AD-A080 743 SOUTHEASTERN CENTER FOR ELECTRICAL ENGINEERING EDUCAT--ETC F/G 20/4
STABILITY ANALYSIS OF THE LOWER BRANCH SOLUTIONS OF THE FALKNER--ETC(U)
JUL 79 G R VERMA, W L HANKEY, S J SCHERR F49620-79-C-0038
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STABILITY ANALYSIS OF THE LOWER BRANCH SOLUTIONS OF THE FALKNER-SKAN EQUATIONS

G. R. VERMA, W. L. HANKEY, S. J. SCHERR

Southeastern Center for Electrical Engineering Education
Orlando, Florida 32809

July 1979

Technical Report AFFDL-TR-79-3116

Final Report for Period May 17 - July 31, 1979

Air Force Flight Dynamics Laboratory
Air Force Wright Aeronautical Laboratories
Air Force Systems Command
Wright-Patterson Air Force Base, OH 45433

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Project Engineer

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Chief, Aerodynamics & Airframe Br.

FOR THE COMMANDER

Peter J. Butkewicz
PETER J. BUTKEWICZ, Colonel, USAF
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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: AFFDL-TR-79-3116

2. GOVT ACCESSION NO.:

3. RECIPIENT'S CATALOG NUMBER:

4. TITLE (and Subtitle): STABILITY ANALYSIS OF THE LOWER BRANCH SOLUTIONS OF THE FALKNER-SKAN EQUATIONS.

5. TYPE OF REPORT & PERIOD COVERED: Final Technical Report. 17 May-31 Jul 79

6. PERFORMING ORG. REPORT NUMBER:

7. AUTHOR(s): G. R. Verma, L. Hankey, S. J. Scherr

Will

8. CONTRACT OR GRANT NUMBER(s): F49620-79-C-0038

9. PERFORMING ORGANIZATION NAME AND ADDRESS: Southeastern Center for Electrical Engineering Education, Orlando, Florida 32809

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 2307 N4 18

11. CONTROLLING OFFICE NAME AND ADDRESS: AIR FORCE FLIGHT DYNAMICS LABORATORY (FXM), WRIGHT-PATTERSON AIR FORCE BASE, OH 45433

12. REPORT DATE: Jul [redacted] 79

13. NUMBER OF PAGES: 41

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):

41

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report):

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):

18. SUPPLEMENTARY NOTES:

19. KEY WORDS (Continue on reverse side if necessary and identify by block number):

Stability Analysis; Boundary Layers; Adverse Pressure Gradient; Lower Branch Solutions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

The stability of the Lower-Branch solutions of the Falkner-Skan boundary layers has been analyzed. A perturbation analysis of these boundary layers was performed resulting in the Rayleigh stability equation. Eigenvalue solutions were obtained for the Rayleigh equation for different adverse pressure gradient values. Propagation velocity and amplification factors were computed over the entire range of unstable wave numbers. All retarded flows were found to be unstable for a small range of frequencies, with the amplification factor increasing as the extent of reversed flow increased.

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S/C 394958

ACKNOWLEDGEMENT

The authors would like to thank the Air Force Systems Command, Air Force Office of Scientific Research, Southern Center for Electrical Engineering Education and Wright-Patterson Air Force Base for providing resources for the senior author to spend the summer of '79 at WPAFB.

We would also like to thank Dr. Charles Jobe for helping with programming phase of the research.

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SECTION I

INTRODUCTION

Self excited oscillations have been experimentally observed in separated flows for over hundred years. Rayleigh [1] in 1880 proved that for inviscid, incompressible flow the unstable velocity profiles must have an inflection point. Tollmein [2] in 1935 showed that for symmetrical velocity distributions, or for velocity distributions of the boundary layer type, the existence of the inflection point implies instability.

Recently Hankey and Shang [3] have examined the self induced pressure oscillations in an open cavity. Their numerical computations compare very well with the previous experimental investigations. Roscoe and Hankey [4] have studied the stability of hyperbolic tangent velocity profile in a compressible fluid, while Hankey, Hunter and Harney [5] have examined the self-sustained oscillations (Buzz) on spiked tipped bodies for large Mach numbers. However, a systematic stability analysis of separated flows has not been undertaken. It is the purpose of this report to conduct a stability analysis of a general class of separated flows (i.e. reversed flow Falkner-Skan) in order to help shed light on the phenomenon of self-excited fluid flows.

SECTION II

OBJECTIVES OF THE RESEARCH EFFORT

The objective of this research effort was to analyze a series of similar separated flows for different values of β and to determine the amplification factors and propagation velocities in all these different cases. Eight cases of different β were identified to be analyzed. These cases were those with reversed flows which contained velocity profiles with inflection points.

SECTION III

MEAN FLOW EQUATIONS

In this report, incompressible flows will be analyzed. In subsequent work we plan to analyze the compressible flows.

The incompressible two-dimensional Navier-Stokes equations are as follows

$$U_t + UU_x + VU_y = -\frac{1}{\rho} P_x + \nu\nabla^2 U \tag{3.1}$$

$$V_t + UV_x + VV_y = -\frac{1}{\rho} P_y + \nu\nabla^2 V \tag{3.2}$$

$$U_x + V_y = 0 \tag{3.3}$$

Applying the boundary layer approximations to the above equations for steady flows results in the following:

$$UU_x + VU_y = U_e U_{ex} + \nu U_{yy} \tag{3.4}$$

$$U_x + V_y = 0 \tag{3.5}$$

These equations may be reduced to one ordinary differential equation for the case where $U_e = cx^m$ by transforming with similarity variables.

$$d\xi = \frac{U_e dx}{\nu} \tag{3.6}$$

$$d\eta = \frac{U_e dy}{\sqrt{2\xi}\,\nu} \tag{3.7}$$

Hence

$$f''' + ff'' = \beta(f'^2 - 1) \tag{3.8}$$

where

$$f'(\eta) = \frac{U}{U_e} \tag{3.9}$$

and

$$\beta = \frac{\xi U_{e\xi}}{U_e} = \frac{2m}{m+1} = \text{constant} \tag{3.10}$$

with boundary conditions

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1 \tag{3.11}$$

Falkner and Skan [6] originally derived this equation for attached flows however, Stewartson [7] discovered a lower branch to these solutions which represented reversed flows from incipient separation to the Chapman solution. Christian, Hankey and Petty [8] have tabulated these solutions for compressible and incompressible flows. It is this wide class of flows (which have inflection points) that are known to be unstable for which we shall now perform a stability analysis.

SECTION IV

PERTURBATION EQUATIONS

Let us assume small perturbations of the form

$$U = \bar{U}\ (y) + \hat{U}(y)\ e^{i\alpha(x - ct)} \tag{4.1}$$

$$V = \qquad \Phi(y)\ e^{i\alpha(x - ct)} \tag{4.2}$$

$$p = P_e(x) + \hat{P}(y)\ e^{i\alpha(x - ct)} \tag{4.3}$$

where $c = c_r + ic_i$ and $\hat{U}$, ϕ and $\hat{P}$ are small in comparison to the mean quantities. If we substitute these values of U, V, and P in equations (3.1), (3.2) and (3.3); retain only the first order terms and assume that Reynolds number Uex/ν is large then the equations (4.1), (4.2) and (4.3) reduce to one single equation

$$\phi'' - (\alpha^2 + \frac{\bar{U}''}{\bar{U} - c})\phi = 0 \tag{4.4}$$

The classical Rayleigh equation with the boundary conditions

$$\phi(0) = 0, \quad \phi(\infty) = 0 \tag{4.5a,b}$$

By transforming the equation from y to the η variable we obtain the following equation

$$\phi_{\eta\eta} - (\bar{\alpha}^2 + \frac{f'''}{f' - c})\phi = 0 \tag{4.6}$$

where

$$\bar{\alpha} = \alpha \frac{dy}{d\eta}$$

By inserting the values of $f'(\eta, \beta)$ into the Rayleigh equation $c(\bar{\alpha}, \beta)$ can be determined as an eigenvalue which satisfies the boundary conditions (4.5a,b).

SECTION V

SOLVING SCHEME

Eigenvalues were determined by a shooting method; starting with a given boundary condition, integrating over the range of η and comparing the result with the outer boundary condition, namely $\phi = 0$ at η_{max}. The process involved minimization of the error in the outer boundary condition which was chosen to be the square of the norm of ϕ, $|\phi|^2 = \phi_R^2 + \phi_i^2$ = SSQ.(See Appendix 3). The integration was done using a fourth-order Runge-Kutta method.

The method of finding eigenvalues utilized a minimization routine written primarily by Roscoe [4]. Starting from a given guess the routine searched along a constant line of c_i with increasing steps until it found a relative minimum of the error. It then used the last three calculated points to determine a parabola, with the c_r value at the vertex used as the latest approximation. Then this value of c_r was held constant and a search along a line of changing c_i was carried out. After a new minimum was found, the quadratic approximation was again used to determine a new value for c_i. The third step involved searching the line connecting the original guess and the new point. After finding a minimum and utilizing quadratic approximation, the error was checked to see if it was less than some preset limit. If not, the routine started again with the latest value used in place of the original guess.

Generally, the routine worked quite well. Most of the search time was attributable to bad guesses and finding the direction in which the search should be continued. An eigenvalue was usually located in a very narrow region of the plane and even though the step size was continually reduced, it was frequently large enough to move the test point out of the acceptable region. For example, the initial guess in one case led to an error of 4.1×10^{13}, however, after only 128 new error computations, the error had been reduced 17 orders of magnitude to 1.9×10^{-4}, while c_r had been changed by 4.25% and c_i had been changed by 3.82%. Convergence was also retarded for small values of c_i, e.g. $|c_i| < .001$. This was concurrent with c_r approaching its limiting value.

The Howard semicircle theorem [9] was used as an aid in determining suitable initial guesses. If c_r is the propagation velocity, α is the wave number, c_i is the amplification factor, and U_{max} and U_{min} are the maximum and minimum values of the range of U, the theorem states

$$[c_r - 1/2(U_{max} + U_{min})]^2 + c_i^2 \leq [1/2(U_{max} - U_{min})]^2.$$

Thus, the complex wave velocity for an unstable mode lies inside the upper semi-circle which has the range of U as diameter.

SECTION VI

RESULTS

Eight cases were computed for β values of -.0001, -.0005, -.002, -.04, -.08, -.12, -.16 and -.19884. For a wide range of $\bar{\alpha}$ values the eigenvalues were ascertained. These values are tabulated in tables B-1 - B-8 in Appendix B. $\bar{\alpha}$ is related to α by the relation

$$\alpha\delta^* = \bar{\alpha}\delta^* \frac{d\eta}{dy} = \bar{\alpha} \int_0^\infty (1 - f')\, d\eta$$

The values of C_r and C_i versus $\alpha\delta^*$ are plotted in figures 1a-1h and 2a-2h. Figure 3a-3h shows Howard's plot (9) for these solutions. Some typical eigenvalues for a series of solutions are also tabulated and plotted in Appendix B.

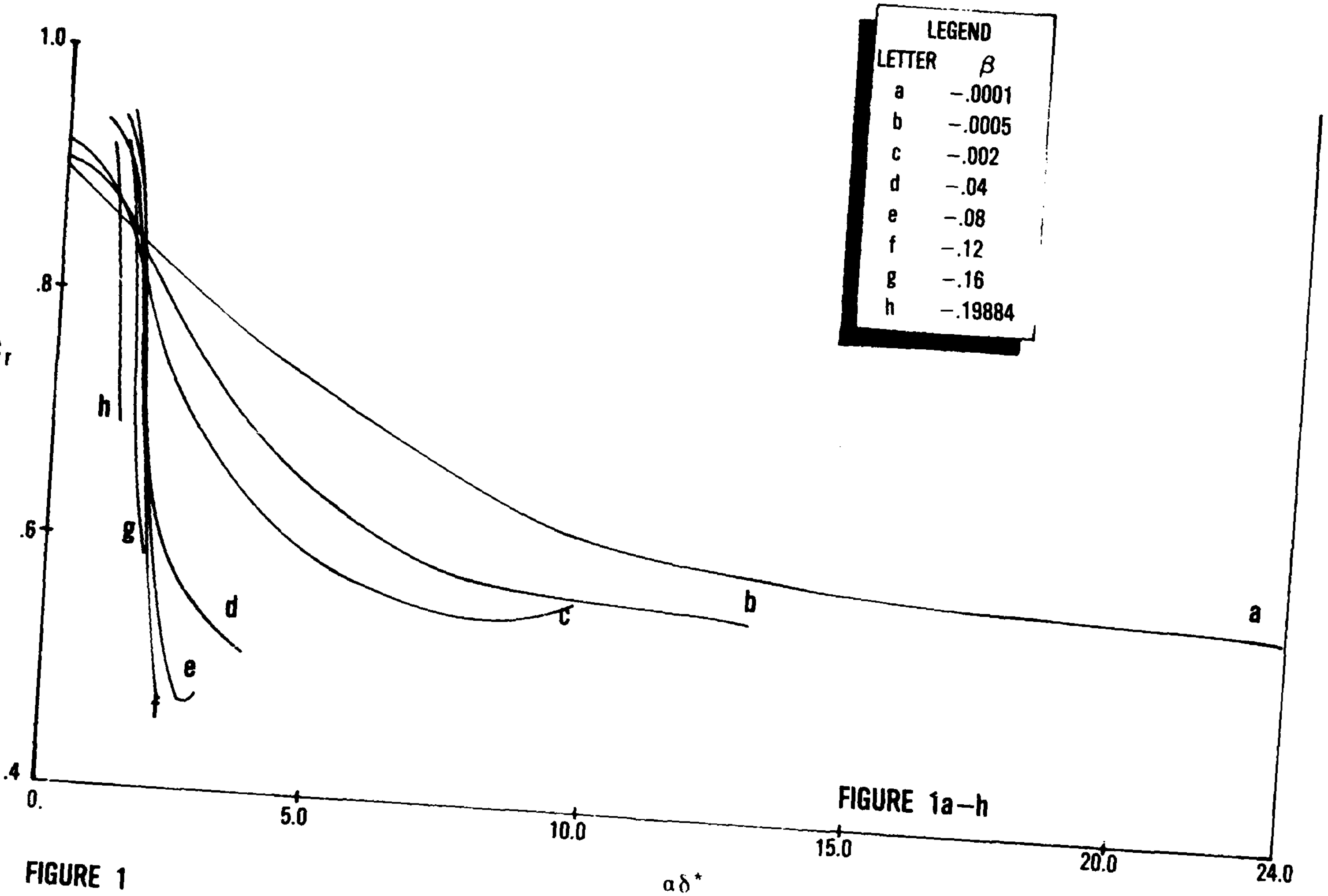

FIGURE 1a–h

FIGURE 1

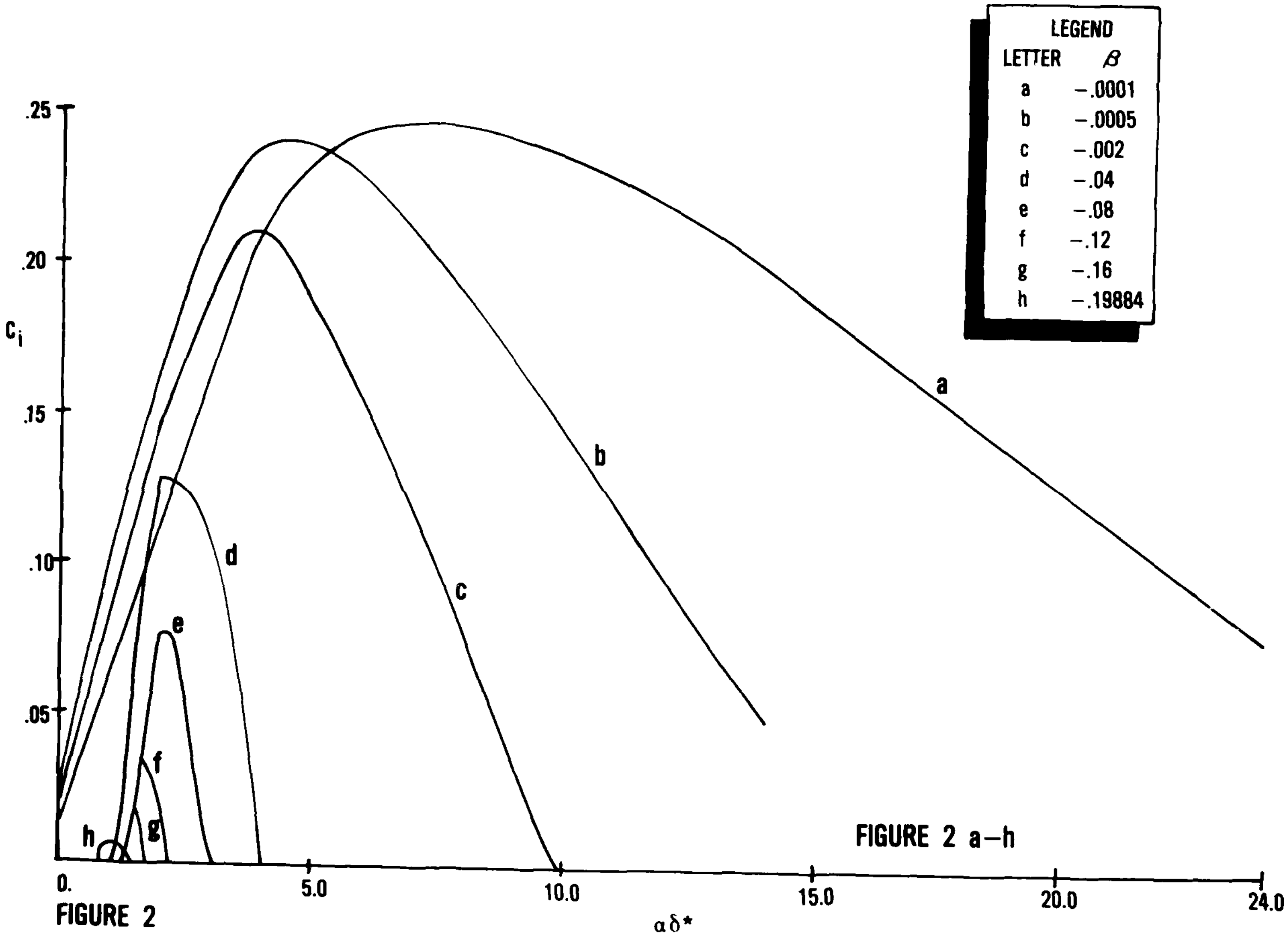

FIGURE 2 a—h

FIGURE 2

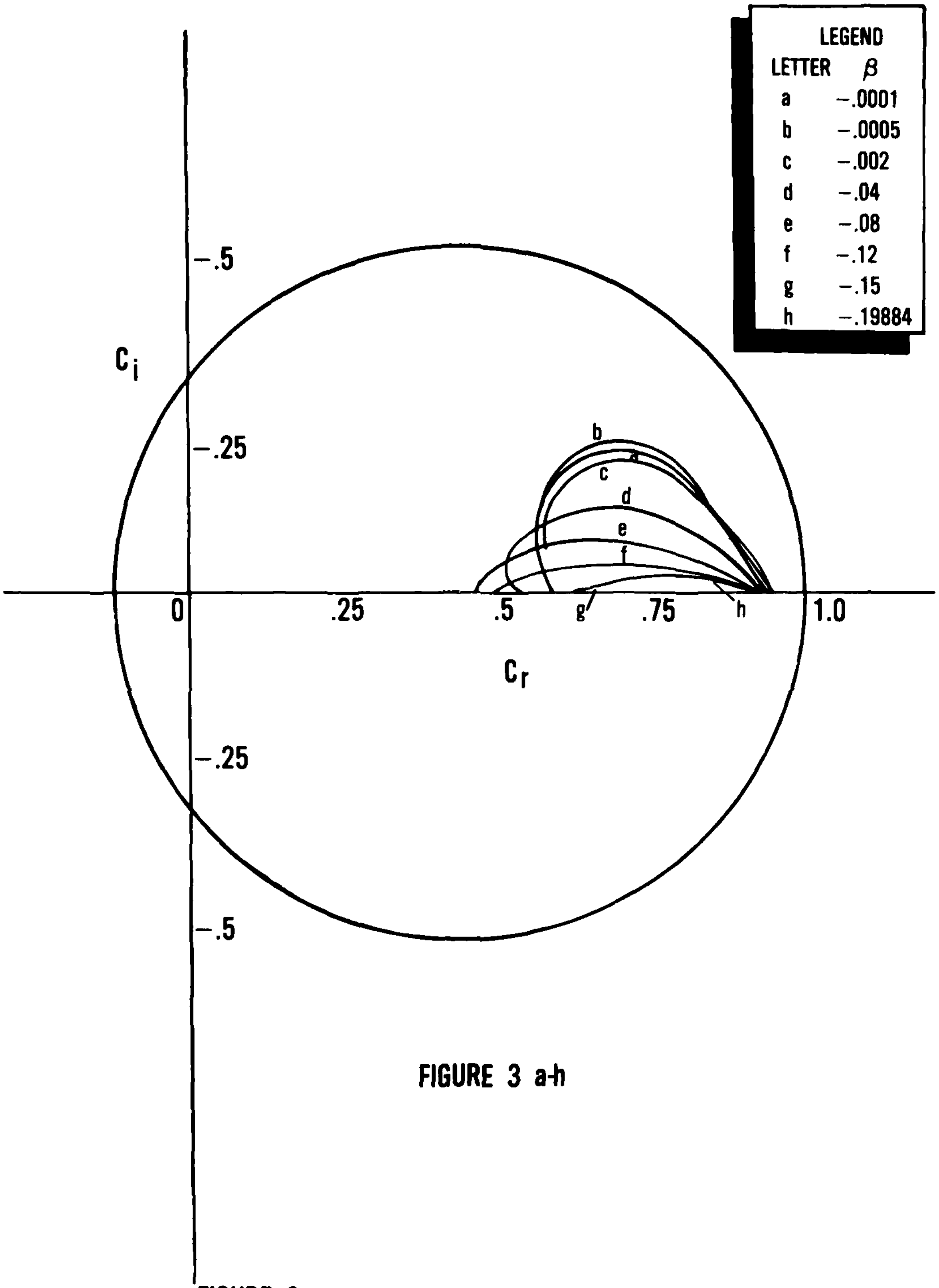

FIGURE 3 a-h

FIGURE 3

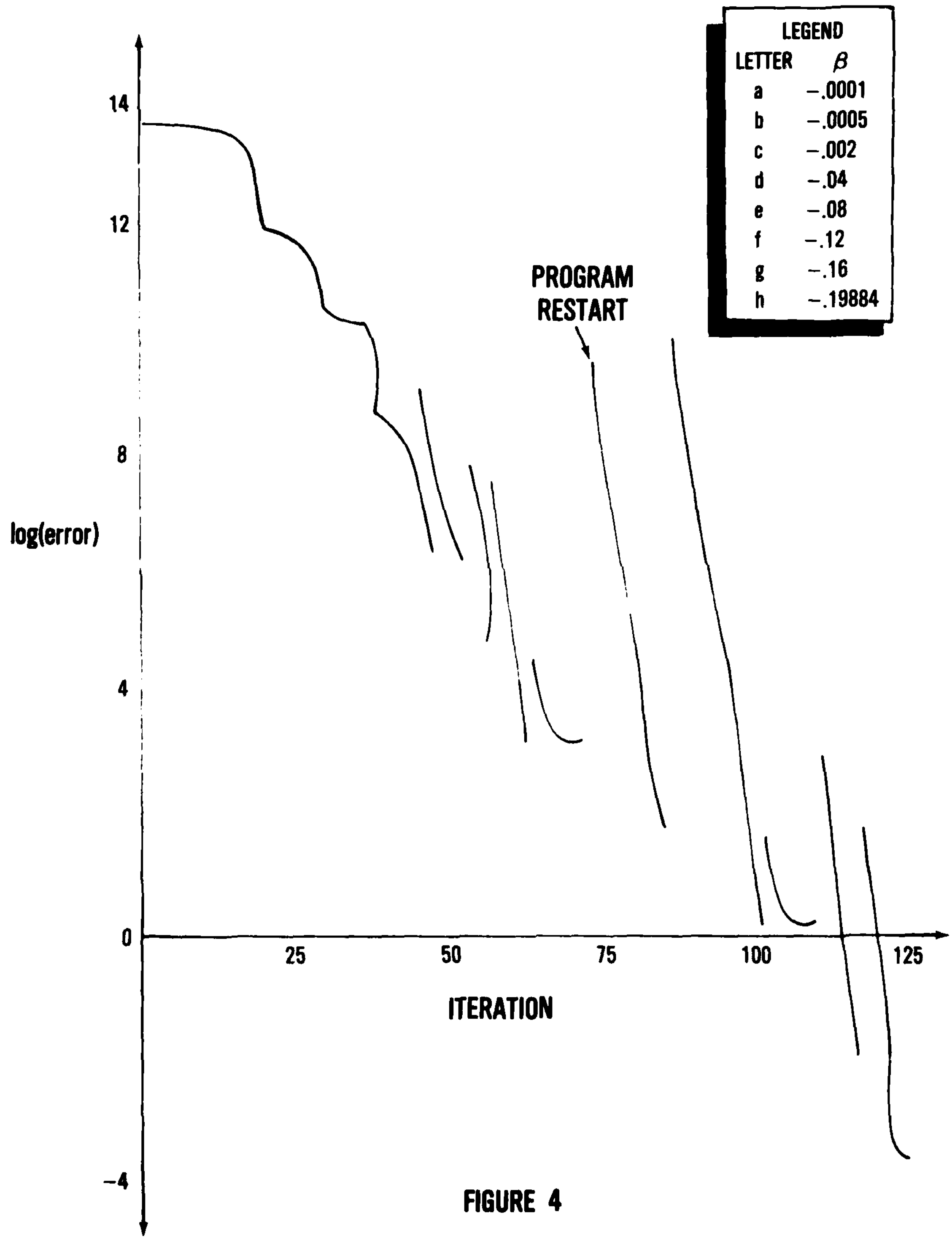

FIGURE 4

FIGURE 4

SECTION VII

CONCLUSIONS

The stability of a series of similar separated flows have been analyzed. Amplification factors and propagation velocities of the disturbances were determined. The results show that a small zone of instability does exist for these flows with inflexion points. The amplifaction factor increases as the extent of the reversed flow increases.

SECTION VIII

RECOMMENDATIONS

Suggestions for follow-on research: We would like to investigate the instability of laminar separated flows under the influence of compressibility. For a hyperbolic tangent velocity profile Roscoe (4) showed the instability to diminish with the increase of Mach number until the Rayleigh instability actually vanished at Mach number $M = 2.5$. The analysis should be repeated for the compressible, adiabatic, Falkner-Skan velocity profiles. We have completed $M = 0$ cases for various values of β, and would like to examine the influence of Mach number for the same values of β. It was observed that for $\beta = -.0001$ and $-.0005$ the convergence at the two ends of the spectrum was very slow. These cases should be analyzed somewhat more thoroughly.

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APPENDIX A

THE HOWARD CIRCLE THEOREM

The Howard semicircle theorem [9] is an extension of the well known fact that if the amplification factor $C_i > 0$ then the propagation velocity C_r must lie in the range of U. Howard was able to restrict the permissible values of C_r and C_i so that the complex wave velocity C is confined to a semicircle which has the range of U as its diameter. If U_{max} and U_{min} are the extrema of the range of U, the theorem states

$$[C_r - 1/2(a + b)]^2 + C_i^2 \leq [1/2(a + b)]^2 \ , \quad C_i > 0$$

where $a = U_{max}$, $b = U_{min}$.

APPENDIX B

EIGENVALUES FROM STABILITY ANALYSIS FOR REVERSED FLOW BOUNDARY LAYERS

EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-1

$\beta = -.0001$

$\bar{\alpha}$	c_r	c_i
0	.90538414714	.025680518247
.01	.91223794353	.071997946349
.02	.89422525454	.11766985541
.03	.87074750211	.12610731927
.04	.8412659309	.14572816584
.05	.80533205	.17071090
.07	.76425576	.21131732
.10	.72900063223	.23958792186
.15	.67639882435	.24398318931
.18	.651890026062	.235453936491
.20	.637798788145	.22696852817
.22	.62587353506	.21686502743
.25	.60392754004	.19700990613
.27	.60253175376	.18702547786
.29	.58908572761	.17144819857
.30	.58616022	.16481124
.31	.58992531643	.16063748234
.32	.58748966823	.15388430488
.35	.58168544052	.13349344451
.40	.57632116036	.099664728940
.41	.56994477164	.090751023415
.42	.56963082632391	.08414875686479

EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-2

$\beta = -.0005$

$\bar{\alpha}$	c_r	c_i
0	.91530377348	.02166761564
.01	.91372171557	.025240777143
.05	.85274628651	.13020125042
.10	.74360436406	.21320430557
.15	.67480093675282	.24234861164626
.20	.63120621598575	.23023864570417
.25	.60282728906963	.20264037260815
.30	.58491658821	.16939693499
.35	.57438769614	.13455531319
.40	.56967589511	.099924636095
.45	.56930353919	.066316584587
.46	.5696486522	.059749391713

EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-3

$\beta = -.002$

$\bar{\alpha}$	C_r	C_i
0	.9237409069	.0095488477066
.01	.92117830521	.010983058744
.05	.88971797	.091967207
.10	.78686024	.15214965
.15	.70466415	.20301679
.20	.64798726	.21161601
.25	.61069242	.19445403
.30	.58711391	.16596060
.35	.57311136	.13306074
.40	.56593472	.098968376
.45	.56371274	.065174535
.50	.56504181973	.032280308591
.55	.56865534872	.00023011047136
.56	.57775580181	$.19174876005(10)^{-10}$
.57	.58685371398	$.61658214056(10)^{-11}$
.58	.58704040582	$.34648626536(10)^{-11}$

EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-4

$\beta = -.04$

$\bar{\alpha}$	C_r	C_i
.12	.9462446953107	.00077598474441
.13	.94480294724055	.00092398603074
.14	.93834475353190	.0019712426475
.15	.92864533	.0041227836
.17	.91398636	.009481253
.20	.88234356	.031932225
.23	.78918067	.079558300
.25	.73325251	.10055163
.30	.63589654	.12879591
.35	.57744885	.12534423
.40	.54449017	.10193011
.42	.53642765	.089694409
.44	.53060095	.076599668
.46	.52663117	.062983805
.48	.52423199	.049089198
.50	.52306536	.035068680
.52	.52292271	.021022046
.54	.52356367	.0069847632
.55	.52416855	$.34183573(10)^{-6}$
.56	.53445687	$.11840627(10)^{-8}$

TABLE B-4 (con't)

$\beta = -.04$

$\bar{\alpha}$	C_r	C_i
.57	.53445745	$.28132369(10)^{-9}$
.58	.53445776	$.10618506(10)^{-9}$
.59	.53445776	$-.16007328(10)^{-10}$
.60	.534457761505	$-.581878617(10)^{-10}$
.61	.53445776292	$-.75694864203(10)^{-10}$

EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-5

$\beta = -.08$

$\bar{\alpha}$	C_r	C_i
.20	.94462051904	.00060442053448
.22	.93168283195	.0023838799717
.25	.91152233835	.007171373662
.27	.89227578	.014429883
.30	.83585489	.036775313
.35	.68705498	.067058934
.40	.57949813	.076982790
.45	.51260920	.067262598
.47	.49694639	.057479918

EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-6

$\beta = -.12$

$\bar{\alpha}$	C_r	C_i
.28	.93771615513	.00057735106118
.30	.92267372951	.0025385188104
.32	.90428630103	.006241760255
.35	.86703422542955	.016184737774036
.37	.82518601942224	.025407066978775
.40	.74110563666309	.033621469860405
.42	.68545236670957	.034160709745084
.45	.60602465101	.029295418399
.47	.54979488217	.020797858572
.50	.47928620449	$.17208372583(10)^{-8}$
.51	.47900001254	$.164119887(10)^{-8}$
.52	.47900001195552	$.2969542445(10)^{-9}$

EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-7

$\beta = -.16$

$\bar{\alpha}$	C_r	C_i
.35	.92539848853	.00064674041646
.37	.90652445594	.0030732291515
.40	.87214263	.0087996030
.42	.839073304	.013608897
.45	.76943755	.016917978
.47	.71714018755	.014067976705
.50	.63945116	.0025996533
.52	.59255631588	-.0054550150066

EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-8

$\beta = -.19884$

$\bar{\alpha}$	C_r	C_i
.37	.92379057577	.00067331237442
.38	.91876397945	.00087323704317
.39	.91329400663	.001065562476
.40	.90688937	.0013915674
.42	.89032638669	.00295916137
.45	.85880806	.0055504066
.47	.83019514805718	.0066806913008495
.50	.77300900	.0043585794
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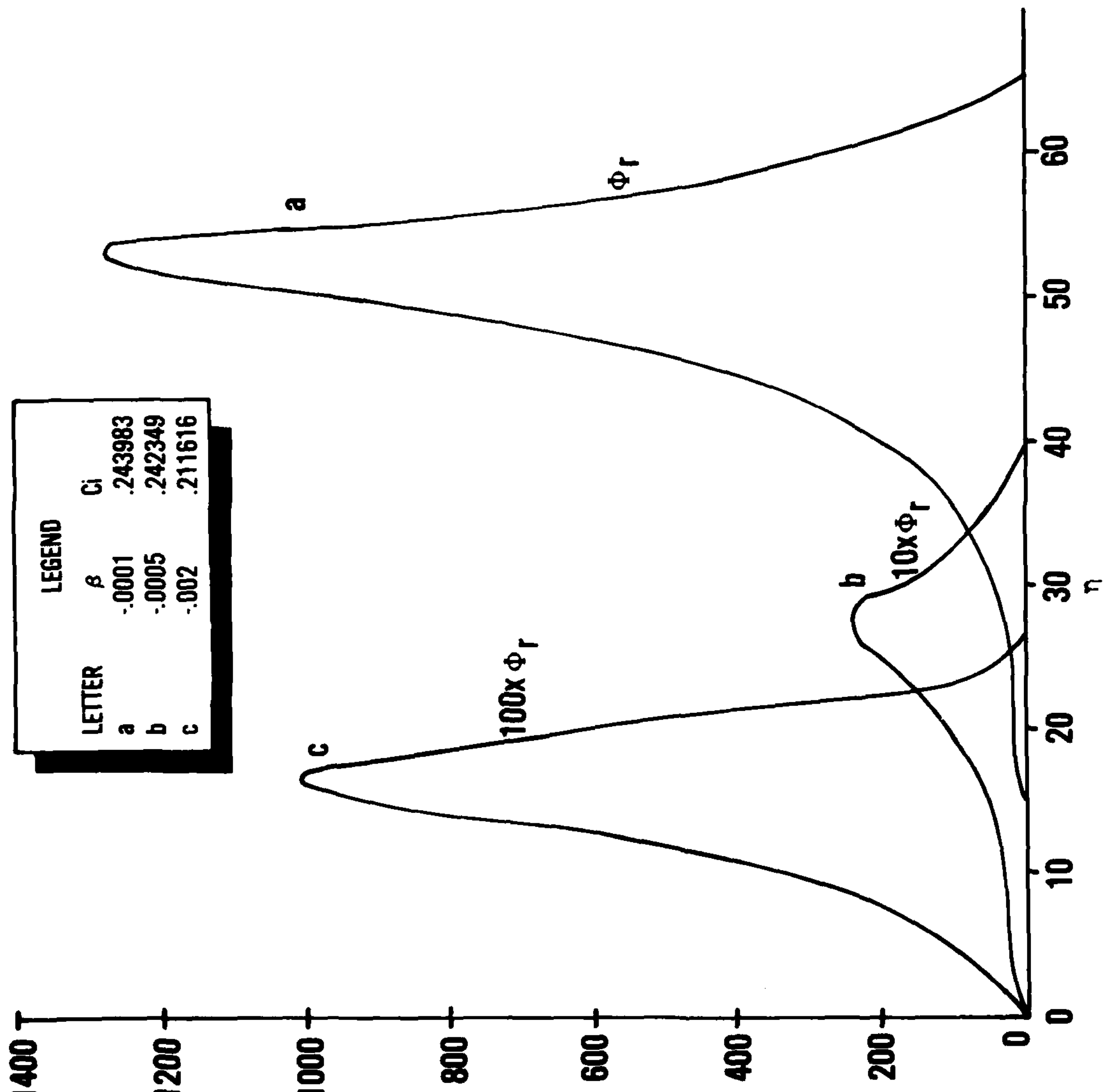
.54	.70412941	$-.19018251(10)^{-7}$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c -.002 .211616
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1200
1000
800
600
400
200
0
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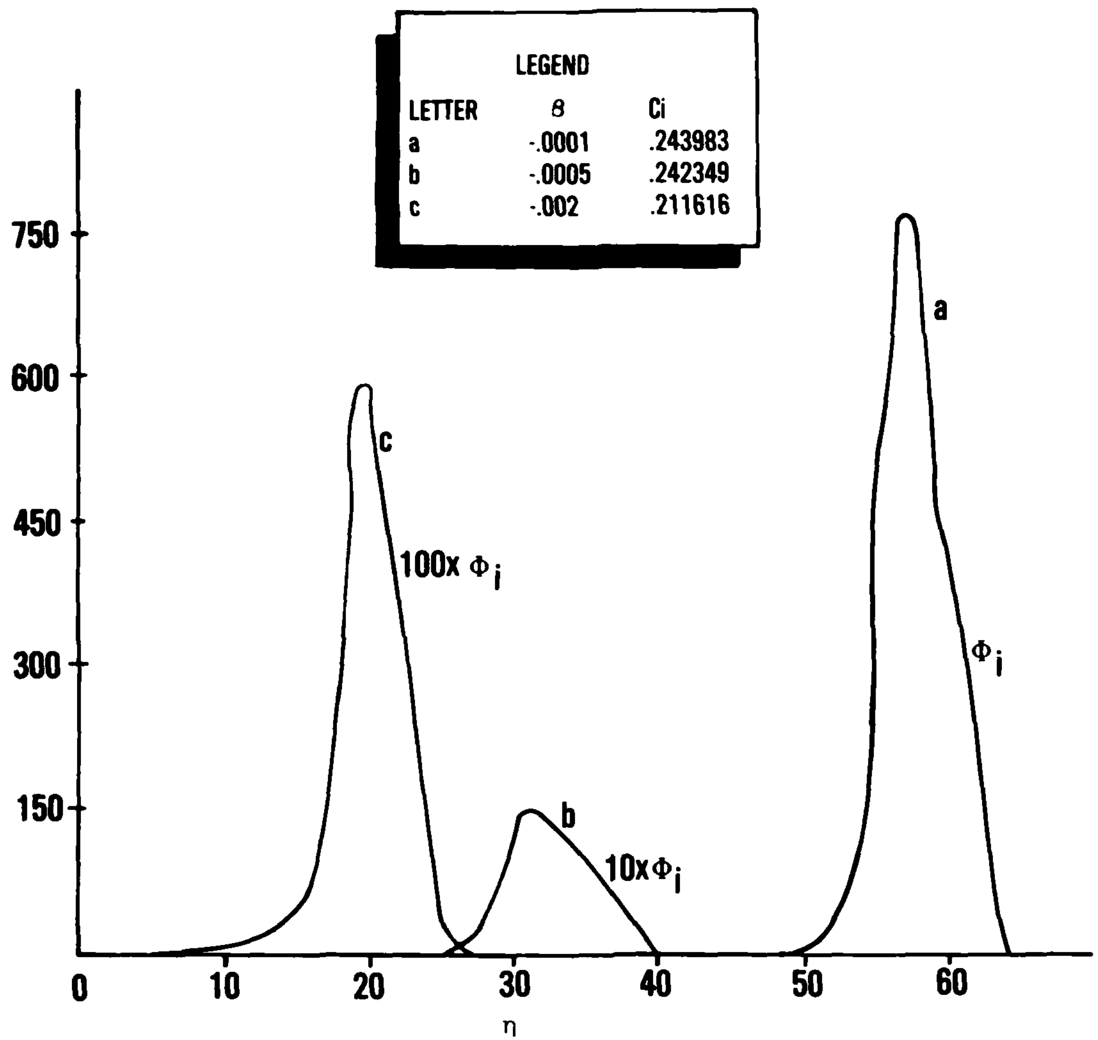
η
a
Φ_r
b
$10x\Phi_r$
c
$100x\Phi_r$

FIGURE B-1



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c -.002 .211616
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600
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300
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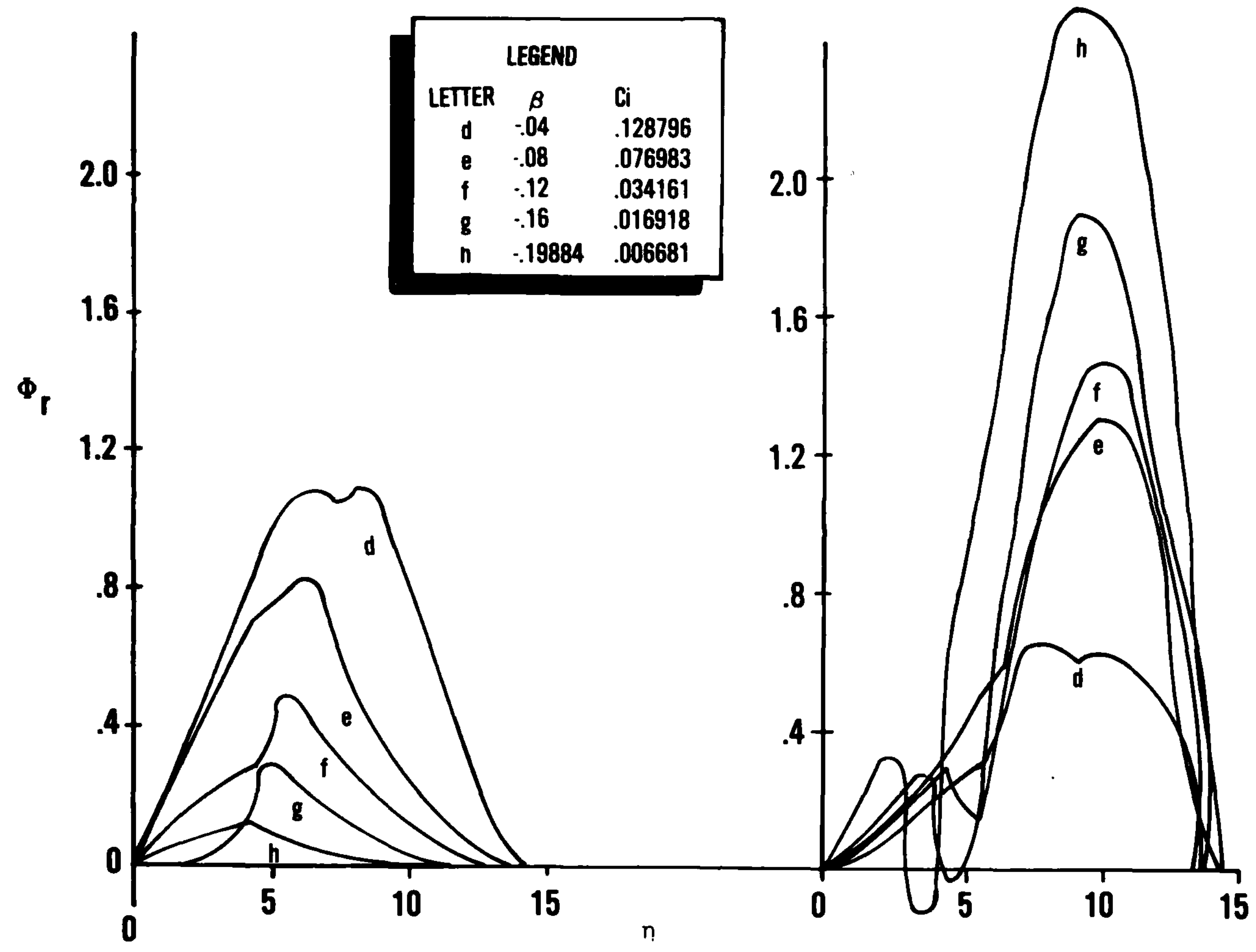
40
50
60
η
a
b
c
100x Φi
10xΦi
Φi

FIGURE B-2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1.2
.8
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5
10
15
η
d
e
f
g
h

FIGURE B-3 - B-4

APPENDIX C
COMPUTER PROGRAM

The following FORTRAN program was used in the search for eigenvalues. The driver program FINDMIN gives the initial guesses for CREAL and CIM and then calls the minimization routine. The error is returned as SSQ. Subroutine MAINFCN does the integration using a system routine RKDF which uses a fourth order Runge-Kutta method. The arguments to RKDF are: X - the independent variable, Y - the dependent variables, N - the number of variables, DX - the step size, and IER - an error return. RKDF also requires a function F which computes the derivatives of the dependent variables and stores them in P.

The minimization routines are fairly general. The equivalence causes the minization to be done with respect to CREAL and CIM. To minimize with respect to CIM and ALPHA the equivalence statement would be: "EQUIVALENCE (CIM, X(1)), (ALPHA, X(2))." Note that the two variables which are equivalenced with X(1) and X(2) must be stored consecutively in memory.

The array Y represents the following quantities. Y(1) = f, Y(2) = f', Y(3) = f", Y(4) = ϕ_R, Y(5) = ϕ_I, Y(6) = ϕ_R', Y(7) = ϕ_I'. Convergence was generally achieved when minimization errors of 10^{-5} to 10^{-9} occurred for most cases.

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      PROGRAM FINDMIN(INPUT,OUTPUT,TAPE5=INPUT,TAPE2=OUTPUT)
        COMMON CREAL,CIM,ALPHA,BETA,P3D
      COMMON /B2/SSQ
       CREAL = .57194967522
       CIM = .033942176531
       ALPHA = .5
      CALL MIN1
      WRITE(2,1)ALPHA,BETA,CREAL,CIM,SSQ
1     FORMAT(50(2H *)/" ALPHA=",E14.8,5X,"BETA=",E14.8/" CREAL=",E14.8,5X
     1X,"CIM=",E14.8,5X,"ERROR=",E14.8/50(2H *))
 100   CONTINUE
      END
      SUBROUTINE MAINFCN
      COMMON CREAL,CIM,ALPHA,BETA,P3D
      COMMON /B2/ SSQ
      DIMENSION Y(7),P(7)
       BETA = -.0005
       Y(3)  = -.0051546
       XEND = 40.
      X = 00.0
      Y(1)  = 0.00
      Y(2)  = 0.0
      Y(4)  = 0.0
      Y(5)  = 0.0
      CCBAR = CREAL*CREAL + CIM*CIM
      BCC = BETA / CCBAR
         FACT R = ALPHA*ALPHA + BCC*CREAL
      R4 = FACTR*FACTR + BCC*BCC*CIM*CIM
      R = SQRT(SQRT(R4))
       GAMMA = 0.5*ATAN2(-BCC*CIM,FACT R )
      Y(6)  = R * COS(GAMMA)
      Y(7)  = R * SIN(GAMMA)
         DX  = 0.05
      N = 7
      CALL F(X,Y,P)
C***    ******************INNER LOOP    INTEGRATION *********************
    2 CONTINUE
        CALL RKDF(X,Y,N,DX,IER)
      IF(X.LE.XEND)  GO TO 2
C***    ******************INNER LOOP    INTEGRATION *********************
       SSQ=Y(5)*Y(5)+Y(4)*Y(4)
      RETURN
       END
      SUBROUTINE F(X,Y,P)
      COMMON CREAL,CIM,ALPHA,BETA,P3D
      DIMENSION  Y(7) , P(7)
      P(1)  = Y(2)
      P(2)  = Y(3)
      P(3)  = BETA*(Y(2) -1.)*(Y(2) + 1.) - Y(1)*Y(3)
      P(4)  = Y(6)
      P(5)  = Y(7)
      U = Y(2)
      UDB = P(3)
      D = UDB/((U-CREAL)**2 + CIM**2)
      B = D*CIM
      A = ALPHA*ALPHA + D*(U-CREAL)
      P(6)  = A*Y(4) - B*Y(5)
      P(7)  = B*Y(4) - A*Y(2)
      RETURN
      END
      SUBROUTINE MIN1
      COMMON CREAL,CIM,ALPHA,BETA,P3D
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      COMMON /B2/ SSQ
      DIMENSION XEST(2,2),X(2),STEP(2)
      EQUIVALENCE(CREAL,X(1)),(CIM,X(2))
       STEP (1) = 1.E-10
       STEP(2) = 1.E-4
      ERSSQ=1.E-3
      P30=0.
 2    CONTINUE
      DY=STEP(1)
      XEST(1,1)=X(1)
      GRAD=1.E-30
C
C    SEARCH ALONG X1-AXIS
C
      CALL MIN2(X,DY,GRAD)
       IF(SSQ.LT.ERSSQ)RETURN
      XEST(1,2)=X(1)
      STEP(1)=DY
      XEST(2,1)=X(2)
      DY=STEP(2)
      GRAD=1.E+30
C
C    SEARXH ALONG X2-AXIS
C
      CALL MIN2(X,DY,GRAD)
       IF(SSQ.LT.ERSSQ)RETURN
      XEST(2,2)=X(2)
      STEP(2)=DY
      GRAD=(XEST(1,2)-XEST(1,1))/(XEST(2,2)-XEST(2,1))
C
C    SEARCH ALONG LINE
C
      CALL MIN2(X,DY,GRAD)
      IF(SSQ.GT.ERSSQ)GOTO2
      RETURN
      END
      SUBROUTINE MIN2(X,STEP,GRAD)
      COMMON /B2/ SSQ
      LOGICAL DIRP,DIRN
      DIMENSION X(2),Y1(3),Y2(3),F(3)
      ERSSQ=1.E-3
C
C    FIND DIRECTION
C
      WRITE(2,200)
 200  FORMAT(11X,"FIND DIRECTION")
      N=0
      SGRAD=SIGN(1.,GRAD)
      CALL MAINFCN
 2    CONTINUE
      DIRP=.FALSE.
      DIRN=.FALSE.
      X1STAR=X(1)
      X2STAR=X(2)
      F(1)=SSQ
      Y1(1)=X(1)
      Y2(1)=X(2)
      WRITE(2,100) SSQ,X(1),X(2)
       IF(SSQ.LT.ERSSQ)RETURN
  100 FORMAT(10X,"ERR= ",E17.11,"CR= ",E17.11," CI= ",E17.11)
 13   CONTINUE
C
C    TRY POSITIVE INCREMENT
C
      STEPX=STEP
      DX=STEPX/SQRT(1.+GRAD**2)
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      X(1)=X1STAR+DX
      X(2)=X2STAR+SGRAD*SQRT(STEPX**2-DX**2)
      CALL MAINFCN
      F(2)=SSQ
      Y1(2)=X(1)
      Y2(2)=X(2)
      IF(F(2)-F(1))9,11,11
9     CONTINUE
C
C     POSITIVE INCREMENT WORKED
C
      DIRP=.TRUE.
      STEPX=2*STEP
      DX=STEPX/SQRT(1.+GRAD**2)
      X(1)=X1STAR+DX
      X(2)=X2STAR+SGRAD*SQRT(STEPX**2-DX**2)
      CALL MAINFCN
      F(3)=SSQ
      Y1(3)=X(1)
      Y2(3)=X(2)
      WRITE(2,100) SSQ,X(1),X(2)
       IF(SSQ.LT.ERSSQ)RETURN
      GOTO14
11    CONTINUE
C
C     TRY NEGATIVE INCREMENT
C
      STEPX=STEP
      DX=STEPX/SQRT(1.+GRAD**2)
      X(1)=X1STAR-DX
      X(2)=X2STAR-SGRAD*SQRT(STEPX**2-DX**2)
      CALL MAINFCN
      F(2)=SSQ
      Y1(2)=X(1)
      Y2(2)=X(2)
      WRITE(2,100) SSQ,X(1),X(2)
       IF(SSQ.LT.ERSSQ)RETURN
      IF(F(2)-F(1))10,12,12
10    CONTINUE
      DIRN=.TRUE.
C
C     NEGATIVE INCREMENT WORKED
C
      STEPX=2.*STEP
      DX=STEPX/SQRT(1.+GRAD**2)
      X(1)=X1STAR-DX
      X(2)=X2STAR-SGRAD*SQRT(STEPX**2-DX**2)
      CALL MAINFCN
      F(3)=SSQ
      Y1(3)=X(1)
      Y2(3)=X(2)
      WRITE(2,100) SSQ,X(1),X(2)
       IF(SSQ.LT.ERSSQ)RETURN
      GOTO14
12     CONTINUE
C
C     NEITHER WORKED.  HALVE STEP
C
      STEP=STEP/2.
      GOTO13
14    CONTINUE
C
C     DIRECTION FOUND
C
      WRITE(2,201)
201   FORMAT(10X,"BRACKET MINIMUM")
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      IF(DIRP)XSIGN=+1
      IF(DIRN)XSIGN=-1
15    CONTINUE
      IF(F(3)-F(2))16,17,17
16    CONTINUE
      N=N+1
      STEPX=N*STEP
      DX=STEPX/SQRT(1.+GRAD**2)
      X(1)=X(1)+XSIGN*DX
      X(2)=X(2)+XSIGN*SGRAD*SQRT(STEPX**2-DX**2)
      Y1(1)=Y1(2)
      Y1(2)=Y1(3)
      Y1(3)=X(1)
      Y2(1)=Y2(2)
      Y2(2)=Y2(3)
      Y2(3)=X(2)
      F(1)=F(2)
      F(2)=F(3)
      CALL MAINFCN
      F(3)=SSQ
      WRITE(2,100) SSQ,X(1),X(2)
       IF(SSQ.LT.ERSSQ)RETURN
      IF(F(3)-F(2))16,17,17
17    CONTINUE
C
C   MINIMUM BRACKETTED
C
C    NOW FIT QUADRATIC
C
      WRITE(2,202)
202   FORMAT(10X,"USE QUADRATIC APPROX FOR MINIMUM")
      IF(ABS(GRAD).GT.0.5E+10)GOTO3
      F1=Y1(1)-Y1(2)
      F2=Y1(1)-Y1(3)
      F3=Y1(2)-Y1(3)
      BIT1=F(1)/F1/F2
      BIT2=-F(2)/F1/F3
      BIT3=F(3)/F2/F3
      CIT1=Y1(1)*(BIT2+BIT3)
      CIT2=Y1(2)*(BIT1+BIT3)
      CIT3=Y1(3)*(BIT1+BIT2)
      X(1)=(CIT1+CIT2+CIT3)/2./(BIT1+BIT2+BIT3)
      IF(ABS(GRAD).LT.1.5E-10)GOTO4
3     CONTINUE
      F1=Y2(1)-Y2(2)
      F2=Y2(1)-Y2(3)
      F3=Y2(2)-Y2(3)
      BIT1=F(1)/F1/F2
      BIT2=-F(2)/F1/F3
      BIT3=F(3)/F2/F3
      CIT1=Y2(1)*(BIT2+BIT3)
      CIT2=Y2(2)*(BIT1+BIT3)
      CIT3=Y2(3)*(BIT1+BIT2)
      X(2)=(CIT1+CIT2+CIT3)/2./(BIT1+BIT2+BIT3)
4     CONTINUE
      CALL MAINFCN
      WRITE(2,100)SSQ,X(1),X(2)
       IF(SSQ.LT.ERSSQ)RETURN
      STEP=STEP/2.
1     CONTINUE
      RETURN
      END
```

LIST OF SYMBOLS

C = $C_r + iC_i$, where C_r and C_i are real and $i = \sqrt{-1}$

C_r propagation velocity

C_i amplification factor

f defined by $\frac{df}{d\eta} = \frac{U}{U_e}$; dimensionless velocity ratio

m pressure gradient parameter (eqn 3.10)

p pressure

U longitudinal velocity component

V transverse velocity component

α wave number

$\bar{\alpha}$ = $\alpha \frac{dy}{d\eta}$

β pressure gradient parameter (eqn 3.10)

δ boundary layer thickness

δ^* displacement thickness

ξ transformed similarity variable (eqn 3.6)

η transformed similarity variable (eqn 3.7)

$\phi(y)$ small perturbation variable for transverse velocity (eqn 4.2)

ν kinematic viscosity

Subscripts

e external flow

x partial derivative with respect to x

y partial derivative with respect to y

η partial derivative with respect to η

$\hat{}$ small perturbation variable function of y

*U.S.Government Printing Office: 1979 — 657-084/355