STABILITY ANALYSIS OF THE LOWER BRANCH SOLUTIONS
OF THE FALKNER-SKAN EQUATIONS

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Chief, Aerodynamics & Airframe Br.

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**Abstract**

The stability of the Lower-Branch solutions of the Falkner-Skan boundary layers has been analyzed. A perturbation analysis of these boundary layers was performed resulting in the Rayleigh stability equation. Eigenvalue solutions were obtained for the Rayleigh equation for different adverse pressure gradient values. Propagation velocity and amplification factors were computed over the entire range of unstable wave numbers. All retarded flows were found to be unstable for a small range of frequencies, with the amplification factor increasing as the extent of reversed flow increased.
ACKNOWLEDGEMENT

The authors would like to thank the Air Force Systems Command, Air Force Office of Scientific Research, Southern Center for Electrical Engineering Education and Wright-Patterson Air Force Base for providing resources for the senior author to spend the summer of '79 at WPAFB.

We would also like to thank Dr. Charles Jobe for helping with programming phase of the research.
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SECTION I

INTRODUCTION

Self excited oscillations have been experimentally observed in separated flows for over hundred years. Rayleigh [1] in 1880 proved that for inviscid, incompressible flow the unstable velocity profiles must have an inflection point. Tollmein [2] in 1935 showed that for symmetrical velocity distributions, or for velocity distributions of the boundary layer type, the existence of the inflection point implies instability.

Recently Hankey and Shang [3] have examined the self induced pressure oscillations in an open cavity. Their numerical computations compare very well with the previous experimental investigations. Roscoe and Hankey [4] have studied the stability of hyperbolic tangent velocity profile in a compressible fluid, while Hankey, Hunter and Harney [5] have examined the self-sustained oscillations (Buzz) on spiked tipped bodies for large Mach numbers. However, a systematic stability analysis of separated flows has not been undertaken. It is the purpose of this report to conduct a stability analysis of a general class of separated flows (i.e. reversed flow Falkner-Skan) in order to help shed light on the phenomenon of self-excited fluid flows.
SECTION II

OBJECTIVES OF THE RESEARCH EFFORT

The objective of this research effort was to analyze a series of similar separated flows for different values of $\beta$ and to determine the amplification factors and propagation velocities in all these different cases. Eight cases of different $\beta$ were identified to be analyzed. These cases were those with reversed flows which contained velocity profiles with inflection points.
SECTION III

MEAN FLOW EQUATIONS

In this report, incompressible flows will be analyzed. In subsequent work we plan to analyze the compressible flows.

The incompressible two-dimensional Navier-Stokes equations are as follows:

\[ \begin{align*}
U_t + U U_x + V U_y &= -\frac{1}{\rho} P_x + \nu V^2 U \\
V_t + U V_x + V V_y &= -\frac{1}{\rho} P_y + \nu V^2 V \\
U_x + V_y &= 0
\end{align*} \tag{3.1} \tag{3.2} \tag{3.3} \]

Applying the boundary layer approximations to the above equations for steady flows results in the following:

\[ \begin{align*}
UU_x + UV_y &= U U_x e^\epsilon x + \nu U_y y \\
U_x + V_y &= 0 \tag{3.4} \tag{3.5}
\end{align*} \]

These equations may be reduced to one ordinary differential equation for the case where \( U_e = c x^m \) by transforming with similarity variables.

\[ \frac{d\xi}{\nu} = \frac{U_e dx}{\nu} \tag{3.6} \]

\[ \frac{d\eta}{\sqrt{2\xi} \nu} = \frac{U_e dy}{\sqrt{2\xi} \nu} \tag{3.7} \]

Hence

\[ f''' + ff'' = \beta (f'^2 - 1) \tag{3.8} \]

where

\[ f'(\eta) = \frac{U}{U_e} \tag{3.9} \]

and

\[ \beta = \frac{U_e\epsilon}{U_e} = \frac{2m}{m + 1} = \text{constant} \tag{3.10} \]

with boundary conditions

\[ f(0) = 0, f'(0) = 0, f'(\infty) = 1 \tag{3.11} \]
Falkner and Skan [6] originally derived this equation for attached flows however, Stewartson [7] discovered a lower branch to these solutions which represented reversed flows from incipient separation to the Chapman solution. Christian, Hankey and Petty [8] have tabulated these solutions for compressible and incompressible flows. It is this wide class of flows (which have inflection points) that are known to be unstable for which we shall now perform a stability analysis.
SECTION IV

PERTURBATION EQUATIONS

Let us assume small perturbations of the form

\[ U = \bar{U}(y) + \hat{U}(y) e^{i\alpha(x - ct)} \]  
\[ V = \hat{\phi}(y) e^{i\alpha(x - ct)} \]  
\[ p = \hat{P}(x) + \hat{P}(y) e^{i\alpha(x - ct)} \]

where \( c = c_r + ic_i \) and \( \bar{U}, \phi \) and \( \hat{P} \) are small in comparison to the mean quantities. If we substitute these values of \( U, V, \) and \( p \) in equations (3.1), (3.2) and (3.3); retain only the first order terms and assume that Reynolds number \( \frac{U_{ex}}{v} \) is large then the equations (4.1), (4.2) and (4.3) reduce to one single equation

\[ 2\pi \left( \alpha^2 + \frac{\bar{U}''}{\bar{U} - c} \right) \phi = 0 \]  

The classical Rayleigh equation with the boundary conditions

\[ \phi(0) = 0, \quad \phi(\infty) = 0 \]  

(4.5a,b)

By transforming the equation from \( y \) to the \( \eta \) variable we obtain the following equation

\[ \phi_{\eta\eta} - (\widetilde{\alpha}^2 + \frac{f'''}{f'' - c}) \phi = 0 \]  

where

\[ \widetilde{\alpha} = \alpha \frac{dy}{d\eta} \]

By inserting the values of \( f'(\eta, \beta) \) into the Rayleigh equation \( c(\widetilde{\alpha}, \beta) \) can be determined as an eigenvalue which satisfies the boundary conditions (4.5a,b).
SECTION V

SOLVING SCHEME

Eigenvalues were determined by a shooting method; starting with a
given boundary condition, integrating over the range of $\eta$ and comparing
the result with the outer boundary condition, namely $\dot{\eta} = 0$ at $\eta_{\text{max}}$.
The process involved minimization of the error in the outer boundary
condition which was chosen to be the square of the norm of $\phi$,
$$\eta_{\text{SSQ}} = \eta_R^2 + \eta_I^2$$ (See Appendix 3). The integration was done using
a fourth-order Runge-Kutta method.

The method of finding eigenvalues utilized a minimization routine
written primarily by Roscoe [4]. Starting from a given guess the
routine searched along a constant line of $c_I$ with increasing steps until
it found a relative minimum of the error. It then used the last three
calculated points to determine a parabola, with the $c_r$ value at the vertex
used as the latest approximation. Then this value of $c_r$ was held constant
and a search along a line of changing $c_I$ was carried out. After a new
minimum was found, the quadratic approximation was again used to determine
a new value for $c_I$. The third step involved searching the line connecting
the original guess and the new point. After finding a minimum and utilizing
quadratic approximation, the error was checked to see if it was less than
some preset limit. If not, the routine started again with the latest value
used in place of the original guess.

Generally, the routine worked quite well. Most of the search time
was attributable to bad guesses and finding the direction in which the
search should be continued. An eigenvalue was usually located in a very
narrow region of the plane and even though the step size was continually reduced, it was frequently large enough to move the test point out of the acceptable region. For example, the initial guess in one case led to an error of $4.1 \times 10^{13}$, however, after only 128 new error computations, the error had been reduced 17 orders of magnitude to $1.9 \times 10^{-4}$, while $c_r$ had been changed by 4.25% and $c_i$ had been changed by 3.82%. Convergence was also retarded for small values of $c_i$, e.g. $|c_i| < 0.001$. This was concurrent with $c_r$ approaching its limiting value.

The Howard semicircle theorem [9] was used as an aid in determining suitable initial guesses. If $c_r$ is the propagation velocity, $\alpha$ is the wave number, $c_i$ is the amplification factor, and $U_{max}$ and $U_{min}$ are the maximum and minimum values of the range of $U$, the theorem states

$$[c_r - 1/2(U_{max} + U_{min})]^2 + c_i^2 \leq [1/2(U_{max} - U_{min})]^2.$$ 

Thus, the complex wave velocity for an unstable mode lies inside the upper semi-circle which has the range of $U$ as diameter.
SECTION VI

RESULTS

Eight cases were computed for \( \beta \) values of -.0001, -.0005, -.002, -.04, -.08, -.12, -.16 and -.19884. For a wide range of \( \bar{\alpha} \) values the eigenvalues were ascertained. These values are tabulated in tables B-1 - B-8 in Appendix B. \( \bar{\alpha} \) is related to \( \alpha \) by the relation

\[
a\delta^* = \frac{\bar{\alpha}^*}{\alpha} = \left(1 - f'\right) \bar{\alpha} \quad \text{for} \quad \delta \to \infty
\]

The values of \( C_r \) and \( C_i \) versus \( \alpha \delta^* \) are plotted in figures 1a-1h and 2a-2h. Figure 3a-3h shows Howard's plot (9) for these solutions. Some typical eigenvalues for a series of solutions are also tabulated and plotted in Appendix B.
LEGEND

LETTER  $\beta$

a  -.0001
b  -.0005
c  -.002
d  -.04
e  -.08
f  -.12
g  -.15
h  -.18884

\[ \begin{array}{c}
C_i \\
-0.5 \\
-0.25 \\
0 \\
0.25 \\
0.5 \\
\end{array} \]

\[ \begin{array}{c}
C_r \\
0 \\
0.25 \\
0.5 \\
0.75 \\
1.0 \\
\end{array} \]

FIGURE 3 a-h

FIGURE 3
Figure 4

**Legend**

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<tr>
<td>b</td>
<td>-0.0005</td>
</tr>
<tr>
<td>c</td>
<td>-0.002</td>
</tr>
<tr>
<td>d</td>
<td>-0.04</td>
</tr>
<tr>
<td>e</td>
<td>-0.08</td>
</tr>
<tr>
<td>f</td>
<td>-0.12</td>
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<tr>
<td>g</td>
<td>-0.16</td>
</tr>
<tr>
<td>h</td>
<td>-0.1984</td>
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SECTION VII

CONCLUSIONS

The stability of a series of similar separated flows have been analyzed. Amplification factors and propagation velocities of the disturbances were determined. The results show that a small zone of instability does exist for these flows with inflexion points. The amplification factor increases as the extent of the reversed flow increases.
SECTION VIII

RECOMMENDATIONS

Suggestions for follow-on research: We would like to investigate the instability of laminar separated flows under the influence of compressibility. For a hyperbolic tangent velocity profile Roscoe (4) showed the instability to diminish with the increase of Mach number until the Rayleigh instability actually vanished at Mach number $M = 2.5$. The analysis should be repeated for the compressible, adiabatic, Falkner-Skan velocity profiles. We have completed $M = 0$ cases for various values of $\beta$, and would like to examine the influence of Mach number for the same values of $\beta$. It was observed that for $\beta = -.0001$ and $-.0005$ the convergence at the two ends of the spectrum was very slow. These cases should be analyzed somewhat more thoroughly.
REFERENCES


APPENDIX A

THE HOWARD CIRCLE THEOREM

The Howard semicircle theorem [9] is an extension of the well known fact that if the amplification factor $C_i > 0$ then the propagation velocity $C_r$ must lie in the range of $U$. Howard was able to restrict the permissible values of $C_r$ and $C_i$ so that the complex wave velocity $C$ is confined to a semicircle which has the range of $U$ as its diameter. If $U_{\text{max}}$ and $U_{\text{min}}$ are the extrema of the range of $U$, the theorem states

$$[C_r - \frac{1}{2}(a + b)]^2 + C_i^2 \leq \left[\frac{1}{2}(a + b)\right]^2, \quad C_i > 0$$

where $a = U_{\text{max}}$, $b = U_{\text{min}}$. 
APPENDIX B

EIGENVALUES FROM STABILITY ANALYSIS
FOR REVERSED FLOW BOUNDARY LAYERS
EIGENVALUES FROM STABILITY ANALYSIS FOR REVERSED FLOW BOUNDARY LAYERS

**TABLE B-1**

\[ \beta = -0.0001 \]

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<th>( \bar{\alpha} )</th>
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<th>( C_1 )</th>
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<td>0</td>
<td>0.90538414714</td>
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EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-2
β = -.0005

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EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-3
\[ \beta = -0.002 \]

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<td>0.56865534872</td>
<td>0.00023011047136</td>
</tr>
<tr>
<td>0.56</td>
<td>0.57775580.81</td>
<td>0.19174876005(10)^{-10}</td>
</tr>
<tr>
<td>0.57</td>
<td>0.58685371398</td>
<td>0.61658214056(10)^{-11}</td>
</tr>
<tr>
<td>0.58</td>
<td>0.58704040582</td>
<td>0.34648626536(10)^{-11}</td>
</tr>
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</table>
EIGENVALUES FROM STABILITY ANALYSIS FOR REVERSED FLOW BOUNDARY LAYERS

TABLE B-4

\( \beta = -0.04 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( C_r )</th>
<th>( C_i )</th>
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<tr>
<td>0.12</td>
<td>0.9462446953107</td>
<td>0.00077598474441</td>
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<tr>
<td>0.13</td>
<td>0.94480294724055</td>
<td>0.00092398603074</td>
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<td>0.14</td>
<td>0.93834475353190</td>
<td>0.0019712426475</td>
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<tr>
<td>0.15</td>
<td>0.92864533</td>
<td>0.0041227836</td>
</tr>
<tr>
<td>0.17</td>
<td>0.91398636</td>
<td>0.009481253</td>
</tr>
<tr>
<td>0.20</td>
<td>0.88234356</td>
<td>0.031932225</td>
</tr>
<tr>
<td>0.23</td>
<td>0.78918067</td>
<td>0.079558300</td>
</tr>
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<td>0.73325251</td>
<td>0.10055163</td>
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<tr>
<td>0.30</td>
<td>0.63589654</td>
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<td>0.57744885</td>
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<tr>
<td>0.40</td>
<td>0.54449017</td>
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<td>0.53642765</td>
<td>0.089694409</td>
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<td>0.021022046</td>
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<tr>
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<td>0.55</td>
<td>0.52416855</td>
<td>3.4183573(10)^{-6}</td>
</tr>
<tr>
<td>0.56</td>
<td>0.53445687</td>
<td>1.1840627(10)^{-8}</td>
</tr>
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\[
\begin{array}{cccc}
\bar{\alpha} & C_r & C_i \\
0.57 & 0.53445745 & 0.28132369(10)^{-9} \\
0.58 & 0.53445776 & 0.10618506(10)^{-9} \\
0.59 & 0.53445776 & -0.16007328(10)^{-10} \\
0.60 & 0.534457761505 & -0.581878617(10)^{-10} \\
0.61 & 0.53445776292 & -0.75694864203(10)^{-10} \\
\end{array}
\]
EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-5
\( \beta = -.08 \)

<table>
<thead>
<tr>
<th>( \bar{\alpha} )</th>
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<th>( C_i )</th>
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<tbody>
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<td>.20</td>
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<td>.27</td>
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<td>.30</td>
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EIGENVALUES FROM STABILITY ANALYSIS FOR REVERSED FLOW BOUNDARY LAYERS

**TABLE B-6**

\[ \beta = -0.12 \]

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<td>0.006241760255</td>
</tr>
<tr>
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<td>0.86703422542955</td>
<td>0.016184737774036</td>
</tr>
<tr>
<td>0.37</td>
<td>0.82518601942224</td>
<td>0.025407066978775</td>
</tr>
<tr>
<td>0.40</td>
<td>0.74110563666309</td>
<td>0.033621469860405</td>
</tr>
<tr>
<td>0.42</td>
<td>0.68545236670957</td>
<td>0.034160709745084</td>
</tr>
<tr>
<td>0.45</td>
<td>0.60602465101</td>
<td>0.029295418399</td>
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<td>0.47</td>
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<td>0.47928620449</td>
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<tr>
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<td>0.47900001254</td>
<td>0.164119887(10)^{-8}</td>
</tr>
<tr>
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<td>0.47900001195552</td>
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**EIGENVALUES FROM STABILITY ANALYSIS FOR REVERSED FLOW BOUNDARY LAYERS**

**TABLE B-7**

\[ \beta = -.16 \]

<table>
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<th>( C_1 )</th>
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<td>.35</td>
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<tr>
<td>.45</td>
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<td>.50</td>
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<tr>
<td>.52</td>
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EIGENVALUES FROM STABILITY ANALYSIS FOR
REVERSED FLOW BOUNDARY LAYERS

TABLE B-8

\( \beta = -0.19884 \)

<table>
<thead>
<tr>
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<th>( C_r )</th>
<th>( C_1 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.38</td>
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<tr>
<td>0.39</td>
<td>0.91329400663</td>
<td>0.001065562476</td>
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<tr>
<td>0.40</td>
<td>0.90688937</td>
<td>0.0013915674</td>
</tr>
<tr>
<td>0.42</td>
<td>0.89032638669</td>
<td>0.00295916137</td>
</tr>
<tr>
<td>0.45</td>
<td>0.85880806</td>
<td>0.0055504066</td>
</tr>
<tr>
<td>0.47</td>
<td>0.83019514805718</td>
<td>0.0066806913008495</td>
</tr>
<tr>
<td>0.50</td>
<td>0.77300900</td>
<td>0.0043585794</td>
</tr>
<tr>
<td>0.52</td>
<td>0.7283639558</td>
<td>0.48854083711(10)^{-6}</td>
</tr>
<tr>
<td>0.53</td>
<td>0.70401641</td>
<td>0.28953714(10)^{-7}</td>
</tr>
<tr>
<td>0.54</td>
<td>0.70412941</td>
<td>-1.9018251(10)^{-7}</td>
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</table>
**LEGEND**

<table>
<thead>
<tr>
<th>LETTER</th>
<th>( \theta )</th>
<th>( \phi_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>.0001</td>
<td>.243983</td>
</tr>
<tr>
<td>b</td>
<td>.0005</td>
<td>.242349</td>
</tr>
<tr>
<td>c</td>
<td>.002</td>
<td>.211616</td>
</tr>
</tbody>
</table>

**FIGURE B-2**
APPENDIX C
COMPUTER PROGRAM

The following FORTRAN program was used in the search for eigenvalues. The driver program FINDMIN gives the initial guesses for CREAL and CIM and then calls the minimization routine. The error is returned as SSQ. Subroutine MAINFCN does the integration using a system routine RKDF which uses a fourth order Runge-Kutta method. The arguments to RKDF are: X - the independent variable, Y - the dependent variables, N - the number of variables, DX - the step size, and IER - an error return. RKDF also requires a function F which computes the derivatives of the dependent variables and stores them in P.

The minimization routines are fairly general. The equivalence causes the minimization to be done with respect to CREAL and CIM. To minimize with respect to CIM and ALPHA the equivalence statement would be: "EQUIVALENCE (CIM, X(1)), (ALPHA, X(2))." Note that the two variables which are equivalenced with X(1) and X(2) must be stored consecutively in memory.

The array Y represents the following quantities. Y(1) = f, Y(2) = f', Y(3) = f'', Y(4) = φ_R, Y(5) = φ_I, Y(6) = φ_R', Y(7) = φ_I'. Convergence was generally achieved when minimization errors of 10^{-5} to 10^{-9} occurred for most cases.
PROGRAM FINDMIN
INPUT,OUTPUT,TAPE5=INPUT,TAPE2=OUTPUT
COMMON CREAL,CIM,ALPHA,BETA,P3D
COMMON /92/SSQ
CREAL = .571945967522
CIM = .033942176531
ALPHA = .5
CALL MINI
WRITE(2,1)ALPHA,BETA,CREAL,CIM,SSQ
100 CONTINUE
END
SUBROUTINE MAINFCN
COMMON CREAL,CIM,ALPHA,BETA,P3D
COMMON /B2/SSQ
DIMENSION Y(7),P(7)
BETA = -.0305
Y(3) = +.051546
XEND = +0.
X = 0.0
Y(1) = U.00
Y(2) = 0.0
Y(4) = 0.0
Y(5) = 0.0
CGBAR = CREAL*CREAL + CIM*CIM
GCC = BETA/CGBAR
FACTR = ALPHA*ALPHA + GCC*CREAL
R = FACTR*FACTR + GCC*GCC*CIM*CIM
R = SQRT(SQRT(R))
GAMMA = 0.5*ATAN2(-GCC*CIM,FACTR)
Y(6) = R * COS(GAMMA)
Y(7) = R * SIN(GAMMA)
DX = 0.05
N = 7
CALL F(X,Y,P)
C*** **************INNER LOOP INTEGRATION ***************
2 CONTINUE
CALL RKDF(X,Y,N,DX,IER)
IF(X<=XEND)GO TO 2
C*** **************INNER LOOP INTEGRATION ***************
SSQ=Y(5)*Y(5)+Y(4)*Y(4)
RETURN
END
SUBROUTINE F(X,Y,P)
COMMON CREAL,CIM,ALPHA,BETA,P3D
DIMENSION Y(7),P(7)
P(1) = Y(1)
P(2) = Y(2)
P(3) = ALPHA*(Y(2) - 1.)*(Y(2) + 1.) - Y(1)*Y(3)
P(4) = Y(6)
P(5) = Y(7)
U = Y(2)
UBB = P(3)
D = UBB/(U-CREAL)**2 + CIM**2
B = D*CIM
A = ALPHA*ALPHA + D*(U-CREAL)
P(6) = A*Y(4) - B*Y(5)
P(7) = B*Y(4) - A*Y(2)
RETURN
END
SUBROUTINE MINI
COMMON CREAL,CIM,ALPHA,BETA,P3D
COMMON /B2/ SSQ
DIMENSION XEST(2,2),X(2),STEP(2)
EQUIVALENCE(CREALX(1),X(1)),(CIM,X(2))
STEP (1) = 1.E-10
STEP (2) = 1.E-4
ERSSQ=1.E-3
P30=0.
CONTINUE
DY=STEP(1)
XEST(1,1)=X(1)
GRAD=1.E-30

SEARCH ALONG X1-AXIS

CALL MIN2(X,DY,GRAD)
IF(SSQ.LT.ERSSQ) RETURN
XEST(1,2)=X(1)
STEP(1)=DY
XEST(2,1)=X(2)
Dy=STEP(2)
GRAD=1.E-30

SEARCH ALONG X2-AXIS

CALL MIN2(X,DY,GRAD)
IF(SSQ.LT.ERSSQ) RETURN
XEST(2,2)=X(2)
STEP(2)=DY
GRAD=(XEST(1,2)-XEST(1,1))/(XEST(2,2)-XEST(2,1))

SEARCH ALONG LINE

CALL MIN2(X,Y,GRAD)
IF(SSQ.LT.ERSSQ) RETURN
END

SUBROUTINE MIN2(X,STEP,GRAD)
COMMON /B2/ SSQ
LOGICAL DIRP,DIRN
DIMENSION X(2),Y(3),F(3)

ERSSQ=1.E-3

FIN D DIRECTION

WRITE(2,200)
FORMAT(11X,"FIND DIRECTION")
N=0
SGRAD=SIGN(1.,GRAD)
CALL MAINFCN
CONTINUE
DIP=.FALSE.
DIN=.FALSE.
X1STAR=X(1)
X2STAK=X(2)
F(1)=SSQ
Y(1)=X(1)
Y(2)=X(2)
WRITE(2,100) SSQ,X(1),X(2)
IF(SSQ.LT.ERSSQ) RETURN
100 FORMAT(10X,"ERR = \",E17.11,",CR = \",E17.11,", CI = \",E17.11)
CONTINUE

TRY POSITIVE INCREMENT

STEPX=STEP
DX=STE PX/SQRT(1.+GRAD**2)
\[ X(1) = X^{\text{STAR}} + DX \]
\[ X(2) = X^{\text{STAR}} + S\text{GRAD} \cdot \text{SORT}(\text{STEP}X^2 - DX^2) \]

**Main Function Call:**

\[ F(2) = \text{SSQ} \]
\[ Y1(2) = X(1) \]
\[ Y2(2) = X(2) \]
\[ \text{IF}(F(2) - F(1)) \]

**POSITIVE INCREMENT WORKED**

\[ \text{DIRP} = \text{TRUE.} \]
\[ \text{STEPX} = 2 \cdot \text{STEP} \]
\[ DX = \text{STEPX} \cdot \text{SORT}(1 + \text{GRAD}^2) \]
\[ X(1) = X^{\text{STAR}} - DX \]

**Main Function Call:**

\[ F(3) = \text{SSQ} \]
\[ Y1(3) = X(1) \]
\[ Y2(3) = X(2) \]

\[ \text{WRITE}(2,10) \text{, SSQ, } X(1), X(2) \]

**If (SSQ < ERSSQ)**

\[ \text{RETURN} \]

**GO TO 12**

**CONTINUE**

**NEGATIVE INCREMENT WORKED**

\[ \text{DIRP} = \text{TRUE.} \]

\[ \text{STEPX} = 2 \cdot \text{STEP} \]
\[ DX = \text{STEPX} \cdot \text{SORT}(1 + \text{GRAD}^2) \]
\[ X(1) = X^\text{STAR} + DX \]

**Main Function Call:**

\[ F(2) = \text{SSQ} \]
\[ Y1(2) = X(1) \]
\[ Y2(2) = X(2) \]

\[ \text{WRITE}(2,10) \text{, SSQ, } X(1), X(2) \]

**If (SSQ < ERSSQ)**

\[ \text{RETURN} \]

**GO TO 12**

**CONTINUE**

**NEITHER WORKED. HALVE STEP**

\[ \text{STEP} = \text{STEP} / 2. \]

**GO TO 13**

**CONTINUE**

**DIRECTION FOUND**

\[ \text{WRITE}(2,201) \]

**FORMAT 110X, "BRACKET MINIMUM"**
IF (DIRP) XSIGN = +1
IF (DIRN) XSIGN = -1

CONTINUE
IF (F(3) - F(2)) 16, 17, 17

CONTINUE
N = N + 1
STEP = STEP/2.

DX = STEP * SQRT(1. + GRAD**2)
X(1) = X(1) + XSIGN * DX
X(2) = X(2) + XSIGN * GRAD * SQRT(STEP**2 - DX**2)

Y1(1) = Y1(2)
Y1(2) = Y1(3)
Y1(3) = Y1(1)
Y2(1) = Y2(2)
Y2(2) = Y2(3)
Y2(3) = Y2(1)
F(1) = F(2)
F(2) = F(3)

CALL MAINFCN
F(3) = SSQ
WRITE (2, 100) SSQ, X(1), X(2)
IF (SSQ .LT. ERSSQ) RETURN
IF (F(3) - F(2)) 16, 17, 17

CONTINUE
MINIMUM BRACKETED
NOW FIT QUADRATIC

WRITE (2, 202)

FCN=AT(10X,"USE QUADRATIC APPROX FOR MINIMUM")
IF (ABS (GRAD) .GT. 5.5E-13) GOTO 3

F1 = Y1(1) - Y1(2)
F2 = Y1(1) - Y1(3)
F3 = Y1(2) - Y1(3)

BIT1 = F(1)/F1/F2
BIT2 = F(2)/F1/F3
BIT3 = F(3)/F2/F3

CIT1 = Y1(1)*BIT2 + BIT3
CIT2 = Y1(2)*BIT1 + BIT2
CIT3 = Y1(3)*BIT1 + BIT2

X(1) = (CIT1 + CIT2 + CIT3)/2. /(BIT1 + BIT2 + BIT3)
IF (ABS (GRAD) .LT. 1.5E-15) GOTO 4

CONTINUE

F1 = Y2(1) - Y2(2)
F2 = Y2(1) - Y2(3)
F3 = Y2(2) - Y2(3)

BIT1 = F(1)/F1/F2
BIT2 = F(2)/F1/F3
BIT3 = F(3)/F2/F3

CIT1 = Y2(1)*BIT2 + BIT3
CIT2 = Y2(2)*BIT1 + BIT2
CIT3 = Y2(3)*BIT1 + BIT2

X(2) = (CIT1 + CIT2 + CIT3)/2. /(BIT1 + BIT2 + BIT3)

CONTINUE

CALL MAINFCN
WRITE (2, 100) SSQ, X(1), X(2)
IF (SSQ .LT. ERSSQ) RETURN
STEP = STEP/2.

CONTINUE
RETURN
END
LIST OF SYMBOLS

\[ C = C_r + iC_i, \text{ where } C_r \text{ and } C_i \text{ are real and } i = \sqrt{-1} \]

\( C_r \)
propagation velocity

\( C_i \)
amplification factor

\( f \)
defined by \( \frac{df}{d\eta} = \frac{U}{U_e} \); dimensionless velocity ratio

\( m \)
pressure gradient parameter (eqn 3.10)

\( p \)
pressure

\( U \)
longitudinal velocity component

\( V \)
transverse velocity component

\( \alpha \)
wave number

\( \bar{\alpha} = \alpha \frac{dy}{d\eta} \)

\( \beta \)
pressure gradient parameter (eqn 3.10)

\( \delta \)
boundary layer thickness

\( \delta^* \)
displacement thickness

\( \xi \)
transformed similarity variable (eqn 3.6)

\( \eta \)
transformed similarity variable (eqn 3.7)

\( \phi(y) \)
small perturbation variable for transverse velocity (eqn 4.2)

\( \nu \)
kinematic viscosity

Subscripts

\( e \)
external flow

\( x \)
partial derivative with respect to \( x \)

\( y \)
partial derivative with respect to \( y \)

\( \eta \)
partial derivative with respect to \( \eta \)

\( \phi(y) \)
small perturbation variable function of \( y \)