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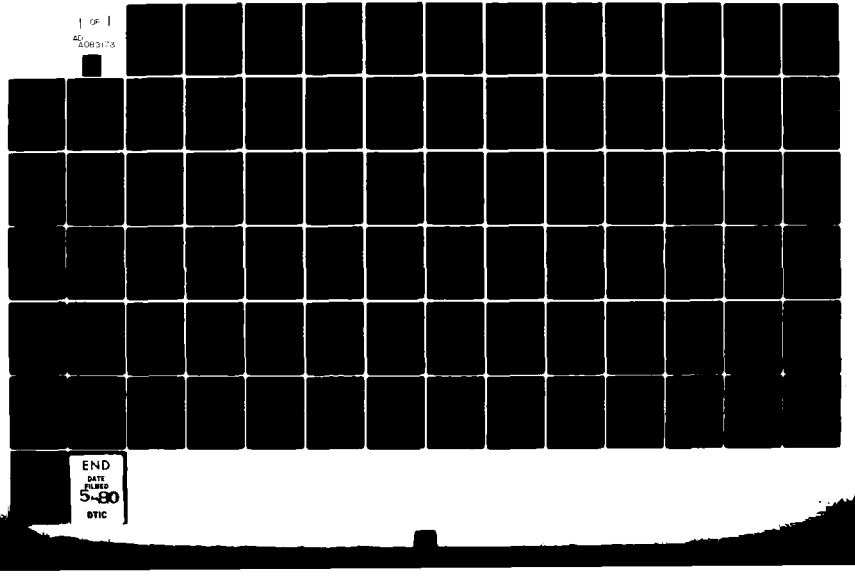

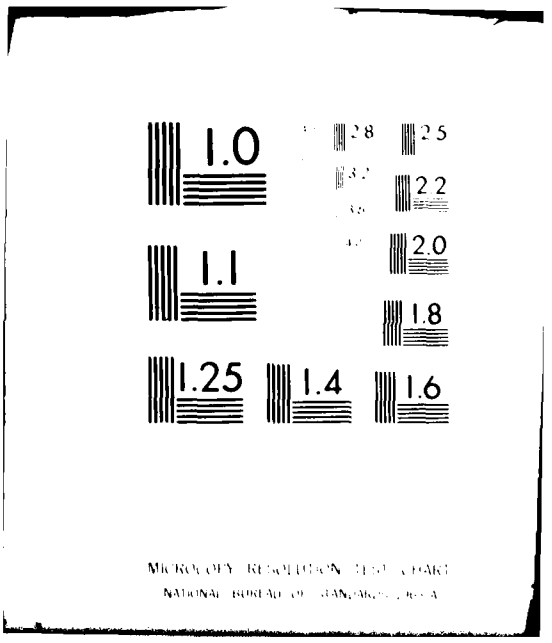
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ELASTICITY THEORY OF COMPOSITES

by

J. R. Willis

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Variational methods for estimating overall properties of composites are reviewed. Bounds that allow for general two-point correlations are emphasized and new explicit formulae are given for a two-component composite. A new general third-order bound is also given. Finally, a variational treatment of wave propagation problems for random composites is presented, of which the preceding static formulation is a limiting case.

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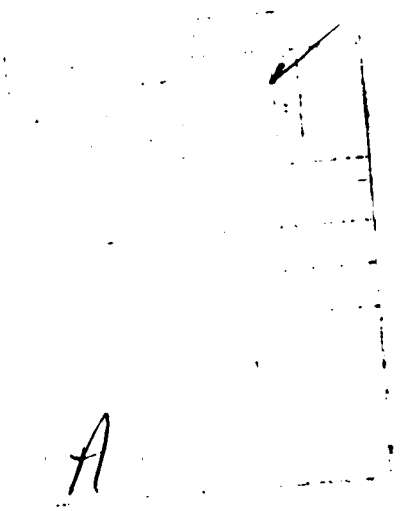

ELASTICITY THEORY OF COMPOSITES

by
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SUMMARY

The determination of the overall elastic moduli of random composites is approached using variational principles. Information on the configuration of a composite is never complete and strict estimates can do no more than take the form of bounds that use whatever statistical information is available. The classical Hashin-Shtrikman bounds are deduced as special cases of bounds that allow explicitly for any two-point correlations and it is shown that these general bounds take a simple form not given before for any two-phase composite. Self-consistent estimates are related to the variational formulation. Bounds of higher order are discussed and equations that define a new optimal bound of third order are given explicitly. The concluding sections treat wave propagation problems from the standpoint of a new variational principle which reduces to the Hashin-Shtrikman principle in the static limit.

1. INTRODUCTION

The subject of this article is the theoretical determination of the overall properties of inhomogeneous materials such as fibre-reinforced plastics. The distinguishing feature of these materials, to which the generic term 'composite' is applied, is that they are strongly inhomogeneous relative to a small length scale (the microscale) and yet, relative to a larger scale (the macroscale) over which variations in applied loads are significant, they appear either homogeneous or else their 'average' properties appear to vary smoothly. Generally, a composite may comprise n different phases, distributed either deterministically or at random, and it will be assumed that the microscale defined by the local phase geometry is sufficiently large for each phase to be treated as a continuum: this is certainly true of fibre-reinforced plastics and it is true also of polycrystalline aggregates. The latter represent a limiting case in which the number of phases is indefinitely large, each crystal orientation being regarded as defining a different phase.

The article will be concerned with the determination of the overall elastic response of a composite, since this particular aspect of material behaviour can be treated with reasonable precision. Some of the methods to be described have wider application (when suitably generalized), for example to the elastoplastic response of a polycrystal, but study of the simpler elastic problem permits some assessment of limitations inherent in these methods, by comparison with results obtained from methods that allow greater precision but are less easily generalized.

The elastic properties of composites have been studied quite intensively during the past twenty years and many of the simpler problems are

at a definitive state of development. A reasonably comprehensive review is contained in the recent book by Christensen (1979). The major gaps in Christensen's presentation concern allowance for random phase geometry when this is not isotropic, and wave propagation in randomly inhomogeneous composites. Progress has, in fact, been made in both of these areas which happen, too, to be of particular interest to the writer. Correspondingly, attention is focussed in this article upon problems for random media, which need not be isotropic. The presentation is self-contained; the longer-established results are deducible as special cases and some of the more important of these are discussed explicitly, though not in the detail given by Christensen. Apart from introductory material, the present review is largely complementary to one recently prepared by the writer (Willis, 1980d). This covered a somewhat broader range of topics and emphasised more strongly composites consisting of a matrix reinforced by inclusions. Distinctive features of the present work are the discussion of bounds, including a new simple formula which allows for arbitrary two-point correlations in a two-phase composite, new optimal bounds of third order and a variational treatment of wave propagation problems.

2. OVERALL PROPERTIES

Underlying the 'overall modulus' concept is the belief that, in a specimen subjected to some system of loading, the stress and strain fields vary in a complicated fashion from point to point but, if some form of 'running average' is taken which, in effect, replaces the fields at $\underset{\sim}{x}$ by weighted means evaluated over some neighbourhood centred on $\underset{\sim}{x}$, whose dimensions are small relative to overall specimen dimensions and yet large relative to the microscale, then the 'averaged' stress and strain fields will vary smoothly relative to the macroscale. It is difficult to define the averaging procedure precisely for a general composite but it corresponds, hopefully, to what might be measured by a strain guage or a transducer. The details of the fields in the neighbourhood of $\underset{\sim}{x}$ are then expected to be determined just by their mean values at $\underset{\sim}{x}$ and the local configuration of the material. These expectations are indeed borne out for composites with a periodic structure, for which the stress and strain fields may be expressed, asymptotically at least, as products of periodic functions with the period of the structure, and amplitudes that vary smoothly on the macroscale. For static problems, these ideas were enunciated by Sanchez Palencia (1974). The Floquet theory approach of Kohn, Krumhansl and Lee (1972) for dispersion relations for periodic composites is essentially similar, except that it does not even require the wavelength associated with the 'amplitude' function to be large.

Overall moduli are defined so that they relate 'average' stress to 'average' strain. Symbolically, if stress and strain tensors are denoted $\underset{\sim}{\sigma}$, $\underset{\sim}{e}$ and their local averages are correspondingly represented as $\overline{\underset{\sim}{\sigma}}$, $\overline{\underset{\sim}{e}}$, then the tensor of overall moduli $\overline{\underset{\sim}{L}}$ is defined so that

$$\bar{\underset{\sim}{\sigma}} = \bar{\underset{\sim}{L}}\,\bar{\underset{\sim}{e}} . \tag{2.1}$$

Strictly, if the relation (2.1) is to apply when $\bar{\underset{\sim}{\sigma}}$, $\bar{\underset{\sim}{e}}$ are generated from arbitrary boundary conditions, then $\bar{\underset{\sim}{L}}$ should be a non-local operator (Kröner, 1977) but, asymptotically, when boundary conditions are such that $\bar{\underset{\sim}{\sigma}}$, $\bar{\underset{\sim}{e}}$ vary smoothly relative to the microscale (and so hardly at all in a neighbourhood of $\underset{\sim}{x}$), $\bar{\underset{\sim}{L}}$ reduces to a tensor of moduli except, of course, in a thin layer close to the boundary of the specimen which is excluded from detailed consideration.

If the validity of (2.1) is accepted, the overall moduli $\bar{\underset{\sim}{L}}$ may be determined experimentally by studying the response of a macroscopically uniform specimen when it is subjected to loads that generate uniform average stress and strain fields. In fact, the tensor of moduli $\bar{\underset{\sim}{L}}$ may be defined in an unambiguous manner by subjecting a specimen that occupies a region V with boundary ∂V to surface displacements that have the form

$$u_i = \bar{e}_{ij} x_j \tag{2.2}$$

relative to a Cartesian basis, the summation convention being implied for repeated suffixes. Application of Gauss' theorem then shows that the volume average of the strain actually produced throughout V is $\bar{\underset{\sim}{e}}$ precisely ($\bar{\underset{\sim}{e}}$ having components $\bar{e}_{ij}$). If the average over V of the stress $\underset{\sim}{\sigma}$ is denoted by $\bar{\underset{\sim}{\sigma}}$, then $\bar{\underset{\sim}{\sigma}}$ depends linearly upon $\bar{\underset{\sim}{e}}$ and defines the tensor $\bar{\underset{\sim}{L}}$ in (2.1). The above discussion assumes, of course, that the phases of the composite are linearly elastic and that there is

perfect adhesion across phase boundaries. It may be noted that the components $\bar{\sigma}_{ij}$ of the mean stress may be determined from measured values of surface traction, through the relation

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_{\partial V} x_i t_j \, dS, \tag{2.3}$$

where V has also been used to denote the volume of the region V to avoid further symbols, and $t_j = \sigma_{jk} n_k$, the normal to ∂V having components n_k. Equation (2.3) follows from Gauss' theorem, together with the requirement that the stress $\underset{\sim}{\sigma}$ should be self-equilibrated; that is,

$$\sigma_{ij,j} = 0 \tag{2.4}$$

except at interphase boundaries, across which $\sigma_{ij} n_j$ is continuous. The latter proviso is included in (2.4) if the derivatives are interpreted in the sense of generalized functions (Gel'fand and Shilov, 1964): this interpretation will be used throughout the sequel to avoid the need for explicit discussion of jump conditions and it should not be assumed that sharp changes in properties are thereby excluded.

An alternative definition for $\underset{\sim}{\bar{L}}$ is obtained by applying to the specimen the traction boundary condition

$$\sigma_{ij} n_j = \bar{\sigma}_{ij} n_j . \tag{2.5}$$

Equation (2.3) then ensures that the average of the actual stress over

V is $\bar{\underset{\sim}{\sigma}}$ and $\bar{\underset{\approx}{L}}$ follows from (2.1), this time with $\bar{\underset{\sim}{e}}$ calculated as the average of the strain $\underset{\sim}{e}$ over V. Again, $\bar{\underset{\sim}{e}}$ can be determined from surface values of displacement alone, by use of the identity

$$\bar{e}_{ij} = \frac{1}{2V}\int_{\partial V}(u_i n_j + u_j n_i)\,dS. \tag{2.6}$$

Thus, either experiment determines $\bar{\underset{\approx}{L}}$ from values that are accessible to measurement, in principle at least. They were both proposed, as precise mathematical definitions for $\bar{\underset{\approx}{L}}$, by Hill (1963a). The two definitions are not equivalent to one another for arbitrarily inhomogeneous bodies and the extent to which they agree in their estimates for $\bar{\underset{\approx}{L}}$ provides a partial check on the validity of the 'overall modulus' concept.

The definitions are interesting for another reason, also developed by Hill (1963a), that, when either of the boundary conditions (2.2) or (2.5) is applied, then the average energy density $\bar{u}$ in the specimen is given by

$$2\bar{u} = \bar{\underset{\sim}{\sigma}}\,\bar{\underset{\sim}{e}} = \bar{\underset{\sim}{e}}\,\bar{\underset{\approx}{L}}\,\bar{\underset{\sim}{e}} \tag{2.7}$$

precisely, again by application of Gauss' theorem combined with one of (2.3), (2.6). The shorthand notation implicit in (2.1) has been extended in (2.7) which, in suffixes, would read

$$2\bar{u} = \bar{\sigma}_{ij}\bar{e}_{ij} = \bar{e}_{ij}\bar{L}_{ijkl}\bar{e}_{kl}. \tag{2.8}$$

In the sequel, suffixes will be avoided wherever possible.

The derivation of (2.7) has assumed the usual symmetry of the tensor of moduli $\underset{\sim}{L}$ which applies to the actual inhomogeneous specimen. Corresponding symmetry of $\bar{\underset{\sim}{L}}$ may be demonstrated by calculating the increment $\delta\bar{U}$ in mean energy corresponding to an increment $\delta\bar{\underset{\sim}{e}}$ in mean strain: the argument that led to (2.7) gives

$$\delta\bar{U} = \bar{\underset{\sim}{\sigma}}\,\delta\bar{\underset{\sim}{e}} = \delta\bar{\underset{\sim}{e}}\,\bar{\underset{\sim}{L}}\,\bar{\underset{\sim}{e}} \tag{2.9}$$

which, when compared with $\delta\bar{U}$ calculated by differentiating (2.7), gives the desired result.

Bounds upon components of the tensor of overall moduli $\bar{\underset{\sim}{L}}$ may now be developed from (2.7) and the classical variational principles for elasticity. For the boundary condition (2.2), the minimum energy principle asserts that

$$2\bar{U} \leq \frac{1}{V}\int_V \underset{\sim}{e}^* \underset{\sim}{L}\, \underset{\sim}{e}^*\, dx \tag{2.10}$$

for a strain field $\underset{\sim}{e}^*$ derived from any displacement field $\underset{\sim}{u}^*$ that conforms with (2.2) on ∂V. The simplest field that can be substituted into (2.10) is $\underset{\sim}{e}^* = \bar{\underset{\sim}{e}}$; this gives

$$\bar{\underset{\sim}{L}} \leq \underset{\sim}{L}_V, \tag{2.11}$$

where $\underset{\sim}{L}_V$ denotes the average of $\underset{\sim}{L}$ over V; it was introduced by Voigt (1889). The symbol '$\leq$' in (2.11) defines an ordering in the sense that the quadratic form $\bar{\underset{\sim}{e}}(\underset{\sim}{L}_V - \bar{\underset{\sim}{L}})\bar{\underset{\sim}{e}}$ is positive semidefinite,

for symmetric second-order tensors $\bar{\underline{\underline{e}}}$. Dually, for the boundary condition (2.2), the complementary energy principle asserts that

$$\frac{2}{V}\int_{\partial V} \underline{t}^* \underline{u}\, dS - \frac{1}{V}\int_V \underline{\underline{\sigma}}^* \underline{\underline{M}}\, \underline{\underline{\sigma}}^*\, d\underline{x} \leq 2\bar{U}, \tag{2.12}$$

where $t_i^* = \sigma_{ij}^* n_j$, $\underline{\underline{\sigma}}^*$ is any self-equilibrated stress field, $u_i = \bar{e}_{ij} x_j$ and $\underline{\underline{M}}$ is the tensor of compliances, inverse to $\underline{\underline{L}}$. The simplest stress field $\underline{\underline{\sigma}}^*$ to substitute into (2.12) is a constant; this given, upon use of Gauss' theorem,

$$2\underline{\underline{\sigma}}^* \bar{\underline{\underline{e}}} - \underline{\underline{\sigma}}^* \underline{\underline{M}}_R \underline{\underline{\sigma}}^* \leq 2\bar{U}, \tag{2.13}$$

where $\underline{\underline{M}}_R$ is the average of $\underline{\underline{M}}$ over V, introduced by Reuss (1929). The left side of (2.13) is maximized by taking

$$\underline{\underline{\sigma}}^* = \underline{\underline{L}}_R \bar{\underline{\underline{e}}}, \tag{2.14}$$

where $\underline{\underline{L}}_R$ is inverse to $\underline{\underline{M}}_R$. It follows then that

$$\underline{\underline{L}}_R \leq \bar{\underline{\underline{L}}} \leq \underline{\underline{L}}_V. \tag{2.15}$$

The result (2.15) is due to Hill (1952). It can also be deduced in a corresponding fashion if $\bar{\underline{\underline{L}}}$ is defined by the boundary condition (2.5). The bounds $\underline{\underline{L}}_R$, $\underline{\underline{L}}_V$ depend only upon the moduli and concentrations of the phases and take no account of how the phases are distributed. The construction of better bounds which make more allowance for the structure

of the composite is considered in later sections. Before proceeding, however, it may be remarked that corresponding mean value theorems can be developed for a body subjected to finite deformation. The relation (2.3) is valid in any continuum but it is useful to have a description that employs Lagrangian variables, based upon a fixed reference state. Relative to such a state, Hill (1972) has shown that an equation of the form (2.3) applies if the Cauchy stress $\underset{\sim}{\sigma}$ is replaced by nominal stress and x_i is interpreted as a Lagrangian coordinate. Alternatively, $\underset{\sim}{\sigma}$ may be replaced by Kirchhoff stress and x_i is interpreted as the current (Eulerian) coordinate, expressed as a function of the Lagrangian coordinates. An equation like (2.6) (not symmetrized) applies to the deformation gradient, and mean Kirchhoff stress, nominal stress and deformation gradient are related exactly as though the fields were uniform. Similar relations apply to rates which allow, for elastic materials, generalizations of (2.7) and (2.9); some implications for nonlinearly elastic composites have been explored by Ogden (1978).

3. STRESS POLARIZATION AND VARIATIONAL PRINCIPLES

Bounds better than (2.15) require the construction of stress and strain fields that make some explicit allowance for the microscopic arrangement of the composite. For this purpose, it proves useful to introduce a homogeneous 'comparison' material with moduli $\underset{\sim}{L}_0$ and to set

$$\underset{\sim}{L} = \underset{\sim}{L}_0 + \delta\underset{\sim}{L} . \tag{3.1}$$

Substitution of (3.1) into the constitutive relation

$$\underset{\sim}{\sigma} = \underset{\sim}{L}\,\underset{\sim}{e} \tag{3.2}$$

and thence into the equilibrium equation (2.4) gives

$$\operatorname{div}\left(\underset{\sim}{L}_0\,\underset{\sim}{e}\right) + \operatorname{div}\left(\delta\underset{\sim}{L}\,\underset{\sim}{e}\right) = \underset{\sim}{0} , \quad \underset{\sim}{x} \in V, \tag{3.3}$$

which is to be solved together with whatever boundary condition is given. The discussion to follow will be phrased in terms of the displacement condition

$$\underset{\sim}{u} = \underset{\sim}{u}_0 , \quad \underset{\sim}{x} \in \partial V, \tag{3.4}$$

with a mind to subsequent specialization to (2.2). If the Green's function $\underset{\sim}{G}$ for the homogeneous comparison material is known, (3.3) and (3.4) may be expressed together in the form

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{u}_0(\underset{\sim}{x}) + \int_V \underset{\sim}{G}(\underset{\sim}{x},\underset{\sim}{x}')\,\mathrm{div}\,(\delta\underset{\sim}{L}\,\underset{\sim}{e})(\underset{\sim}{x}')\,d\underset{\sim}{x}', \tag{3.5}$$

where $\underset{\sim}{u}_0(\underset{\sim}{x})$ is the field that the boundary condition (3.4) would generate by itself in the homogeneous comparison body. An integral equation for the strain $\underset{\sim}{e}(\underset{\sim}{x})$ is now deduced by integration by parts, followed by differentiating with respect to $\underset{\sim}{x}$. This gives

$$\underset{\sim}{e}(\underset{\sim}{x}) = \underset{\sim}{e}_0(\underset{\sim}{x}) - \int_V \underset{\sim}{\Gamma}(\underset{\sim}{x},\underset{\sim}{x}')\,\delta\underset{\sim}{L}(\underset{\sim}{x}')\,\underset{\sim}{e}(\underset{\sim}{x}')\,d\underset{\sim}{x}', \tag{3.6}$$

where $\underset{\sim}{e}_0$ is the strain associated with $\underset{\sim}{u}_0$ and $\underset{\sim}{\Gamma}(\underset{\sim}{x},\underset{\sim}{x}')$ has components

$$\Gamma_{ijkl}(\underset{\sim}{x},\underset{\sim}{x}') = \partial^2 G_{jk}(\underset{\sim}{x},\underset{\sim}{x}')/\partial x_i\,\partial x_l'\,\Big|_{(ij)(kl)}. \tag{3.7}$$

It has a singularity of order $|\underset{\sim}{x}-\underset{\sim}{x}'|^{-3}$ which is to be interpreted in the sense of generalized functions (Gel'fand and Shilov, 1964). In the sequel, (3.6) will be written even more briefly as

$$\underset{\sim}{e} = \underset{\sim}{e}_0 - \underset{\sim}{\Gamma}\delta\underset{\sim}{L}\,\underset{\sim}{e}. \tag{3.8}$$

If the components of $\delta\underset{\sim}{L}$ are small, equation (3.8) can be solved by iteration and the resulting series, suitably truncated, can be substituted into the energy and complementary energy principles. This has been developed in detail by Dederichs and Zeller (1973); the validity of the bounds is, of course, independent of convergence of the perturbation series.

It is possible, however, to make rather better use of the work

implicit in the construction of the bounds of Dederichs and Zeller (1973), by considering a form for $\underset{\sim}{e}$ that contains some parameters which may be chosen optimally. It is helpful for this purpose (as well as for other, subsequent, developments) to introduce the stress polarization $\underset{\sim}{\tau}$ so that

$$\underset{\sim}{\tau} = \delta\underset{\sim}{L}\,\underset{\sim}{e} = (\underset{\sim}{L} - \underset{\sim}{L}_0)\,\underset{\sim}{e} \tag{3.9}$$

and

$$\underset{\sim}{\sigma} = \underset{\sim}{L}_0\,\underset{\sim}{e} + \underset{\sim}{\tau}. \tag{3.10}$$

Equation (3.8) gives

$$\underset{\sim}{e} = \underset{\sim}{e}_0 - \underset{\sim}{\Gamma}\,\underset{\sim}{\tau} \tag{3.11}$$

and, for any choice $\underset{\sim}{\tau}^*$ of $\underset{\sim}{\tau}$, equations (3.11) and (3.10) generate strain and stress fields $\underset{\sim}{e}^*$ and $\underset{\sim}{\sigma}^*$ that are, respectively, compatible with boundary conditions and self-equilibrated. If the mean energy density in the actual body is called $\bar{U}$, the minimum energy principle gives

$$2\bar{U} \leq (\underset{\sim}{e}^*, \underset{\sim}{L}_0\,\underset{\sim}{e}^*) + (\underset{\sim}{e}^*, \delta\underset{\sim}{L}\,\underset{\sim}{e}^*), \tag{3.12}$$

where the 'inner product' is defined so that

$$(\underset{\sim}{f}, \underset{\sim}{g}) = \frac{1}{V}\int_V \underset{\sim}{f}\,\underset{\sim}{g}\,d\underset{\sim}{x}. \tag{3.13}$$

The first term on the right side of (3.12) is simplified by employing (3.10) and (3.11) for $\underset{\sim}{\sigma}^*$, $\underset{\sim}{e}^*$ and $\underset{\sim}{\tau}^*$ and using the 'virtual work' equality

$$(\underset{\sim}{\sigma}_1, \underset{\sim}{e}_2) = 0 \tag{3.14}$$

for any self-equilibrated stress field $\underset{\sim}{\sigma}_1$ and strain field $\underset{\sim}{e}_2$ compatible with zero boundary displacements. Then, in terms of $\underset{\sim}{\tau}^*$, (3.12) becomes

$$2(\bar{U} - \bar{U}_0) \leq (\underset{\sim}{\tau}^*, \underset{\sim}{\Gamma}\underset{\sim}{\tau}^*) + (\underset{\sim}{e}_0, \delta\underset{\sim}{L}\,\underset{\sim}{e}_0) - 2(\underset{\sim}{e}_0, \delta\underset{\sim}{L}\,\underset{\sim}{\Gamma}\underset{\sim}{\tau}^*) + (\underset{\sim}{\Gamma}\underset{\sim}{\tau}^*, \delta\underset{\sim}{L}\,\underset{\sim}{\Gamma}\underset{\sim}{\tau}^*), \tag{3.15}$$

where $2\bar{U}_0 = (\underset{\sim}{e}_0, \underset{\sim}{L}_0\underset{\sim}{e}_0)$ The inequality (3.15) will be used in full later. Clearly, the last term is the one whose evaluation is the most complicated. A variational principle in which it does not appear can, in fact, be generated by noting that the exact polarization $\underset{\sim}{\tau}$ must satisfy the equation

$$[(\delta\underset{\sim}{L})^{-1} + \underset{\sim}{\Gamma}]\underset{\sim}{\tau} = \underset{\sim}{e}_0, \tag{3.16}$$

from (3.9) and (3.11). This equation is self-adjoint and hence the actual field $\underset{\sim}{\tau}$ extremizes the functional

$$F(\underset{\sim}{\tau}^*) = (\underset{\sim}{\tau}^*, (\delta\underset{\sim}{L})^{-1}\underset{\sim}{\tau}^*) + (\underset{\sim}{\tau}^*, \underset{\sim}{\Gamma}\underset{\sim}{\tau}^*) - 2(\underset{\sim}{\tau}^*, \underset{\sim}{e}_0). \tag{3.17}$$

This principle was deduced from the field equations by Hashin and Shtrikman (1962a) and directly from (3.16) by Willis (1977). Here, however, it seems particularly apt to give a derivation which corresponds to that of Hill (1963b). This shows how the Hashin-Shtrikman principle is related to the classical energy principles. Reverting to (3.15), therefore, the offending last term is rewritten

$$\left(\underset{\sim}{\Gamma}\underset{\sim}{\tau}^*, \delta\underset{\sim}{L}\,\underset{\sim}{\Gamma}\underset{\sim}{\tau}^*\right) = \left(\left[(\delta\underset{\sim}{L})^{-1}+\underset{\sim}{\Gamma}\right]\underset{\sim}{\tau}^*-\underset{\sim}{e}_0, \delta\underset{\sim}{L}\left(\left[(\delta\underset{\sim}{L})^{-1}+\underset{\sim}{\Gamma}\right]\underset{\sim}{\tau}^*-\underset{\sim}{e}_0\right)\right)$$

$$-\left(\underset{\sim}{\tau}^*, (\delta\underset{\sim}{L})^{-1}\underset{\sim}{\tau}^*\right) - 2\left(\underset{\sim}{\tau}^*, \underset{\sim}{\Gamma}\underset{\sim}{\tau}^*\right) + 2\left(\underset{\sim}{e}_0, \underset{\sim}{\tau}^*+\delta\underset{\sim}{L}\,\underset{\sim}{\Gamma}\underset{\sim}{\tau}^*\right) - \left(\underset{\sim}{e}_0, \delta\underset{\sim}{L}\,\underset{\sim}{e}_0\right); \tag{3.18}$$

since the exact $\underset{\sim}{\tau}$ satisfies (3.16), the first term on the right of (3.18) is zero when $\underset{\sim}{\tau}^* = \underset{\sim}{\tau}$ and is small when $\underset{\sim}{\tau}^*$ is a good approximation to $\underset{\sim}{\tau}$. Hence, (3.15) gives

$$2(\bar{U}-\bar{U}_0) \le -\left(\underset{\sim}{\tau}^*, (\delta\underset{\sim}{L})^{-1}\underset{\sim}{\tau}^*\right) - \left(\underset{\sim}{\tau}^*, \underset{\sim}{\Gamma}\underset{\sim}{\tau}^*\right) + 2\left(\underset{\sim}{\tau}^*, \underset{\sim}{e}_0\right) + \left(\underset{\sim}{\varepsilon}^*, \delta\underset{\sim}{L}\,\underset{\sim}{\varepsilon}^*\right), \tag{3.19}$$

where $\underset{\sim}{\varepsilon}^*$ has been written for the "error" $\left[(\delta\underset{\sim}{L})^{-1}+\underset{\sim}{\Gamma}\right]\underset{\sim}{\tau}^*-\underset{\sim}{e}_0$. If, now, $\underset{\sim}{L}_0$ is chosen in such a way that $\delta\underset{\sim}{L}$ is negative definite at each point of V, discarding the last term in (3.19) gives

$$2(\bar{U}-\bar{U}_0) \le -F(\underset{\sim}{\tau}^*). \tag{3.20}$$

The extreme value of $F(\underset{\sim}{\tau}^*)$ is $2(\bar{U}_0-\bar{U})$, whatever $\underset{\sim}{L}_0$ is chosen. The inequality (3.19) shows that the Hashin-Shtrikman principle embodied in (3.20) is weaker than the corresponding classical principle, through

neglect of the term quadratic in $\underset{\sim}{\varepsilon}^*$. Hill (1963b) made no use of the operator equation (3.16) but instead gave $\underset{\sim}{\varepsilon}^*$ in the form

$$\underset{\sim}{\varepsilon}^* = (\delta \underset{\sim}{L})^{-1}(\underset{\sim}{\sigma}^* - \underset{\sim}{L}\,\underset{\sim}{e}^*). \tag{3.21}$$

The complementary energy principle

$$2(\underset{\sim}{\sigma}^*, \underset{\sim}{e}_0) - (\underset{\sim}{\sigma}^*, \underset{\sim}{M}\underset{\sim}{\sigma}^*) \leq 2\bar{U} \tag{3.22}$$

may be expanded similarly: with the notation

$$\underset{\sim}{M} = \underset{\sim}{M}_0 + \delta \underset{\sim}{M} \tag{3.23}$$

so that $\underset{\sim}{M}_0$ is inverse to $\underset{\sim}{L}_0$, and some use of the identity

$$\underset{\sim}{L}_0 \delta \underset{\sim}{M} + \delta \underset{\sim}{L}\,(\underset{\sim}{M}_0 + \delta \underset{\sim}{M}) = \underset{\sim}{0}, \tag{3.24}$$

it follows that

$$-\mathcal{F}(\underset{\sim}{\tau}^*) - (\underset{\sim}{L}_0 \underset{\sim}{\varepsilon}^*, \delta \underset{\sim}{M}\, \underset{\sim}{L}_0 \underset{\sim}{\varepsilon}^*) \leq 2(\bar{U} - \bar{U}_0). \tag{3.25}$$

If $\underset{\sim}{L}_0$ is chosen so that $\delta \underset{\sim}{M}$ is negative definite (or, equivalently, so that $\delta \underset{\sim}{L}$ is positive definite), (3.25) implies

$$-\mathcal{F}(\underset{\sim}{\tau}^*) \leq 2(\bar{U} - \bar{U}_0). \tag{3.26}$$

The traction boundary value problem can be treated in a similar way.

Its duality with the displacement problem is best displayed by introducing a strain polarization $\underset{\sim}{\eta}$, so that

$$\underset{\sim}{\eta} = \underset{\sim}{M}_0 \underset{\sim}{\tau} \quad \text{and} \quad \underset{\sim}{e} = \underset{\sim}{M}_0 \underset{\sim}{\sigma} - \underset{\sim}{\eta}. \tag{3.27}$$

If tractions are specified that would generate the stress field $\underset{\sim}{\sigma}_0$ in the comparison material, an operator $\underset{\sim}{\Delta}_1$ may be defined so that

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}_0 + \underset{\sim}{\Delta}_1 \underset{\sim}{\eta}, \tag{3.28}$$

which is analogous to (3.11). The corresponding analogue of (3.16) is

$$[\underset{\sim}{\Delta}_1 + (\delta \underset{\sim}{M})^{-1}] \underset{\sim}{\eta} = -\underset{\sim}{\sigma}_0, \tag{3.29}$$

with which the functional

$$\mathcal{G}_1(\underset{\sim}{\eta}^*) = -(\underset{\sim}{\eta}^*, [\underset{\sim}{\Delta}_1 + (\delta \underset{\sim}{M})^{-1}] \underset{\sim}{\eta}^*) - 2(\underset{\sim}{\eta}^*, \underset{\sim}{\sigma}_0) \tag{3.30}$$

may be associated. The minimum energy principle and the complementary energy principle now in effect change roles. The analogue of (3.12), from the complementary principle, is

$$2\bar{U} \leq (\underset{\sim}{\sigma}^*, \underset{\sim}{M}_0 \underset{\sim}{\sigma}^*) + (\underset{\sim}{\sigma}^*, \delta \underset{\sim}{M} \underset{\sim}{\sigma}^*) \tag{3.31}$$

which, expanded in terms of $\underset{\sim}{\eta}^*$, is

$$2(\bar{u}-\bar{u}_0) \le -(\underset{\sim}{\eta}^*, \underset{\sim}{\Delta}_1 \underset{\sim}{\eta}^*) + (\underset{\sim}{\sigma}_0, \delta\underset{\sim}{M}\,\underset{\sim}{\sigma}_0) + 2(\underset{\sim}{\sigma}_0, \delta\underset{\sim}{M}\,\underset{\sim}{\Delta}_1 \underset{\sim}{\eta}^*) + (\underset{\sim}{\Delta}_1 \underset{\sim}{\eta}^*, \delta\underset{\sim}{M}\,\underset{\sim}{\Delta}_1 \underset{\sim}{\eta}^*). \tag{3.32}$$

This, finally, can be subjected to the reasoning that produced (3.19), to give

$$2(\bar{u}-\bar{u}_0) \le -\mathcal{G}_1(\underset{\sim}{\eta}^*) + (\underset{\sim}{L}_0 \underset{\sim}{\varepsilon}^*, \delta\underset{\sim}{M}\,\underset{\sim}{L}_0 \underset{\sim}{\varepsilon}^*), \tag{3.33}$$

so that

$$2(\bar{u}-\bar{u}_0) \le -\mathcal{G}_1(\underset{\sim}{\eta}^*) \tag{3.34}$$

whenever $\underset{\sim}{M}_0$ is chosen so that $\delta\underset{\sim}{M}$ is negative definite (or, equivalently, so that $\delta\underset{\sim}{L}$ is positive definite). The "error" $\underset{\sim}{\varepsilon}^*$ in (3.33) still conforms to (3.21); in terms of $\underset{\sim}{\eta}^*$,

$$\underset{\sim}{L}_0 \underset{\sim}{\varepsilon}^* = -\left[\underset{\sim}{\Delta}_1 + (\delta\underset{\sim}{M})^{-1}\right]\underset{\sim}{\eta}^* - \underset{\sim}{\sigma}_0. \tag{3.35}$$

It is, of course, possible to express the functional $\mathcal{F}$ in terms of $\underset{\sim}{\eta}^*$ and $\mathcal{G}_1$ in terms of $\underset{\sim}{\tau}^*$, with $\underset{\sim}{\tau}^*$ and $\underset{\sim}{\eta}^*$ related by $(3.27)_1$. For example, equation (3.16) can be written

$$\left[\underset{\sim}{L}_0(\delta\underset{\sim}{L})^{-1}\underset{\sim}{L}_0 + \underset{\sim}{L}_0\left[\underset{\sim}{\Gamma}\,\underset{\sim}{L}_0\right]\right]\underset{\sim}{\eta} = \underset{\sim}{\sigma}_0 \tag{3.36}$$

from which follows, using a rearrangement of (3.24), an equation of the form of (3.29), with the operator $\underset{\sim}{\Delta}_1$ replaced by $\underset{\sim}{\Delta}$, where

$$\underset{\sim}{\Delta} = \underset{\sim}{L}_0 - \underset{\sim}{L}_0 \underset{\sim}{\Gamma} \underset{\sim}{L}_0 . \tag{3.37}$$

Correspondingly, $\mathcal{F}(\underset{\sim}{\tau}^*) = \mathcal{G}(\underset{\sim}{\eta}^*)$, where $\mathcal{G}(\underset{\sim}{\eta}^*)$ is defined by (3.30) but with $\underset{\sim}{\Delta}_1$ replaced by $\underset{\sim}{\Delta}$. Dually, $\mathcal{G}_1(\underset{\sim}{\eta}^*) = \mathcal{F}_1(\underset{\sim}{\tau}^*)$, where $\mathcal{F}_1(\underset{\sim}{\tau}^*)$ is defined by (3.17), with $\underset{\sim}{\Gamma}$ replaced by $\underset{\sim}{\Gamma}_1$, where

$$\underset{\sim}{\Gamma}_1 = \underset{\sim}{M}_0 - \underset{\sim}{M}_0 \underset{\sim}{\Delta}_1 \underset{\sim}{M}_0 . \tag{3.38}$$

Finally, there are duals of (3.25) and (3.26), obtained from the minimum energy principle for the traction problem:

$$-\mathcal{G}_1(\underset{\sim}{\eta}^*) - (\underset{\sim}{\varepsilon}^*, \delta \underset{\sim}{L}\, \underset{\sim}{\varepsilon}^*) \leq 2(\bar{U} - \bar{U}_0) \tag{3.39}$$

and

$$-\mathcal{G}_1(\underset{\sim}{\eta}^*) \leq 2(\bar{U} - \bar{U}_0) \tag{3.40}$$

whenever $\delta \underset{\sim}{L}$ is negative definite everywhere in V.

4. INCLUSION PROBLEMS

For any composite that comprises a matrix reinforced (or weakened) by inclusions, a fundamental problem is to determine the way in which any particular inclusion perturbs the field that the matrix would support in its absence. Interaction effects between inclusions are significant but the problem of finding the effect of a single inclusion in an infinite homogeneous matrix provides an essential building block for considering a general composite. This basic problem is discussed now. The integral equations (3.8) or (3.16) are relevant and it is natural to identify the comparison material with the matrix. With a slight distortion of earlier notation, therefore, the matrix is taken to have moduli $\underset{\sim}{L}_0$ while the inclusion, which occupies a region D, is taken to be uniform, with moduli $\underset{\sim}{L}$. The field $\underset{\sim}{e}_0$ is now the strain field that would be present if the matrix contained no inclusion and the term involving $\underset{\sim}{\Gamma}$ represents the perturbation of this field produced by the inclusion. $\delta\underset{\sim}{L}$ is zero except over D and (3.8) is now an integral equation only for $\underset{\sim}{x} \in D$; elsewhere, it provides a representation for the strain field $\underset{\sim}{e}$. Equivalently, $\underset{\sim}{\tau}$ is non-zero only over D and equation (3.16) applies only for $\underset{\sim}{x} \in D$. Either of these equations could be studied equally well but there is some later advantage in concentrating upon (3.16).

The region D occupied by the inclusion will be assumed to be bounded. The three-dimensional problem will be considered first but afterwards a short discussion will be given of the corresponding two-dimensional problem for which the inclusion may be an infinite cylinder. The matrix is infinite in extent and so the operator $\underset{\sim}{\Gamma}$ that appears in (3.16) is obtained, via (3.7), from the infinite-body Green's function

$\underset{\sim}{G}^{\infty}$. This satisfies, in matrix notation, the differential equation

$$\underset{\sim}{L}_0(\nabla)\underset{\sim}{G}^{\infty} + \underset{\sim}{I}\,\delta(\underset{\sim}{x}) = \underset{\sim}{0}\,, \tag{4.1}$$

where $\underset{\sim}{L}_0(\nabla)$ is a differential operator whose symbol $\underset{\sim}{L}_0(\underset{\sim}{\xi})$ has components

$$[\underset{\sim}{L}_0(\underset{\sim}{\xi})]_{ik} = (\underset{\sim}{L}_0)_{ijkl}\,\xi_j\,\xi_l\,. \tag{4.2}$$

Equation (4.1) can be solved by noting the plane-wave decomposition

$$\delta(\underset{\sim}{x}) = -\frac{1}{8\pi^2}\int_{|\underset{\sim}{\xi}|=1} dS\;\delta''(\underset{\sim}{\xi}\cdot\underset{\sim}{x}) \tag{4.3}$$

given by Gel'fand and Shilov (1964). Solving the differential equation (4.1) with $\delta(\underset{\sim}{x})$ replaced by $\delta''(\underset{\sim}{\xi}\cdot\underset{\sim}{x})$ and then performing the superposition implied by (4.3) gives

$$\underset{\sim}{G}^{\infty}(\underset{\sim}{x}) = \frac{1}{8\pi^2}\int_{|\underset{\sim}{\xi}|=1} dS\,[\underset{\sim}{L}_0(\underset{\sim}{\xi})]^{-1}\delta(\underset{\sim}{\xi}\cdot\underset{\sim}{x}). \tag{4.4}$$

The components of $\underset{\sim}{\Gamma}^{\infty}$ (so called to correspond with $\underset{\sim}{G}^{\infty}$) now follow using (3.7):

$$\Gamma^{\infty}_{ijkl}(\underset{\sim}{x},\underset{\sim}{x}') = \frac{-1}{8\pi^2}\int_{|\underset{\sim}{\xi}|=1} dS\;\xi_i\,[\underset{\sim}{L}_0(\underset{\sim}{\xi})]^{-1}_{jk}\,\xi_l\,\delta''[\underset{\sim}{\xi}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')]\Big|_{(ij)(kl)} \tag{4.5}$$

and substitution of (4.5) into (3.16) gives

$$(\delta \underset{\sim}{L})^{-1}\underset{\sim}{\tau}(\underset{\sim}{x}) - \frac{1}{8\pi^2}\int_{|\underset{\sim}{\xi}|=1} dS\ \underset{\sim}{\tilde{\Gamma}}''(\underset{\sim}{\xi})\,\underset{\sim}{\check{\tau}}''(\underset{\sim}{\xi}.\underset{\sim}{x},\underset{\sim}{\xi}) = \underset{\sim}{e}_0(\underset{\sim}{x}),\ \underset{\sim}{x}\in D, \tag{4.6}$$

where

$$\underset{\sim}{\check{\tau}}''(p,\underset{\sim}{\xi}) = \frac{\partial^2}{\partial p^2}\,\underset{\sim}{\check{\tau}}(p,\underset{\sim}{\xi}) = \frac{\partial^2}{\partial p^2}\int \underset{\sim}{\tau}(\underset{\sim}{x}')\,\delta(p-\underset{\sim}{\xi}.\underset{\sim}{x}')\,d\underset{\sim}{x}'. \tag{4.7}$$

$\underset{\sim}{\check{\tau}}(p,\underset{\sim}{\xi})$ is called the Radon transform of $\underset{\sim}{\tau}(\underset{\sim}{x})$. In (4.6), the term $\underset{\sim}{\tilde{\Gamma}}''(\underset{\sim}{\xi})$ represents the part of the integrand of (4.5) that depends upon $\underset{\sim}{\xi}$ only; it is actually the Fourier transform of $\underset{\sim}{\Gamma}''$.

It is not clear whether (4.6) is superior to (3.16) in general but it is particularly useful when D is an ellipsoid. To illustrate this, consider first the case for which D represents a sphere of radius a, centred at the origin. If $\underset{\sim}{\tau}$ is taken constant over D, its Radon transform is obtained as $\underset{\sim}{\tau}$ times the area of intersection of the plane $\underset{\sim}{\xi}.\underset{\sim}{x} = p$ with D; that is,

$$\underset{\sim}{\check{\tau}}(p,\underset{\sim}{\xi}) = \underset{\sim}{\tau}\,\pi(a^2-p^2)H(a^2-p^2). \tag{4.8}$$

The second derivative that appears in (4.6) is then constant when $|\underset{\sim}{\xi}|=1$ and $\underset{\sim}{x}\in D$, so that the left side of (4.6) is constant over D. If $\underset{\sim}{e}_0(\underset{\sim}{x})$ is taken constant, therefore, $\underset{\sim}{\tau}$ may be taken constant and must satisfy the algebraic equation

$$\left[(\delta \underset{\sim}{L})^{-1} + \underset{\sim}{P}\right]\underset{\sim}{\tau} = \underset{\sim}{e}_0, \tag{4.9}$$

where the constant tensor $\underset{\sim}{P}$ is given by

$$\underset{\sim}{P} = \frac{1}{4\pi} \int_{|\underset{\sim}{\xi}|=1} dS \, \underset{\sim}{\tilde{\Gamma}}^{\infty}(\underset{\sim}{\xi}). \tag{4.10}$$

This result was first given by Kneer (1965). The case for which D is the ellipsoid

$$D = \{ \underset{\sim}{x} : \underset{\sim}{x}^T \underset{\sim}{A}^T \underset{\sim}{A} \underset{\sim}{x} < 1 \} \tag{4.11}$$

can be reduced to the one just considered by the dual transformations

$$\underset{\sim}{y} = \underset{\sim}{A}\,\underset{\sim}{x}, \quad \underset{\sim}{\xi} = \underset{\sim}{A}^T \underset{\sim}{\zeta} \quad ; \tag{4.12}$$

this reproduces equation (4.9) except that $\underset{\sim}{P}$ is now given as

$$\underset{\sim}{P} = \frac{1}{4\pi|\underset{\sim}{A}|} \int_{|\underset{\sim}{\xi}|=1} dS \, \underset{\sim}{\tilde{\Gamma}}^{\infty}(\underset{\sim}{\xi}) [\underset{\sim}{\xi}^T (\underset{\sim}{A}^T \underset{\sim}{A})^{-1} \underset{\sim}{\xi}]^{-3/2}. \tag{4.13}$$

This result was given first by Khatchaturyan (1967) for a generally anisotropic matrix. For an isotropic matrix, the result was derived using methods of potential theory by Eshelby (1957), who expressed $\underset{\sim}{P}$ in terms of elliptic integrals. Eshelby (1961) showed further that, for an inclusion in an isotropic matrix, a polynomial field $\underset{\sim}{e}_0(\underset{\sim}{x})$ would generate a polynomial field within the inclusion: the corresponding result for a generally anisotropic matrix was given by Asaro and Barnett (1975) and Willis (1975), essentially by demonstrating that, if $\underset{\sim}{\tau}(\underset{\sim}{x})$ is a polynomial of degree n in $\underset{\sim}{x}$, then $\underset{\sim}{\check{\tau}}(p, \underset{\sim}{\xi})$ is a polynomial of degree $n+2$ in p.

The tensor $\underset{\sim}{P}$ given by (4.13) has generally to be computed. However,

for illustration, it will be evaluated for a spherical inclusion in an isotropic matrix. Thus, $\underset{\sim}{L}_0$ is taken to have components

$$(\underset{\sim}{L}_0)_{ijkl} = \kappa \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\right), \tag{4.14}$$

so that it has bulk modulus κ and shear modulus μ. It is helpful to use the symbolic notation

$$\underset{\sim}{L}_0 = (3\kappa, 2\mu) \tag{4.15}$$

introduced by Hill (1965a), in which the product of two isotropic tensors $\underset{\sim}{A} = (3\kappa_A, 2\mu_A)$ and $\underset{\sim}{B} = (3\kappa_B, 2\mu_B)$ is given by

$$\underset{\sim}{A}\,\underset{\sim}{B} = \left((3\kappa_A)(3\kappa_B), (2\mu_A)(2\mu_B)\right). \tag{4.16}$$

It is easily deduced from (4.14) that

$$\left[\underset{\sim}{L}_0(\underset{\sim}{\xi})\right]^{-1}_{ik} = \left[|\underset{\sim}{\xi}|^2\delta_{ik} - \left(\frac{3\kappa+\mu}{3\kappa+4\mu}\right)\xi_i\xi_k\right]/\mu|\underset{\sim}{\xi}|^4 \tag{4.17}$$

and the components of $\underset{\sim}{\tilde{\Gamma}}^{\infty}$ follow correspondingly. The tensor $\underset{\sim}{P}$ is isotropic and its 'moduli' κ_P, μ_P may be obtained from the scalars

$$P_{iikk} = 9\kappa_P, \quad P_{ijij} = 3\kappa_P + 10\mu_P, \tag{4.18}$$

for each of which the integrand of (4.10) reduces to a constant. Completing the details gives

$$(3k_p, 2\mu_p) = \left(\frac{1}{3k+4\mu}, \frac{3(k+2\mu)}{5\mu(3k+4\mu)}\right). \tag{4.19}$$

No other special case is particularly simple. However, some reduction is possible when the inclusion is long and thin, so that it approximates to a cylinder with elliptic cross-section. If its principal axes are aligned with coordinate axes and the 3-axis is the one of greatest length, the matrix $\underset{\sim}{A}$ may be taken diagonal, with entries $(1/a_1, 1/a_2, \varepsilon)$, where ε is small. The integral in (4.13) may now be transformed by projection onto the circular cylinder of unit cross-section defined by $\xi_1^2+\xi_2^2=1, -\infty < \xi_3 < \infty$. The element of area on this cylinder is $ds\, d\xi_3$, s representing arc length around a cross-section, and the element of solid angle dS is given by

$$dS = ds\, d\xi_3 \left(1+\xi_3^2\right)^{-3/2}. \tag{4.20}$$

The function $\underset{\sim}{\tilde{\Gamma}}^{\infty}(\underset{\sim}{\xi})$ is unchanged by the projection because it is homogeneous of degree zero. The variable ξ_3 is now replaced by $\xi_3' = \xi_3/\varepsilon$ and (4.13) takes the form

$$\underset{\sim}{P} = \frac{a_1 a_2}{4\pi} \oint_{\xi_1^2+\xi_2^2=1} ds \int_{-\infty}^{\infty} d\xi_3' \; \underset{\sim}{\tilde{\Gamma}}^{\infty}(\underset{\sim}{\xi}) \left[a_1^2\xi_1^2 + a_2^2\xi_2^2 + \xi_3'^2\right]^{-3/2} \left(1+\varepsilon^2\xi_3'^2\right)^{-3/2}. \tag{4.21}$$

The integral with respect to ξ_3' may be estimated asymptotically by setting $\varepsilon = 0$ directly; performing the resulting elementary integration with respect to ξ_3' then gives

$$\underset{\sim}{P} \sim \frac{a_1 a_2}{2\pi} \oint_{\xi_1^2+\xi_2^2=1} ds \; \underset{\sim}{\tilde{\Gamma}}^{\infty}(\xi_1, \xi_2, 0)\left[a_1^2\xi_1^2 + a_2^2\xi_2^2\right]^{-1}. \tag{4.22}$$

Equation (4.22) offers a solution to the cylindrical inclusion problem that is an alternative to the usual one involving complex variables.

Reverting now to composites, suppose that a composite comprises a matrix with moduli $\underset{\sim}{L}_0$, which contains n different types of ellipsoidal inclusion, with moduli $\underset{\sim}{L}_r$, $r = 1, 2 \cdots n$, distributed at volume concentrations c_r, where each c_r is small. The overall moduli of the composite may be estimated by imposing a uniform strain $\bar{\underset{\sim}{e}}$ at infinity. The mean stress $\bar{\underset{\sim}{\sigma}}$ in the composite is given by the formula

$$\bar{\underset{\sim}{\sigma}} = \underset{\sim}{L}_0 \bar{\underset{\sim}{e}} + \sum_{r=1}^{n} c_r \underset{\sim}{\tau}_r, \tag{4.23}$$

where $\underset{\sim}{\tau}_r$ represents the mean polarization in the rth phase. To lowest order in c_r, the inclusions do not interact and $\underset{\sim}{\tau}_r$ may be estimated as the solution of equation (4.9), with $\underset{\sim}{e}_0 = \bar{\underset{\sim}{e}}$, $\delta\underset{\sim}{L} = \underset{\sim}{L}_r - \underset{\sim}{L}_0$ and $\underset{\sim}{P} = \underset{\sim}{P}_r$, the value appropriate to an inclusion of rth type. The low-concentration estimate of the tensor of overall moduli then follows, using (2.1), as

$$\bar{\underset{\sim}{L}} \sim \underset{\sim}{L}_0 + \sum_{r=1}^{n} c_r \left[(\underset{\sim}{L}_r - \underset{\sim}{L}_0)^{-1} + \underset{\sim}{P}_r\right]^{-1}. \tag{4.24}$$

This result is (asymptotically) exact. It is affected by the symmetry of the individual inclusions but not by their relative positions because

they are by definition far apart. Estimates that apply at any concentration are considered in the following sections.

5. THE SELF-CONSISTENT METHOD

Consider now a composite comprising a matrix containing n different types of inclusions, not necessarily ellipsoidal, with tensors of moduli $\underset{\sim}{L}_r$ and volume concentrations c_r, $r = 1, 2 \cdots n$, not necessarily small. A comparison material different from the matrix will be used, so the matrix is now regarded as phase $n+1$, with moduli $\underset{\sim}{L}_{n+1}$ and volume concentration c_{n+1}. Let the average values of the stress and strain over the rth phase be denoted $\underset{\sim}{\sigma}_r$, $\underset{\sim}{e}_r$. Then

$$\underset{\sim}{\sigma}_r = \underset{\sim}{L}_r \underset{\sim}{e}_r , \tag{5.1}$$

$$\bar{\underset{\sim}{e}} = \sum_{r=1}^{n+1} c_r \underset{\sim}{e}_r \tag{5.2}$$

and

$$\bar{\underset{\sim}{\sigma}} = \sum_{r=1}^{n+1} c_r \underset{\sim}{\sigma}_r = \sum_{r=1}^{n+1} c_r \underset{\sim}{L}_r \underset{\sim}{e}_r . \tag{5.3}$$

Eliminating $c_{n+1} \underset{\sim}{e}_{n+1}$ between (5.2) and $(5.3)_2$ gives

$$\bar{\underset{\sim}{\sigma}} = \underset{\sim}{L}_{n+1} \bar{\underset{\sim}{e}} + \sum_{r=1}^{n} c_r \left(\underset{\sim}{L}_r - \underset{\sim}{L}_{n+1} \right) \underset{\sim}{e}_r \tag{5.4}$$

so that, if $\bar{\underset{\sim}{e}}$ is prescribed and

$$\underset{\sim}{e}_r = \underset{\sim}{A}_r \bar{\underset{\sim}{e}} , \tag{5.5}$$

the tensor of overall moduli $\bar{\underset{\sim}{L}}$ is given by

$$\underset{\sim}{\bar{L}} = \underset{\sim}{L}_{n+1} + \sum_{r=1}^{n} c_r (\underset{\sim}{L}_r - \underset{\sim}{L}_{n+1}) \underset{\sim}{A}_r . \tag{5.6}$$

The problem now is to estimate the tensors $\underset{\sim}{A}_r$. An approximate method for performing this estimation is to embed a single inclusion of rth type in an infinite matrix composed of the overall material. If the inclusion is an ellipsoid, $\underset{\sim}{A}_r$ follows from the results of the preceding section, with $\underset{\sim}{L}_0 = \underset{\sim}{\bar{L}}$ and $\delta \underset{\sim}{L} = \underset{\sim}{L}_r - \underset{\sim}{L}_0$. The polarization $\underset{\sim}{\tau}_r$ is found from equation (4.9) with $\underset{\sim}{e}_0 = \underset{\sim}{\bar{e}}$ and then equation (3.9) gives

$$\underset{\sim}{A}_r = (\underset{\sim}{L}_r - \underset{\sim}{\bar{L}})^{-1} [(\underset{\sim}{L}_r - \underset{\sim}{\bar{L}})^{-1} + \underset{\sim}{P}_r]^{-1} = [\underset{\sim}{I} + \underset{\sim}{P}_r (\underset{\sim}{L}_r - \underset{\sim}{\bar{L}})]^{-1} . \tag{5.7}$$

If the rth inclusion is not an ellipsoid, the solution of the inclusion problem is less straightforward. It is convenient for the discussion to follow, however, to treat (5.7) as defining $\underset{\sim}{P}_r$ in such cases. Substituting (5.7) into (5.6) now gives an algebraic equation for $\underset{\sim}{\bar{L}}$, since it appears on the right side both directly and implicitly through the tensors $\underset{\sim}{P}_r$. The estimate for $\underset{\sim}{\bar{L}}$ that results from solving this equation is usually termed the self-consistent estimate. The prescription was developed in essentially this form by Hill (1965c) and Budiansky (1965). It allows for inclusion shape and also approximately for interactions because the inclusion is embedded in the 'overall' material. It also reduces correctly to (4.24) in the limit of low concentrations. A possible objection is that the inclusions actually interface with 'overall' material rather than with the matrix. One method of overcoming this is to surround the inclusion by a 'shell' of matrix material and then to embed this new compound inclusion in an infinite matrix of 'overall'

material. For spherical inclusions, it is reasonable to take the shell as spherical and to choose its radius so that the inclusion concerned occupies exactly its share of the matrix. Kerner (1956) solved this problem for isotropic phases, exactly for an imposed dilatation and approximately for an imposed shear. The exact solution for an imposed shear has been provided by Christensen and Lo (1979).

Suppose now that the composite has no clearly defined matrix phase or, equivalently, that $c_{n+1} = 0$. The prescription (5.6) fails in this case but, if the shapes of the phases can still be distinguished, it remains possible to estimate the tensors $\underset{\sim}{A}_r$ by solving inclusion problems. Three possible prescriptions for estimating $\bar{\underset{\sim}{L}}$ are now available. Equation (5.2) requires that

$$\sum_{r=1}^{n} c_r \left[\underset{\sim}{I} + \underset{\sim}{P}_r \left(\underset{\sim}{L}_r - \bar{\underset{\sim}{L}}\right)\right]^{-1} = \underset{\sim}{I}, \tag{5.8}$$

equation $(5.3)_2$ requires

$$\sum_{r=1}^{n} c_r \underset{\sim}{L}_r \left[\underset{\sim}{I} + \underset{\sim}{P}_r \left(\underset{\sim}{L}_r - \bar{\underset{\sim}{L}}\right)\right]^{-1} = \bar{\underset{\sim}{L}} \tag{5.9}$$

and finally, $\bar{\underset{\sim}{\sigma}} = \bar{\underset{\sim}{L}}\,\bar{\underset{\sim}{e}} + \Sigma c_r \underset{\sim}{\tau}_r$ implies

$$\sum_{r=1}^{n} c_r \left[\left(\underset{\sim}{L}_r - \bar{\underset{\sim}{L}}\right)^{-1} + \underset{\sim}{P}_r\right]^{-1} = \underset{\sim}{0}. \tag{5.10}$$

Equations (5.8) to (5.10) are not all equivalent, for (5.8) can be expressed in the form

$$\sum_{r=1}^{n} c_r \underset{\sim}{P}_r \left[\left(\underset{\sim}{L}_r - \bar{\underset{\sim}{L}}\right)^{-1} + \underset{\sim}{P}_r\right]^{-1} = \underset{\sim}{0} \tag{5.11}$$

and equation (5.10) is equivalent to

$$\left\{\sum_{r=1}^{n} c_r \underset{\sim}{L}_r \left[\underset{\sim}{I} + \underset{\sim}{P}_r \left(\underset{\sim}{L}_r - \bar{\underset{\sim}{L}}\right)\right]^{-1}\right\}\left\{\sum_{s=1}^{n} c_s \left[\underset{\sim}{I} + \underset{\sim}{P}_s \left(\underset{\sim}{L}_s - \bar{\underset{\sim}{L}}\right)\right]^{-1}\right\}^{-1} = \bar{\underset{\sim}{L}}. \tag{5.12}$$

It is clear from these relations that equations (5.8) to (5.10) become equivalent when $\underset{\sim}{P}_r = \underset{\sim}{P}$ for all r, that is, when all phases can be modelled by similar ellipsoids. The prescriptions (5.8) to (5.10) were proposed variously by Hershey (1954), Kröner (1958) and Eshelby (1961) for estimating the overall moduli of a polycrystal by treating each grain as a sphere. Their equivalance was proved, essentially as above, by Hill (1965a). It is true also that, when $\underset{\sim}{P}_r = \underset{\sim}{P}$ for all r, equation (5.6) can be manipulated into any of the forms (5.8) to (5.10), with n replaced by $n+1$. This was noted for spheres in an isotropic matrix by Budiansky (1965) and generally by Hill (1965c).

The great advantage of the self-consistent method is its simplicity. Furthermore, it will be seen when examples are discussed later that it often yields results of acceptable accuracy. There are, however, some situations with which it cannot be expected to cope. If, for example, an isotropic matrix contained spherical isotropic inclusions, arranged on a lattice, the simple self-consistent method would estimate the overall moduli as isotropic even though they should plainly reflect the symmetry of the lattice. Although exact methods are available for periodic composites, real composites are usually not perfectly periodic, though they may well display some directional bias 'on average'; the inadequacy of

the self-consistent method would remain in this case also. The section that follows presents a method that makes systematic allowance for such phase geometry.

6. BOUNDS FOR OVERALL MODULI

If every detail of the geometry of a composite were known, overall moduli could, in principle, be calculated exactly by solving the relevant boundary value problem. In practice, however, except in cases such as those displaying periodicity, a complete solution could not even be computed because of the detail that it would contain. Also, in practice, only certain 'average' features of the microstructure will be known. If, for example, only the moduli and mean concentrations of the phases are known, the only completely accurate statement that can be made is embodied in the inequalities (2.15); that is, that the Reuss and Voigt estimates bound the overall moduli. If more information is available, however, it should be possible to provide better bounds. This information is likely to be statistical in character and it is necessary now to make a slight digression to discuss random media and introduce thereby the quantities that will be employed.

6.1 Random Media

A random medium is one of a family, any member of which may be characterized by a label α that belongs to a sample space $\mathcal{J}$. The value of α is taken as defining the medium completely. It has associated with it a probability density $p(\alpha)$ which is defined over $\mathcal{J}$. The sample space $\mathcal{J}$ could be of low dimension, for example if the random medium were known to be perfectly periodic, except for uncertainty over the position of any one representative cell, or it could be of infinite dimension, with α representing the values of r at each individual point of the medium. For an n-phase material, it is convenient to introduce the indicator function $f_r(\underset{\sim}{x})$, that takes the value 1 if

$\underset{\sim}{x}$ lies in phase r and zero otherwise; it depends, of course, on α. The mean value, or ensemble average, of $f_r(\underset{\sim}{x})$ defines the probability $P_r(\underset{\sim}{x})$ of finding phase r at $\underset{\sim}{x}$. Thus,

$$P_r(\underset{\sim}{x}) = \langle f_r(\underset{\sim}{x}) \rangle = \int_J f_r(\underset{\sim}{x}, \alpha) p(\alpha)\, d\alpha. \tag{6.1}$$

Likewise, the probability $P_{rs}(\underset{\sim}{x}, \underset{\sim}{x}')$ of finding simultaneously phase r at $\underset{\sim}{x}$ and phase s at $\underset{\sim}{x}'$ is

$$P_{rs}(\underset{\sim}{x}, \underset{\sim}{x}') = \langle f_r(\underset{\sim}{x}) f_s(\underset{\sim}{x}') \rangle = \int_J f_r(\underset{\sim}{x}, \alpha) f_s(\underset{\sim}{x}', \alpha) p(\alpha)\, d\alpha. \tag{6.2}$$

Probabilities involving more points are defined similarly. If, now, phase r has moduli $\underset{\sim}{L}_r$, $r = 1, 2 \cdots n$, the value $\underset{\sim}{L}(\underset{\sim}{x})$ of the tensor of moduli at $\underset{\sim}{x}$ can be written

$$\underset{\sim}{L}(\underset{\sim}{x}) = \sum_{r=1}^{n} \underset{\sim}{L}_r f_r(\underset{\sim}{x}) \tag{6.3}$$

and this has, using (6.1), mean value

$$\langle \underset{\sim}{L}(\underset{\sim}{x}) \rangle = \sum_{r=1}^{n} \underset{\sim}{L}_r P_r(\underset{\sim}{x}). \tag{6.4}$$

Similarly,

$$\langle \underset{\sim}{L}(\underset{\sim}{x}) \otimes \underset{\sim}{L}(\underset{\sim}{x}') \rangle = \sum_{r=1}^{n} \sum_{s=1}^{n} \underset{\sim}{L}_r \otimes \underset{\sim}{L}_s P_{rs}(\underset{\sim}{x}, \underset{\sim}{x}'). \tag{6.5}$$

The individual components in (6.5) represent two-point correlation functions.

Correlation functions for more points follow a similar pattern; clearly, they are expressible in terms of $P_r(\underset{\sim}{x})$, $P_{rs}(\underset{\sim}{x}, \underset{\sim}{x}')$ and so on, for any physical quantities whose values are given for each phase.

The foregoing description is useful for developing statements 'on average', taken over many specimens produced, perhaps, by some manufacturing process with which the probability $p(\alpha)$ is associated. A statistically uniform material is one for which the probabilities $P_r(\underset{\sim}{x})$ $P_{rs}(\underset{\sim}{x}, \underset{\sim}{x}')$ etc. are insensitive to translations; thus $P_r(\underset{\sim}{x})$ reduces to a constant, $P_{rs}(\underset{\sim}{x}, \underset{\sim}{x}')$ becomes a function of $(\underset{\sim}{x} - \underset{\sim}{x}')$ only, and so on. For such a medium, it is usual to make an assumption of ergodic type, that local configurations occur over any one specimen with the frequency with which they occur over a single neighbourhood in an ensemble of specimens. In this case, the probability P_r is identified with the volume average of the function $f_r(\underset{\sim}{x})$ and so represents the volume concentration c_r of phase r, while $P_{rs}(\underset{\sim}{x}, \underset{\sim}{x}')$ becomes the average of $f_r(\underset{\sim}{x} + \underset{\sim}{x}'')f_s(\underset{\sim}{x}' + \underset{\sim}{x}'')$ as $\underset{\sim}{x}''$ ranges over a large volume. Strictly, these notions require the medium to be infinite but they provide useful empirical definitions in practice. Clearly, it will be possible to determine multipoint probabilities only up to some relatively low order. Correspondingly, Kröner (1977) proposed classifying materials as uniform 'of grade k' if probabilities involving up to k points were known to be translation-invariant. He classified other properties (such as isotropy) similarly.

For future use, the two-point probabilities $P_{rs}(\underset{\sim}{x}, \underset{\sim}{x}')$ are now discussed for a two-phase material. They satisfy the relations

$$P_1(\underset{\sim}{x}) + P_2(\underset{\sim}{x}) = 1,$$

$$P_{11}(\underset{\sim}{x}, \underset{\sim}{x}') + P_{12}(\underset{\sim}{x}, \underset{\sim}{x}') = P_1(\underset{\sim}{x}) = P_{11}(\underset{\sim}{x}', \underset{\sim}{x}) + P_{21}(\underset{\sim}{x}', \underset{\sim}{x}),$$

$$P_{12}(\underset{\sim}{x}', \underset{\sim}{x}) + P_{22}(\underset{\sim}{x}', \underset{\sim}{x}) = P_2(\underset{\sim}{x}) = P_{21}(\underset{\sim}{x}, \underset{\sim}{x}') + P_{22}(\underset{\sim}{x}, \underset{\sim}{x}') \qquad (6.6)$$

always. Therefore, when the medium is statistically uniform (up to grade two),

$$P_{11} = c_1^2 + c_1 c_2 f, \quad P_{12} = c_1 c_2 (1 - f), \quad P_{22} = c_2^2 + c_1 c_2 f, \qquad (6.7)$$

where f is some function that depends upon $(\underset{\sim}{x} - \underset{\sim}{x}')$ only which, furthermore, is an even function of its argument. If, in addition, the medium has no long-range order, $P_{rs} \sim c_r c_s$ at large separations of $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ and f tends to zero as $|\underset{\sim}{x} - \underset{\sim}{x}'|$ tends to infinity. If the distribution of phases is isotropic (up to grade 2), then f is a function of $|\underset{\sim}{x} - \underset{\sim}{x}'|$ only; it should be noted that this statement makes no reference to the elastic symmetries of the phases themselves so that the composite could still be elastically anisotropic.

Suppose now that the composite comprises a matrix in which are embedded inclusions of known shapes and orientations. All mean values for this composite follow from knowledge of the probability of finding any specified number of inclusions in any specified region. This will not be discussed in detail but, for purposes of later application, if the matrix contains just a single population of inclusions, centred at $\underset{\sim}{x}_A$, $A = 1, 2 \ldots$, P_A will be defined as the probability density for

finding an inclusion centred at $\underset{\sim}{x}_A$, P_{AB} will denote the joint probability density for finding different inclusions centred at $\underset{\sim}{x}_A$, $\underset{\sim}{x}_B$ and so on. Statistical uniformity, isotropy, etc. for such a composite is expressed in terms of P_A, P_{AB} etc. in a similar manner to that described above for P_r, P_{rs} etc.

6.2 Hashin-Shtrikman Bounds

In most applications, it is unlikely that information of grade higher than two (in Kröner's terminology) will be available. Therefore, although formal extensions are possible (and will be outlined later), attention will be focussed now on the best bounds that can be obtained using two-point correlations only. Inequalities such as (3.15) which are derived from the classical energy principles are excluded immediately for, written explicitly, the last term in (3.15) is

$$V^{-1}\int d\underset{\sim}{x}\int d\underset{\sim}{x}'\,\underset{\sim}{\Gamma}(\underset{\sim}{x},\underset{\sim}{x}')\underset{\sim}{\tau}^*(\underset{\sim}{x}')\delta\underset{\sim}{L}(\underset{\sim}{x})\int d\underset{\sim}{x}''\,\underset{\sim}{\Gamma}(\underset{\sim}{x},\underset{\sim}{x}'')\underset{\sim}{\tau}^*(\underset{\sim}{x}''),$$

which involves points taken three at a time. The Hashin-Shtrikman inequalities (3.20), (3.26) were constructed to contain no such term and bounds can be obtained from them by considering trial fields of the form

$$\underset{\sim}{\tau}^*(\underset{\sim}{x}) = \sum_{r=1}^{n} \underset{\sim}{\tau}_r(\underset{\sim}{x}) f_r(\underset{\sim}{x}), \tag{6.8}$$

for an n-phase material, where the $\underset{\sim}{\tau}_r$ are functions of $\underset{\sim}{x}$ alone, independent of the configuration: any relaxation of this restriction would introduce terms into $F(\underset{\sim}{\tau}^*)$ that would involve more than two points, so that (6.8) is the most general trial field that can be allowed.

Substituting (6.8) into (3.17) now gives

$$\mathcal{F}(\underset{\sim}{\tau}^*) = V^{-1}\left\{\sum_{r=1}^{n}\int d\underset{\sim}{x}\, f_r(\underset{\sim}{x})\underset{\sim}{\tau}_r(\underset{\sim}{x})\left[(\underset{\sim}{L}_r-\underset{\sim}{L}_0)^{-1}\underset{\sim}{\tau}_r(\underset{\sim}{x})-2\,\underset{\sim}{e}_0(\underset{\sim}{x})\right]\right.$$

$$\left.+\sum_{r=1}^{n}\sum_{s=1}^{n}\int d\underset{\sim}{x}\, f_r(\underset{\sim}{x})\underset{\sim}{\tau}_r(\underset{\sim}{x})\int d\underset{\sim}{x}'\,\underline{\Gamma}(\underset{\sim}{x},\underset{\sim}{x}')\underset{\sim}{\tau}_s(\underset{\sim}{x}')f_s(\underset{\sim}{x}')\right\}. \tag{6.9}$$

This depends upon the sample α, as it must. Further simplification is possible subject to the assumption that the functions f_r vary rapidly over regions that are still sufficiently small relative to the macro-scale for the functions $\tau_r(\underset{\sim}{x}), \underline{\Gamma}(\underset{\sim}{x},\underset{\sim}{x}')$ to be treated as effectively constant. It is credible (though not rigorously proven) in this case that the value of $\mathcal{F}(\underset{\sim}{\tau}^*)$ is independent of α and so identical with its mean value, or ensemble average. Then, (6.9) reduces to

$$\mathcal{F}(\underset{\sim}{\tau}^*) = V^{-1}\left\{\sum_{r=1}^{n}\int d\underset{\sim}{x}\, P_r(\underset{\sim}{x})\underset{\sim}{\tau}_r(\underset{\sim}{x})\left[(\underset{\sim}{L}_r-\underset{\sim}{L}_0)^{-1}\underset{\sim}{\tau}_r(\underset{\sim}{x})-2\,\underset{\sim}{e}_0(\underset{\sim}{x})\right]\right.$$

$$\left.+\sum_{r=1}^{n}\sum_{s=1}^{n}\int d\underset{\sim}{x}\,\underset{\sim}{\tau}_r(\underset{\sim}{x})\int d\underset{\sim}{x}'\,\underline{\Gamma}(\underset{\sim}{x},\underset{\sim}{x}')\underset{\sim}{\tau}_s(\underset{\sim}{x}')P_{rs}(\underset{\sim}{x},\underset{\sim}{x}')\right\}. \tag{6.10}$$

This is extremized when

$$(\underset{\sim}{L}_r-\underset{\sim}{L}_0)^{-1}\underset{\sim}{\tau}_r(\underset{\sim}{x})P_r(\underset{\sim}{x})+\sum_{s=1}^{n}\int d\underset{\sim}{x}'\,\underline{\Gamma}(\underset{\sim}{x},\underset{\sim}{x}')\underset{\sim}{\tau}_s(\underset{\sim}{x}')P_{rs}(\underset{\sim}{x},\underset{\sim}{x}')$$

$$=\underset{\sim}{e}_0(\underset{\sim}{x}),\quad r=1,2\cdots n, \tag{6.11}$$

to give the extreme value $-(\langle \underset{\sim}{\tau} \rangle, \underset{\sim}{e}_0), \langle \underset{\sim}{\tau} \rangle$ denoting the ensemble average of $\underset{\sim}{\tau}(\underset{\sim}{x})$.

It may be noted that, if the integral equation (3.16) is averaged conditionally upon phase r being present at $\underset{\sim}{x}$, it gives

$$(\underset{\sim}{L}_r - \underset{\sim}{L}_0)^{-1} \langle \underset{\sim}{\tau}(\underset{\sim}{x}) \rangle_r^r + \sum_{s=1}^{n} \int d\underset{\sim}{x}' \, \underset{\sim}{\Gamma}(\underset{\sim}{x}, \underset{\sim}{x}') \langle \underset{\sim}{\tau}(\underset{\sim}{x}') \rangle_{rs}^s \, P_{s|r}(\underset{\sim}{x}', \underset{\sim}{x})$$

$$= \underset{\sim}{e}_0(\underset{\sim}{x}), \tag{6.12}$$

where $\langle \underset{\sim}{\tau}(\underset{\sim}{x}) \rangle_r^r$ denotes the expectation value of $\underset{\sim}{\tau}(\underset{\sim}{x})$, conditional upon phase r being present at $\underset{\sim}{x}$, $\langle \underset{\sim}{\tau}(\underset{\sim}{x}') \rangle_{rs}^s$ denotes the expectation value of $\underset{\sim}{\tau}(\underset{\sim}{x}')$, conditional upon phases r and s being present at $\underset{\sim}{x}$, $\underset{\sim}{x}'$ respectively and $P_{s|r}(\underset{\sim}{x}', \underset{\sim}{x})$ is the corresponding conditional probability for finding phase s at $\underset{\sim}{x}'$, so that

$$P_{rs}(\underset{\sim}{x}, \underset{\sim}{x}') = P_{s|r}(\underset{\sim}{x}', \underset{\sim}{x}) P_r(\underset{\sim}{x}). \tag{6.13}$$

Equation (6.12) reduces to (6.11) if $\langle \underset{\sim}{\tau} \rangle_r^r$ is identified with $\underset{\sim}{\tau}_r$ and, in addition,

$$\langle \underset{\sim}{\tau}(\underset{\sim}{x}') \rangle_{rs}^s = \langle \underset{\sim}{\tau}(\underset{\sim}{x}') \rangle_s^s. \tag{6.14}$$

Equation (6.14) is not, of course, strictly correct, unless the composite has some special structure such as periodicity. It is an example of the 'quasicrystalline approximation', introduced as an ad hoc approximation by Lax (1952); its appearance from the variational principle at

least partially explains its success.

The discussion so far has been rather general; it is desirable now to specialize to the case of a statistically uniform medium and to the boundary condition (2.2), so that $\underset{\sim}{e}_0 = \bar{\underset{\sim}{e}}$, to obtain bounds upon overall moduli. The body has already been assumed to be large relative to the microstructure and it might be expected that $\underset{\sim}{\Gamma}(\underset{\sim}{x}, \underset{\sim}{x}')$ could be replaced by its infinite-body form (4.5). This is not quite correct because this is of order $|\underset{\sim}{x}|^{-3}$ at large $|\underset{\sim}{x}|$ so that the integral in (6.11) would converge only conditionally. It may be noted, however, that the mean value of the exact $\underset{\sim}{\Gamma}$ over V is zero, since it is associated with zero boundary displacement. Consequently, for any field $\underset{\sim}{\tau}$ that oscillates rapidly about a mean $\bar{\underset{\sim}{\tau}}$, $\underset{\sim}{\Gamma}\underset{\sim}{\tau} = \underset{\sim}{\Gamma}(\underset{\sim}{\tau} - \bar{\underset{\sim}{\tau}})$ exactly and, in this latter form, a convergent integral is obtained if $\underset{\sim}{\Gamma}$ is now replaced by $\underset{\sim}{\Gamma}^{\infty}$. This does not prove that this prescription is correct, but further arguments for its plausibility have been advanced by Korringa (1973), Willis and Acton (1976) and Willis (1977), which are also developed slightly differently in the review of Willis (1980d). These essentially hinge upon representing $\underset{\sim}{\Gamma}$ in terms of the infinite-body operator as above, plus an additional term whose influence will be felt only in a layer close to the boundary ∂V. In this approximation, the operator $\underset{\sim}{\Gamma}$ is insensitive to translations and the functions $\underset{\sim}{\tau}_r(\underset{\sim}{x})$ reduce to constants $\underset{\sim}{\tau}_r$ that satisfy the algebraic equations

$$(\underset{\sim}{L}_r - \underset{\sim}{L}_0)^{-1}\underset{\sim}{\tau}_r + \sum_{s=1}^{n}\int d\underset{\sim}{x}'\,\underset{\sim}{\Gamma}^{\infty}(\underset{\sim}{x}')\left[p_{s|r}(\underset{\sim}{x}') - c_s\right]\underset{\sim}{\tau}_s = \bar{\underset{\sim}{e}},$$

$$r = 1, 2 \cdots n, \qquad (6.15)$$

where $P_{s|r}(\underset{\sim}{x}')$ represents the probability of finding phase s at $\underset{\sim}{x}'$, given phase r at the origin. The concentration c_s is its mean value. Having solved the system (6.15), to give

$$\underset{\sim}{\tau}_r = \underset{\sim}{S}_r \underset{\sim}{\bar{e}}, \tag{6.16}$$

say, the extreme value of $f(\underset{\sim}{\tau}^*)$ is expressible in the form

$$f(\underset{\sim}{\tau}^*) = \underset{\sim}{\bar{e}} (\underset{\sim}{L}_0 - \underset{\sim}{\tilde{L}}) \underset{\sim}{\bar{e}}, \tag{6.17}$$

where the estimate

$$\underset{\sim}{\tilde{L}} = \underset{\sim}{L}_0 + \sum_{r=1}^{n} c_r \underset{\sim}{S}_r \tag{6.18}$$

for the overall moduli bounds $\underset{\sim}{\bar{L}}$ from below if $\underset{\sim}{L}_r - \underset{\sim}{L}_0$ is positive definite for each r, and from above if $\underset{\sim}{L}_r - \underset{\sim}{L}_0$ is negative definite for each r.

The solution (6.18) takes a particularly simple form if all of the functions $P_{s|r}(\underset{\sim}{x}')$ are isotropic and so depend upon $|\underset{\sim}{x}'|$ only. The result given in Section 4, that the integral of $\underset{\sim}{\Gamma}^{\infty}$ over any sphere is $\underset{\sim}{P}$, independently of its radius, implies that the integral of $\underset{\sim}{\Gamma}^{\infty}$ over any shell $|\underset{\sim}{x}'| = r > 0$ is zero. The integrals in (6.15) may therefore be replaced by integrals over a small sphere centred at the origin, within which $P_{s|r}$ differs negligibly from its value at the origin, namely, δ_{rs}. Evaluating the integral then gives

$$[(\underset{\sim}{L}_r - \underset{\sim}{L}_0)^{-1} + \underset{\sim}{P}]\underset{\sim}{\tau}_r - \underset{\sim}{P}\langle \underset{\sim}{\tau} \rangle = \underset{\sim}{\bar{e}}, \tag{6.19}$$

where $\langle \underset{\sim}{\tau} \rangle = \sum c_s \underset{\sim}{\tau}_s$. It follows now that

$$\langle \underset{\sim}{\tau} \rangle = \left\{ \sum c_r \left[\underset{\sim}{I} + (\underset{\sim}{L}_r - \underset{\sim}{L}_0) \underset{\sim}{P} \right]^{-1} \right\}^{-1} \sum c_s \left[\underset{\sim}{I} + (\underset{\sim}{L}_s - \underset{\sim}{L}_0) \underset{\sim}{P} \right]^{-1} (\underset{\sim}{L}_s - \underset{\sim}{L}_0) \tag{6.20}$$

and, correspondingly,

$$\underset{\sim}{\tilde{L}} = \left\{ \sum c_r \left[\underset{\sim}{I} + (\underset{\sim}{L}_r - \underset{\sim}{L}_0) \underset{\sim}{P} \right]^{-1} \right\}^{-1} \sum c_s \left[\underset{\sim}{I} + (\underset{\sim}{L}_s - \underset{\sim}{L}_0) \underset{\sim}{P} \right]^{-1} \underset{\sim}{L}_s . \tag{6.21}$$

The estimate (6.21) was originally produced by Hashin and Shtrikman (1962a, b, 1963) from their variational principle and by a different but equivalent method by Walpole (1966a,b). Both derivations relied upon the isotropy of the phase geometry from an early stage, however, and the equations (6.15), which allow for any two-point probabilities, were derived much more recently, by Willis (1977). It was also observed by Willis (1977) that the same formal simplification occurs if the function has 'ellipsoidal' symmetry, such as would be obtained if an isotropic distribution were subjected to an affine transformation. Overall moduli are still bounded by (6.21) except that the tensor $\underset{\sim}{P}$ is now defined by (4.13).

Equations (6.15) can also be solved for a two-phase composite with arbitrary geometry. In terms of the function h introduced in equations (6.7), let

$$\underset{\sim}{P}' = \int d\underset{\sim}{x}' \, \underset{\sim}{\Gamma}(\underset{\sim}{x}') h(\underset{\sim}{x}') . \tag{6.22}$$

Equations (6.15) now take the form of (6.19), with $n = 2$ and $\underset{\sim}{P}$ replaced by $\underset{\sim}{P}'$ and their solution yields the estimate (6.21), similarly modified, for $\underset{\sim}{\tilde{L}}$.

Willis (1977) suggested that the estimates (6.21), with $\underset{\sim}{P}$ taking 'ellipsoidal' form, might be appropriate to discuss a composite containing aligned ellipsoids including, as limiting cases, thin needles and flat cracks. Justification was provided by Willis (1978), through explicit study of a composite containing inclusions, whose distribution was described by the probability densities P_A, $P_{B|A}$ etc. for the positions of their centres. If $P_{B|A}$ was 'ellipsoidal', (6.21) was reproduced. Willis (1980a) also generated a result rather like the one above with $\underset{\sim}{P}$ replaced by $\underset{\sim}{P}'$, for a matrix containing a single population of aligned inclusions of any shape, distributed arbitrarily. Only the 'ellipsoidal' example has been worked out in detail, however.

6.3 Bounds for Compliances

Bounds for compliances are obtained by inversion of $\underset{\sim}{\tilde{L}}$, to produce corresponding estimates $\underset{\sim}{\tilde{M}}$ for $\underset{\sim}{\bar{M}}$. These can also be obtained directly, however, by considering the traction boundary value problem, with boundary condition (2.5). The bounds follow by extremizing the functional $g_1(\underset{\sim}{\eta}^*)$ given by (3.30): their structure is exactly like that for $\underset{\sim}{\bar{L}}$, except that $(\underset{\sim}{L}_r - \underset{\sim}{L}_0)$ is replaced by $(\underset{\sim}{M}_0 - \underset{\sim}{M}_r)$ and $\underset{\sim}{\Gamma}$ is replaced by $-\underset{\sim}{\Delta}_1$. The infinite-body form for the operator $\underset{\sim}{\Delta}_1$ can be deduced by noting that, when V is large, equations (3.16) and (3.29) must be consistent, with $\underset{\sim}{\tau} = \underset{\sim}{L}_0 \underset{\sim}{\eta}$, $\underset{\sim}{\varepsilon}_0 = \underset{\sim}{\bar{\varepsilon}}$ and $\underset{\sim}{\sigma}_0 = \underset{\sim}{\bar{\sigma}} = \underset{\sim}{L}_0 \underset{\sim}{\bar{\varepsilon}} + \langle \underset{\sim}{\tau} \rangle$. The result is that, for large V, $\underset{\sim}{\Delta}_1$ is defined by

$$\underset{\sim}{\Delta}_1 : \underset{\sim}{\eta} \rightarrow (\underset{\sim}{L}_0 - \underset{\sim}{L}_0 \underset{\sim}{\Gamma}^{\infty} \underset{\sim}{L}_0)(\underset{\sim}{\eta} - \underset{\sim}{\bar{\eta}}). \tag{6.23}$$

Then, for a composite with "ellipsoidal' symmetry, for example,

$$\tilde{\underline{\underline{M}}} = \left\{ \sum c_r \left[\underline{\underline{I}} + (\underline{\underline{M}}_0 - \underline{\underline{M}}_r)\underline{\underline{Q}} \right]^{-1} \right\}^{-1} \sum c_s \left[\underline{\underline{I}} + (\underline{\underline{M}}_0 - \underline{\underline{M}}_s)\underline{\underline{Q}} \right]^{-1} \underline{\underline{M}}_s , \tag{6.24}$$

where

$$\underline{\underline{Q}} = \underline{\underline{L}}_0 \underline{\underline{P}} \underline{\underline{L}}_0 - \underline{\underline{L}}_0 . \tag{6.25}$$

It can be verified directly, using (3.24), that $\tilde{\underline{\underline{L}}}$ and $\tilde{\underline{\underline{M}}}$, as given by (6.21) and (6.24), are inverses.

6.4 Bounds of Higher Order

The bounds that are given by (6.18) are not the best that are obtainable from the trial field (6.8) for $\underline{\tau}^*$, since they were derived from the Hashin-Shtrikman principle rather than the stronger classical principles. The penalty for using the latter is that three-point probabilities will also be involved. Still,assuming that these are known, (6.8) is now substituted into the minimum energy principle (3.15), to give

$$2(\bar{U} - \bar{U}_0) \le V^{-1} \int d\underline{x} \left\{ \underline{e}_0(\underline{x}) \sum_r (\underline{L}_r - \underline{L}_0) f_r(\underline{x}) \underline{e}_0(\underline{x}) \right.$$

$$\left. + \sum_r \sum_s \left(\underline{\tau}_r(\underline{x}) \underline{A}_{rs} \left(\tilde{\underline{\tau}}_s - 2(\underline{L}_s - \underline{L}_0)\underline{e}_0 \right) + \underline{B}_{rs}(\tilde{\underline{\tau}}_r, \tilde{\underline{\tau}}_s) \right) \right\}, \tag{6.26}$$

where

$$\underset{\sim}{A}_{rs} : \underset{\sim}{\tau}_s \rightarrow f_r(\underset{\sim}{x}) \int d\underset{\sim}{x}' \underset{\sim}{\Gamma}(\underset{\sim}{x}, \underset{\sim}{x}') f_s(\underset{\sim}{x}') \underset{\sim}{\tau}_s(\underset{\sim}{x}') \tag{6.27}$$

and

$$\underset{\sim}{B}_{rs} :$$

$$(\underset{\sim}{\tau}_r, \underset{\sim}{\tau}_s) \rightarrow \sum_t \int d\underset{\sim}{x}' \underset{\sim}{\tau}_r(\underset{\sim}{x}') f_r(\underset{\sim}{x}') \underset{\sim}{\Gamma}(\underset{\sim}{x}', \underset{\sim}{x})(\underset{\sim}{L}_t - \underset{\sim}{L}_0) f_t(\underset{\sim}{x}) \int d\underset{\sim}{x}'' \underset{\sim}{\Gamma}(\underset{\sim}{x}, \underset{\sim}{x}'') f_s(\underset{\sim}{x}'') \underset{\sim}{\tau}_s(\underset{\sim}{x}''). \tag{6.28}$$

Specializing now to a statistically uniform medium, the integrals that appear in (6.27) and (6.28) may be evaluated as in the preceding section: for (6.27), $f_s(\underset{\sim}{x}')$ is replaced by $f_s(\underset{\sim}{x}') - c_s$ and $\underset{\sim}{\Gamma}$ is replaced by $\underset{\sim}{\Gamma}^{\infty}$, and similar replacements are made in (6.28). The expectation values of the operators $\underset{\sim}{A}_{rs}$, $\underset{\sim}{B}_{rs}$ then become translation-invariant and, if $\underset{\sim}{e}_0(\underset{\sim}{x})$ is taken equal to $\bar{\underset{\sim}{e}}$, the functions $\underset{\sim}{\tau}_r$ may be taken constant. If the right side of (6.26) is now replaced by its expectation value, it follows by extremizing that

$$\bar{\underset{\sim}{e}}\, \bar{\underset{\sim}{L}}\, \bar{\underset{\sim}{e}} \leq \bar{\underset{\sim}{e}}\, \underset{\sim}{L}_V\, \bar{\underset{\sim}{e}} - \sum_r \sum_s \underset{\sim}{\tau}_r \langle \underset{\sim}{A}_{rs} \rangle (\underset{\sim}{L}_r - \underset{\sim}{L}_0) \bar{\underset{\sim}{e}}, \tag{6.29}$$

where $\underset{\sim}{L}_V$ is the Voigt estimate $\sum c_r \underset{\sim}{L}_r$ and $\underset{\sim}{\tau}_r$, $r = 1, 2 \cdots n$ satisfy the equations

$$\sum_s \left(\langle \underset{\sim}{A}_{rs} \rangle + \langle \underset{\sim}{B}_{rs} \rangle \right) \underset{\sim}{\tau}_s = \sum_s \langle \underset{\sim}{A}_{rs} \rangle (\underset{\sim}{L}_s - \underset{\sim}{L}_0) \bar{\underset{\sim}{e}}, \tag{6.30}$$

with

$$\langle \underset{\sim}{A}_{rs} \rangle = \int d\underset{\sim}{x}' \, \underset{\sim}{\Gamma}^{*}(\underset{\sim}{x}') \left(P_{rs}(\underset{\sim}{0}, \underset{\sim}{x}') - c_r c_s \right) \tag{6.31}$$

and

$$\langle \underset{\sim}{B}_{rs} \rangle = \sum_t \int d\underset{\sim}{x}' \, \underset{\sim}{\Gamma}^{*}(\underset{\sim}{x}') (\underset{\sim}{L}_t - \underset{\sim}{L}_0) \int d\underset{\sim}{x}'' \, \underset{\sim}{\Gamma}^{*}(\underset{\sim}{x}'')$$

$$\left[P_{rst}(\underset{\sim}{x}', \underset{\sim}{x}'', \underset{\sim}{0}) - c_r P_{st}(\underset{\sim}{x}'', \underset{\sim}{0}) - c_s P_{tr}(\underset{\sim}{0}, \underset{\sim}{x}') + c_r c_s c_t \right]. \tag{6.32}$$

Following Beran and Molyneux (1966) and Dederichs and Zeller (1973), Kröner (1977) generated an 'optimized' third-order bound, essentially by substituting into (3.15) the trial field

$$\underset{\sim}{\tau}^{*} = \delta \underset{\sim}{L} \, \underset{\sim}{e}_1 \,, \tag{6.33}$$

in which the constant tensor $\underset{\sim}{e}_1$ was arbitrary. This possibility is included within (6.8) and the bound (6.29) is correspondingly tighter. The work required to produce the Beran-Molyneux bound is only marginally less than is needed to produce (6.29), since it requires evaluation of $\langle \underset{\sim}{A}_{rs} \rangle$ and $\langle \underset{\sim}{B}_{rs} \rangle$.

Kröner (1977) also produced a tighter bound for a special type of material that he termed 'disordered'. Its distinctive feature (which he describes in terms of the moduli and topology being uncorrelated) is that it permits the replacement of the awkward function $\underset{\sim}{\Gamma}(\underset{\sim}{x})$ in (6.31) and

(6.32) by $\underset{\sim}{P}\,\delta(\underset{\sim}{x})$, where the constant tensor $\underset{\sim}{P}$ is defined by (4.10). For (6.31), this is legitimate so long as ρ_{rs} is isotropic, but for (6.32) it requires more justification: Kröner (1977) considers the replacement justified so long as $\underset{\sim}{L}_0$ is identified with $\underset{\sim}{L}_V$, when $\langle \delta \underset{\sim}{L} \rangle = \underset{\sim}{0}$. Without arguing for or against this prescription, it is simply remarked that its adoption yields simple forms for $\langle \underset{\sim}{A}_{rs} \rangle \cdot \langle \underset{\sim}{B}_{rs} \rangle$ for which (6.29) and (6.30) lead to the upper bound (6.21), previously derived from the Hashin-Shtrikman principle, except that now $\underset{\sim}{L}_0$ is identified with $\underset{\sim}{L}_V$.

Analogous lower bounds of third order can be obtained from (3.22); no explicit development will be given.

Bounds that involve still more statistical information can be found by allowing trial fields more complex than (6.8). For example, the trial field

$$\underset{\sim}{\tau}^*(\underset{\sim}{x}) = f_r(\underset{\sim}{x})\Big[\underset{\sim}{\tau}_r(\underset{\sim}{x}) + \sum_s \int \underset{\sim}{\varphi}_{rs}(\underset{\sim}{x}, \underset{\sim}{x}') f_s(\underset{\sim}{x}')\, d\underset{\sim}{x}'\Big], \tag{6.34}$$

in which the functions $\underset{\sim}{\tau}_r(\underset{\sim}{x})$ and $\underset{\sim}{\varphi}_{rs}(\underset{\sim}{x}, \underset{\sim}{x}')$ are to be determined optimally, introduces correlations involving up to four points in the Hashin-Shtrikman principle, and up to five points in the classical principles. These are not considered further, except to remark that Willis (1978) employed a polarization similar to (6.34) for a matrix containing a single set of inclusions and obtained, by considering only the terms of low order, a variational estimate for the c^2-coefficient in the low-concentration expansion for $\overline{\underset{\sim}{L}}$, c being the volume concentration of the inclusions.

6.5 Further Comments on the Self-Consistent Method

The Hashin-Shtrikman estimate (6.18) for $\underset{\sim}{\bar{L}}$ provides a lower bound when $\underset{\sim}{L}_0$ is sufficiently small and an upper bound when $\underset{\sim}{L}_0$ is sufficiently large. At intermediate values of $\underset{\sim}{L}_0$, it does not yield a bound, but it is still a variational estimate and it might be postulated that this estimate will be closest to $\underset{\sim}{\bar{L}}$ when $\underset{\sim}{L}_0 = \underset{\sim}{\bar{L}}$, for then the mean polarization will be zero. An estimate of self-consistent type is therefore obtained by postulating that $\underset{\sim}{\tilde{L}}$ is identical to $\underset{\sim}{\bar{L}}$ when $\underset{\sim}{L}_0 = \underset{\sim}{\bar{L}}$. That is,

$$\underset{\sim}{\tilde{L}}(\underset{\sim}{\bar{L}}) = \underset{\sim}{\bar{L}} \tag{6.35}$$

or, equivalently,

$$\Sigma c_r \underset{\sim}{S}_r = \underset{\sim}{0} \qquad (\underset{\sim}{L}_0 = \underset{\sim}{\bar{L}}). \tag{6.36}$$

The prescription (6.35) of course is not exact because $\underset{\sim}{\tilde{L}}$ is estimated from a piecewise-constant polarization field. If the composite has isotropic (or, more generally 'ellipsoidal') phase geometry, $\underset{\sim}{\tilde{L}}$ is given by (6.21) and it is easily seen that the self-consistent prescription (6.35) yields any of (5.8), (5.9) or (5.10), with $\underset{\sim}{P}_r = \underset{\sim}{P}$ for all r. The interesting point is that the prescription (6.35) allows for two-point statistics in a general way and makes no reference to particular inclusion shapes. Thus, it provides some justification for the use of (5.8) to (5.10) for composites such as isotropic polycrystals, in which the grains are certainly not all spheres, and, in addition, offers an

extension of the self-consistent method to composites in which the microstructure displays spatial bias.

7 EXAMPLES

The application of the preceding theory will now be illustrated for some simple examples.

7.1 An Isotropic Mixture of Isotropic Phases

In the notation of Hill (1965a), introduced at equation (4.15), let a composite have n phases, with moduli $\underset{\sim}{L}_r = (3k_r, 2\mu_r)$, $r = 1, 2 \cdots n$, distributed statistically uniformly and isotropically. It is reasonable to take the comparison material as isotropic so that $\underset{\sim}{L}_0 = (3k, 2\mu)$, say. The Hashin-Shtrikman bounds for this composite then follow from (6.21), with the tensor $\underset{\sim}{P}$ given by (4.19). Explicitly,

$$\tilde{k} = \left\{ \sum_r c_r \frac{(3k+4\mu)}{(3k_r+4\mu)} \right\}^{-1} \sum_s c_s \frac{(3k+4\mu)\,k_s}{(3k_s+4\mu)} ,$$

$$\tilde{\mu} = \left\{ \sum_r c_r \frac{5\mu(3k+4\mu)}{6\mu_r(k+2\mu)+\mu(9k+8\mu)} \right\}^{-1} \sum_s c_s \frac{5\mu(3k+4\mu)\,\mu_s}{6\mu_s(k+2\mu)+\mu(9k+8\mu)} . \tag{7.1}$$

Equations (7.1) yield upper bounds for $\bar{k}$, $\bar{\mu}$ whenever k, μ are chosen larger than k_r, μ_r for each r and the smallest upper bounds are obtained by setting $k = k_g = \mathrm{Max}(k_r)$, $\mu = \mu_g = \mathrm{Max}(\mu_r)$. Similarly, the greatest lower bounds are found by setting $k = k_\ell = \mathrm{Min}(k_r)$, $\mu = \mu_\ell = \mathrm{Min}(\mu_r)$. These bounds were given by Hashin and Shtrikman (1963), with the implicit restriction that k_g, μ_g had both to be obtained from the same phase, and k_ℓ, μ_ℓ similarly. This restriction was removed by Walpole (1966a).

Equations (7.1) also provide self-consistent equations for the

determination of $\bar{\kappa}$, $\bar{\mu}$, if $\tilde{\kappa}$ and κ are identified with $\bar{\kappa}$ and $\tilde{\mu}$ and μ are identified with $\bar{\mu}$. For a two-phase material, elimination of $\bar{\kappa}$ from equations (7.1) yields a quartic equation for $\bar{\mu}$. This will not be given but a difficulty with the self-consistent method will be highlighted by considering the special case $\kappa_1 = \mu_1 = 0$, so that the composite represents a porous medium whose matrix phase has moduli κ_2, μ_2. It is easy to show in this case that

$$\bar{\kappa} = \frac{4(1-c_1)\kappa_2\bar{\mu}}{3c_1\kappa_2 + 4\bar{\mu}}, \tag{7.2}$$

while $\bar{\mu}$ satisfies the quadratic equation

$$16\bar{\mu}^2 + \bar{\mu}\left[\kappa_2(3-c_1) - 4\mu_2(4-5c_1)\right] - 3\mu_2\kappa_2(1-2c_1) = 0. \tag{7.3}$$

If, now, $\kappa_2 \to \infty$, equations (7.2) and (7.3) have solutions

$$\bar{\kappa} = \frac{4(1-c_1)(1-2c_1)\mu_2}{c_1(3-c_1)},$$

$$\bar{\mu} = \frac{3(1-2c_1)\mu_2}{(3-c_1)}. \tag{7.4}$$

The corresponding Hashin-Shtrikman upper bounds are found from (7.1) with $\kappa = \kappa_2 \sim \infty$, $\mu = \mu_2$:

$$\tilde{K} = 4\mu_2(1-c_1)/3c_1 \quad , \quad \tilde{\mu} = 3\mu_2(1-c_1)/(3+2c_1). \tag{7.5}$$

The Hashin-Shtrikman lower bounds, obtained with $K=\mu=0$, are zero.

Equations (7.4) and (7.5) agree at low values of c_1 but the self consistent estimates (7.4) reduce to zero at a pore concentration equal to 1/2. This is unrealistic and demonstrates the need for caution in applying the self-consistent method in extreme cases. Hill (1965c) and Budiansky (1965) both discussed the example presented in this section with $n=2$, in the context of a matrix containing spherical inclusions but, with the interpretation of Section 6.5, any phase geometry is allowed, so long as it is isotropic.

For a two-phase composite for which K_g, μ_g are both derived from one phase and K_ℓ, μ_ℓ are both derived from the other, the bounds obtained from (7.1) agree with the estimates of overall moduli given by Kerner (1956), which were obtained by self consistently embedding into a matrix of 'overall' material a composite inclusion consisting of a sphere of one material encased in a spherical shell of the other, the radii being chosen to give the correct volume ratios of each phase. Either bound can be reproduced, by interchanging the two phases. Christensen and Lo (1979) noted that Kerner's solution for an imposed shear violated a continuity condition, though his solution for imposed dilation or compression was correct. They provided the true solution for the shear case and demonstrated that it estimates the shear modulus as lying between the Hashin-Shtrikman bounds. For extreme differences between phase moduli, it may be remarked that the model of Kerner and Christensen

and L_0 is not necessarily more reliable than the simple self-consistent model. In surrounding an inclusion by a uniform shell of matrix, it is likely to underestimate interactions between inclusions while, to judge from (7.4), the simple self-consistent model appears to overestimate them.

7.2 Polycrystals

A polycrystal may be regarded as a limiting case of an n-phase material in which $n \to \infty$, each phase being associated with a different crystal orientation. Relative to crystallographic axes, any one crystal has moduli $\underset{\sim}{L}_c$, say, and the moduli $\underset{\sim}{L}_r$ of phase r are obtained from $\underset{\sim}{L}_c$ by a rotation.

Consider now an isotropic polycrystal, in which each crystal orientation is equally likely and all of the functions P_{rs} are isotropic. Estimates for the overall moduli $\underset{\sim}{\tilde{L}}$ are obtained from (6.21) with taken isotropic. The manipulations are easier to perform if (6.21) is written in the alternative form

$$\underset{\sim}{\tilde{L}} = \underset{\sim}{L}_0 + \left[\underset{\sim}{I} - \langle \underset{\sim}{T} \rangle \underset{\sim}{P}\right]^{-1} \langle \underset{\sim}{T} \rangle, \tag{7.6}$$

where $\langle \underset{\sim}{T} \rangle = \Sigma c_r \underset{\sim}{T}_r$

and $$\underset{\sim}{T}_r = \left[(\underset{\sim}{L}_r - \underset{\sim}{L}_0)^{-1} + \underset{\sim}{P}\right]^{-1}. \tag{7.7}$$

Since all orientations are equally likely, the tensor $\langle \underset{\sim}{T} \rangle$ is isotropic, so that $\langle \underset{\sim}{T} \rangle = (3k_T, 2\mu_T)$. The bulk and shear 'moduli' associated with $\langle \underset{\sim}{T} \rangle$ can be found from its scalar invariants, as given by

(4.18) for the isotropic tensor $\underset{\sim}{P}$. The corresponding scalars calculated for $\underset{\sim}{T}_r$ are independent of rotation and so may be calculated for any one orientation, say that for which $\underset{\sim}{L}_r = \underset{\sim}{L}_c$. Once K_T and μ_T have been found in this way, (7.16) gives the bulk and shear moduli $\tilde{K}$, $\tilde{\mu}$ in the form

$$\tilde{K} = K + K_T/(1 - 9K_p K_T),$$

$$\tilde{\mu} = \mu + \mu_T/(1 - 4\mu_p \mu_T), \tag{7.8}$$

where $\underset{\sim}{L}_0 = (3K, 2\mu)$. If $\underset{\sim}{L}_c$ is cubic, so that

$$\sigma_{kk} = 3K_c e_{kk}, \quad \sigma_{11} - \sigma_{22} = 2\mu_c (e_{11} - e_{22}), \quad \sigma_{12} = 2\mu_c' e_{12}, \tag{7.9}$$

with equations for other components obtained by cyclic permutation of the suffixes, it follows that

$$(\underset{\sim}{L}_c)_{iikk} = 9K_c, \quad (\underset{\sim}{L}_c)_{ijij} = 3K_c + 4\mu_c + 6\mu_c'. \tag{7.10}$$

Then, if the notation $\underset{\sim}{L}_c = (3K_c, 2\mu_c, 2\mu_c')$ introduced by Walpole (1966b) is employed, products and inverses can be worked out directly and, if $\underset{\sim}{T}_c$ represents $\underset{\sim}{T}_r$ referred to the cube axes,

$$\underset{\sim}{T}_c = \left(\left[\frac{1}{3(\kappa_c - \kappa)} + 3\kappa_p \right]^{-1}, \left[\frac{1}{2(\mu_c - \mu)} + 2\mu_p \right]^{-1}, \left[\frac{1}{2(\mu_c' - \mu)} + 2\mu_p \right]^{-1} \right) \quad (7.11)$$

Then, from (7.10) and (4.18),

$$3\kappa_T = \left[\frac{1}{3(\kappa_c - \kappa)} + 3\kappa_p \right]^{-1},$$

$$2\mu_T = \frac{4}{5}\left[\frac{1}{2(\mu_c - \mu)} + 2\mu_p \right]^{-1} + \frac{6}{5}\left[\frac{1}{2(\mu_c' - \mu)} + 2\mu_p \right]^{-1}. \quad (7.12)$$

Bounds for $\bar{\kappa}, \bar{\mu}$ follow by specifying κ, μ appropriately. The unique choice for κ is κ_c, which gives $\kappa_T = 0$ and $\tilde{\kappa} = \kappa_c$, *exactly*. μ, on the other hand, may take either of the values μ_c, μ_c', the greater giving an upper bound and the lower a lower bound for $\bar{\mu}$. When $\mu = \mu_c$, $\tilde{\mu} = \tilde{\mu}_1$, where

$$\tilde{\mu}_1 = \mu_c + 3\left(\frac{5}{\mu_c' - \mu_c} + 8\mu_p \right)^{-1} \quad (7.13)$$

and, when $\mu = \mu_c'$, $\tilde{\mu} = \tilde{\mu}_2$, where

$$\tilde{\mu}_2 = \mu_c' + 2\left(\frac{5}{\mu_c - \mu_c'} + 12\mu_p' \right)^{-1}. \quad (7.14)$$

In these equations, μ_p and μ_p' are obtained from $\underset{\sim}{P}$, evaluated with

$\mu = \mu_c$ and μ_c', respectively. The bounds (7.13), (7.14) were first given by Hashin and Shtrikman (1962b).

A self-consistent estimate may be obtained from equation (6.23), which implies that $K_T = \mu_T = 0$. The first of these gives $\bar{K} = K_c$, exactly, while the second can be manipulated to give

$$8\bar{\mu}^3 + (9K_c + 4\mu_c)\bar{\mu}^2 - 3\mu_c'(K_c + 4\mu_c)\bar{\mu} - 6K_c\mu_c\mu_c' = 0. \quad (7.15)$$

This result was first given by Hershey (1954) in the form of a quartic equation with factors (7.15) and $(8\bar{\mu} + 9K_c)$. He assumed spherical grains and applied the prescription (5.9).

Anisotropic polycrystals require a computation. Kneer (1965) studied textured polycrystals, in which grains were modelled as spheres but not all crystal orientations were equally likely. This required the evaluation of $\underset{\sim}{P}$ for an anisotropic comparison material, using (4.10). Examples for which the actual phase geometry is anisotropic do not yet appear to have been studied, though the development of Section 6 now makes this possible.

7.3 A Composite Containing Long Aligned Fibres

The prototype of the composite now to be considered consists of a matrix reinforced by long parallel fibres, distributed randomly on any cross-section. Such a composite is transversely isotropic and was first studied systematically by Hill (1964). For present purposes, there is nothing to be gained from immediate specialization to this extent and so an

n-phase composite will be considered, with each phase transversely isotropic with symmetry axis parallel to the 3-axis, and with two-point probabilities $P_{rs}(\underset{\sim}{x}-\underset{\sim}{x}')$ that depend only upon $(x_\alpha - x_\alpha')(x_\alpha - x_\alpha')$. Greek suffixes taking the values 1, 2 only. The composite may thus be treated as two-dimensional, or else may be regarded as the limit of a composite with 'spheroidal' symmetry. In either case, estimates of overall moduli are obtained from (6.21), with the tensor $\underset{\sim}{P}$ taking the form (4.16) with $a_1 = a_2$.

The tensor of moduli $\underset{\approx}{L}$ is most usefully expressed, in the notation of Hill (1964, 1965b) and Walpole (1969), as

$$\underset{\approx}{L} = (2k, l, l', n, 2m, 2p) \tag{7.16}$$

when $\underset{\sim}{\sigma} = \underset{\approx}{L}\,\underset{\sim}{e}$ implies

$$\begin{aligned}
\tfrac{1}{2}(\sigma_{11}+\sigma_{22}) &= k(e_{11}+e_{22}) + l e_{33}, \\
\sigma_{33} &= l'(e_{11}+e_{22}) + n e_{33}, \\
(\sigma_{11}-\sigma_{22}) &= 2m(e_{11}-e_{22}), \quad \sigma_{12} = 2m e_{12}, \\
\sigma_{23} &= 2p e_{23}, \quad \sigma_{31} = 2p e_{31}.
\end{aligned} \tag{7.17}$$

In this notation, the product $\underset{\approx}{L}_1\underset{\approx}{L}_2$ of two such tensors is given by

$$\underset{\sim}{L}_1 \underset{\sim}{L}_2 = \left(4k_1k_2+2l_1l_1', 2k_1l_2+l_1n_2, 2l_1'k_2+n_1l_2', 2l_1'l_2+n_1n_2, 4m_1m_2, 4p_1p_2\right) \tag{7.18}$$

and

$$\underset{\sim}{L}^{-1} = \left(n/2\Delta, -l/2\Delta, -l'/2\Delta, k/\Delta, 1/2m, 1/2p\right), \tag{7.19}$$

where

$$\Delta = kn - ll'. \tag{7.20}$$

When $\underset{\sim}{L}$ is a tensor of elastic moduli, it has the symmetry $l = l'$ but products do not share this property. The moduli k, m are the plane-strain bulk and shear moduli. Other useful quantities are the plane-strain Poisson's ratio ν and the longitudinal Young's modulus E. These are

$$\nu = l/2k, \quad E = n - l^2/k. \tag{7.21}$$

Positive-definiteness of $\underset{\sim}{L}$ requires that all of k, m, E and p are positive.

Taking the comparison material as transversely isotropic now, with components as in (7.16) with $l = l'$, it is fairly easy derived from (4.16) that

$$\underset{\sim}{P} = \left(1/2(k+m), 0, 0, 0, (2m+k)/4m(k+m), 1/4p\right); \tag{7.22}$$

this was given by Walpole (1969) and Laws and McLaughlin (1978). All that is required now is routine substitution into (6.21). Walpole (1969) gives the results in the form

$$\tilde{k} = \left[\Sigma c_r/(k_r+m)\right]^{-1} - m, \quad \tilde{l} = \left[\Sigma c_r l_r/(k_r+m)\right]/\left[\Sigma c_r/(k_r+m)\right],$$

$$\tilde{n} = \Sigma c_r n_r + \left\{\Sigma c_r l_r \left[\Sigma c_s (l_s-l_r)/(k_s+m)\right]/(k_r+m)\right\}/\left[\Sigma c_r/(k_r+m)\right],$$

$$\tilde{\kappa} = \left\{\Sigma c_r/\left[\kappa_r + mk/(k+2m)\right]\right\}^{-1} - mk/(k+2m),$$

$$\tilde{p} = \left[\Sigma c_r/(p_r+p)\right]^{-1} - p, \qquad (7.23)$$

from which both bounds and self-consistent estimates follow. Equivalent formulae, together with useful variants involving E and ν, were given by Hill (1964, 1965b) for a two-phase composite, except that he obtained no bound for $\bar{m}$. Hashin (1979) recently reconsidered the two-phase problem and derived estimates of moduli from a 'composite cylinder' model similar to the spherical model of Kerner (1956). It generated as estimates the bounds that are obtained from (7.23).

7.4 Composite Containing Short Fibres

Fibre-reinforced composites frequently contain fibres whose length to thickness ratio is of the order of 10 to 100. The analysis of the preceding section therefore applies only approximately and some assessment of its range of validity is called for. Such an assessment can be made if the fibres are modelled as aligned spheroids. This allows a fairly straightforward self-consistent analysis and, if the distribution of the

fibres is taken as 'spheroidal', bounds can also be constructed using (6.21). The self-consistent approach requires the evaluation of $\underset{\sim}{P}$ for a transversely isotropic matrix. Although the integrals can be worked out analytically, their form is still sufficiently complicated to require numerical evaluation by computer except, of course, that for long fibres, $\underset{\sim}{P}$ approaches the value (7.22). Laws and McLaughin (1979) have carried out the required computations for a glass reinforced polyester resin. Their results will be discussed later. A complete set of bounds has not been found but, in the course of a study of waves in composites, Willis (1980b,c) obtained as a by-product the estimate (6.21) for 'spheroidal' $\underset{\sim}{P}$, when both phases were isotropic and the comparison material was identified with one of them. Again, general results could be obtained only with a computation but some general feeling for the influence of fibre length can be gained by considering the asymptotic form taken by $\underset{\sim}{P}$, for a spheroid whose thickness to length ratio is ε, where $\varepsilon \ll 1$. Willis (1980b) gave the formula

$$\underset{\sim}{P} \sim \left(\frac{3}{2(3K+4\mu)}, 0, 0, 0, \frac{3K+7\mu}{4\mu(3K+4\mu)}, \frac{1}{4\mu} \right)$$

$$+ \frac{\varepsilon^2 \ln(2/\varepsilon)}{2\mu(3K+4\mu)} \left(3K+2\mu, -(3K+\mu), -(3K+\mu), 2(3K+4\mu), -3\mu, -3K/2 \right)$$

$$+ \frac{3\varepsilon^2}{4\mu(3K+4\mu)} \left(-(3K-\mu), 3K+\mu, 3K+\mu, -6(K+\mu), (3K-11\mu)/2, 5K+2\mu/3 \right). \tag{7.24}$$

Russel (1973) did not give $\underset{\sim}{P}$ but derived an equivalent result calculated

from general formulae of Eshelby (1957). If $\underset{\sim}{L}_c$ is identified with one of the phases, say $\underset{\sim}{L}_0 = \underset{\sim}{L}_1$, it is advantageous to use (6.21) in the equivalent form

$$\underset{\sim}{\tilde{L}} = \underset{\sim}{L}_1 + c_2 \left[(\underset{\sim}{L}_2 - \underset{\sim}{L}_1)^{-1} + c_1 \underset{\sim}{P} \right]^{-1}, \tag{7.25}$$

whose similarity with (4.24) may be noted. Clearly, now, the error involved in disregarding terms of order ε^2 is small so long as $(\underset{\sim}{L}_2 - \underset{\sim}{L}_1)^{-1}$ is of the same order as μ^{-1}, for example. Within this range, fibres with an aspect ratio of 10 or more may be regarded as long. If, however, $\underset{\sim}{L}_2$ is much stiffer than $\underset{\sim}{L}_1$ so that $(\underset{\sim}{L}_2 - \underset{\sim}{L}_1)^{-1}$ is small, terms of order ε^2 make a significant contribution because the 0(1) part of $\underset{\sim}{P}$ is singular. The extreme case is that of phase 2 being rigid, when (7.25) gives

$$\underset{\sim}{\tilde{L}} = \underset{\sim}{L}_1 + c_2 \underset{\sim}{P}^{-1} / c_1 , \tag{7.26}$$

in which a singular result is obtained if the terms of order ε^2 are ignored. Explicitly,

$$\underset{\sim}{\tilde{L}} = \underset{\sim}{L}_1 + \frac{c_2 \mu}{c_1 \varepsilon^2 [\ln(2/\varepsilon) - 9(K+\mu)/2(3K+4\mu)]} (0, 0, 0, 1, 0, 0). \tag{7.27}$$

The example treated by Laws and McLaughlin (1979) falls in the intermediate range, the ratio of Young's moduli for E glass and their polyester

resin being around 20, and the results of their calculations differed appreciably from the infinite-fibre predictions, for aspect ratios up to about 100. Clearly a similar situation would occur for many other composites.

Real composites contain imperfectly aligned fibres of variable length. Laws and McLaughin (1978) made some approximate allowance for these features but no rigorous study has yet been made. One possibility would be to obtain both bounds and self-consistent estimates from (7.25), but with $\underset{\sim}{P}$ replaced by the tensor $\underset{\sim}{P}'$ defined by (6.22). The difficult task, of course, would be that of realistically estimating the function h which defines the two-point probabilities.

7.5 A Body Containing Aligned Cracks

A body containing aligned penny-shaped cracks, whose centres are distributed uniformly at number density n_2, can reasonably be treated as a limiting case of a composite with 'spheroidal' symmetry, the thickness to diameter ratio of the spheroids being $\varepsilon \ll 1$. Willis (1980b) gave the result

$$\underset{\sim}{P} \sim \left(0, 0, 0, 3/(3\kappa+4\mu), 0, 1/2\mu\right)$$

$$+ \frac{\pi\varepsilon}{\mu(3\kappa+4\mu)}\Big((3\kappa+11\mu)/8, -3(\kappa+\mu)/8, -3(\kappa+\mu)/8, (3\kappa-\mu)/4, 9(\kappa+3\mu)/16, 9(\kappa+2\mu)/8\Big) \tag{7.28}$$

so that, if $\underset{\sim}{L}_0$ is identified with $\underset{\sim}{L}_1$ ($\underset{\sim}{L}_2 = \underset{\sim}{0}$ for the cracks), substitution of (7.28) into (7.25) yields an upper-bound estimate for $\bar{\underset{\sim}{L}}$.

Again, the presence of the term of order ε in (7.28) is crucial, since without it the inverse in (7.25) does not exist. The result was given by Willis (1980c). It is complicated when expressed in terms of $\tilde{\underset{\sim}{\underset{\sim}{L}}}$ but takes a much simpler form when expressed in terms of the inverse $\tilde{\underset{\sim}{\underset{\sim}{M}}}$:

$$\tilde{\underset{\sim}{\underset{\sim}{M}}} = \underset{\sim}{\underset{\sim}{M}}_1 + \frac{4(n_2 a^3)(3K+4\mu)}{3\mu}\left(0, 0, 0, 1/(3K+\mu), 0, 2/9(K+2\mu)\right), \tag{7.29}$$

where the relation $c_2 = 4\pi n_2 a^3 \varepsilon/3$ between volume concentration and number density has been employed and the limit $\varepsilon \to 0$ has been taken with number density fixed. Equation (7.29) confirms that the cracks interact only with the stress components $\sigma_{13}, \sigma_{23}, \sigma_{33}$, as they should. Hoenig (1979) has performed self-consistent calculations for the same configuration. His estimates agree with (7.29) at low concentrations; more generally, (7.29) bounds the compliances from below.

8. WAVE PROPAGATION

Problems involving wave propagation necessarily involve spatial variation and effective properties cannot be defined through global averages, in the way outlined for static problems in Section 2. Instead, at least for random composites, equations which govern the ensemble average $\langle \underset{\sim}{u} \rangle$ of the solution to any particular boundary value problem are sought or, alternatively, $\langle \underset{\sim}{u} \rangle$ is constructed directly. Work published to date has relied on one of two methods. The first involves solving the dynamical analogue of (3.8) by perturbation theory. If this is done in the most elementary manner, it leads to a series some of whose individual terms contain integrals that diverge, which are then grouped together in a way that eliminates the divergence by a technique known as renormalization. This is a well-known technique in many branches of physics; its application to composites has recently been discussed by McCoy (1979). Alternatively, the equations can be solved by the method of smoothing introduced by Karal and Keller (1964), in which an essential intermediate step is to represent $\underset{\sim}{u}' = \underset{\sim}{u} - \langle \underset{\sim}{u} \rangle$ in terms of $\langle \underset{\sim}{u} \rangle$. The statistically fluctuating term $\underset{\sim}{u}'$ can then be eliminated from the starting equation to give an equation for $\langle \underset{\sim}{u} \rangle$ itself. Karal and Keller (1964) truncated their formal iterative solution after one iteration to obtain results valid for weakly inhomogeneous media. McCoy (1973), on the other hand, considered the series to arbitrary order, to obtain formal expressions containing correlation functions of all orders. Since these are not generally known, however, the method is most useful for weakly inhomogeneous media. An entirely different method has been applied to composites consisting of a matrix with distributed inclusions. Each inclusion is

treated as a scatterer, which scatters the field incident upon it. The complication is that the field incident upon any one inclusion has contributions from the scattered fields of every other inclusion. The scattered fields have always been represented explicitly as eigenfunction expansions (typically involving Hankel functions), so that the method appears to be restricted to an isotropic matrix, at least in the form presented. After some complex manipulations, equations for the coefficients defining the fields scattered from the inclusions are derived. These are ensemble averaged (essentially as in (6.12)) and the system is closed by making the quasicrystalline approximation of Lax (1952) (essentially as in (6.14)). Particular examples of this approach are provided in the papers of Bose and Mal (1973, 1974), Datta (1977, 1978), Twersky (1977), Varadan, Varadan and Pao (1978) and Varadan and Varadan (1979).

A compressed and preliminary account will now be given of a variational approach. It is a natural extension of that already given for static problems and offers a unified treatment for materials such as polycrystals (for which only perturbation theory has been available) and matrix-inclusion composites (for which only the multiple scattering method has been available), without restriction to an isotropic matrix.

8.1 Variational Formulation

The equation of motion for any continuum may be given, in the absence of body forces, in the form

$$\operatorname{div} \underset{\sim}{\sigma} = \dot{\underset{\sim}{p}} \,, \tag{8.1}$$

where $\underset{\sim}{p}$ denotes momentum density and the superposed dot represents

differentiation with respect to time, t. Equation (8.1) is supplemented by the constitutive relations

$$\underset{\sim}{\sigma} = \underset{\sim}{L}\,\underset{\sim}{e} \quad , \quad \underset{\sim}{p} = \rho\,\dot{\underset{\sim}{u}} \, , \tag{8.2}$$

where $\underset{\sim}{L}$, ρ are the tensor of moduli and the density respectively, together with suitable boundary and initial conditions. The similarity between equations $(8.2)_{1,2}$ suggests introducing a comparison material with moduli $\underset{\sim}{L}_0$ and density ρ_0 and defining a momentum polarization $\underset{\sim}{\pi}$ so that

$$\underset{\sim}{\pi} = (\rho - \rho_0)\,\dot{\underset{\sim}{u}} \, , \quad \underset{\sim}{p} = \rho_0\,\dot{\underset{\sim}{u}} + \underset{\sim}{\pi} \, , \tag{8.3}$$

to parallel (3.9), (3.10) for $\underset{\sim}{\tau}$. Substituting (3.10) and $(8.3)_2$ into (8.1) then gives an equation for the displacement that would be generated in the comparison material by the body force $\operatorname{div} \underset{\sim}{\tau} - \underset{\sim}{f}$. Willis (1980b) showed that its solution could be expressed, in terms of the Green's function $\underset{\sim}{G}$ for the comparison body, in the form

$$\underset{\sim}{u} = \underset{\sim}{u}_0 - \underset{\sim}{S}\,\underset{\sim}{\tau} - \underset{\sim}{M}\,\underset{\sim}{\pi} \, , \tag{8.4}$$

generalizing (3.5), where the operators $\underset{\sim}{S}$, $\underset{\sim}{M}$ are defined by

$$\underset{\sim}{S} : \tau_{ij}(\underset{\sim}{x}, t) \rightarrow \int dt' \int_V d\underset{\sim}{x}'\, S_{pij}(\underset{\sim}{x}, t, \underset{\sim}{x}', t')\, \tau_{ij}(\underset{\sim}{x}', t') \, , \tag{8.5}$$

$$\underset{\sim}{M} : \pi_i(\underset{\sim}{x}, t) \rightarrow \int dt' \int_V d\underset{\sim}{x}' M_{pi}(\underset{\sim}{x}, t, \underset{\sim}{x}', t') \pi_i(\underset{\sim}{x}', t') \tag{8.6}$$

and

$$S_{pij}(\underset{\sim}{x}, t, \underset{\sim}{x}', t') = \frac{\partial G_{pi}(\underset{\sim}{x}, t, \underset{\sim}{x}', t')}{\partial x'_j}\bigg|_{(ij)}, \tag{8.7}$$

$$M_{pi}(\underset{\sim}{x}, t, \underset{\sim}{x}', t') = -\frac{\partial G_{pi}(\underset{\sim}{x}, t, \underset{\sim}{x}', t')}{\partial t'}. \tag{8.8}$$

The displacement $\underset{\sim}{u}_o$ that appears in (8.4) satisfies the elastic wave equation (without body forces) in the comparison material, together with the prescribed boundary and initial conditions. The initial conditions, incidentally, must be prescribed displacement and momentum density, not velocity: the distinction is usually irrelevant but (8.4) relies upon it.

Combination of (8.4) with the definitions (3.9), $(8.3)_1$ of $\underset{\sim}{\tau}$, $\underset{\sim}{\pi}$ now generates the operator equations

$$\begin{aligned}
(\underset{\sim}{L} - \underset{\sim}{L}_o)^{-1} \underset{\sim}{\tau} + \underset{\sim}{S}_x \underset{\sim}{\tau} + \underset{\sim}{M}_x \underset{\sim}{\pi} &= \underset{\sim}{e}_o, \\
(\rho - \rho_o)^{-1} \underset{\sim}{\pi} + \underset{\sim}{S}_t \underset{\sim}{\tau} + \underset{\sim}{M}_t \underset{\sim}{\pi} &= \dot{\underset{\sim}{u}}_o,
\end{aligned} \tag{8.9}$$

where $\underset{\sim}{e}_o$ is the strain associated with $\underset{\sim}{u}_o$. $\underset{\sim}{S}_x$, $\underset{\sim}{M}_x$ have kernels

$$(\underset{\sim}{S}_x)_{pqij} = \frac{\partial^2 G_{pi}}{\partial x_q \partial x'_j}\Big|_{(pq)(ij)},$$
$$(\underset{\sim}{M}_x)_{pqi} = -\frac{\partial^2 G_{pi}}{\partial x_q \partial t'}\Big|_{(pq)} \tag{8.10}$$

and $\underset{\sim}{S}_t$, $\underset{\sim}{M}_t$ are the derivatives with respect to t of $\underset{\sim}{S}$, $\underset{\sim}{M}$. Equations $(8.9)_1$ and $(8.10)_1$ generalize (3.16) and (3.7). The object now is to produce a variational principle, analogous to the Hashin-Shtrikman principle, that is equivalent to equations (8.9). One possibility is to construct an analogue of Hamilton's principle, using a Lagrangian that is quadratic in $\underset{\sim}{\tau}$, $\underset{\sim}{\pi}$. Such a principle was displayed by Willis (1980d) but an alternative, which appears to be better, is to obtain a principle, similar to that of Gurtin (1964), which is tailored to the type of initial conditions that are actually imposed. To develop this, let $\underset{\sim}{u}_1$, $\underset{\sim}{p}_1$, $\underset{\sim}{\sigma}_1$, $\underset{\sim}{e}_1$ be displacement, momentum, stress and strain fields generated by polarizations $\underset{\sim}{\tau}_1$, $\underset{\sim}{\pi}_1$ with homogeneous initial and boundary conditions, so that they satisfy (8.1), (3.10), $(8.3)_2$ (but not necessarily (8.2)) and define fields with a suffix 2 similarly. It is now easy to show that the bilinear form

$$\{(\underset{\sim}{\tau}_1, \underset{\sim}{\pi}_1), (\underset{\sim}{\tau}_2, \underset{\sim}{\pi}_2)\} = \frac{1}{V}\int_V d\underset{\sim}{x}\,(\underset{\sim}{\tau}_1 * \underset{\sim}{e}_2 + \underset{\sim}{\pi}_1 * \dot{\underset{\sim}{u}}_2), \tag{8.11}$$

where $*$ denotes the ordinary operation of convolution with respect to time, is symmetric and it follows immediately that equations (8.9) are equivalent to the variational principle

$$\delta \mathcal{F}(\underset{\sim}{\tau}, \underset{\sim}{\pi}) = 0, \tag{8.12}$$

where

$$\mathcal{F}(\underset{\sim}{\tau},\underset{\sim}{\pi}) = \frac{1}{V}\int d\underset{\sim}{x}\left\{\underset{\sim}{\tau}*\left[(\underset{\sim}{\mathsf{L}}-\underset{\sim}{\mathsf{L}}_0)^{-1}\underset{\sim}{\tau}-\underset{\sim}{e}-\underset{\sim}{e}_0\right]\right.$$
$$\left.+\,\underset{\sim}{\pi}*\left[(\rho-\rho_0)^{-1}\underset{\sim}{\pi}-\dot{\underset{\sim}{u}}-\dot{\underset{\sim}{u}}_0\right]\right\}, \qquad (8.13)$$

$\underset{\sim}{e}$ and $\dot{\underset{\sim}{u}}$ being expressed in terms of $\underset{\sim}{\tau}$, $\underset{\sim}{\pi}$ via (8.4). The functional (8.13) is a generalization of (3.17).

The principle (8.12) may now be employed to generate approximate solutions by the procedure given in Section 6. If $\mathcal{F}$ is replaced by its expectation value in (8.12), this equation generates exactly 'hierarchy equations' such as (6.12), by suitable choice of variations. These are of no use as they stand but the observation that they can be obtained in this way encourages the production of approximations generated from trial fields $\underset{\sim}{\tau}^*$, $\underset{\sim}{\pi}^*$. The natural extension of (6.8) is

$$\underset{\sim}{\tau}^*(\underset{\sim}{x},t) = \sum_{r=1}^{n}\underset{\sim}{\tau}_r(\underset{\sim}{x},t)f_r(\underset{\sim}{x}),\quad \underset{\sim}{\pi}^*(\underset{\sim}{x},t) = \sum_{r=1}^{n}\underset{\sim}{\pi}_r(\underset{\sim}{x},t)f_r(\underset{\sim}{x}); \qquad (8.14)$$

the extremum of $\langle\mathcal{F}(\underset{\sim}{\tau}^*,\underset{\sim}{\pi}^*)\rangle$ is obtained when $\underset{\sim}{\tau}_r$, $\underset{\sim}{\pi}_r$ satisfy

$$(\underset{\sim}{\mathsf{L}}_r-\underset{\sim}{\mathsf{L}}_0)^{-1}\underset{\sim}{\tau}_r+\sum_s\left(\underset{\sim}{\mathsf{S}}_x P_{s|r}\underset{\sim}{\tau}_s+\underset{\sim}{\mathsf{M}}_x P_{s|r}\underset{\sim}{\pi}_s\right)=\underset{\sim}{e}_0,$$

$$(\rho_r-\rho_0)^{-1}\underset{\sim}{\pi}_r+\sum_s\left(\underset{\sim}{\mathsf{S}}_t P_{s|r}\underset{\sim}{\tau}_s+\underset{\sim}{\mathsf{M}}_t P_{s|r}\underset{\sim}{\pi}_s\right)=\dot{\underset{\sim}{u}}_0. \qquad (8.15)$$

These equations generalize (6.11) and are the same as would be produced from the exact 'hierarchy equations' by making the quasicrystalline assumption (6.14) with regard to both $\underset{\sim}{\tau}$ and $\underset{\sim}{\pi}$. Similar equations were produced by Willis (1980c), for a matrix containing inclusions. In that work, however, they were derived making the quasicrystalline assumption explicitly, without reference to a variational principle. More general approximate equations could also be obtained from (8.12), whose full implications are still under investigation. The article is concluded, however, with a short discussion of the properties of (8.15).

8.2 Wave Speeds

The polarizations (8.14) generate a mean wave

$$\langle \underset{\sim}{u} \rangle = \underset{\sim}{u}_0 - \sum_s \left(\underset{\sim}{S} \rho_s \underset{\sim}{\tau}_s + \underset{\sim}{M} \rho_s \underset{\sim}{\pi}_s \right), \tag{8.16}$$

obtained by taking the mean of (8.4). Correspondingly, using (8.16), equations (8.15) can be recast in the form

$$\left(\underset{\sim}{L}_r - \underset{\sim}{L}_0\right)^{-1} \underset{\sim}{\tau}_r + \sum_s \left[\underset{\sim}{S}_x \left(\rho_{s|r} - \rho_s\right) \underset{\sim}{\tau}_s + \underset{\sim}{M}_x \left(\rho_{s|r} - \rho_s\right) \underset{\sim}{\pi}_s \right] = \langle \underset{\sim}{\varepsilon} \rangle,$$

$$\left(\rho_r - \rho_0\right)^{-1} \underset{\sim}{\pi}_r + \sum_s \left[\underset{\sim}{S}_t \left(\rho_{s|r} - \rho_s\right) \underset{\sim}{\tau}_s + \underset{\sim}{M}_t \left(\rho_{s|r} - \rho_s\right) \underset{\sim}{\pi}_s \right] = \langle \dot{\underset{\sim}{u}} \rangle. \tag{8.17}$$

An elementary and important problem for the ordinary elastodynamic equations

is to determine the speeds with which plane waves may travel in an infinite, homogeneous medium. The corresponding problem for (8.15) is to determine possible plane-wave solutions, for which

$$\underset{\sim}{\tau}_r(\underset{\sim}{x},t) = \underset{\sim}{\tau}_r \exp[-i(k\underset{\sim}{n}\cdot\underset{\sim}{x}+\omega t)],$$

$$\underset{\sim}{\pi}_r(\underset{\sim}{x},t) = \underset{\sim}{\pi}_r \exp[-i(k\underset{\sim}{n}\cdot\underset{\sim}{x}+\omega t)] \tag{8.18}$$

with $\underset{\sim}{\tau}_r$, $\underset{\sim}{\pi}_r$ constants, in an infinite, statistically uniform medium. Thus, plane-wave solutions of (8.15) are required, when $\underset{\sim}{u}_0 = \underset{\sim}{0}$, the probabilities P_r, $P_{s|r}$ are translation-invariant and the operators $\underset{\sim}{S}$, $\underset{\sim}{M}$ take limiting forms corresponding to infinite V. The limit is easily taken for the equivalent equations (8.16), (8.17). Equation (8.16) gives (with $\underset{\sim}{u}_0 = \underset{\sim}{0}$)

$$\langle\underset{\sim}{u}\rangle = -\tilde{\underset{\sim}{S}}(k\underset{\sim}{n},\omega)\langle\underset{\sim}{\tau}\rangle - \tilde{\underset{\sim}{M}}(k\underset{\sim}{n},\omega)\langle\underset{\sim}{\pi}\rangle, \tag{8.19}$$

where

$$\langle\underset{\sim}{\tau}\rangle = \sum_s c_s \underset{\sim}{\tau}_s \exp[-i(k\underset{\sim}{n}\cdot\underset{\sim}{x}+\omega t)],$$

$$\langle\underset{\sim}{\pi}\rangle = \sum_s c_s \underset{\sim}{\pi}_s \exp[-i(k\underset{\sim}{n}\cdot\underset{\sim}{x}+\omega t)] \tag{8.20}$$

and $\tilde{\underset{\sim}{S}}$, $\tilde{\underset{\sim}{M}}$ are the Fourier transforms of the operators $\underset{\sim}{S}$, $\underset{\sim}{M}$. Equation (8.17), implies, when V is large,

$$(\underline{\underline{L}}_r - \underline{\underline{L}}_0)^{-1}\underline{\tau}_r + \sum_s \int d\underline{x}'\, \underline{\bar{S}}^{\infty}_x(\underline{x}', \omega)(\rho_{s|r}(\underline{x}') - c_s)\, e^{-ik\underline{\eta}\cdot\underline{x}'}\underline{\tau}_s$$
$$+ \sum_s \int d\underline{x}'\, \underline{\underline{\bar{M}}}^{\infty}_x(\underline{x}', \omega)(\rho_{s|r}(\underline{x}') - c_s)\, e^{-ik\underline{\eta}\cdot\underline{x}'}\underline{\pi}_s$$
$$= -\underline{\underline{\tilde{S}}}_x(k\underline{\eta}, \omega)\sum_s c_s\underline{\tau}_s - \underline{\underline{\tilde{M}}}_x(k\underline{\eta}, \omega)\sum_s c_s\underline{\pi}_s, \tag{8.21}$$

where $\underline{\bar{S}}^{\infty}_x(\underline{x}, \omega)$, $\underline{\underline{\bar{M}}}^{\infty}_x(\underline{x}, \omega)$ are time-reduced operators obtained from the time-reduced infinite-body Green's function $\underline{\underline{\bar{G}}}^{\infty}(\underline{x}, \omega)$. Equation $(8.17)_2$ can be expanded similarly. The choice $\underline{x} = \underline{0}$ could be made in (8.21) because of the translation-invariance of $\rho_{s|r}$.

Further progress now depends upon finding a suitable representation for the Green's function $\underline{\underline{\bar{G}}}^{\infty}$. This satisfies the equation

$$\underline{\underline{L}}_0(\nabla)\underline{\underline{\bar{G}}}^{\infty} + \rho_0\omega^2\underline{\underline{\bar{G}}}^{\infty} + \underline{\underline{I}}\,\delta(\underline{x}) = \underline{0}, \tag{8.22}$$

which may be solved by the method that was explained for the special case (4.1): Willis (1980b) showed in detail that

$$\underline{\underline{\bar{G}}}^{\infty}(\underline{x}, \omega) = \frac{1}{8\pi^2}\sum_{r=1}^{3}\int_{|\underline{\xi}|=1} dS\, \frac{\underline{u}^r(\underline{\xi})[\underline{u}^r(\underline{\xi})]^T}{\rho_0 c_r^2}\left\{\delta(\underline{\xi}\cdot\underline{x}) + \frac{i\omega}{2c_r}\, e^{i\omega|\underline{\xi}\cdot\underline{x}|/c_r}\right\}, \tag{8.23}$$

where $c_r(\underline{\xi})$, $\underline{u}^r(\underline{\xi})$ satisfy

$$\left[\underline{\underline{L}}_0(\underline{\xi}) - \rho_0 c_r^2\,\underline{\underline{I}}\right]\underline{u}^r = \underline{0}, \tag{8.24}$$

so that they represent the speeds and polarizations of plane waves that can propagate in the direction $\underset{\sim}{\xi}$ in the comparison material. In (8.23), the eigenvectors $\underset{\sim}{u}^r$ are taken as normalized. The limiting case $\omega = 0$ reproduces the static formula (4.4).

Attention will now be restricted to long waves or equivalently, to small ω and a lowest-order approximate solution of (8.21) and the corresponding equation for $\underset{\sim}{\Pi}$ will be constructed by retaining only terms of order zero in k or ω. Amongst the operators $\underset{\sim}{\bar{S}}^{\infty}_x$, $\underset{\sim}{\bar{M}}^{\infty}_x$, $\underset{\sim}{\bar{S}}^{\infty}_t$, $\underset{\sim}{\bar{M}}^{\infty}_t$, only the first has a term of order zero in ω, since the other three involve at least one time derivative. Furthermore, $\underset{\sim}{\bar{S}}^{\infty}_x \sim \underset{\sim}{\Gamma}^{\infty}$, defined by (4.5). The Fourier transforms $\underset{\sim}{\tilde{S}}^{\infty}_x$ etc., on the other hand, are all homogeneous of degree zero and are retained exactly. Equation (8.21) now reduces to

$$\left(\underset{\sim}{L}_r - \underset{\sim}{L}_0\right)^{-1} \underset{\sim}{\tau}_r + \sum_s \int d\underset{\sim}{x}' \, \underset{\sim}{\Gamma}^{\infty}(\underset{\sim}{x}')\left(\rho_{s/r}(\underset{\sim}{x}') - c_s\right) \underset{\sim}{\hat{\tau}}_s = \langle \underset{\sim}{e} \rangle \tag{8.25}$$

while the corresponding equation for $\underset{\sim}{\Pi}$ becomes

$$\left(\rho_r - \rho_0\right)^{-1} \underset{\sim}{\Pi}_r = -i\omega \langle \underset{\sim}{u} \rangle, \tag{8.26}$$

where $\langle \underset{\sim}{e} \rangle$ is the strain associated with $\langle \underset{\sim}{u} \rangle$ and both are evaluated at $\underset{\sim}{x} = \underset{\sim}{0}, t = 0$. Equations (8.25) are identical with (6.15). Therefore, by definition of $\underset{\sim}{\tilde{L}}$, they imply

$$\langle \underset{\sim}{\tau} \rangle = \left(\underset{\sim}{\tilde{L}} - \underset{\sim}{L}_0\right) \langle \underset{\sim}{e} \rangle \tag{8.27}$$

(each side having the exponential dependence restored). Also, from equations (8.26),

$$\langle \underset{\sim}{\pi} \rangle = (\bar{\rho} - \rho_0)\langle \dot{\underset{\sim}{u}} \rangle, \tag{8.28}$$

where $\bar{\rho} = \Sigma c_r \rho_r$ is the mean density of the composite . Equations (8.27), (8.28) show that the mean polarizations $\langle \underset{\sim}{\tau} \rangle$, $\langle \underset{\sim}{\pi} \rangle$ are those that would be generated relative to $\underset{\sim}{L}_0$, ρ_0, if the mean wave $\langle \underset{\sim}{u} \rangle$ propagated in a homogeneous medium with moduli $\tilde{\underset{\sim}{L}}$ and density $\bar{\rho}$. This conclusion can also be reached by algebraic manipulation. Retention of terms of higher order in the equations produces both disposition and attenuation. Attenuation in a matrix containing inclusions has been studied by Willis (1980c). A more complete study of the present system is being conducted and will be reported elsewhere.

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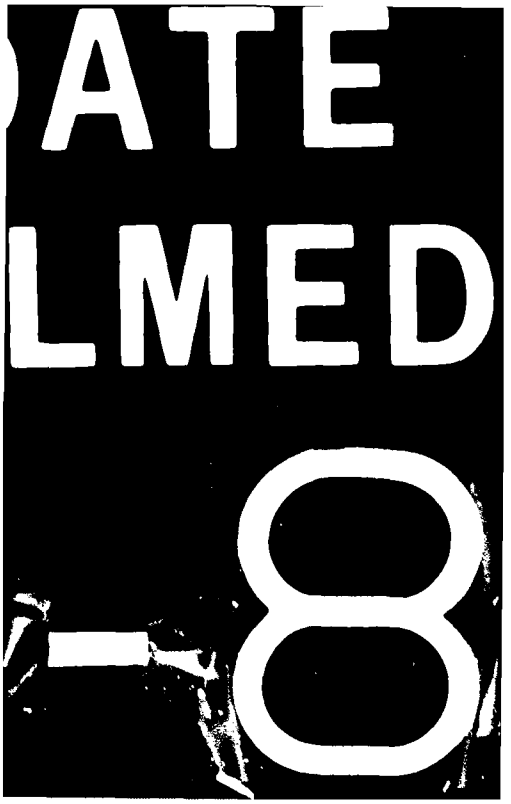
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