RADC-TR-80-30
Final Technical Report
February 1980

BAYESIAN RELIABILITY THEORY FOR REPAIRABLE EQUIPMENT

Southeastern Center for Electrical Engineering Education

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: RADC-TR-80-30	2. GOVT ACCESSION NO.
	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): BAYESIAN RELIABILITY THEORY FOR REPAIRABLE EQUIPMENT	5. TYPE OF REPORT & PERIOD COVERED: Final Technical Report, 1 Jun 79 — 30 Sep 79
	6. PERFORMING ORG. REPORT NUMBER: N/A
7. AUTHOR(s): Theodore S. Bolis	8. CONTRACT OR GRANT NUMBER(s): F30602-78-C-0148
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Southeastern Center for Electrical Engineering Education, 7300 Lake Ellenor Drive, Orlando FL 32809	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 62702F, 233802P7
11. CONTROLLING OFFICE NAME AND ADDRESS: Rome Air Development Center (RBET), Griffiss AFB NY 13441	12. REPORT DATE: February 1980
	13. NUMBER OF PAGES: 25
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Same	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE: N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Same

18. SUPPLEMENTARY NOTES

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Reliability
Statistics
Reliability Testing
Bayesian Testing
Bayesian Statistics

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The report introduces an assumption called instantaneous resurrection which holds that a repaired equipment is returned to the state of an equipment of similar age which has not yet failed, rather than to the state of a new equipment. Under this assumption, the number of failures of an equipment in time t is a Poisson process with leading function -log R(t), where R(t) is the reliability function of the equipment. This fact can be used in the estimation of parameters of various prior (over)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) distributions via the appropriate Poisson mixtures. The case of the two-way truncated Gamma prior and its limiting forms is discussed in the report.

ACKNOWLEDGEMENTS

The author expresses his thanks to Anthony Coppola and all the personnel of the R&M Techniques Section of RADC for their congeniality and support during the completion of this effort.

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ABSTRACT

A repairement assumption called instantaneous resurrection is introduced and it is shown that, under this assumption, the number of failures of an equipment in time t is a Poisson process with leading function -log R(t), where R(t) is the reliability function of the equipment. This fact is used in the estimation of parameters of prior distributions via the appropriate Poisson mixtures. The case of the two-way truncated Gamma prior and its limiting forms is handled in some detail. A preliminary form of a goodness of fit test in the case of varying operational time in attribute failure data is presented.

EVALUATION

The work described in the report extends previous work by Dr. Bolis which provided guides to structuring Bayesian Reliability Test Plans assuming an inverted gamma distribution of the prior information and a constant failure rate of the tested equipment. The report considers cases where these assumptions do not hold. The previous work is currently being incorporated into a standard for Bayesian Reliability Tests. The results presented in this report will be used for modifying the standard.

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0. INTRODUCTION

In this report we introduce a repairment assumption, which we call resurrection. Under this assumption when an equipment fails and is repaired, the repair does not bring the equipment to a brand new stage but to a stage of a functioning equipment of the same type and age that had no failures. This assumption may be deemed as a simplication of the actual circumstances, but to our opinion, it is a lot more preferable than the renewal assumption.

We show in theorem 1.2.1 that, under the instantaneous resurrection assumption, the number of failures of an equipment in time t is a non-homogeneous Poisson process with leading function - log R(t), where R(t) is the reliability function of the equipment. If we assume a suitable form of a prior distribution of the parameters of the reliability function, the resulting Poisson mixture (marginal distribution) allows the estimation of the prior distribution from prior data. When the prior is determined, Bayesian acceptance sampling plans can be constructed. The above form the content of Section 1.

In Section 2 we consider the case of constant failure rate, where the resurrection and renewal assumptions coincide. The two-way truncated inverted Gamma distribution as well as all of its limiting forms (truncated Pareto, uniform, etc.) is assumed and the methods of estimating its parameters (mainly the method of moments) are presented.

In Section 3 we indicate how the methods of Section 2 can be carried out (with minor adjustments) to the case of variable failure rate. As an example, the Weibull time to failure is considered.

In Section 4 we construct a statistic, akin to the χ^2 statistic, to test goodness of fit in the case the failure data do not exhibit the same total time of operation. The exact mean and variance of this statistic as well as the cumulants of its asymptotic distribution are computed. We do not know how these quantities are affected when parameters are estimated from the sample used in the construction of this statistic.

This report is of a theoretical nature. It is of interest to check the resurrection assumption by using failure data of repairable equipment with known distribution of time to failure other than exponential. Also, it would be of interest to see the amount of improvement of fit if the two-way truncated inverted Gamma prior distribution is used on the mean time to failure of electronic equipment in comparison with the use of the full inverted Gamma. Further, the question of settling the behaviour of the statistic S of Section 4 when parameters are estimated is important. Some future effort may be dedicated to these questions.

1. GENERAL THEORY

1.1 THE DISTRIBUTION OF THE TIME TO FAILURE. Let $f(t)$, $t \geq o$ be the density function of the time to first failure of an equipment. We assume that the function f is continuous. The function

$$R(t) = \int_t^\infty f(x)dx, \quad t \geq o,$$

is called the reliability function of the equipment under discussion. It is the probability that the equipment will first fail after time t. The failure rate $\lambda(t)$ (or hazard rate, or force of mortality) of this equipment is defined by

$$\lambda(t) = -\frac{d}{dt} \log R(t) = f(t)/R(t).$$

Another measure of reliability is the mean time to failure. In the general case of an aging repairable equipment, it is more precise to call this measure the mean time to first failure. If $\lim tR(t) = 0$, $t \to \infty$, this mean is given by

$$(1.1.1) \quad \theta_1 = \int_0^\infty t\, f(t)dt = \int_0^\infty R(t)dt.$$

1.2 THE DISTRIBUTION OF ATTRIBUTE FAILURE DATA. When a repairable equipment is put to life testing and a failure occurs, two simple and opposite assumptions can be made with regard to its state after repair: The repair either brought the equipment to the start of a brand new one (renewal assumption) or to the start of an equipment of a failure-free equipment of the same age (resurrection assumption). The first case has been extensively studied in the literature. Although the actual situation lies somewhere between these two extreme cases, we believe that the resurrection assumption is more realistic in the case of nonconstant failure rate.

When a repairable equipment is tested, we shall assume instantaneous repair, i.e., either the repair time is negligible or it is not included as test time. Two quantities enter in such testing: the test time T and the number of failures r during the time T. We can either fix the test time T in advance and observe the random variable r (type I censoring) or test the equipment until a specified number of failures occur (type II censoring). While a type I censored test has obvious advantages (test time and cost well controlled), a type II censoring has theoretical as well as practical interest. For example, if only r repairs are possible, then one would be interested in the distribution of the waiting time for the (r+1) th failure.

1.2.1 TYPE I CENSORING UNDER THE INSTANTANEOUS RESURRECTION ASSUMPTION. It is surprising that the distribution of the number of failures r in constant time T under the instantaneous resurrection assumption is always a Poisson distribution with mean $-\log R(T)$. We formalize and prove this result in the following:

THEOREM 1.2.1. Under the instantaneous resurrection assumption, the distribution of the number of failures of an equipment with reliability function $R(t)$ in a fixed time T is the Poisson distribution with mean $-\log R(T)$.

PROOF: The density function of the time t_1 of the first failure is of course $f(t_1)$, $t_1 \geq o$. The cumulative distribution function of the time t_2 of the second failure is given by

$$\text{Prob}\,\{t_2 \leq x \mid t_2 > t_1\} = \text{Prob}\,\{t_1 < t_2 \leq x\} \,/\, \text{Prob}\,\{t_2 > t_1\} = (F(x) - F(t_1))/R(t_1),$$

$$x > t_1,$$

where F denotes the cumulative distribution function of the time to failure. At this point we used the instantaneous resurrection assumption. Thus, the density function of t_2 is $f(t_2)/R(t_1)$, $t_2 > t_1$. Similarly, the density function of the time t_i of the ith failure is $f(t_i)/R(t_{i-1})$, $t_i > t_{i-1}$, $1 < i \leq r$. Finally, the probability that no failures occurred between times t_r and T is $R(T)/R(t_r)$. Thus the joint density function of the times $t_1, t_2, \ldots, t_r$ with $o \leq t_1 < t_2 < \ldots < t_r \leq T$ is

$$f(t_1)\,\frac{f(t_2)}{R(t_1)}\;\cdots\;\frac{f(t_r)}{R(t_{r-1})}\,\frac{R(T)}{R(t_r)} = R(T)\,\lambda(t_1)\,\ldots\,\lambda(t_r).$$

Therefore, the probability that r failures occurred in fixed time T is

$$P_T(r) = R(T) \int_o^T \int_{t_1}^T \cdots \int_{t_{r-1}}^T \lambda(t_1)\,\lambda(t_2)\,\ldots\,\lambda(t_r)\,dt_r \ldots dt_1$$

$$= R(T) \int_o^T \int_{t_1}^T \cdots \int_{t_{r-2}}^T \lambda(t_1)\,\ldots\,\lambda(t_{r-1}) \log \frac{R(t_{r-1})}{R(T)}\,dt_{r-1} \ldots dt_1$$

$$= \frac{1}{r!}\,R(T)\,\log^r\,(1/R(T)),\quad r = o, 1, \ldots \qquad \text{q.e.d.}$$

More generally, let $\xi(t)$ be the number of failures of the equipment in time t. By reasoning as in theorem 1.2.1, we obtain

$$(1.2.1)\quad \text{Prob}\,\{\xi(t) - \xi(s) = r\} = \frac{1}{r!}\,\frac{R(t)}{R(s)}\,\log^r\,(R(s)/R(t)),\; t > s,$$

i.e. the stochastic process $\xi(t)$ is a non-homogeneous Poisson process with leading function $-\log R(t)$ (cf. page 46 of (1)).

1.2.2. TYPE II CENSORING UNDER INSTANTANEOUS RESURRECTION. Let T denote the waiting time until the rth failure (r fixed). We shall show that the random variable $-\log R(T)$ has the Gamma distribution with shape parameter r and scale parameter 1. In view of theorem 1.2.1 this does not seem surprising.

THEOREM 1.2.2. _Under the instantaneous resurrection assumption, the density function of the waiting time T until the rth failure is given by_

$$(1.2.2) \quad f_r(T) = \frac{1}{(r-1)!} f(T) \log^{r-1} (1/R(T)), \; T \geq o.$$

PROOF: By reasoning along lines similar to those of the proof of theorem 1.2.1 we obtain that the cumulative distribution function of T is given by

$$\text{Prob } \{T \leq x\} = \int_o^x \int_o^x \cdots \int_{t_{r-1}}^x \lambda(t_1) \ldots \lambda(t_{r-1}) f(t_r) \, dt_r \ldots dt_1 .$$

By differentiating this function with respect to x and replacing x by T we obtain (1.2.2). q.e.d.

The average waiting time to the rth failure will be called the _mean time to rth failure_ and it will be denoted by θ_r.

Thus,

$$(1.2.3) \quad \theta_r = \frac{1}{(r-1)!} \int_0^\infty T \log^{r-1} (1/R(T)) f(T) \, dT \quad r=1, 2, \ldots$$

This agrees with the formula (1.1.1).

1.2.3 THE INSTANTANEOUS RENEWAL ASSUMPTION. The distribution of attribute failure data under the instantaneous renewal assumption is well worked out in the literature (cf. e.g. (1), (2), (3)). In the type I situation we have

$$(1.2.4) \quad P_T(r) = \int_o^T \int_{t_1}^T \cdots \int_{t_{r-1}}^T f(t_1) f(t_2-t_1) \cdots R(T-t_r) \, dt_r \ldots dt_1$$

whereas in the type II case we obtain

$$(1.2.5) \quad f_r(T) = \int_o^T \int_{t_1}^T \cdots \int_{t_{r-2}}^T f(t_1) f(t_2-t_1) \cdots f(T-t_{r-1}) \, dt_{r-1} \cdots dt_1$$

$$T \geq o, \; r=2, 3, \ldots ,$$

$$f_1(T) = f(T).$$

If we set $F_o(T) = 1$, $F_r(T) = \int_o^T f_r(t)dt$, $r = 1, 2, \ldots$ $(T \geq o)$, we get

$$P_T(r) = F_r(T) - F_{r+1}(T).$$

In general, the integrations occurring in (1.2.4) and (1.2.5) are difficult to carry out in closed form. It is well known that the instantaneous resurrection assumption is equivalent to the instantaneous renewal assumption if and only if the distribution of the time to failure is exponential.

1.2.4 CENSORING WITHOUT REPLACEMENT. Suppose that n identical equipments are put to life testing until a beforehand specified time T. Suppose further that r equipments failed at times $t_1 \le t_2 \le \ldots \le t_r \le T$. The joint density of these times is given by

$$\frac{n!}{(n-r)!} f(t_1) \ldots f(t_r) R(T)^{n-r} \quad o \le t_1 \le \ldots \le t_r \le T$$

and the probability function of the numer of failures r is binomial:

$$(1.2.6) \quad P_T(r) = \binom{n}{r} (1-R(T))^r R(T)^{n-r} \quad r = 0, 1, \ldots, n.$$

Similarly, if n equipments are tested until the rth failure occurs (when an equip- fails, it is discarded), the times of failure $0 \le t_1 \le \ldots t_{r-1} \le t_r = T$ have joint density function

$$\frac{n!}{(n-r)!} f(t_1) \ldots f(t_{r-1}) f(T) R(T)^{n-r}, \; o \le t_1 \le \ldots \le t_{r-1} \le T$$

and the density function of T is given by

$$(1.2.7) \quad f_r(T) = r \binom{n}{r} f(T) (1-R(T))^{r-1} R(T)^{n-r}, \; T \ge o$$

i.e. the random variable R(T) has the Beta distribution with parameters n-r+1 and r.

1.3 BAYESIAN ESTIMATION TECHNIQUES OF RELIABILITY CHARACTERISTICS. Usually the time to failure distribution depends on a (possibly vector-valued) parameter α, i.e. f(t) is of the form

$$f(t|\alpha), \; t \ge o, \quad \alpha \varepsilon G_1 \subset R^k$$

where G_1 is the range of the parameter α (usually a nice subset of the Euclidean space R^k). Knowledge of the parameter α entails complete knowledge of the reliability characteristics of the equipment. Classical estimation techniques apply to failure data of the different kinds mentioned in the previous section 1.2 to yield point estimates and confidence regions for α. Also tests of hypothesis (reliability demonstration) can be devised for α. These latter tests can be acceptance tests with their power functions delineating the producer's and consumer's risks. Here, we confine ourselves to Bayesian techniques.

1.3.1 DETERMINATION OF THE PRIOR DISTRIBUTION. The choice of the form of the prior distribution of α is made on the basis of computational tractability, sufficient flexibility to accommodate a wide variety of distribution shapes and, if possible, on whether it gives rise to conjugate pairs, i.e., on whether the posterior distribution of α given the failure data will be of the same form as the prior. Once the form of the prior has been decided upon, it remains to determine its parameters. We shall assume that prior failure data of the equipment in question are available. Since α is unobservable, the parameters of the prior can be estimated from the marginal distribution of the available failure data, provided that this marginal distribution is uniquely determined by the prior. In this case, we say that the prior distribution is identifiable by the marginal distribution of the data.

Let us be more specific. Let

$$g(\alpha;\beta),\ \alpha\epsilon G_1\subset R^k,\ \beta\epsilon G_2\subset R^m$$

be the prior density of α, where β is a (possibly vector-valued) parameter to be determined (estimated). If the available failure date are attribute data of the type I censoring, then the marginal distribution in question will be

$$(1.3.1)\quad \int_{G_1} p_T(r|\alpha)\, g(\alpha;\beta)\, d\alpha$$

where $P_T(r|\alpha)$ is given either by theorem 1.2.1 or by (1.2.4) or by (1.2.6) depending on the nature of the available data. The parameter β can be estimated by classical methods under the hypothesis of identifiability. The appropriateness of the prior can be checked by a χ^2 - goodness of fit test. If the data are of the type II censoring (a rare situation), a similar procedure can be applied.

The procedure described so far in this section was initiated by Schafer et.al. (4) for the case of exponential time to failure with an inverted Gamma prior distribution for the mean time to failure. The method of moments was used in (4) whereas Goel (5) used the maximum likelihood method in the same situation. When the available failure data are of the form (r_i, T_i), i=1, 2, ... n (the ith equipment had r_i failures in time T_i) it was proposed by us in (6) that in order to estimate the parameter(s) of the prior distribution, the following generalized likelihood function

$$(1.3.2)\quad \prod_i \int_{G_1} P_{T_1}(r_i|\alpha)\, g(\alpha;\beta)\, d\alpha$$

should be maximized. Since this method coincides with the maximum likelihood method in case all T_i are the same, we called it the generalized maximum likelihood method. A preliminary form of a goodness of fit test for this situation is discussed in Section 4.

1.3.2. BAYESIAN ACCEPTANCE PLANS. In this section we assume that the prior distribution is known. We further assume that equipments with values of the parameter α in some subset G_0 of G_1 are acceptable. We confine ourselves to acceptance test plans of the form (T^*, r^*), i.e., if in test time T^* the number of failures of the equipment is less than or equal to r^*, the equipment is accepted. If this number of failures is greater than r^*, the equipment is rejected. The quantities

$$p = \text{Prob } \{\alpha \varepsilon G_0 | r > r^*\}$$

$$q = \text{Prob } \{\alpha \notin G_0 | r \leq r^*\}$$

are called the producer's and the consumer's risk respectively. These risks are usually referred to as posterior risks (cf. Goel and Joglekar (5)). These risks are given by the following formulas

$$(1.3.3) \quad p = \sum_{r=r^*+1}^{\infty} \int_{G_0} P_{T^*} (r|\alpha)\, g(\alpha;\beta)\, d\alpha \Big/ \sum_{r=r^*+1}^{\infty} \int_{G_1} P_{T^*} (r|\alpha)\, g(\alpha;\beta)\, d\alpha$$

$$(1.3.4) \quad q = \sum_{r=0}^{r^*} \int_{G_1 \setminus G_0} P_{T^*}(r|\alpha)\, g(\alpha;\beta)\, d\alpha \Big/ \sum_{r=0}^{r^*} \int_{G_1} P_{T^*} (r|\alpha) g(\alpha;\beta)\, d\alpha$$

Once the consumer decides on q and G_0, he can determine T^* and r^* from (1.3.4) (with T^* smallest possible) and thus obtain an acceptance test plan. The equation (1.3.3) will then determine the producer's risk.

2. EXPONENTIAL TIME TO FAILURE

2.1 THE EXPONENTIAL DISTRIBUTION. The most widely used distribution for the time to failure is the exponential distribution. This distribution is deemed appropriate for electronic equipment. We will not enter into the discussion on when its use is appropriate and when its use constitutes a misuse.

The density function is given by

$$(2.1.1) \quad f(t|\theta) = \frac{1}{\theta} \exp(-t/\theta) \quad t \geq o, \quad \theta > o,$$

the reliability function is

$$(2.1.2) \quad R(t|\theta) = \exp(-t/\theta)$$

and the failure rate is constant

$$(2.1.3) \quad \lambda(t) = 1/\theta \quad t \geq o.$$

It follows from (1.1.1) that the mean time to failure is θ, i.e.,

$$(2.1.4) \quad \theta_1 = \theta.$$

Because of the strong Markov property of the exponential distribution

$$(2.1.5) \quad \text{Prob}\{t>a+b|t>a\} = \text{Prob}\{t>b\} \qquad (a, b>o),$$

the instantaneous resurrection and the instantaneous renewal assumptions lead to the same distributions of the attribute failure data. Thus, theorem 1.2.1 and formula (1.2.4) yield

$$(2.1.6) \quad P_T(r|\theta) = \frac{1}{r!} (T/\theta)^r \exp(-T/\theta),$$

whereas theorem 1.2.2 and formula (1.2.5) yield

$$(2.1.7) \quad f_r(T|\theta) = \frac{1}{\Gamma(r)} (T^{r-1}/\theta^r) \exp(-T/\theta).$$

Also formula (1.2.1) is reduced to

$$\text{Prob}\{\xi(t) - \xi(s) = r\} = \frac{1}{r!} \left(\frac{t-s}{\theta}\right)^r \exp(-(t-s)/\theta),$$

i.e., the stochastic process $\xi(t)$ is, in this case, a <u>homogeneous</u> Poisson process with constant $1/\theta$. The formula (1.2.3) yields

$$(2.1.8) \quad \theta_r = r\theta$$

Because of (2.1.5) the mean time to failure equals the mean time between failures. The formula (2.1.8) tells us that the mean time to the rth failure is r times the mean time to failure.

2.2. THE TRUNCATED INVERTED GAMMA PRIOR DISTRIBUTION AND ITS LIMIT FORMS.

The parameter of the exponential distribution is the mean time to failure θ. In the set-up of section 1.3, $k=1$, $\alpha=\theta$, $G_1 = (o, +\infty)$. We propose as a prior distribution the truncated inverted gamma distribution

$$(2.2.1)\quad g(\theta;\gamma,\lambda, a, b) = \frac{\gamma^{\lambda}\,\theta^{-(\lambda+1)} \exp(-\gamma/\theta)}{\Gamma(\lambda,\gamma/b) - \Gamma(\lambda,\gamma/a)},\ a<\theta<b$$

where $\gamma>o$ and

$$(2.2.2)\quad \Gamma(\lambda, x) = \int_x^{+\infty} u^{\lambda-1} e^{-u}\, du,\ x\geq o.$$

If $x=o$ in (2.2.2), we must have $\lambda>o$ and, in this case, $\Gamma(\lambda,o) = \Gamma(\lambda)$

If $\lambda>o$, $\Gamma(\lambda, x)$ can be computed either by numerical integration by using the formula

$$(2.2.3)\quad \Gamma(\lambda, x) = \Gamma(\lambda) - \int_o^x u^{\lambda-1} e^{-u}\, du$$

or by using Pearson's tables (7), where the function

$$(2.2.4)\quad I(x, p) = \frac{1}{\Gamma(p+1)} \int_o^{x\sqrt{p+1}} u^p e^{-u}\, du$$

is tabulated. From (2.2.3) and (2.2.4) we obtain

$$(2.2.5)\quad \Gamma(\lambda, x) = \Gamma(\lambda)\,(1-I\,(x/\sqrt{\lambda}, \lambda-1))$$

The recurrence relation

$$(2.2.6)\quad \Gamma(\lambda+1, x) = x^{\lambda} e^{-x} + \lambda\,\Gamma(\lambda, x),$$

the asymptotic formula

$$\Gamma(\lambda,x) \sim x^{\lambda-1} e^{-x} \left(1 + \frac{\lambda-1}{x} + \frac{(\lambda-1)(\lambda-2)}{x^2} + \dots\right) \text{ as } x\to\infty$$

as well as the limit

$$\lim_{\lambda\to\infty} \frac{\Gamma(\lambda,\lambda x)}{\Gamma(\lambda)} = \begin{cases} o \text{ if } x>1 \\ 1/2 \text{ if } x=1 \\ 1 \text{ if } -.27846\ldots<x<1 \end{cases}$$

shed some light into the behavior of this function (cf. for example Chapter 6 of (8)).

The kth moment about zero of the distribution (2.2.1) is

$$(2.2.7)\quad \mu_k' = \gamma^k \frac{\Gamma(\lambda-k, \gamma/b) - \Gamma(\lambda-k, \gamma/a)}{\Gamma(\lambda, \gamma/b) - \Gamma(\lambda, \gamma/a)},\ k=o, 1, \dots$$

We now look at some interesting limiting forms of (2.2.1).

Case (A): $a \to o$. In this case (2.2.1) takes the form

$$(2.2.1A)\quad g(\theta;\gamma,\lambda,o,b) = \gamma^{\lambda}\theta^{-(\lambda+1)}\exp(-\gamma/\theta)/\Gamma(\lambda,\gamma/b)$$

$$o<\theta<b,\ \gamma>o$$

and its moments about zero are

$$(2.2.7A)\quad \mu_k' = \gamma^k\,\Gamma(\lambda-k,\gamma/b)/\Gamma(\lambda,\gamma/b),\ k=o,1,\ldots$$

Case (B): $b \to \infty$. Here, the formulas (2.2.1) and (2.2.7) yield

$$(2.2.1B)\quad g(\theta;\gamma,\lambda,a,\infty) = \frac{\gamma^{\lambda}\theta^{-(\lambda+1)}\exp(-\gamma/\theta)}{\Gamma(\lambda)-\Gamma(\lambda,\gamma/a)},\ a<\theta,\ \gamma>o,\ \lambda>o$$

$$(2.2.7B)\quad \mu_k' = \gamma^k\,\frac{\Gamma(\lambda-k)-\Gamma(\lambda-k,\gamma/a)}{\Gamma(\lambda)-\Gamma(\lambda,\gamma/a)},\ o\le k<\lambda$$

Case (AB): $a \to o$ and $b \to \infty$. This is the case of the inverted Gamma distribution.

$$(2.2.1AB)\quad g(\theta;\gamma,\lambda,o,\infty) = \gamma^{\lambda}\theta^{-(\lambda+1)}\exp(-\gamma/\theta)/\Gamma(\lambda),\ \theta>o,\ \gamma>o,\ \lambda>o;$$

$$(2.2.1AB)\quad \mu_k' = \gamma^k/(\lambda-1)(\lambda-2)\ldots(\lambda-k),\ o\le k<\lambda.$$

Case (Γ): $\gamma \to o$. In this case we obtain

$$(2.2.1\Gamma)\quad g(\theta;o,\lambda,a,b) = \lambda\theta^{-(\lambda+1)}/(a^{-\lambda}-b^{-\lambda}),\ \text{if } \lambda\neq 0,$$

$$(2.2.1\Gamma\Lambda_0)\quad g(\theta;o,o,a,b) = 1/\theta\log(b/a),\ o<a<\theta<b.$$

This is a two-ways truncated Pareto distribution. If $\lambda<o$, b can be taken equal to $+\infty$ in (2.2.1Γ), whereas if $\lambda<o$, a can be zero. If $\lambda=-1$ (2.2.1Γ) yields the uniform distribution. Thus

$$(2.2.1\Gamma\Lambda_{-1})\quad g(\theta;o,-1,a,b) = 1/(b-a),\ o\le a<\theta<b.$$

If $b \to \infty$ in ($2.2.1\Gamma\Lambda_{-1}$) we obtain an improper distribution.

The moments are easily obtained:

$$(2.2.7\Gamma)\quad \mu_k' = \begin{cases} \dfrac{\lambda}{\lambda-k}\,(a^{-(\lambda-k)}-b^{-(\lambda-k)})/(a^{-\lambda}-b^{-\lambda}), & \text{if } k\neq\lambda \\ \lambda\log(b/a)/(a^{-\lambda}-b^{-\lambda}), & \text{if } k=\lambda, \end{cases}$$

$$(2.2.7\Gamma\Lambda_o)\ \mu_k' = \frac{1}{k}\ (b^k - a^k)\ /\ \log\ (b/a), k=1,\ 2,\ \ldots\ ,$$

$$(2.2.7\Gamma\Lambda_+B)\ \mu_k' = \lambda a^k/(\lambda - k), o\leq k<\lambda,$$

$$(2.2.7\Gamma\Lambda_-A)\ \mu_k' = b^k\ /\ (k-\lambda)\ , k=o,\ 1,\ \ldots\ .$$

($\Gamma\Lambda_+B$ means that in 2.2.7Γ, λ is positive and $b\to\infty$ whereas $\Gamma\Lambda_-A$ means that λ is negative and $a\to o$. The uniform case is included in 2.2.7Γ with λ =-1).

2.2.1 ESTIMATION OF THE PARAMETERS. By virtue of (2.1.6) and (2.2.1), the marginal distribution (cf. formula (1.3.1)) has the form

$$(2.2.8)\quad P_T\ (r;\gamma,\ \lambda,\ a,\ b) = \gamma^\lambda\ T^r\ [\Gamma(\lambda+r,(T+\gamma)/b)-\Gamma(\lambda+r,(T+\gamma)/a)]/$$

$$r!\ (T+\gamma)^{\lambda+r}\ [\Gamma(\lambda,\gamma/b) - \Gamma(\lambda,\ \gamma/b)]$$

In terms of the incomple Gamma ratio (the function given by (2.2.4)), (2.2.8) takes the form

$$P_T(\gamma;\gamma,\ \lambda,\ a,b) = \left(\frac{\gamma}{T+\gamma}\right)^\lambda \left(\frac{T}{T+\gamma}\right)^r \binom{\lambda+r-1}{r} \frac{I((T+\gamma)/a\sqrt{\lambda},\ \lambda+r-1)-I((T+\gamma)/b\sqrt{\lambda},\ \lambda+r-1)}{I(\gamma/a\sqrt{\lambda},\ \lambda-1) - I\ (\gamma/b\sqrt{\lambda},\ \lambda-1)}$$

The (descending) factorial moments of this compound distribution are

$$(2.2.9)\quad \mu_{(k)} = (T/\gamma)^k\ [\Gamma(\lambda+k,\gamma/b)-\Gamma(\lambda+k,\gamma/a)]/[\Gamma(\lambda,\gamma/b) - \Gamma(\lambda,\gamma/a)]$$

$$k=o,\ 1,\ \ldots$$

There are four parameters to be estimated. If we have a sample with T fixed, the method of moments can be used. Let $m_{(k)}$ denote the factorial sample moments. Thus, by matching moments we get

$$(2.2.10)\quad m_{(k)} = (T/\gamma)^k\ [\Gamma(\lambda+k;\ \gamma/b)-\Gamma(\lambda+k,\gamma/a)]/[\Gamma(\lambda,\gamma/b)-\Gamma(\lambda,\gamma/a], k=1,2,3,4\ .$$

By dividing the kth equation in (2.2.10) by the (k-1)th (remembering that $m_{(o)} = \mu_{(o)} = 1$) and using the recursion (2.2.6) we obtain

$$(2.2.11)\quad m_{(k)}/m_{(k-1)} - (\lambda+k-1)T/\gamma = (T/\gamma)[(\gamma/b)^{\lambda+k-1}e^{-\gamma/b} - (\gamma/a)^{\lambda+k-1}e^{-\gamma/a}] \div$$

$$[\Gamma(\lambda+k-1,\ \gamma/b)-\Gamma(\lambda+k-1,\ \gamma/a)],\ k=1,2,3,4\ .$$

We set $M_k = m_{(k)} - m_{(k-1)}\ (\lambda+k-1)\ T/\gamma$ and consecutively divide the the equations (2.2.11). In this way, we obtain the equivalent system

$$M_{k+1}/M_k = \frac{T}{b}\ [1-(b/a)^{\lambda+k}\ \exp[(b^{-1}-a^{-1})\gamma]/(1-(b/a)^{\lambda+k-1}\ \exp[(b^{-1}-a^{-1})\gamma])$$

$$k=1,2,3,$$

(2.2.12)

$$M_1=(T/\gamma)(\gamma/b)^{\lambda}e^{-\gamma/b}[1-(b/a)^{\lambda}\exp\ [(b^{-1}-a^{-1})\gamma]]/[\Gamma(\lambda,\gamma/b)-\Gamma(\lambda,\gamma/a)].$$

A little more algebra with the first three of (2.2.12) yields

$$abM_3 - (a+b)\ TM_2 + T^2M_1 = 0$$

$$abM_4 - (a+b)\ TM_3 + T^2M_2 = 0$$

$$(2.2.13)\quad (b/a)^{\lambda+1}\ \exp\ [\gamma(b^{-1}-a^{-1})] = (bM_2 - TM_1)/(aM_2-TM_1)$$

The first two of these yield

$$ab = T^2(M_1M_3 - M_2^2)/(M_2M_4 - M_3^2) \equiv T^2B(\gamma,\lambda,T)$$

$$a+b = T(M_1M_4 - M_2M_3)/(M_2M_4 - M_3^2) \equiv TA(\gamma,\lambda,T)$$

which imply

$$(2.2.14)\quad a = \tfrac{1}{2}T[A - (A^2 - 4B)^{\frac{1}{2}}] \qquad b = \tfrac{1}{2}T[A+(A^2 - 4B)^{1/2}].$$

Thus we have a and b exclusively in terms of γ and λ. If we substitute these values of a and b in (2.2.13) and the fourth of (2.2.12), we obtain two equations in γ and λ which can be solved numerically.

We now describe a simpler estimation method. We can take as a and b the lower and upper confidence bounds for θ as estimated from the failure data. If the data are r_i, $i=1, \ldots, n$, and if $r=\Sigma r_i$, then $\hat{\theta} = nT/r$ and a two sided confidence interval for $\hat{\theta}$ at confidence level $1-\alpha$ is

$$(2.2.15)\quad \left[\frac{2r\hat{\theta}}{\chi^2_{1-\alpha/2}\ (2r+2)},\ \frac{2r\hat{\theta}}{\chi^2_{\alpha/2}\ (2r)}\right]$$

(cf. p. 181 of (2)).

We can take α very small and put a equal to the lower confidence bound and b equal to the upper confidence bound. Then, we can proceed to the computation of γ and λ.

It is easy to write down the maximum likelihood equations or the generalized maximum likelihood equations for this situation. They turn out to be very complicated to work with although this complication is considerably reduced if we assume that a and b are known. The main disadvantage is that numerical integration is necessary in order to compute integrals of the form

$$\psi(\lambda, x) = \int_x^{+\infty} u^{\lambda-1} \log u\, e^{-u}\, du = \Gamma(\lambda)\psi(\lambda) - \int_0^x u^{\lambda-1} \log u e^{-u}\, du.$$

However, if λ is assumed known, this difficuly is overcome. If λ and γ are assumed known, the situation is quite manageable. We omit the details.

In the situation of type II censored failure data we obtain

$$(2.2.16)\quad f_r(T; \gamma, \lambda, a,b) = \frac{r}{T} P_T(r; \gamma, \lambda, a, b)$$

where $P_T(r; \gamma, \lambda, a, b)$ is given by (2.2.8). The moments about zero are

$$(2.2.17)\quad \mu'_k = \gamma^k \frac{\Gamma(r+k)}{\Gamma(r)} \frac{\Gamma(\lambda-K, \gamma/b) - \Gamma(\lambda-k, \gamma/a)}{\Gamma(\lambda, \gamma/b) - \Gamma(\lambda, \gamma/a)}$$

An analysis similar to that of the case of type I data can be carried out in using the method of moments to estimate the parameters. Incidentally, (2.2.16) shows that the generalized likelihood equations will have the same form as that of type I data (since r/T does not depend on the parameters).

We observe that the posterior distribution is again a truncated inverted Gamma distribution with parameters $T+\gamma$, $\lambda+r$, a, b, i.e.

$$h_T(\theta|r) = h_r(\theta|T) = (T+\gamma)^{\lambda+r}\theta^{-(\lambda+r+1)} \exp[-(T+\gamma)/\theta] \div$$

$$[\Gamma(\lambda+r, (T+\gamma)/b) - \Gamma(\lambda+r, (T+\gamma)/a)].$$

The posterior mean is given by (2.2.7) for k=1, γ replaced by $T+\gamma$ and λ replaced by $\lambda+r$.

We now turn to the more hospitable limiting cases of the truncated inverted Gamma prior distribution.

<u>Case</u> (A): $a \to o$. The marginal distribution for type I attribute data is

$$(2.2.8A)\quad P_T(r;\gamma, \lambda, o, b) = \gamma^{\lambda} T^{r}\, \Gamma(\lambda+r, (T+\gamma/b)/r!(T+\gamma)^{\lambda+r}\, \Gamma(\lambda, \gamma/b)$$

and the equating of sample factorial moments with the theoretical ones yields

$$(2.2.10A)\quad m_{(k)} = (T/\gamma)^k\, \Gamma(\lambda+k, \gamma/b) / \Gamma(\lambda, \gamma/b) \quad k=1,2,3.$$

By eliminating b we obtain the quadratic equation

$$(2.2.18)\quad (\gamma m_{(2)} - (\lambda+1)Tm_{(1)})^2 = (\gamma m_{(1)} - \lambda T)(\gamma m_{(3)} - (\lambda+2)Tm_{(2)})$$

from which γ can be expressed as a simple function of λ, i.e. $\gamma = \gamma(\lambda)$. The parameter b is given by

$$(2.2.19) \quad b = T(\gamma m_{(1)} - \lambda T)/(m_{(2)}\gamma - (\lambda+1) m_{(1)} T) \equiv b(\lambda)$$

Thus we have both γ and b expressed in terms of λ and we only have to solve the equation (first of (2.2.10A))

$$(2.2.20) \quad [\gamma(\lambda)m_{(1)} - \lambda T]\ \Gamma(\lambda,\ \gamma(\lambda)/b(\lambda)-T\ [\gamma(\lambda)/b(\lambda)]^{\lambda} \exp(-\gamma(\lambda)/b(\lambda)=o$$

This can be done by using interpolative methods and Pearson's tables (7) for the values of $\Gamma(\lambda, x)$ (in conjunction with formula (2.2.5)). We omit the details for type II data.

Case (B): $b \to \infty$. In this case the method of moments yields the following: The quadratic (2.2.18) remains unchanged, in (2.2.19) b is replaced by a and (2.2.20) is replaced by

$$(2.2.19)[\gamma(\lambda)\ m_{(1)} - \lambda T]\Gamma(\lambda,\gamma(\lambda)/a(\lambda))+T\ [\gamma(\lambda)/a(\lambda)]^{\lambda}\exp(-\gamma(\lambda)/a(\lambda)=o\ .$$

Case (AB): $a \to o$ and $b \to \infty$. This the classical case. The marginal distribution for type I data is a negative binomial distribution. The moment estimates as well as the maximum likelihood estimates of γ and λ are well known. In (6) we give a sufficient condition for the existence of the generalized maximum likelihood estimates of γ and λ when the test time for each equipment is not constant.

Case (Γ): $\gamma \to o$. The marginal distribution for type I data (with replacement) is

$$(2.2.8\Gamma) \quad P_T(r;\ o,\ \lambda,\ a,b) = \lambda T^{-\lambda}\ [\Gamma(\lambda+r,\ T/b) - \Gamma(\lambda+r,\ T/a)]/r!(a^{-\lambda}-b^{-\lambda}).$$

where $\lambda/(a^{-\lambda}-b^{-\lambda})$ must be replaced by $1/\log(b/a)$ if $\lambda=o$. If $\lambda \neq o, -1, -2, -3,$ the method of moments yields

$$(2.2.10\Gamma) \quad m_{(k)} = \frac{\lambda T^k}{\lambda+k}\ (a^{-(\lambda+k)}-b^{-(\lambda+k)})/(a^{-\lambda}-b^{-\lambda}),\ k=1,2,3.$$

From these equations we obtain

$$a = \frac{T}{2}\ [A-(A^2 - 4B)^{1/2}],\ b = \frac{T}{2}\ [A + (A^2 - 4B)^{1/2}]$$

where

$$A=[m_{(3)}\ \lambda(\lambda+3) - m_{(2)}\ (\lambda+1)\ (\lambda+2)]/\ [m_{(1)}m_{(3)}\ (\lambda+1)\ (\lambda+3)-m^2_{(2)}\ (\lambda+2)^2]$$

$$B=[m_{(2)}\ \lambda(\lambda+2)-m_{(1)}\ (\lambda+1)^2]/[m_{(1)}\ m_{(3)}\ (\lambda+1)\ (\lambda+3)\ -m^2_{(2)}\ (\lambda+2)^2].$$

These values of a and b can be substituted into the first of (2.2.10Γ) (which can be written in the form

$$(b/a)^{\lambda+1} = [(\lambda+1)\, bm_{(1)} - \lambda T]/\ [(\lambda+1)\, am_{(1)} - \lambda T]$$

and then solve numerically for λ.

If λ=o, we get the two equations

$$m_{(1)} = T(a^{-1} - b^{-1})/\log (b/a),\ m_{(2)} = T^2\ (a^{-2} - b^{-2})/2 \log (b/a)$$

which yield

$$am_{(1)}\ (2m_{(2)} - aTm_{(1)}) \log \left(\frac{2m(2)}{aTm_{(1)}} - 1\right) = 2T\ (m_{(2)} - aTm_{(1)})$$

$$b = 2m_{(2)}/m_{(1)}T - a.$$

If λ=1 (the case of uniform distribution), we get

$$m_{(1)} = \frac{T}{b-a} \log {}^{b}/_{a},\ m_{(2)} = T^2/ab$$

which can easily be solved numerically.

If λ = -2 we get

$$m_{(1)} = 2T/(a+b),\ m_{(2)} = 2T^2 \log ({}^{b}/a)/(b^2 - a^2).$$

Finally, if λ = -3 we get

$$m_{(1)} = 3T\ (a+b)/2\ (a^2+ab+b^2),\ m_{(2)} = 3T^2/(a^2+ab+b^2) \text{ which}$$

imply that a and b are the two roots of the quadratic equation

$$m^2_{(2)}\ x^2 - 2m_{(1)}\ m_{(2)}\ T\ x + (4m^2_{(1)} - 3m_{(2)})\ T^2 = o.$$

<u>Case</u> (ΓA): $\gamma \to o$, $a \to o$. In this case the distribution is defined if $\lambda < o$ and the method of moments is applicable only if $\lambda < -2$. Under this assumption we have

$$m_{(1)} = \lambda T/(\lambda+1)\ b,\ m_{(2)} = \lambda T^2/(\lambda+2)\ b^2$$

which imply that

$$\lambda = 1 - [\, m_{(2)}/(m_{(2)} - m^2_{(1)}\,]^{1/2}, \quad b = \lambda T/(\lambda+1)\, m_{(1)}$$

provided that $m^2_{(1)} < m_{(2)} < \frac{9}{8}\, m^2_{(1)}$.

<u>Case</u> (ΓB): $\gamma \to o$, $b \to \infty$, $\lambda > o$. In this case we get the estimates

$$\lambda = 1 + [m_{(2)}/(m_{(2)} - m^2_{(1)})]^{\frac{1}{2}} \quad , \quad a = \lambda T/(\lambda+1)\, m_{(1)}$$

provided that $m_{(2)} > m^2_{(1)}$.

Before closing this section, we observe that the question of identifiability of Poisson or Gamma mixtures is affirmatively settled by methods akin to the proof of uniqueness of Laplace transforms.

2.2.2. ACCEPTANCE PLANS BASED ON POSTERIOR RISKS. Let θ_o be the minimum acceptable mean time to failure. In the notation of section 1.3.2, $G_o = [\theta_o, b]$, $G_1 = [a, b]$ and the equation (1.3.4) is reduced to

$$(2.2.22) \quad q = \sum_{r=o}^{r^*} P_{T^*}(r; \gamma, \lambda, a, \theta_o) / \sum_{r=o}^{r^*} P_{T^*}(r; \gamma, \lambda, a, b)$$

from which (r*, T*) with r* or T* minimum can be computed if a solution exists.

2.2.3. OTHER PRIOR DISTRIBUTIONS. Numerous types of distributions (e.g. truncated normal, log-normal, etc.) can be tried as prior distributions of θ. There is an extensive literature on the resulting Poisson mixtures. Because of lack of space, we will not deal with them in this report. Most of the information can be found in the encyclopaedic work of N.I. Johnson and S. Kotz (9), Chapter 8.

3. THE CASE OF NONCONSTANT FAILURE RATE.

Under the instantaneous resurrection hypothesis, which is usable in the case of repairable aging equipment, the whole procedure of using prior distributions for acceptance plans is reduced to either Poisson mixtures (theorem 1.2.1) or to transformed Gamma mixtures (theorem 1.2.2) just as in the case of censored tests without replacement the theory of Binomial mixtures (formula (1.2.6)) and transformed Beta mixtures (formula (1.2.7)) is the relevant set-up.

As an example, consider the Weibull time-to-failure distribution

$$(3.1)\quad f(t|\theta, c) = c\theta^{-1}(t/\theta)^{c-1} \exp(-(t/\theta)^{c}), \; t \geq o, \; \theta > o, \; c > o.$$

The reliability function is

$$(3.2)\quad R(t|\theta, c) = \exp(-(t/\theta)^{c})$$

and the failure rate is given by

$$(3.3)\quad \lambda(t) = c\theta^{-1}(t/\theta)^{c-1}.$$

By the theorem 1.2.1, the distribution of the type I censored failure data under the instantaneous resurrection assumption is given by

$$(3.4)\quad p_T(r|\theta, c) = \frac{1}{r!}(T/\theta)^{cr} \exp(-(T/\theta)^{c})$$

By supposing c known and assuming a Gamma prior distribution of θ^{-c}, we obtain a situation almost exactly the same as in the case of exponential time to failure. The difference is that θ now is not the mean time to failure. The mean time to failure is given by

$$(3.5)\quad \theta_1 = \theta\,\Gamma(c^{-1} + 1).$$

i.e. there is a change by a factor of $\Gamma(c^{-1} + 1)$. In the formula for the marginal distribution as well as in the formulas of the moment estimators, maximum likelihood estimators or generalized maximum likelihood estimators the only change is the replacement of T by T^{c}. An additional minor adjustment has to be made to the formula determining the acceptance test plans.

Further results on Bayesian techniques in connection with the Weibull distribution can be found in (10).

4. TOWARD A GOODNESS OF FIT TEST WHEN THE GENERALIZED MAXIMUM LIKELIHOOD ESTIMATORS ARE USED.

When the available attribute failure data do not exhibit uniform time of operation, the only way available for the estimation of the parameters of the prior seems to be the generalized maximum likelihood method (cf. section 1.3.1, formula (1.3.2)). In order to test goodness of fit in this case, the ordinary χ^2 will not do simply because the usual statistic is not asymptotically χ^2 - distributed in this case.

Suppose that we have decided the ranges of the cells and let $C_1, C_2, \ldots, C_k$ be these cells (e.g., suppose that all equipments, which had no failures belong to cell C_1, the equipments which had one or two failures belong to cell C_2 etc.). Suppose further that the sample size is n and that the ith equipment had r_i failures in time T_i, $i=1, 2, \ldots n$. Let P_{ij} be the probability that the ith equipment belongs to the cell C_j, i.e.

$$(4.1) \quad P_{ij} = \sum_{r \varepsilon C_j} P_{T_i}(r) \quad i=1, 2, \ldots, n \ ; \ j=1, 2, \ldots, k$$

Obviously,

$$(4.2) \quad \sum_{j=1}^{k} P_{ij} = 1.$$

We set

$$(4.3) \quad P_j = \frac{1}{n} \sum_{i=1}^{n} P_{ij}.$$

Then, obviously

$$(4.4) \quad \sum_{j=1}^{k} P_j = 1,$$

and the expected cell frequencies are nP_j, $j=1, \ldots, k$.

Let y_j, $j=1, \ldots, k$ be the observed cell frequencies. The joint moment generating function of $y_1, \ldots, y_k$ is

$$\prod_{i=1}^{n} \sum_{j=1}^{k} P_{ij} e^{t_j}.$$

Therefore, the joint moment generating function of the random variables $x_j = y_j - np_j$ is

$$(4.5) \quad \psi(t_1, \ldots, t_k) = \exp\left(-n \sum_{j=1}^{k} P_j t_j\right) \prod_{i=1}^{n} \sum_{j=1}^{k} P_{ij} e^{t_j}$$

From the McLaurin expension of this function we obtain the following joint moments that are of interest to us (the indices range from 1 to k).

(4.6) $m_{aa} = nP_a - \Sigma\, P_{ia}^2,\ m_{ab} = -\Sigma\, P_{ia}P_{ib},\ (a \neq b)$

(4.7) $m_{aaaa} = 3n^2P_a^2 + nP_a - 6nP_a\, \Sigma P_{ia}^2 - 7\Sigma P_{ia}^2 + 12\, \Sigma P_{ia}^2 +$

$+3(\Sigma P_{ia}^2)^2 - 6\Sigma P_{ia}^4$

(4.8) $m_{aabb} = n^2P_aP_b - \Sigma P_{ia}P_{ib} - nP_a\, \Sigma P_{ib}^2 - nP_b\, \Sigma P_{ia}^2 + 2\, \Sigma P_{ia}P_{ib}^2 +$

$+ 2\Sigma P_{ia}^2P_{ib} + (\Sigma P_{ia}^2)\,(\Sigma P_{ib}^2) + 2(\Sigma P_{ia}P_{ib})^2 - 6\Sigma P_{ia}^2P_{ib}^2 .$

We consider the statistic

(4.8) $S = \Sigma\, x_j^2/w_j,\ w_j = (nP_j - \sum_{i=1}^{n} P_{ij}^2)/(1-P_j)$

The mean of this statistic is given by

(4.9) $E(S) = \sum_{j=1}^{k} m_{jj}/w_j = \sum_{j=1}^{k} (1-P_j) = k - \sum_{j=1}^{k} P_j = k-1$

where (4.6), (4.8) and (4.4) have been used. The variance is given by

(4.10) $Var(S) = E(S^2) - E(S)^2 = \sum_{a=1}^{k} m_{aaaa}/w_a^2 + 2 \sum_{a \neq b} m_{aabb}/w_aw_b - (k-1)^2$

where the values of m_{aaaa} and m_{aabb} are given by (4.7) and (4.8) respectively. The Chebyshev inequality can be used to obtain a (weak) probabilistic statement about the goodness of fit.

If we adjust 4.5 because of the weights w_j (i.e., if we consider the joint moment generating function of the variables $x_j/\sqrt{w_j}$, j=1, ..., k) take its logarithm and expand into Taylor series about the origin, we obtain that for $n \to \infty$, the variables are asympotically normally distributed with variance-covariance matrix of rank k-1

$$A = (m_{ab}/\sqrt{w_aw_b}).$$

The non-zero eigenvalues of this matrix are not all equal to 1 (in general) and therefore the asymptotic distribution of the statistic S is not a χ^2 distribution. The cumulants of the asymptotic distribution of S are given by (cf. Chapter 29, Section 3 of (11)).

(4.11) $\kappa_m = 2^{m-1}\,(m-1)!\ tr(A^m)\quad m=1, 2, \ldots$

Thus, in particular

$$\kappa_1 = u_1' = E(S) = tr(A) = k-1$$

$$\kappa_2 = \mu_2 = Var(S) = 2tr\,(A^2)$$

$$\kappa_3 = \mu_3 = 8tr(A^3)$$

$$\kappa_4 = \mu_4 - 3\mu_2^2 = 48tr(A^4) \text{ etc.}$$

It is sometimes more convenient to approximate the variance of S this way than compute it through (4.10).

We did not study the effect on S of the estimating of parameters from the data used to form S. Hopefully something not very far from the classical case of "reduction of degrees of freedom" will occur.

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