p) &= \int_{a|p|}^{\infty} \exp\{i(-yq - t\omega_+(p, q))\} (R_+(y, p, \lambda) - R_+(p, \lambda))^* a(p, q) dq \\ &= \int_{-\infty}^{-a|p|} \exp\{i(yq - t\omega_+(p, q))\} (R_+(y, p, \lambda) - R_+(p, \lambda))^* a(p, -q) dq \end{aligned} \quad (3.31)$$

Parseval's formula in $L_2(R^2)$, applied to (3.28), gives

$$\int_{R^2} |v_a(t, x, y) - v_a^0(t, x, y) - v_a^1(t, x, y)|^2 dx = \frac{c^2(\infty)}{2\pi} \int_{R^2} |w(t, y, p)|^2 dp$$

Integrating this over $y \geq 0$ gives

$$\begin{aligned} \|v_a(t, \cdot) - v_a^0(t, \cdot) - v_a^1(t, \cdot)\|_{L_2(R_+^3)} &= \frac{c(\infty)}{(2\pi)^{1/2}} \|w(t, \cdot)\|_{L_2(R_+^3)} \\ &\leq \frac{c(\infty)}{(2\pi)^{1/2}} \left(\|w^0(t, \cdot)\|_{L_2(R_+^3)} + \|w^1(t, \cdot)\|_{L_2(R_+^3)} \right) \end{aligned} \quad (3.32)$$

by (3.29) and the triangle inequality.

The estimate (3.32) implies that to prove (3.7) it is sufficient to prove that $w^0(t, \cdot) \rightarrow 0$ and $w^1(t, \cdot) \rightarrow 0$ in $L_2(R_+^3)$ when $t \rightarrow \infty$. To this end the integrals in (3.30) and (3.31) will be estimated by the method

of stationary phase as formulated in the Appendix. To apply the method define

$$r = \sqrt{t^2 + y^2}, \quad t = r \sin \alpha, \quad y = r \cos \alpha \quad \text{where } 0 < \alpha < \pi/2$$

and

$$\theta(p, q, \alpha) = q \cos \alpha - \omega_+(p, q) \sin \alpha \quad (3.33)$$

Then the integrals in (3.30) and (3.31) have the form (A.1) with $s = q$ ($m = 1$) and $\xi = (p, \alpha)$ ($n = 3$). Moreover,

$$\nabla_q \theta(p, q, \alpha) = \cos \alpha - U(p, q) \sin \alpha$$

where

$$U(p, q) = c(\infty)q / \sqrt{|p|^2 + q^2}$$

is the group speed associated with the dispersion relation $\omega = \omega_+(p, q)$.

Clearly $\nabla_q \theta(p, q, \alpha) = 0$ if and only if

$$y/t = \cot \alpha = U(p, q)$$

Note that $\cot \alpha > 0$ for $0 < \alpha < \pi/2$. In the case of (3.31), $U(p, q) < 0$ on the interval of integration and there are no points of stationary phase. In the case of (3.30), $a|p| \leq q < \infty$ and hence $0 < U(p, q) < c(\infty)$. Thus there is a unique point of stationary phase if $0 < \cot \alpha < c(\infty)$, or $\alpha_0 < \alpha < \pi/2$ where $\cot \alpha_0 = c(\infty)$, and no point of stationary phase if $0 < \alpha \leq \alpha_0$. The point of stationary phase is

$$q = \tau(p, \alpha) = |p| \cot \alpha / (c^2(\infty) - \cot^2 \alpha)^{1/2} \quad (3.34)$$

when $0 < \cot \alpha < c(\infty)$. The corresponding approximation to $w^0(t, y, p)$ is,

from (A.11),

$$w^\infty(t, y, p) = \chi(t, y) (2\pi)^{1/2} g(r, p, q, \alpha) \exp \{i(yq - t\omega_+(p, q)) - \pi/4\} / (tU_q(p, q))^{1/2} \quad (3.35)$$

$$= \chi(t, y) (2\pi)^{1/2} g(r, p, q, \alpha) \exp \{i(yq - t\omega_+(p, q)) - \pi/4\} / (tU_q(p, q))^{1/2}$$

where $\chi(t, y)$ is the characteristic function of the set $\{(t, y): 0 < y < c(\infty)t\}$,

$$g(r, p, q, \alpha) = (I_+(r \cos \alpha, p, \lambda(p, q)) - 1)^* a(p, q) \quad (3.36)$$

$$U_q(p, q) = c(\infty) |p|^2 / (|p|^2 + q^2)^{3/2}$$

and q is given by (3.34) with $\cot \alpha = y/t$. These results and Theorems A.1 and A.2 imply the

Lemma. For every $a(p, q) \in C_0^\infty(C_+)$ there exists a constant $M = M(a)$ such that

$$|w^0(t, y, p) - w^\infty(t, y, p)| \leq M/(t^2 + y^2)^{3/4} \quad (3.37)$$

for all $t > 0$, $y > 0$ and $p \in \mathbb{R}^2$. Similarly, for each positive integer k there exists a constant $M_k = M_k(a)$ such that

$$|w^1(t, y, p)| \leq M_k/(t^2 + y^2)^{k/2} \quad (3.38)$$

for all $t > 0$, $y > 0$ and $p \in \mathbb{R}^2$.

Proof of the Lemma. The hypothesis $a \in C_0^\infty(C_+)$ implies that $\text{supp } a = K_0$ is a compact subset of C_+ . Let K and K_1 denote its orthogonal projections on the q -axis and the p -plane, respectively. Then the functions $w^0(t, y, p)$, $w^1(t, y, p)$ and $w^\infty(t, y, p)$ are all zero for $p \in \mathbb{R}^2 - K_1$. Hence it is enough to verify (3.37) and (3.38) for $p \in K_1$.

Consider first the integral in (3.31). It has the form (A.1) with $q \in O = R_-$ (the negative real numbers), $(p, \alpha) \in O'$ $= (R^2 - \{0\}) \times (0, \pi/2)$ and g given by

$$g(r, p, q, \alpha) = [R_+(r \cos \alpha, p, \lambda(p, q)) - R_+(p, \lambda(p, q))]^* a(p, -q) \quad (3.39)$$

Hypothesis (A.2) is evident from (3.33), while (A.3) follows from (3.39) and well-known properties of hypergeometric functions. Moreover, $\text{supp } g(r, \cdot, \cdot, \alpha) \subset K$ for all $r > 0$ and $0 < \alpha < \pi/2$ and hence (A.4) holds. The uniform estimates (A.6) may be derived from the integral representations of the hypergeometric functions - see [3, pp. 77-79]. Finally, $\theta(p, q, \alpha)$ has no points of stationary phase for $g \in K \subset R_-$. Thus Theorem A.1 is applicable to $w^1(t, y, p)$ and (3.38) follows.

Next consider the integral in (3.30). For $0 < \alpha = \text{ctn}^{-1}(y/t) \leq \alpha_0$ there are no points of stationary phase $q \in K$. Moreover, $\tilde{w}^\infty(t, y, p) = 0$ on this set. Hence Theorem A.1 is applicable and implies that (3.37) holds for $y \geq c(\infty)t$. On the other hand, for $\alpha_0 < \alpha < \pi/2$ or $0 < y < c(\infty)t$ there is exactly one point of stationary phase q given by (3.34) and hence (A.8) holds. Moreover,

$$\theta_q''(p, q, \alpha) = -U_q(p, q) \sin \alpha \neq 0 \text{ for } q \in K, (p, \alpha) \in O'$$

which implies (A.9). Finally, (A.10) follows from properties of the hypergeometric functions, as in the preceding cases. Thus Theorem A.2 is applicable and implies that (3.37) holds for $0 < y < c(\infty)t$.

Proof of Theorem 3.1 (concluded). The lemma implies that if $a \in C_0^\infty(C_+)$ then $w^0(t, \cdot)$ and $w^1(t, \cdot)$ tend to zero in $L_2(R_+^3)$ when $t \rightarrow \infty$ and thereby proves Theorem 3.1 in this case. Indeed, integrating (3.38) gives

$$\begin{aligned} \|w^1(t, \cdot)\|_{L_2(R_+^3)}^2 &= \int_0^\infty \int_{K_1} |w^1(t, y, p)|^2 dp dy \\ &\leq C_k^2 |K_1| \int_0^\infty (t^2 + y^2)^{-k} dy = C_k' t^{1-2k} \end{aligned} \quad (3.40)$$

where $|K_1|$ is the measure of K_1 and $C_k' = C_k^2 |K_1| \int_0^\infty (1 + \xi^2)^{-k} d\xi$. Now consider w^0 . The lemma implies that

$$w^0(t, y, p) = w^\infty(t, y, p) + q(t, y, p)$$

where

$$|q(t, y, p)| \leq M/(t^2 + y^2)^{3/4}$$

Thus

$$\|w^0(t, \cdot)\|_{L_2(R_+^3)} \leq \|w^\infty(t, \cdot)\|_{L_2(R_+^3)} + \|q(t, \cdot)\|_{L_2(R_+^3)} \quad (3.41)$$

Moreover,

$$\|q(t, \cdot)\|_{L_2(R_+^3)} \leq C'/t^2$$

by the calculation used to prove (3.40) with $k = 3/2$. It remains to show that $\|w^\infty(t, \cdot)\|_{L_2(R_+^3)} \rightarrow 0$ when $t \rightarrow \infty$. Direct calculation using (3.35) and (3.36) gives

$$\begin{aligned} \|w^\infty(t, \cdot)\|_{L_2(R_+^3)}^2 &= \int_0^{c(\infty)t} \int_{K_1} |w^\infty(t, y, p)|^2 dp dy \\ &= 2\pi \int_0^{c(\infty)t} \int_{K_1} |I_+(y, p, \lambda(p, q)) - 1|^2 |a(p, q)|^2 t^{-1} U_q(p, q)^{-1} dp dy \end{aligned} \quad (3.42)$$

where $q = \tau(p, \text{ctn}^{-1}(y/t))$ is given by (3.34). Since (3.34) defines the solution q of $y/t = U(p, q)$ it is natural to make the change of variable

$y = t U(p, q)$, $dy = t U_q(p, q) dq$ (p fixed) in (3.42). The result is

$$\begin{aligned} \|w^\infty(t, \cdot)\|_{L_2(R_+^3)} \\ = 2\pi \int_{K_1} \int_K |I_+(t U(p, q), p, \lambda(p, q)) - 1|^2 |g(p, q)|^2 dq dp \end{aligned} \quad (3.43)$$

since $\text{supp } g \subset K_1 \times K$. Finally the representation $I_+(y, p, \lambda) = (2/H)^{1/2} \exp(iq_+ y) \phi_2(y, p, \lambda)$ implies that

$$I_+(t U(p, q), p, \lambda(p, q)) \rightarrow 1 \text{ when } t \rightarrow \infty$$

uniformly for $(p, q) \in K_1 \times K$. It follows by Lebesgue's dominated convergence theorem that $\|w^\infty(t, \cdot)\|_{L_2(R_+^3)} = o(1)$, $t \rightarrow \infty$. In fact, a careful estimate of the hypergeometric function ϕ_u gives the stronger estimate

$$\|w^\infty(t, \cdot)\|_{L_2(R_+^3)} = O(e^{-\mu t}), \quad t \rightarrow \infty$$

where $\mu = \mu(a)$ is a positive constant.

The arguments given above, applied to v_a , v_b and v_c , show that the conclusion (3.27) of Theorem 3.1 holds for all h such that $\hat{h}_- \in C_0^\infty(C_+ \cup C_0 \cup C_-)$. Moreover, this set is dense in $L_2(R^3)$ and hence $\Omega_-^* C_0^\infty(C_+ \cup C_0 \cup C_-)$ is dense in $\mathcal{H}_f = P_f \mathcal{H}$ by Theorem 2.2. These facts can be used to extend (3.27) to all $h \in \mathcal{H}$, provided the mappings $U(t): \mathcal{H}_f \rightarrow L_2(R^3)$ and $U_0(t): \mathcal{H}_f \rightarrow L_2(R^3)$ defined by $U(t)h_f = \exp(-it A^{1/2})h_f$ and $U_0(t)h_f = v_f^0(t, \cdot)$ are uniformly bounded for all $t \in \mathbb{R}$. This density argument has been given in many places; see, for example, [11, Ch. 2] or [13, p. 260]. To prove the uniform boundedness note that the $L_2(R^3)$ and \mathcal{H} norms are equivalent:

$$c(y_0) \|h\|_{\mathcal{H}} \leq \|h\|_{L_2(\mathbb{R}^3)} \leq c(-\infty) \|h\|_{\mathcal{H}}$$

and hence

$$\|U(t)h_f\|_{L_2(\mathbb{R}^3)} \leq c(-\infty) \|U(t)h_f\|_{\mathcal{H}} = c(-\infty) \|h_f\|_{\mathcal{H}}$$

for all $t \in \mathbb{R}$. Similarly, (3.24), (3.25) and (3.26) imply the estimates

$$\begin{aligned} \|U_0(t)h_f\|_{L_2(\mathbb{R}^3)} &= \|v_f^0(t, \cdot)\|_{L_2(\mathbb{R}^3)} \\ &\leq (\|v_f^0(t, \cdot)\|_{L_2(\mathbb{R}_+^3)}^2 + \|v_f^0(t, \cdot)\|_{L_2(\mathbb{R}_-^3)}^2)^{1/2} \\ &\leq (\|h^+\|_{L_2(\mathbb{R}^3)}^2 + \|h^-\|_{L_2(\mathbb{R}^3)}^2)^{1/2} \\ &= (\|\hat{h}^+\|_{L_2(\mathbb{R}^3)}^2 + \|\hat{h}^-\|_{L_2(\mathbb{R}^3)}^2)^{1/2} \\ &= (c^2(\infty) \|\hat{h}_-\|_{L_2(\mathbb{R}_+^3)}^2 + c^2(-\infty) \|\hat{h}_-\|_{L_2(\mathbb{R}_-^3)}^2)^{1/2} \\ &\leq c(-\infty) \|\hat{h}_-\|_{L_2(\mathbb{R}^3)} = c(-\infty) \|h_f\|_{\mathcal{H}} \end{aligned}$$

for all $t \in \mathbb{R}$. This completes the proof of Theorem 3.1.

Theorem 3.1 implies corresponding asymptotic estimates for the free component $u_f(t, \cdot) = P_f u(t, \cdot) = \operatorname{Re} \{v_f(t, \cdot)\}$ of the acoustic potential $u(t, \cdot)$. Indeed, if $u_f^0(t, \cdot)$ is defined by

$$u_f^0(t, x, y) = \operatorname{Re} \{v_f^0(t, x, y)\} \quad (3.44)$$

then Theorem 3.1 and the elementary inequality $|\operatorname{Re} z| \leq |z|$ imply

Corollary 3.2. For all $h \in \mathcal{K}$ one has

$$\lim_{t \rightarrow \infty} \|u_f(t, \cdot) - u_f^0(t, \cdot)\|_{\mathcal{K}} = 0 \quad (3.45)$$

If the initial state h has derivatives in \mathcal{K} then $u_f(t, \cdot)$ and $u_f^0(t, \cdot)$ have the same derivatives in \mathcal{K} and (3.45) can be strengthened to include these derivatives. In particular, one has

Corollary 3.3. For all $h \in L_2^1(\mathbb{R}^3) = D(A^{1/2})$ one has

$$\lim_{t \rightarrow \infty} \|D_j u_f(t, \cdot) - D_j u_f^0(t, \cdot)\|_{\mathcal{K}} = 0, \quad j = 0, 1, 2, 3 \quad (3.46)$$

where $D_0 = \partial/\partial t$, $D_1 = \partial/\partial x_1$, $D_2 = \partial/\partial x_2$ and $D_3 = \partial/\partial y$. Note that (3.46) is equivalent to convergence in energy:

$$\lim_{t \rightarrow \infty} E(u_f - u_f^0, \mathbb{R}^3, t) = 0$$

Corollary 3.3 can be proved by applying the method of this section to the derivatives $D_j v_f(t, \cdot)$ ($j = 0, 1, 2, 3$) which are given by integrals of the same form as (3.2). Detailed proofs for the case of the Pekeris profile were given in [13].

§4. Transient Guided Waves.

The asymptotic behavior for $t \rightarrow \infty$ of the guided component $u_g(t, \cdot) = P_g u(t, \cdot)$ is derived in this section. $u_g(t, \cdot)$ is a sum in \mathcal{H} of mutually orthogonal partial waves $u_k(t, \cdot) = P_k u(t, \cdot) = \text{Re} \{v_k(t, \cdot)\}$, $k \geq 1$. The starting point for the analysis is the integral representation

$$v_k(t, x, y) = \int_{\Omega_k} \psi_k(x, y, p) \exp \{-it \omega_k(|p|)\} \tilde{h}_k(p) dp \quad (4.1)$$

where

$$\tilde{h}_k(p) = \int_{R^3} \psi_k(x, y, p)^* h(x, y) c^{-2}(y) dx dy$$

and the integrals converge in \mathcal{H} and $L_2(\Omega_k)$, respectively. The integral in (4.1) can be written

$$v_k(t, x, y) = \frac{1}{2\pi} \int_{\Omega_k} \exp \{i(x \cdot p - t\omega_k(|p|))\} \psi_k(y, p) \tilde{h}_k(p) dp \quad (4.2)$$

This is an oscillatory integral that can be estimated by the method of stationary phase of the Appendix when $\tilde{h}_k \in C_0^\infty(\Omega_k)$. To apply the method define

$$r = \sqrt{t^2 + |x|^2}$$

$$t = r \xi_0, \quad x_1 = r \xi_1, \quad x_2 = r \xi_2$$

$$\xi = (\xi_0, \xi_1, \xi_2) \in S^2 \subset R^3$$

where S^2 denote the unit sphere in R^3 . Then (4.2) takes the form A.1

with $m = 2$, $n = 3$

$$\theta_k(p, \xi, y) = \xi_1 p_1 + \xi_2 p_2 - \xi_0 \omega_k(|p|) \quad (4.3)$$

$$g_k(r, p, \xi, y) = (2\pi)^{-1} \psi_k(y, p) \tilde{h}_k(p) \quad (4.4)$$

Note that θ_k is independent of y and g_k is independent of r and ξ . They have been written in the forms (4.3) and (4.4) to emphasize the applicability of the results of the Appendix.

The phase function (4.3) has a point of stationary phase if and only if

$$U_k(|p|)p/|p| = x/t \quad (4.5)$$

where

$$U_k(|p|) = D_{|p|} \omega_k(|p|)$$

is the group speed associated with the dispersion relation $\omega = \omega_k(|p|)$.

The analysis will be simplified by the following

Proposition. If the parameters defining $c(y)$, equation (1.1), satisfy

$$8K \geq M \quad (4.6)$$

then each of the functions $U_k(|p|)$, $k = 1, 2, \dots$, is a strictly decreasing function on $p_k \leq |p| < \infty$ such that $U_k(p_k) = c(\infty)$ and $U_k(|p|) \rightarrow c(y_0)$ when $|p| \rightarrow \infty$.

A proof of the proposition may be derived from the parametric representation of the dispersion relation [4]. The calculation is too long to give here.

For simplicity of presentation condition (4.6) is assumed in the remainder of this paper. If $8K < M$ then $U'_k(|p|)$ may have a finite number of zeros. This complicates the form of the asymptotic wave functions but is otherwise tractable. In the case of the Pekeris profile, treated in [13], the functions $U'_k(|p|)$ have exactly one zero on $|p| \geq p_k$.

By calculating the Hessian θ''_k one can show that

$$r^2 |\det \theta''_k(p, \xi, y)| = t^2 U_k(|p|) |U'_k(|p|)|/|p|$$

and

$$\operatorname{sgn} \theta''_k(p, \xi, y) = 0$$

In particular, each point of stationary phase is non-degenerate, by the Proposition. Moreover, (4.5) implies that θ_k has a point p of stationary phase if and only if

$$|x|/t = U_k(|p|) \quad (4.7)$$

Now (4.7) has a unique solution $|p|$ if $|x|/t$ lies in the range of $U_k(|p|)$; that is if

$$c(y_0) < |x|/t < c(\infty) \quad (4.8)$$

The point of stationary phase is then given by

$$p = Q_k(|x|/t)x/|x| \text{ for } x \in \mathbb{R}^2 - \{0\}, t > 0 \quad (4.9)$$

where Q_k is the inverse function to U_k . It is a monotone decreasing function that maps $(c(y_0), c(\infty))$ onto (p_k, ∞) . By Theorem A.2 the point

(4.9) makes a contribution

$$v_k^\infty(t, x, y, p) = \frac{|p|^{1/2} \exp \{i(|x||p| - t\omega_k(|p|))\} \psi_k(y, p) \tilde{h}_k(p)}{t \{U_k(|p|) |U'_k(|p|)|\}^{1/2}} \quad (4.10)$$

to the integral in (4.2) when (t, x) satisfies (4.8). For $|x|/t$ outside the interval (4.8) there is no point of stationary phase. Thus the stationary phase approximation to $v_k(t, x, y)$ is given by

$$v_k^\infty(t, x, y) = \chi(|x|/t) v_k^\infty(t, x, y, Q_k(|x|/t)x/|x|) \quad (4.11)$$

where χ is the characteristic function of $(c(y_0), c(\infty))$ and one has

Theorem 4.1. For all $h \in \mathcal{H}$ such that $\tilde{h}_k \in C_0^\infty(\Omega_k)$ there exists a constant $C = C_k(h)$ such that

$$|v_k(t, x, y) - v_k^\infty(t, x, y)| \leq C/t^2 \quad (4.12)$$

for all $t > 0$, $x \in \mathbb{R}^2 - \{0\}$ and $y \in \mathbb{R}$.

This result can be proved by application of Theorems A.1 and A.2. The proof is similar to that of the Lemma in §3 and will not be recorded here. If \tilde{h}_k is not a smooth function then Theorems A.1 and A.2 are not applicable and the estimate (4.12) may fail. However, the definition (4.10), (4.11) is applicable to all $h \in \mathcal{H}$. More precisely, one can prove

Theorem 4.2. For all $h \in \mathcal{H}$, all $t > 0$ and $k = 1, 2, \dots$ one has

$$v_k^\infty(t, \cdot) \in \mathcal{H},$$

and

$$\|v_k^\infty(t, \cdot)\|_{\mathcal{H}} = \|\tilde{h}_k\|_{L_2(\Omega_k)} = \|P_k h\|_{\mathcal{H}}$$

Moreover, the mapping $t \rightarrow v_k^\infty(t, \cdot)$ is continuous from R_+ to \mathcal{H} and

$$\lim_{t \rightarrow \infty} \|v_k(t, \cdot) - v_k^\infty(t, \cdot)\|_{\mathcal{H}} = 0 \quad (4.13)$$

The proofs of these properties are the same as those for the Pekeris profile, given in [13], and are not reproduced here. On defining

$$u_k^\infty(t, x, y) = \operatorname{Re} \{v_k(t, x, y)\}$$

one has

Corollary 4.3. For all $h \in \mathcal{H}$ and $k = 1, 2, 3, \dots$

$$\lim_{t \rightarrow \infty} \|u_k(t, \cdot) - u_k^\infty(t, \cdot)\|_{\mathcal{H}} = 0$$

If $h \in L_2^1(R^3)$ then $u_k(t, \cdot) \in L_2^1(R^3)$ and asymptotic wave functions for the first derivatives of u_k can be constructed. Indeed, if $\tilde{h}_k \in C_0^\infty(\Omega_k)$ then the first derivatives of v_k are given by

$$D_t v_k(t, x, y) = \frac{1}{2\pi} \int_{\Omega_k} \exp \{i(x \cdot p - t\omega_k(|p|))\} \psi_k(y, p) (-i\omega_k(|p|)) \tilde{h}_k(p) dp$$

$$D_j v_k(t, x, y) = \frac{1}{2\pi} \int_{\Omega_k} \exp \{i(x \cdot p - t\omega_k(|p|))\} \psi_k(y, p) (ip_j) \tilde{h}_k(p) dp \quad (j = 1, 2)$$

$$D_y v_k(t, x, y) = \frac{1}{2\pi} \int_{\Omega_k} \exp \{i(x \cdot p - t\omega_k(|p|))\} D_y \psi_k(y, p) \tilde{h}_k(p) dp$$

These integrals have the same form as the integral (4.2) for $v_k(t, x, y)$.

The corresponding asymptotic wave functions are defined by

$$v_{k0}^\infty(t, x, y, p) = (-i\omega_k(|p|)) v_k^\infty(t, x, y, p)$$

$$v_{kj}^\infty(t, x, y, p) = (ip_j) v_k^\infty(t, x, y, p) \quad (j = 1, 2)$$

$$v_{k3}^{\infty}(t, x, y, p) = D_y v_k^{\infty}(t, x, y, p)$$

and

$$v_{kj}^{\infty}(t, x, y) = \chi(|x|/t) v_{kj}^{\infty}(t, x, y, Q_k(|x|/t)x/|x|) \quad (j = 0, 1, 2, 3)$$

The analogue of Theorem 4.2 is

Theorem 4.4. For all $h \in L_2^1(\mathbb{R}^3)$, all $t > 0$ and $k = 1, 2, 3, \dots$

one has

$$v_{k0}^{\infty}(t, \cdot) \in \mathcal{H}, \quad v_{kj}^{\infty}(t, \cdot) \in L_2(\mathbb{R}^3), \quad j = 1, 2, 3$$

$$\|v_{k0}^{\infty}(t, \cdot)\|_{\mathcal{H}}^2 + \sum_{j=1}^3 \|v_{kj}^{\infty}(t, \cdot)\|_{L_2(\mathbb{R}^3)}^2 = 2 \|A^{1/2} h_k\|_{\mathcal{H}}^2$$

and

$$\lim_{t \rightarrow \infty} \|D_j v_k(t, \cdot) - v_{kj}^{\infty}(t, \cdot)\|_{\mathcal{H}} = 0 \quad (j = 0, 1, 2, 3)$$

Finally if

$$u_{kj}^{\infty}(t, x, y) = \operatorname{Re} \{v_{kj}^{\infty}(t, x, y)\} \quad (j = 0, 1, 2, 3)$$

then one has

Corollary 4.5. For all $h \in L_2^1(\mathbb{R}^3)$ and $k = 1, 2, 3, \dots$

$$\lim_{t \rightarrow \infty} \|D_j u_k(t, \cdot) - u_{kj}^{\infty}(t, \cdot)\|_{\mathcal{H}} = 0$$

Theorem 4.4 may be proved by the technique used for the Pekeris profile in [13].

§5. The Asymptotic Distribution of Energy for Large Times.

The total energy of the acoustic field $u(t, x, y)$, given by (1.16), is constant for $t \geq T$. The same is true of the partial waves u_f , u_g and u_k , $k = 1, 2, \dots$. Moreover, the eigenfunction expansion of [4] implies that $\{P_f, P_1, P_2, \dots\}$ is a complete family of orthogonal projections that reduces A . It follows that

$$\|A^{1/2} h\|_{\mathcal{H}}^2 = \|A^{1/2} h_f\|_{\mathcal{H}}^2 + \sum_{k=1}^{\infty} \|A^{1/2} h_k\|_{\mathcal{H}}^2 \quad (5.1)$$

which may be interpreted as an energy partition theorem. The energies of the partial waves can be calculated from the initial state h , or source function $f(t, x, y)$, by means of the eigenfunction expansion theorem. The relationship between h and f is given by (1.9). Hence

$$\hat{h}_-(p, q) = i \lambda^{-1/2} (p, q) \hat{f}_-(-\lambda^{1/2} (p, q), p, q)$$

where

$$\hat{f}_-(\omega, p, q) = \int_{-\infty}^{\infty} \int_{R^3} \exp(-i\omega\tau) \phi_-(x, y, p, q)^* f(\tau, x, y) c^{-2}(y) dy dx$$

and similarly

$$\tilde{h}_k(p) = i \omega_k^{-1}(|p|) \tilde{f}_k(-\omega_k(|p|), p)$$

where

$$\tilde{f}_k(\omega, p) = \int_{-\infty}^{\infty} \int_{R^3} \exp(-i\omega\tau) \psi_k(x, y, p)^* f(\tau, x, y) c^{-2}(y) dy dx$$

It follows that the partial energies are given by

$$E(u_f, R^3, t) = \|A^{1/2} h_f\|_{\mathcal{H}}^2 = \int_{R^3} |\hat{f}_-(-\lambda^{1/2}(p, q), p, q)|^2 dp dq, \quad t \geq T$$

and

$$E(u_k, R^3, t) = \|A^{1/2} h_k\|_{\mathcal{H}}^2 = \int_{\Omega_k} |\tilde{f}_k(-\omega_k(|p|), p)|^2 dp, \quad t \geq T$$

The theorems of §3 and §4 make possible the calculation of asymptotic energy distributions in bounded and unbounded subsets of R^3 . Only the principal results are formulated here since the proofs are the same as for the case of the Pekeris profile given in [13].

The notation

$$E^\infty(u, K) = \lim_{t \rightarrow \infty} E(u, K, t)$$

will be used whenever the limit exists. A first result is the transiency of all waves with finite energy in an Epstein duct:

$$E^\infty(u, K) = 0 \text{ for all compact sets } K \subset R^3$$

Next let C^+ (resp. C^-) denote a cone in R_+^3 (resp. R_-^3). Then Corollary 3.3 and the results of [11] imply

$$\begin{aligned} E^\infty(u_f, C^\pm) &= c^2(\pm\infty) \int_{C^\pm} (|p|^2 + q^2) |\hat{h}_-(p, q)|^2 dp dq \\ &= \int_{C^\pm} |\hat{f}_-(-\lambda^{1/2}(p, q), p, q)|^2 dp dq \end{aligned}$$

Now define the cone C_ε by

$$C_\varepsilon = \{(x, y): |y - y_0| \leq \varepsilon |x|\}, \varepsilon > 0$$

Then, in contrast to the preceding result, one has

$$\varepsilon^\infty(u_k, C_\varepsilon) = \varepsilon(u_k, R^3, t) = \|A^{1/2} h_k\|_{\mathcal{H}}^2 \text{ for every } \varepsilon > 0$$

and for $k = 1, 2, 3, \dots$ (see [13, Theorem 5.5]).

Finally, if

$$S = \{(x, y): x \in R^2 \text{ and } y_1 < y < y_2\}$$

then it can be shown by means of Corollary 4.5 that

$$\varepsilon^\infty(u_k, S) = \int_{\Omega_k} |\tilde{f}_k(-\omega_k(|p|), p)|^2 \left(\int_{y_1}^{y_2} \psi_k^2(y, p) c^{-2}(y) dy \right) dp$$

§6. Concluding Remarks. Examination of the proofs in [4] and in this paper reveals that the methods should apply to a large class of sound speed profiles $c(y)$. As remarked in [4], the principal obstacle to such a generalization is the lack of information about the p -dependence of the eigenfunctions and eigenvalues. In the cases of the Pekeris and Epstein profiles this information was obtained from explicit constructions of the eigenfunctions in terms of known functions.

Appendix. Estimates of Oscillatory Integrals with Parameters.

The method of stationary phase provides asymptotic estimates for $r \rightarrow \infty$ of oscillatory integrals of the form

$$I(r) = \int_0 \exp \{i r \theta(s)\} g(s) ds, \quad 0 \subset \mathbb{R}^m$$

The method is needed in §3 and §4 above to estimate the oscillatory integrals in equations (3.30), (3.31) and (4.2). In (4.2) the integrand g contains a parameter y in addition to the variables of integration. In (3.30) and (3.31) both θ and g contain parameters and, in addition, g contains the large parameter r . In both cases estimates that are uniform in the parameters are needed. This suggests the study of oscillatory integrals of the form

$$I(r, \xi) = \int_0 \exp \{i r \theta(s, \xi)\} g(r, s, \xi) ds \quad (\text{A.1})$$

where $r > 0$, $s \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^n$. $\theta(s, \xi)$ is a real-valued phase function and it is assumed that there are open sets $0 \subset \mathbb{R}^m$ and $0' \subset \mathbb{R}^n$ such that

$$D_s^\delta \theta(s, \xi) \in C(0 \times 0') \text{ for all multi-indices } \delta \quad (\text{A.2})$$

$$D_s^\delta g(r, s, \xi) \in C(R_+ \times 0 \times 0') \text{ for all multi-indices } \delta \quad (\text{A.3})$$

where R_+ denotes the positive real numbers. Moreover, it is assumed that there is a compact set $K \subset 0$ such that

$$\text{supp } g \subset R_+ \times K \times 0' \quad (\text{A.4})$$

Estimates of $I(r, \xi)$ for $r \rightarrow \infty$ are sought which are uniform in ξ on compact subsets of \mathcal{O}' . For large values of r the exponential in (A.1) is highly oscillatory except near critical points of the phase function $\theta(s, \xi)$. Two estimates of $I(r, \xi)$ are given here, corresponding to the cases of no critical points and one critical point, respectively. The first case is formulated as

Theorem A.1. Assume that

$$\nabla_s \theta(s, \xi) = (\partial\theta/\partial s_1, \dots, \partial\theta/\partial s_m) \neq 0 \text{ for all } (s, \xi) \in K \times \mathcal{O}' \quad (\text{A.5})$$

Moreover, assume that for each compact set $K' \subset \mathcal{O}'$, each $r_0 > 0$ and each positive integer k there exists a constant $M = M(K, K', r_0, k) > 0$ such that

$$|D_s^\delta g(r, s, \xi)| \leq M \text{ for all } r \geq r_0, s \in K, \xi \in K' \text{ and } |\delta| \leq k \quad (\text{A.6})$$

Then there exists a constant $C = C(K, K', r_0, k, g) > 0$ such that

$$|I(r, \xi)| \leq C r^{-k} \text{ for all } r \geq r_0 \text{ and } \xi \in K' \quad (\text{A.7})$$

In the second case considered here $\theta(s, \xi)$ has a unique non-degenerate critical point $s = \tau(\xi)$ for each $\xi \in \mathcal{O}'$. It is formulated as

Theorem A.2. Assume that there is a function $\tau \in C^\infty(\mathcal{O}', \mathcal{O})$ such that, for all $\xi \in \mathcal{O}'$,

$$\nabla_s \theta(s, \xi) = 0 \text{ if and only if } s = \tau(\xi) \quad (\text{A.8})$$

Moreover, assume that the Hessian $\theta''(s, \xi) = (\partial^2 \theta(s, \xi) / \partial s_j \partial s_k)$ satisfies

$$\det \theta''(\tau(\xi), \xi) \neq 0 \text{ for all } \xi \in \mathcal{O}' \quad (\text{A.9})$$

In addition, assume that for each compact set $K' \subset \mathcal{O}'$ and each $r_0 > 0$ there exists a constant $M = M(K, K', r_0)$ such that

$$|D_s^\delta g(r, s, \xi)| \leq M \text{ for all } r \geq r_0, s \in K, \xi \in K' \text{ and } |\delta| \leq m + 5 \quad (\text{A.10})$$

Then there exists a constant $C = C(K, K', r_0, g) > 0$ such that if $q(r, \xi)$ is defined by

$$I(r, \xi) = (2\pi)^{m/2} \frac{\exp\{ir \theta(\tau(\xi), \xi) + i\frac{\pi}{4} \operatorname{sgn} \theta''(\tau(\xi), \xi)\} g(r, \tau(\xi), \xi)}{r^{m/2} |\det \theta''(\tau(\xi), \xi)|^{1/2}} + q(r, \xi) \quad (\text{A.11})$$

then

$$|q(r, \xi)| \leq C r^{-m/2-1} \text{ for all } r \geq r_0 \text{ and } \xi \in K' \quad (\text{A.12})$$

In (A.11), $\operatorname{sgn} \theta''(s, \xi)$ is the signature of the real symmetric matrix $\theta''(s, \xi)$.

The uniform estimates given above are due, in essence, to M. Matsumura [6]. No proofs are offered because the theorems can be proved by following Matsumura's proofs and recognizing that under the hypotheses formulated above his estimates are uniform for $\xi \in K'$.

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13. ABSTRACT <u>Abstract.</u> Transient acoustic wave propagation is analyzed for the case of an unlimited plane-stratified fluid having constant density and sound speed $c(y)$ at depth y given by the Epstein profile $c^{-2}(y) = K \operatorname{sech}^2(y/H) + L \cosh(y/H) + M$ The acoustic potential is a solution of the wave equation $\partial_t^2 u - c^2(y) (\partial_x^2 u + \partial_y^2 u + \partial_z^2 u) = f(t, x, y)$ where $x = (x_1, x_2)$ are horizontal coordinates and $f(t, x, y)$ characterizes the wave sources. The principal results of the analysis show that u is the sum of a free component, which behaves like a diverging spherical wave for large t , and a guided component which is approximately localized in a region $ y - y_0 \leq h$ and propagates outward in horizontal planes like a diverging cylindrical wave.		

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