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by

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Lemoine (1977) studies sojourn times in acyclic, Jackson networks as part of his survey of equilibrium results. In his paper, he argues that a particular customer has a sojourn time at node i that is independent of his remaining sojourn time in the network given that the customer transfer from node i to node j. This assumption is then used to derive a set of equations involving the Laplace-Stieltjes transform of the sojourn times at each node from which several properties of sojourn times are determined.

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Abstract

~~Lemoine (1977)~~ studies we make an sojourn times in acyclic, Jackson networks as part of ~~his~~ a survey of equilibrium results. ~~In his paper, he argues~~ Argument is made that a particular customer has a sojourn time at node i that is independent of his remaining sojourn time in the network given that the customer transfer from node i to node j. This assumption is then used to derive a set of equations involving the Laplace-Stieltjes transform of the sojourn times at each node from which several properties of sojourn times are determined.

In a private communication, Mitrani (1978) argues that Lemoine's independence assumption is in error and provides a heuristic argument based on a four node, acyclic, Jackson network. From this, Mitrani concludes that the sojourn time is such networks is still an unsolved problem. We make Mitrani's argument rigorous, using some new results on sojourn times in Jackson networks.

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Continuation of Abstract.

In a private communication, Mitrani (1978) argues that Lemoine's independence assumption is in error and provides a heuristic argument based on a four node, acyclic, Jackson network. From this, Mitrani concludes that the sojourn time in such networks is still an unsolved problem. We make Mitrani's argument rigorous, using some new results on sojourn times in Jackson networks.

SOME RESULTS ON SOJOURN TIMES

IN ACYCLIC JACKSON NETWORKS

1. Background. Lemoine (1977) analyzes the sojourn time of a customer in an acyclic Jackson network. His analysis relies on the sojourn time of a particular customer at server i being independent of the customer's remaining sojourn time in the network, given the customer transfers from server i to server j. Using this independence assumption, Lemoine derives the Laplace-Stieltjes transform for the sojourn time of an arbitrary customer in the network.

In a private communication, Mitrani (1978) challenges Lemoine's assertion that the sojourn time at server i and the remaining sojourn time in the network are independent given that the customer transfers from i to j. In particular he concludes that the distribution of the sojourn time of the customer in an acyclic Jackson network is still an unsolved problem.

We resolve these issues in this paper. Through the use of a counter-example, we substantiate Mitrani's claim. This work culminates with Theorem 4.3. In section 5 we will summarize our results very briefly and point out related literature. We assume the reader is familiar with Jackson (1957) and Lemoine (1977).

2. Statement of Problem. In this section we construct a three node, acyclic Jackson network with the property that a customer's remaining sojourn time is not independent of his sojourn time in node 1, given that he goes through node 2. The network has the following form.

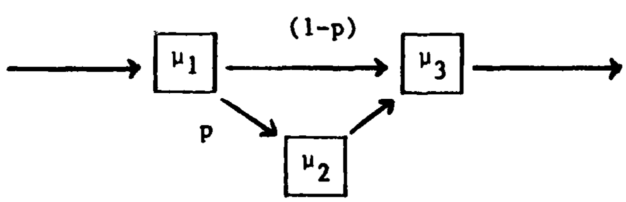

Figure 2.1

There is a Poisson stream with rate λ arriving at node 1. When a customer leaves node 1 he goes directly to node 3 with probability $(1-p)$, or goes to node 2 with probability p. If the customer goes to node 2, he goes directly to node 3 when his service is completed. The customer leaves the network after service at node 3. Service times at node i are exponentially distributed with parameter μ_i.

To show that Lemoine's argument in section 1 is faulty, it suffices to show that the sojourn time in the first queue, T_1, and the total sojourn time in the second and third queue, $T_2 + T_3$, are dependent random variables for a customer who goes through the second queue. For such a customer we will show that for certain values of λ, μ_1, μ_2, μ_3, p, and t; the expected sojourn time in the third queue given that the sojourn time in the first queue is t, $E[T_3|T_1=t]$, is greater than the unconditional expected sojourn time in the third queue $E[T_3]$. By Theorem 3.3, below, the expected sojourn time in the second queue is independent of T_1. Thus, for the above parameters,

$$\begin{aligned} E(T_3 + T_2|T_1 = t) &= E(T_3|T_1 = t) + E(T_2) \\ &> E(T_3) + E(T_2) = E(T_3 + T_2). \end{aligned} \tag{1}$$

This fact verifies that the remaining sojourn time (after leaving queue 1) and the sojourn time in queue 1 are not independent. In turn this verifies the Mitrani conjecture.

3. Preliminary Results. In order to formally show that the sojourn times in the first and third queues are dependent, we need the following results. Theorem 3.3 is proven in more generality than is necessary for the problem at hand, but gives an intuitive reason for believing that sojourn times in Jackson networks are not independent in general.

Let J be an acyclic Jackson network with nodes $(1,2,\cdots,N)$ and switching matrix P. Let

λ_i = exogenous arrival rate to node i,

α_i = total arrival rate to node i (which is given by the solution of the traffic equation),

μ_i = service rate at node i.

$M_i(s)$ = the number of customers at node i at time s,

For a fixed customer ("our customer") who goes through nodes $(r_1,r_2,\cdots,r_k)$, let

a_i = the time of arrival at node i,

t_i = the time of departure from node i,

$T_i = t_i - a_i$ = the sojourn time of our customer at node i,

$D_i(s)$ = the number of departures from node i in the interval $(t_i,t_i + s]$,

$D_i'(s)$ = the number of departures from node i in the interval $(a_i,a_i + s]$.

<u>Lemma 3.1</u> Let J be a steady state, acyclic, Jackson network. Suppose our customer enters node j from outside the system (an exogenous arrival). Then

$$P(D_j'(s) = k|T_j = t) = \frac{(\alpha_j s)^k e^{-\alpha_j s}}{k!}, \quad \forall s \in (0,T_j).$$

<u>Proof</u>.

$$P(D_j'(s) = k|T_j = t) = \sum_{m=k}^{\infty} P(D_j'(s) = k|T_j = t,\ M_j(a_j^-) = m)P(M_j(a_j^-) = m|T_j = t)$$

$$= \sum_{m=k}^{\infty} \binom{m}{k}\left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^{m-k} \frac{\dfrac{\mu_j(\mu_j t)^m e^{-\mu_j t}}{m!}\left(1 - \dfrac{\alpha_j}{\mu_j}\right)\left(\dfrac{\alpha_j}{\mu_j}\right)^m}{(\mu_j-\alpha_j)e^{-(\mu_j-\alpha_j)t}}$$

$$= \frac{e^{-\alpha_j t}}{k!}\left(\frac{s}{t-s}\right)^k \sum_{m=k}^{\infty} \frac{(\alpha_j(t-s))^m}{(m-k)!}$$

$$= \frac{(\alpha_j s)^k e^{-\alpha_j s}}{k!}. \quad \square$$

Using Theorem 1.18 Çinlar (1975:p.77), one immediately has

Corollary 3.2. The departure process from node j is a Poisson process with rate α_j in (a_j, t_j) for any T_j.

This result also follows from the Sojourn Time Theorem in Burke (1972).

Theorem 3.3. Suppose an exogenous arrival enters node j (of a steady state, acyclic, Jackson network) at time a_j. Then

$$P(M_1(t_j^-) = k_1, M_2(t_j^-) = k_2, \cdots, M_j(t_j^-) = k_j + 1, \cdots, M_N(t_j^-) = k_N \mid T_j = t)$$
$$= \frac{(\alpha_j t)^{k_j} e^{-\alpha_j t}}{k_j!} \prod_{i \neq j} \left(1 - \frac{\alpha_i}{\mu_i}\right)\left(\frac{\alpha_i}{\mu_i}\right)^{k_i} \tag{2}$$

Proof. Let J_2 be the set of nodes which can be reached after leaving node j either directly or after passing through other nodes in J_2. Let $J_1 = (\{j\} \cup J_2)^c$. Without loss of generality, assume the nodes in J_1 are numbered $1, \cdots, j-1$ and the nodes in J_2 are $j+1, \cdots, N$. We will call a customer flow a departure process from J_1 if it originates at a node $k \in J_1$ and either leaves the system entirely or goes to some node $m \notin J_1$.

At time a_j^- the network is in steady state. Hence,

$$P\{M_1(a_j^-) = k_1, \cdots, M_N(a_j^-) = k_N\} = \prod_{i=1}^{N} \left(1 - \frac{\alpha_i}{\mu_i}\right)\left(\frac{\alpha_i}{\mu_i}\right)^{k_i}$$

The arrival processes to J_1 after time a_j are Poisson processes independent of T_j. In addition, the service times of customers in J_1 after time a_j are independent of T_j.

Thus, $\forall\, s \in (a_j, \infty)$

$$P\{M_1(s) = k_1, \cdots, M_{j-1}(s) = k_{j-1} \mid T_j = t\} = \prod_{i=1}^{j-1} \left(1 - \frac{\alpha_i}{\mu_i}\right)\left(\frac{\alpha_i}{\mu_i}\right)^{k_i} .$$

In particular for $s = t_j^-$, we have

$$P\{M_1(t_j^-) = k_1,\cdots,M_{j-1}(t_j^-) = k_{j-1}|T_j = t\} = \prod_{i=1}^{j-1}\left(1 - \frac{\alpha_i}{\mu_i}\right)\left(\frac{\alpha_i}{\mu_i}\right)^{k_i} .$$

Since the network is acyclic, by Beutler and Melamed (1978) the departure processes from J_1 before time t_j are mutually independent Poisson processes, and independent of the state of the queues in J_1 at time t_j^-. Hence,

$$P\{M_j(t_j^-) = k_j + 1|T_j = t,\ M_1(t_j^-) = k_1,\cdots,M_{j-1}(t_j^-) = k_{j-1}\} = \frac{e^{-\alpha_j t}(\alpha_j t)^{k_j}}{k_j!} .$$

Let $D_j'(s)$ be the number of departures from node j in the interval $(a_i,a_i+s]$. For $s < T_j$, $D_j'(s)$ is completely determined by $M_j(a_j^-)$, T_j and the service times at j. Since $M_j(a_j^-)$, T_j and the service times at queue j are independent of the departure processes from J_1 in the interval (a_j,t_j^-), $D_j'(s)$ is independent of the departure processes from J_1 during $(a_j,\ t_j^-)$. From Corollary 3.2, $D_j'(s)$ is a Poisson process. Thus $D_j'(s)$ and the departure processes from J_1 before time t_j are mutually independent Poisson processes. Thus the arrival processes to J_2 are independent of T_j and we conclude that:

$$P\{M_{j+1}(t_j^-) = k_{j+1},\cdots,\ M_N(t_j^-) = k_N|T_j = t,M_1(t_j^-) = k_1,\cdots,\ M_j(t_j^-) = k_j + 1\}$$
$$= \prod_{i=j+1}^{N}\left(1 - \frac{\alpha_i}{\mu_i}\right)\left(\frac{\alpha_i}{\mu_i}\right)^{k_i} .$$

Combining these conditional probabilities, yields our result. $\square$

Since an exogenous arrival sees a steady state queue length distribution, it is well known that

$$P(T_j \leq t) = 1 - e^{-(\mu_j-\alpha_j)t} . \qquad (3)$$

By unconditioning (2) by (3) over all possible values of t, one gets the usual Jackson steady state results.

4. Main Results. Our main result is the construction of an acyclic Jackson network where a customer's sojourn times at the queues in his path are not independent random variables. It should be clear from the construction that acyclic networks with independent sojourn times are the exception and not the rule.

Consider the network described in 2. Choose $\lambda > 0$, $p \in (0,1)$, $\mu_2 > 0$, $\mu_3 > 0$ so that $p\lambda < \mu_2$ and $\lambda < \mu_3$. From Disney, et al. (1973), the arrival stream to node 2 is Poisson with rate $p\lambda$ and the stream of customers going directly from node 1 to node 3 is Poisson with rate $(1 - p)\lambda$. Thus $E(T_3)$ does not depend on the value of μ_1. We will show that with a suitable choice of μ_1 and t we can make $E(T_3|T_1 = t)$ arbitrarily large. In particular we can make $E(T_3|T_1 = t) > E(T_3)$ so T_1 and T_3 are not independent.

Fix r > 0 and choose

$$\mu_1 > \mathrm{Max}\left\{ \lambda, \left(r + \frac{1}{\mu_2 - p\lambda}\right) \frac{\mu_3\mu_2^2}{(\mu_2-p\lambda)(1-e^{-\mu_2}-\mu_2 e^{-\mu_2})(1-p)} \right\}. \tag{4}$$

The values of λ, μ_1, μ_2, μ_3 and p assure that a steady state exists. Now fix $\varepsilon > 0$ and choose n* large enough so that

$$\sum_{j=0}^{n^*} \frac{(\mu_1 s)^j e^{-\mu_1 s}}{j!} > 1 - \varepsilon, \ \forall s \in (0,1]. \tag{5}$$

Since (5) is bounded below at s = 1, it suffices to choose n* so that

$$\sum_{j=0}^{n^*} \frac{\mu_1^j e^{-\mu_1}}{j!} > 1 - \varepsilon.$$

Given ε and n^*, now choose t large enough so that

$$P(M_1(t_1^+) > n^* | T_1 = t) > 1 - \varepsilon. \tag{6}$$

This can be done since

$$P(M_1(t_1^+) > n^* | T_1 = t) = \sum_{j=n^*+1}^{\infty} \frac{(\lambda t)^j e^{-\lambda t}}{j!} \to 1 \text{ as } t \to \infty.$$

Let

D = number of departures from node 1 in $(t_1, a_3]$.

D^* = number of departures from node 3 in $(t_1, a_3]$.

<u>Lemma 4.1</u> $E(D|T_1 = t) > (1 - \varepsilon)^2 (\mu_2 - p\lambda) \frac{\mu_1}{\mu_2^2} (1-e^{-\mu_2} - \mu_2 e^{-\mu_2})$.

<u>Proof</u>. $E(D|T_1 = t) = \sum_{j=0}^{\infty} E(D|T_1 = t, M_2(t_1^-) = j) P(M_2(t_1^-) = j | T_1 = t)$.

By Theorem 3.3, $P(M_2(t_1^-) = j | T_1 = t) = (\frac{p\lambda}{\mu_2})^j (1 - \frac{p\lambda}{\mu_2})$, so

$$E(D|T_1 = t) = \sum_{j=0}^{\infty} E(D|T_1 = t,\ M_2(t_1^-) = j) (\frac{p\lambda}{\mu_2})^j (1 - \frac{p\lambda}{\mu_2})$$

$$> (1 - \frac{p\lambda}{\mu_2}) E(D|T_1 = t,\ M_2(t_1^-) = 0) = \frac{\mu_2 - p\lambda}{\mu_2} \int_0^{\infty} E(D_1(s)|T_1 = t) \mu_2 e^{-\mu_2 s} ds$$

$$= (\mu_2 - p\lambda) \int_0^{\infty} \sum_{j=0}^{\infty} E(D_1(s)|T_1 = t, M_1(t_1^+) = j) P(M_1(t_1^+) = j | T_1 = t) e^{-\mu_2 s} ds.$$

Since $D_1(s)$ is conditionally independent of T_1 given M_1, we have

$$E(D|T_1 = t) > (\mu_2 - p\lambda) \int_0^\infty \sum_{j=0}^\infty E(D_1(s)|M_1(t_1^+) = j)\ P(M_1(t_1^+) = j|T_1 = t)e^{-\mu_2 s} ds$$

$$> (\mu_2 - p\lambda) \int_0^\infty \sum_{j=n^*+1}^\infty E(D_1(s)|M_1(t_1^+) = j)P(M_1(t_1^+) = j|T_1 = t)e^{-\mu_2 s} ds$$

and since $E(D_1(s)|M_1(t_1^+) = j)$ is an increasing function of j,

$$\geq (\mu_2 - p\lambda) \int_0^\infty \sum_{j=n^*+1}^\infty E(D_1(s)|M_1(t_1^+) = n^*)P(M_1(t_1^+) = j|T_1 = t)e^{-\mu_2 s} ds,$$

and by (6) this is

$$> (1 - \varepsilon)\ (\mu_2 - p\lambda) \int_0^\infty E(D_1(s)|M_1(t_1^+) = n^*)e^{-\mu_2 s} ds$$

$$> (1 - \varepsilon)\ (\mu_2 - p\lambda) \int_0^1 E(D_1(s)|M_1(t_1^+) = n^*)e^{-\mu_2 s} ds.$$

Note that, $E(D_1(s)|M_1(t_1^+) = n^*) = \sum_{j=1}^\infty j\ P(D_1(s) = j|M_1(t_1^+) = n^*)$

$$> \sum_{j=1}^{n^*} j\ P(D_1(s) = j|M_1(t_1^+) = n^*)$$

$$= \sum_{j=1}^{n^*} \frac{(\mu_1 s)^j e^{-\mu_1 s}}{(j-1)!}$$

which by (5) is $> (1 - \varepsilon)\mu_1 s$. Thus,

$$E(D|T_1 = t) > (1 - \varepsilon)^2(\mu_2 - p\lambda) \int_0^1 \mu_1 s e^{-\mu_2 s} ds$$

$$= (1 - \varepsilon)^2(\mu_2 - p\lambda) \frac{\mu_1}{\mu_2^2} (1 - e^{-\mu_2} - \mu_2 e^{-\mu_2}). \quad \square$$

<u>Lemma 4.2</u> $E(D^*|T_1 = t) \leq \mu_3/(\mu_2 - p\lambda)$.

<u>Proof</u>. $E(D^*|T_1 = t) \leq E(\mu_3 T_2|T_1 = t)$.

By Theorem 3.3 and (3) we have

$$E(T_2|T_1 = t) = \int_0^\infty (\mu_2 - p\lambda) s e^{-(\mu_2 - p\lambda)s} ds, \quad \text{so}$$

$$E(D^*|T_1 = t) \leq \mu_3(\mu_2 - p) \int_0^\infty s e^{-(\mu_2 - p\lambda)s} ds = \mu_3/(\mu_2 - p\lambda). \quad \square$$

<u>Theorem 4.3</u> For the chosen values of λ, μ_1, μ_2, μ_3, p and t we have $E(T_3|T_1 = t) \geq r$.

<u>Proof</u>. $E(M_3(a_3^+)|T_1 = t) = (1 - p)E(D|T_1 = t) - E(D^*|T_1 = t)$

$$+ E(M_2(t_1^-)|T_1 = t) + E(M_3(t_1^-)|T_1 = t) + 1$$

$$> (1 - p)\, E(D|T_1 = t) - E(D^*|T_1 = t).$$

Thus by lemmas 4.1 and 4.2 we have

$$E(M_3(a_3^+)|T_1 = t) > \frac{(1-p)(1-\varepsilon)^2(\mu_2-p\lambda)\mu_1}{\mu_2^2} (1 - e^{-\mu_2} - \mu_2 e^{-\mu_2}) - \frac{\mu_3}{\mu_2-p\lambda} \,. \quad \text{Thus}$$

$$E(T_3|T_1 = t) = E(M_3(a_3^+)|T_1 = t)/\mu_3$$

$$> \frac{(1-p)(1-\varepsilon)^2(\mu_2-p\lambda)\mu_1}{\mu_2^2\mu_3} (1 - e^{-\mu_2} - \mu_2 e^{-\mu_2}) - \frac{1}{\mu_2-p\lambda} \,.$$

By (4) we have $E(T_3|T_1 = t) > (1 - \varepsilon)^2(r + \frac{1}{\mu_2 - p\lambda}) - \frac{1}{\mu_2 - p\lambda}$, and since ε was arbitrary, we have $E(T_3|T_1 = t) \geq r$. $\square$

Since r was arbitrary we can choose μ_1 and t so that $E(T_3|T_1 = t) > E(T_3)$. Thus the sojourn times at node 1 and 3 are dependent for customers that go through node 2.

5. Summary. We have shown in section 4 that the sojourn times T_1 and T_3 are not independent for the network considered here. This proves the Mitrani point. Independence, of course, is sufficient to Lemoine's formula (18) as occurs, for example, in tandem queues as studied by Reich (1963). The key difference between the Lemoine result and the Reich result is that Lemoine's network allows customers to overtake each other while Reich does not. This overtaking feature allows events occurring at earlier periods of time (outputs from queue 1 that occur after our man has left queue 1) to influence events at later periods of time (the sojourn time of our man at queue 3). Clearly, if the network does not allow overtaking (e.g. if the graph of the network has at most one path from node i to node j for every i,j) then one can use results such as Reich's to show that sojourn times at nodes are mutually independent and exponentially distributed random variables.

Results related to ours have previously been discussed by Burke (1969 and 1972) and the references therein for tandem queues with multiservers. The problem discussed by Burke and that discussed by us seems to have the same underlying behavior. That is, if overtaking can occur in acyclic Jackson networks then, in general, sojourn times will not be independent.

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