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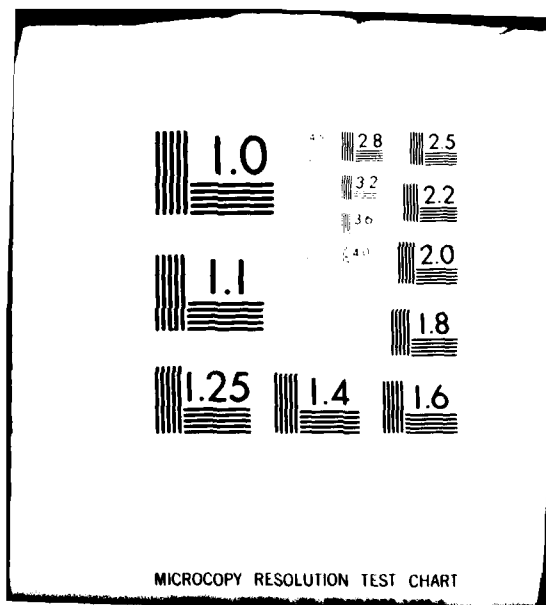
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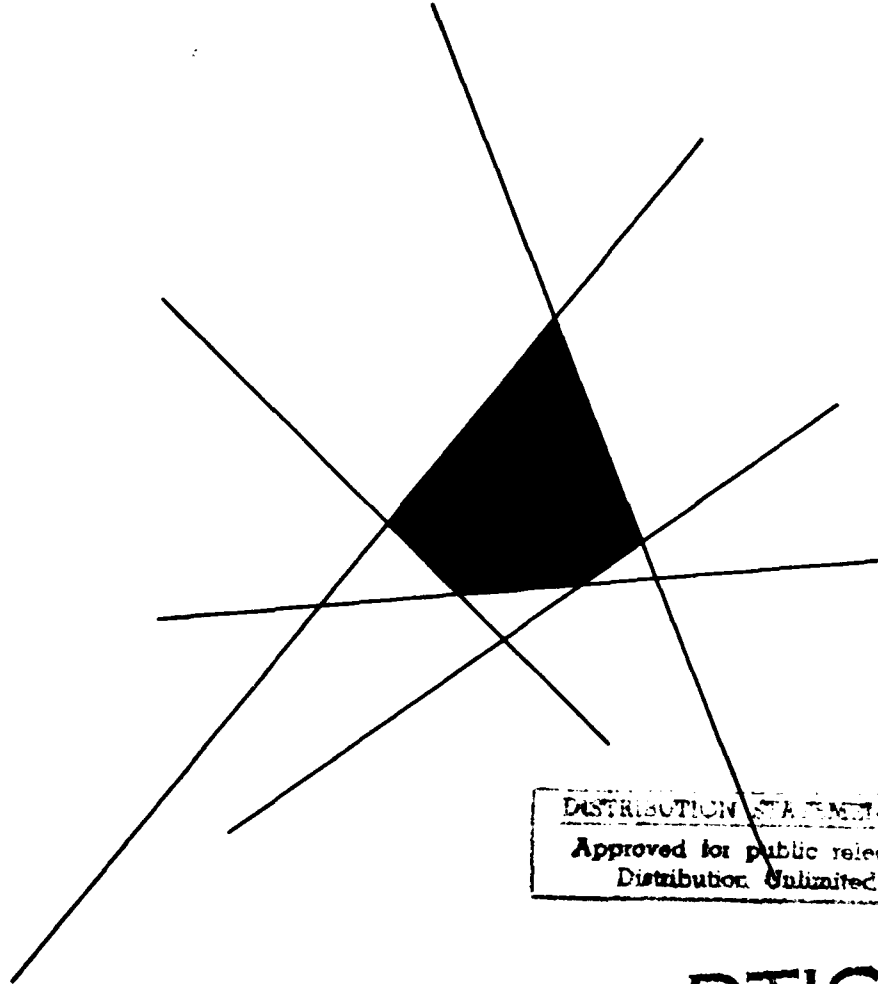
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THE OBSERVED HAZARD AND MULTICOMPONENT SYSTEMS

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ABSTRACT

*the integral of  $R(t)$  at from the  
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Let  $X$  denote the life of some system. We define the observed hazard rate at time  $t$ , call it  $R(t)$ , as the instantaneous probability (density) of failure of  $X$  at time  $t$  given survival up to  $t$  and given a complete description of the system state at  $t$ . We conjecture

that the total observed hazard--namely,  $\int_0^X R(t)dt$ --is an exponential

random variable with mean 1 and verify this for the special case when  $X$  is the distribution of system life of an  $n$  component system having an arbitrary monotone structure function.



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# THE OBSERVED HAZARD AND MULTICOMPONENT SYSTEMS

by

Mark Brown and Sheldon M. Ross

## 0. INTRODUCTION

Let  $X$  denote the survival time of some item and suppose  $X$  has distribution function  $F$  and density  $f$ . Then, the hazard rate function of  $X$ --call it  $\lambda(t)$ --is defined by

$$\lambda(t) = \frac{f(t)}{1 - F(t)} .$$

As

$$\lambda(t)dt \approx P\{t < X < t + dt \mid X > t\} ,$$

we can interpret  $\lambda(t)$  as the instantaneous probability (density) that an item of age  $t$  will fail.

The distribution  $F$  can be expressed as

$$1 - F(t) = \exp \left\{ - \int_0^t \lambda(s) ds \right\} ,$$

implying that

$$1 - F(X) = \exp \left\{ - \int_0^X \lambda(s) ds \right\} .$$

Now, as is well-known,  $1 - F(X)$  has a uniform distribution on  $(0,1)$  and as the negative logarithm of such a random variable has an exponential distribution with mean 1, it follows that

$$\int_0^X \lambda(s) ds \sim \text{Exponential } (1) ,$$

or, in words, the total hazard experienced by the item is exponentially distributed with mean 1.

In the above, the hazard rate at time  $t$  was defined to be the probability (density) of failure at  $t$  given survival up to that time. Now, however, let us suppose that we define the observed hazard at time  $t$ --call it  $R(t)$ --to be a random variable which represents the actual probability (density) of death at time  $t$  given not only the fact of survival up to time  $t$  but also a complete description of the "state" of the item at that time. (Such a quantity would, in general, be a random variable as it would be a function of the "state" of the item at time  $t$  and the state would itself generally be a random variable.) For instance, for a given individual,  $R(t)$  would denote the probability (density) of failure at time  $t$  given the life history of the individual up to time  $t$ .

We conjecture that  $\int_0^X R(t) dt$  is also an exponentially distributed random variable and in the following section, we verify this conjecture in the case of an  $n$  component coherent system in which components function for a random time and then fail.



### 1. COHERENT SYSTEMS AND THE OBSERVED HAZARD

We are given a system consisting of  $n$  components each of which is at all times either working or failed. In addition, we suppose that whether or not the system is working is solely determined as a function --call it  $\phi$ --of the component states. That is, letting  $x_i$  equal 1 or 0 according to whether or not the  $i^{\text{th}}$  component is working, then we suppose that there exists a nondecreasing binary function  $\phi$  such that

$$\phi(\underline{x}) = \phi(x_1, \dots, x_n) = \begin{cases} 1 & \text{if system works under state vector } \underline{x} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now that component  $i$  is initially working and will work for a random time having distribution  $F_i$  at which time it will fail,  $i = 1, \dots, n$ . Once a component has failed, it remains failed from that time on. Let  $x_i(t)$  equal 1 if component  $i$  is working at time  $t$  and 0 otherwise and define the random sets  $C(t)$  by

$$C(t) = \{i : \phi(1_i, \underline{x}(t)) = 1, \phi(0_i, \underline{x}(t)) = 0\}$$

where  $\phi(1_i, \underline{x}) = \phi(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$  and  $\phi(0_i, \underline{x}) = \phi(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ . In words,  $C(t)$  is the set of critical components at time  $t$ , where a component is critical at some time if its failure at that time would cause the system to go from working to failed.

If we let  $L$  denote the length of system life, then assuming independence of components, the total observed hazard experienced by the system during its lifetime could be expressed as

$$\int_0^L \sum_{i \in C(t)} \lambda_i(t) dt = \text{total system hazard}$$

where  $\lambda_i(t)$  is the (usual) hazard rate function of component  $i$  at time  $t$ . We now show that this random quantity is exponentially distributed with mean 1.

Theorem:

The total system hazard is an exponential random variable having mean 1.

Proof:

The proof is by induction on  $n$ . As the observed hazard rate is equal to the (usual) component hazard rate when  $n = 1$ , the result follows in this case. So assume the result for any system of  $n - 1$  components and consider an  $n$  component system. Say a component is a 1-component minimal cut set if its failure guarantees system failure even if all other components are working. We consider three cases.

Case 1:

There do not exist any 1-component minimal cut sets.

In this case, the observed hazard rate will be 0 until a component fails. At this point, the remaining hazard will be exponentially distributed with mean 1 by the induction hypothesis. Hence, the result follows in this case.

Case 2:

There exists exactly one 1-component minimal cut set--say  $\{1\}$ .

In this case, let  $L_1$  denote the life of component 1 and let  $T$  denote the first time any of the components 2 through  $n$  fail. Now, conditional on  $T = t$ , the total observed hazard can be expressed as

$$\int_0^{L_1} \lambda_1(s) dt \quad \text{if} \quad \int_0^{L_1} \lambda_1(s) ds \leq \int_0^t \lambda_1(s) ds$$

$$\int_0^t \lambda_1(s) ds + \text{Exp}(1) \quad \text{if} \quad \int_0^{L_1} \lambda_1(s) ds > \int_0^t \lambda_1(s) ds$$

where we have used the induction hypothesis in writing that the remaining observed hazard starting at time  $T = t$  and assuming that component 1 has not yet failed is exponentially distributed with mean 1. Thus from the above, we see that, given  $T = t$ , the total observed hazard has the same distribution as the random variable defined by

$$E_1 \quad \text{if} \quad E_1 < c$$

$$c + E_2 \quad \text{if} \quad E_2 > c$$

where  $c$  is a constant and  $E_1, E_2$  are independent exponential random variables each having mean 1. Such a random variable is easily seen to be also exponential with mean 1.

Case 3:

There exists at least two 1-component minimal cut sets--say {1} and {2} .

In this case, we can combine components 1 and 2 into a single component which fails when either one of them fails and the result follows from the induction hypothesis.

Remark:

The above proof goes through in an identical manner even when the component lifetimes are dependent. Of course, the observed hazard rate at time  $t$  would no longer be  $\sum_{i \in C(t)} \lambda_i(t)$  but would have to be suitably modified.

## 2. SOME FINAL REMARKS AND A HEURISTIC ARGUMENT

- (i) Whereas we have only established that the total observed hazard experienced by a system is exponentially distributed with mean 1 for the rather special system described in Section 1, we believe that this result holds with tremendous generality. (Another system in which we have been able to verify it is when events occur in accordance with some arbitrary point process and each event has a random nonnegative damage associated with it. The system is said to fail the first time the total cumulative damage exceeds some specified value.)
- (ii) An interesting sidelight about the system of Section 1 is that it is well known that if all component life distributions are IFR (increasing failure rate) in the sense that  $\lambda_i(t)$  is a monotone nondecreasing function for all  $i = 1, \dots, n$ , then it need not be the case that the system hazard rate is also increasing. However, it easily follows in this case of IFR component life distributions that the observed system hazard rate--namely,  $\sum_{i \in C(t)} \lambda_i(t)$ --increases up to the time of system failure.
- (iii) A general definition of the random hazard can be given along the following lines: Let  $\{F_s, 0 \leq s < \infty\}$  denote an increasing family of sigma fields and let  $X$  denote a stopping time defined on this family. Let

$$R(t) = \lim_{h \rightarrow 0} \frac{P\{t < X < t + h \mid F_t\}}{h}$$

where we assume the above limit exists almost surely.

We now claim that, when it is well defined,

$$(2.1) \quad \int_0^X R(t) dt \text{ has an exponential distribution with mean 1.}$$

We now present a heuristic argument of the above.

Heuristic Argument of (2.1)

Let  $H(t) = \int_0^t R(s) ds$ . We wish to argue that  $H(X)$  is exponential with mean 1 and to do so, we shall argue that its failure rate function--call it  $\lambda(s)$ --is identically 1. To show that  $\lambda(s) \equiv 1$ , let us condition on the event that  $H(X) \geq s$  and on the values of  $T_s$  and  $R(T_s)$  where  $T_s$  is defined to be the time at which  $H$  is equal to  $s$ --that is,  $H(T_s) = s$ . Now given  $H(X) \geq s$ ,  $T_s$ ,  $R(T_s) = \lambda_s$

$$X \geq T_s + \epsilon \text{ with probability } 1 - \epsilon \lambda_s + o(\epsilon).$$

Hence,

$$H(X) \geq H(T_s + \epsilon) \text{ with probability } 1 - \epsilon \lambda_s + o(\epsilon).$$

But

$$\begin{aligned} H(T_s + \epsilon) &= H(T_s) + \epsilon \lambda_s + o(\epsilon) \\ &= s + \epsilon \lambda_s + o(\epsilon) \end{aligned}$$

and so

$H(X) \geq s + \epsilon \lambda_s + o(\epsilon)$  with probability  $1 - \epsilon \lambda_s + o(\epsilon)$

which, for  $\lambda_s > 0$ , is roughly equivalent to

$H(X) \geq s + \delta$  with probability  $1 - \delta + o(\delta)$ .

Thus, independent of  $T_s$ ,  $\lambda_s$ , given that  $H(X) \geq s$ , it has probability  $\delta + o(\delta)$  of failing during the next  $\delta$  units of hazard. Thus,

$$P\{H(X) < s + \delta \mid H(X) \geq s\} = \delta + o(\delta).$$

Dividing the above by  $\delta$  and letting  $\delta$  go to 0 "proves" that the failure rate function of  $H(X)$  is identically one.