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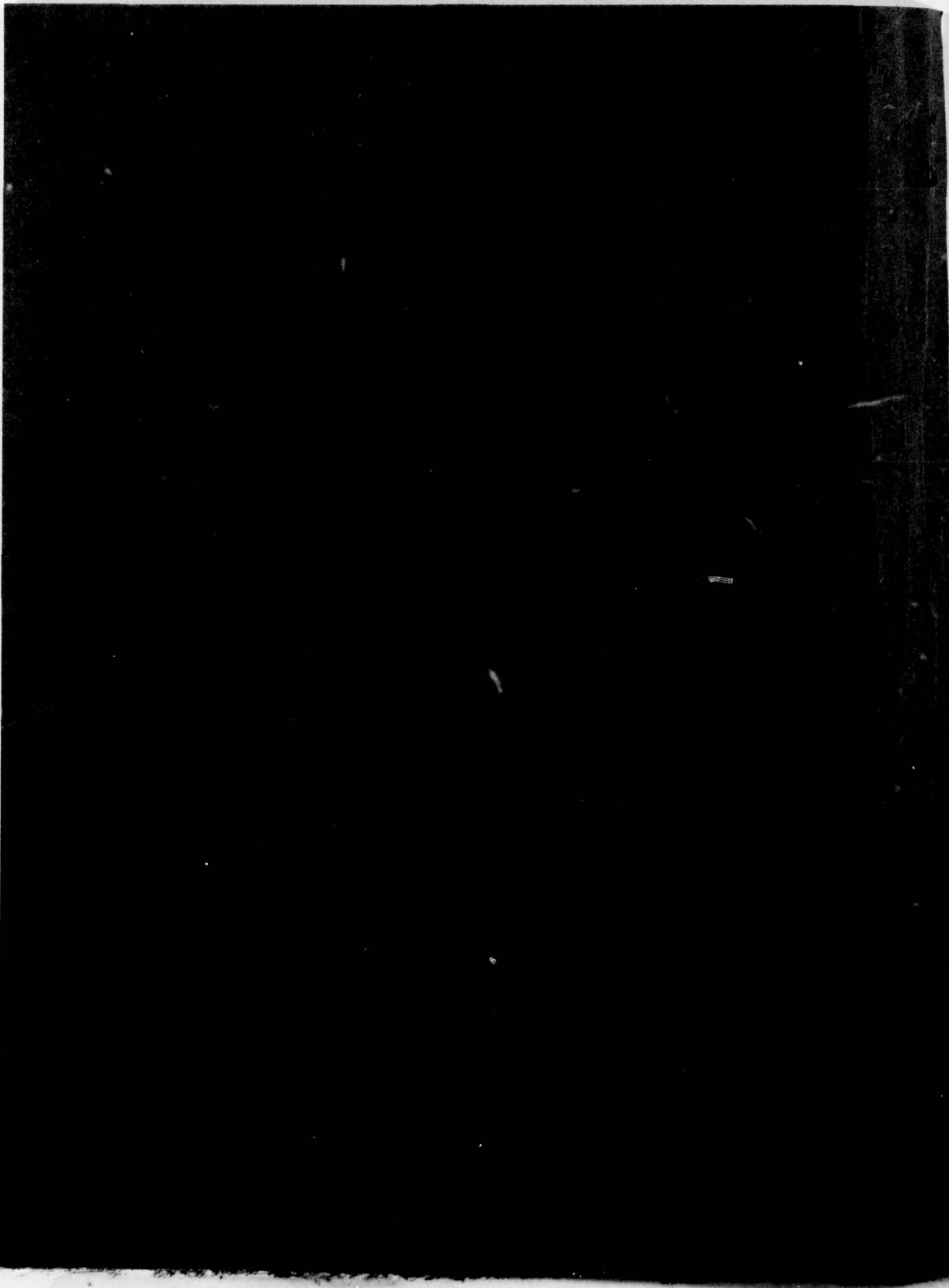
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The magnetic induction is calculated for two configurations consisting of: (1) an infinitesimally thin prolate spheroidal current band carrying a stationary current surrounding a ferromagnetic prolate spheroidal shell, and (2) an infinitesimally thin band internal to a ferromagnetic prolate spheroidal shell. The ferromagnetic body is assumed to be linear and homogeneous. The reduction of the solutions to that of a prolate spheroidal

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current band in free space is shown when the permeability of the ferro-
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TABLE OF CONTENTS

iii

## LIST OF FIGURES

iv

## LIST OF SYMBOLS

| | |
|---|---|
| $\overline{A}$ | Vector potential function |
| $A_{\psi}$ | Psi ($\psi$) component of $\overline{A}$ |
| $A_{\psi I}$ | Psi component of $\overline{A}$ in region I |
| $A_{\psi II}$ | Psi component of $\overline{A}$ in region II |
| $A_{\psi III}$ | Psi component of $\overline{A}$ in region III |
| $A_{\psi IV}$ | Psi component of $\overline{A}$ in region IV |
| $\overline{B}$ | Magnetic flux density or magnetic induction |
| $\overline{B}_1$ | Magnetic flux density in medium 1 |
| $\overline{B}_2$ | Magnetic flux density in medium 2 |
| $B_{\eta}$ | Eta component of the magnetic flux density |
| $B_{\theta}$ | Theta component of the magnetic flux density |
| $B_{\eta I}$ | Eta component of $\overline{B}$ in region I |
| $B_{\eta II}$ | Eta component of $\overline{B}$ in region II |
| $B_{\eta III}$ | Eta component of $\overline{B}$ in region III |
| $B_{\eta IV}$ | Eta component of $\overline{B}$ in region IV |
| $B_{n1}$ | Normal component of $\overline{B}$ in medium 1 |
| $B_{n2}$ | Normal component of $\overline{B}$ in medium 2 |
| $\overline{H}$ | Magnetic field intensity |
| $\overline{H}_1$ | Magnetic field intensity in medium 1 |
| $\overline{H}_2$ | Magnetic field intensity in medium 2 |
| $H_{t1}$ | Tangential component of $\overline{H}$ in medium 1 |
| $H_{t2}$ | Tangential component of $\overline{H}$ in medium 2 |

| | |
|---|---|
| $\overline{J}$ | Electric current density |
| $J_\eta$ | Eta component of J |
| $J_\theta$ | Theta component of J |
| $J_\psi$ | Psi component of J |
| $\overline{J}_s$ | Surface current density |
| $J$ | Magnitude of $\overline{J}_s$ |
| $\chi_m$ | Magnetic susceptibility |
| $\mu$ | Magnetic permeability |
| $\mu_r$ | Relative magnetic permeability |
| $\mu_o$ | Permeability of free space |
| $\mu_1$ | Permeability |
| $\mu_2$ | Permeability |
| $\overline{n}_{12}$ | Unit vector normal to interface; directed from medium 1 into medium 2 |
| $x, y, z$ | Rectangular coordinates |
| $\eta, \theta, \psi$ | Prolate spheroidal coordinates |
| $\overline{e}_\eta$ | Unit normal vector in eta direction |
| $\overline{e}_\theta$ | Unit normal vector in theta direction |
| $\overline{e}_\psi$ | Unit normal vector in azimuthal direction |
| $\overline{\nabla} \cdot$ | Divergence operator |
| $\overline{\nabla} \times$ | Curl operator |
| $\nabla^2$ | Scalar Laplacian operation |
| ✿ | Vector Laplacian operator |
| $p$ | Integer from one to infinity |

| | |
|---|---|
| $p_p^m(\cos\theta)$ | Associated Legendre function of the first kind |
| $Q_p^m(\cos\theta)$ | Associated Legendre function of the second kind |
| $\maltese\,\overline{A}_\psi$ | Psi vector component of the vector Laplacian of $\overline{A}$ in prolate spheroidal coordinates |
| $(\overline{\nabla}\mathrm{x}\overline{A})_\eta$ | Eta component of the curl of A |
| $(\overline{\nabla}\mathrm{x}\overline{A})_\theta$ | Theta component of the curl of A |
| $\theta_1,\theta_2$ | Angles describing the limits of the current band |
| $\eta_1,\eta_2,\eta_3$ | Constants |
| $\left.\begin{array}{l}A_p,B_p,C_p\\[6pt]D_p,E_p,F_p\end{array}\right\}$ | Constants in the general solution of $\maltese\,\overline{A}$ for current band outside the body where p=1 to $\infty$ |
| $\left.\begin{array}{l}H_p,I_p,K_p\\[6pt]L_p,M_p,N_p\end{array}\right\}$ | Constants in the general solution of $\maltese\,\overline{A}$ for the current band inside the body where p=1 to $\infty$ |
| $\xi$ | Variable equal to $\cosh\eta$ |
| $\nu$ | Variable equal to $\cos\theta$ |
| $e_1,e_2,e_3$ | Metric coefficients for a prolate spheroidal coordinate system |
| $P_p^\Delta$ | Variable use for simplification |
| $Q_p^\Delta$ | Variable use for simplification |
| $\left.\begin{array}{l}C_1,\ C_2\\[6pt]K_1,\ K_2\\[6pt]A',\ B'\\[6pt]A,\ B\end{array}\right\}$ | Constants |

$G_p$      Constants in current expansion where p=1 to $\infty$

$\nabla^2 \overline{A}_\psi$      Psi vector component of the vector Laplacian of $\overline{A}$ in prolate spheroidal coordinates

$$\left( = \frac{(sinh^2\eta + sin^2\theta)^{-1/2}}{a^2 (sinh\eta \; sin\theta)} \; \overline{e}_\psi \left( \frac{\partial}{\partial\eta} \frac{1}{sinh\eta} \frac{\partial}{\partial\eta} (sinh\eta \; A_\psi) \right. \right.$$

$$\left. \left. + \frac{\partial}{\partial\theta} \frac{1}{sin\theta} \frac{\partial}{\partial\theta} (sin\theta \; A_\psi) \right) \right)$$

$\nabla^2 A_\psi$      Psi component of the vector Laplacian of $\overline{A}$ in prolate spheroidal coordinates

$$\left( = \frac{\left( sinh^2\eta + sin^2\theta \right)^{-1/2}}{a^2 (sinh\eta \; sin\theta)} \left( \frac{\partial}{\partial\eta} \frac{1}{sinh\eta} \frac{\partial}{\partial\eta} (sinh\eta \; A_\psi) \right. \right.$$

$$\left. \left. + \frac{\partial}{\partial\theta} \frac{1}{sin\theta} \frac{\partial}{\partial\theta} (sin\theta \; A_\psi) \right) \right)$$

## EXECUTIVE SUMMARY

## THE MAGNETIC INDUCTION OF THE SYSTEM CONSISTING OF A COIL AND A FERROMAGNETIC PROLATE SPHEROIDAL SHELL

OBJECTIVE

The objective of this work was to derive solutions to static ferro-
magnetic problems that included both current-carrying coils and linear
ferromagnetic bodies. The solutions are intended for comparison with
solutions to ferromagnetic problems obtained by various numerical tech-
niques such as the finite difference method, the finite element method,
and the integral equation iterative solution method.

APPROACH

After deriving the governing differential equation from Maxwell's .
equations for classical magnetostatic field theory, the method of separa-
tion of variables was employed to obtain the problem solution.

RESULTS

The magnetic induction was calculated for two geometries (configura-
tions) of a ferromagnetic prolate spheroidal shell and a current-carrying
conductor. The first case was for an infinitesimally thin current band
carrying a stationary current and surrounding the spheroidal shell. The
second case was for a spheroidal shell surrounding an infinitesimally thin
current band. The ferromagnetic bodies were assumed to be linear and
homogeneous. The reduction of the solutions to that of a current band in
free space is shown when the permeability of the ferromagnetic spheroidal
shell is allowed to approach that of free space.

RECOMMENDATIONS

It is recommended that the derived solutions be programmed on a digi-
tal computer for direct comparison of these results to those obtained by
various numerical methods. There are plans to implement these recommenda-
tions during the fiscal years 1979 and 1980.

## ABSTRACT

The magnetic induction is calculated for two configurations consisting of: (1) an infinitesimally thin prolate spheroidal current band carrying a stationary current surrounding a ferromagnetic prolate spheroidal shell, and (2) an infinitesimally thin band internal to a ferromagnetic prolate spheroidal shell. The ferromagnetic body is assumed to be linear and homogeneous. The reduction of the solutions to that of a prolate spheroidal current band in free space is shown when the permeability of the ferromagnetic prolate spheroidal shell is allowed to approach that of free space.

## ADMINISTRATIVE INFORMATION

## INTRODUCTION

As noted in Brown and Baker's report,[1*] exact analytical solutions of Maxwell's equations using classical formulations have been limited to body shapes and inhomogeneities that conform to a few separable coordinate systems. Modern digital computers with large computational and storage capabilities permit many electromagnetic field problems to be solved by using a numerical solution to the governing differential or integral equations under a suitable choice of boundary conditions. The numerical solutions of Maxwell's equations, when used with a complete description of the electric and magnetic sources and the constitutive laws of the media, can be used to describe completely the electric and magnetic fields produced by the source, including nonsymmetric geometries, nonsymmetric source distributions, and spatially varying media parameters.

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*A list of references is given on page 71.

The motivation for this work arose out of the need for solutions to static ferromagnetic problems that could be used for comparison with numerical methods.

## PROLATE SPHEROIDAL COORDINATE SYSTEM

The prolate spheroidal coordinate system can be formed by rotating the two-dimensional elliptic coordinate system, whose traces in a plane are confocal ellipses and hyperbolas, about the major axis of the ellipses.[2,3]

Flammer[3] notes that it is customary to make the z-axis the axis of revolution. Figure 1 depicts the three-dimensional prolate spheroidal coordinate system. In this case the coordinate surfaces are: prolate spheroids for $\eta$ = constant; hyperboloids of two sheets for $\theta$ = constant; meridian planes for $\psi$ = constant. The prolate spheroidal coordinates shown on Figure 1 are related to rectangular coordinates by the following transformation equations:

$$x = a \, \sinh\eta \, \sin\theta \, \cos\psi \qquad (1a)$$

$$y = a \, \sinh\eta \, \sin\theta \, \sin\psi \qquad (1b)$$

$$z = a \, \cosh\eta \, \cos\theta \qquad (1c)$$

where

$$o \leq \eta < \infty$$

$$o \leq \theta \leq \pi$$

$$o \leq \psi < 2\pi$$

We have denoted the interfocal distance by 2a and the prolate spheroidal coordinates by $(\eta, \theta, \psi,)$.

PROLATE SPHEROIDS, η = CONST
HYPERBOLOIDS, θ = CONST
MERIDIAN PLANES, ψ = CONST
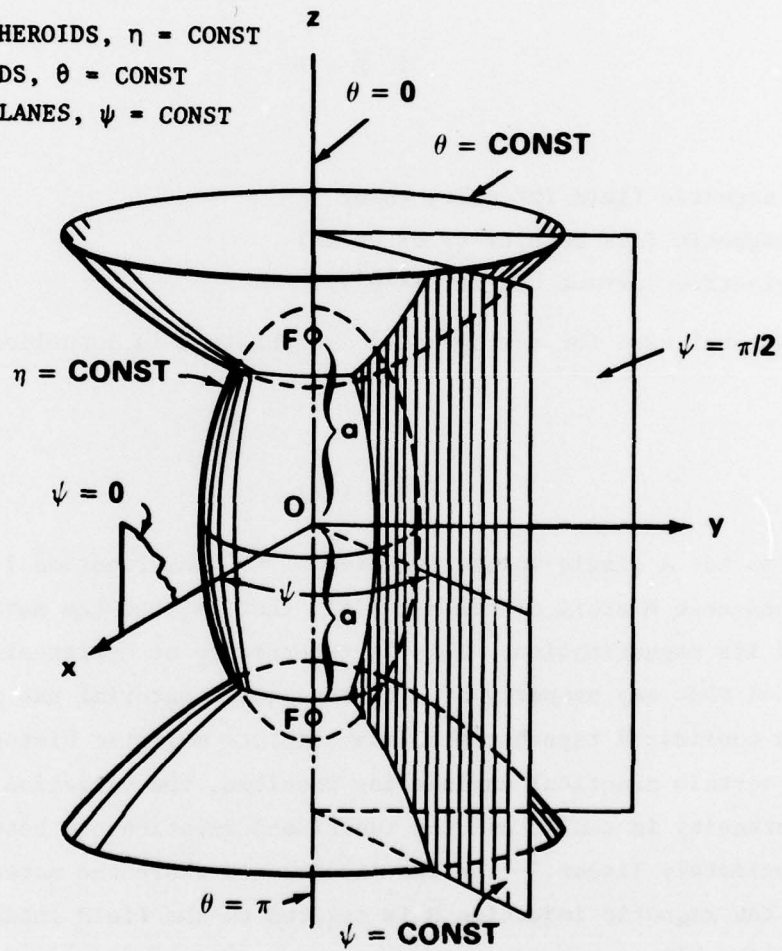
Figure 1 - Prolate Spheroidal Coordinate System

3

## BASIC EQUATIONS

We can start with Maxwell's equations for classical magnetostatic field problems

$$\overline{\nabla} \times \overline{H} = \overline{J} \qquad (2a)*$$

$$\overline{\nabla} \cdot \overline{B} = 0 \qquad (2b)$$

where

$\overline{H}$ magnetic field intensity (A/m)

$\overline{B}$ magnetic flux density (T or Wb/m$^2$)

$\overline{J}$ electric current density (A/m$^2$) .

In the general case for ferromagnetic materials $\overline{B}$ is a nonlinear function of $\overline{H}$

$$\overline{B} = f(\overline{H}) \qquad (3)$$

where $\overline{B}$ is not a single valued function of $\overline{H}$. The function $f(\overline{H})$ depends on the magnetic history of the material, that is, how the material attained its magnetization. This is referred to as hysteresis. It is also noted that any property of a ferromagnetic material has meaning only if it is considered together with its complete magnetic history.

In certain practical engineering problems, the variation in the magnetic intensity is small, and the functional relationship between $\overline{B}$ and $\overline{H}$ is approximately linear.[1] For the linear case where the material is isotropic, the magnetic induction $\overline{B}$ is related to the field intensity $\overline{H}$ by the relationship

---

*The del operator $\nabla$ is defined with respect to the rectangular coordinate system and is strictly valid in a rectangular coordinate system only. Very often $\overline{\nabla}\times$ and $\overline{\nabla}\cdot$ are used as equivalent symbols for curl and divergence generally. This use is followed in this report.

4

$$\overline{B} = \mu_o \ (\chi_m + 1) \ \overline{H} = \mu_o \mu_r \overline{H} = \mu \overline{H} \tag{4}$$

where

$\chi_m$ magnetic susceptibility (dimensionless)

$\mu$ magnetic permeability (henry/meter)

$\left(\chi_m + 1\right) =$ $\mu_r$ relative permeability (dimensionless)

$\mu_o$ free space permeability ($4\pi \times 10^{-7}$ henry/meter).

This report assumes that the ferromagnetic body has isotropic and linear material properties. The divergenceless nature of the magnetic flux density in conjunction with the fact that the divergence of the curl of any vector function is zero allows the introduction of the magnetic vector potential field $(\overline{A})$

$$\overline{B} = \overline{\nabla} \times \overline{A} \tag{5}$$

where $\overline{A}$ is the magnetostatic vector potential function in weber/meter. The substitution of Equation (5) into Equation (2a) gives the fundamental equation of the vector potential of the magnetostatic field.

$$\frac{1}{\mu} \ \overline{\nabla} \times (\overline{\nabla} \times \overline{A}) - (\overline{\nabla} \times \overline{A}) \times \overline{\nabla} \frac{1}{\mu} = \overline{J} \tag{6}$$

For homogeneous materials, as assumed in this report, the magnetic permeability is spatially invariant. Hence,

$$\overline{\nabla} \frac{1}{\mu} = 0 \tag{7}$$

5

and Equation (6) reduces to

$$\overline{\nabla} \times (\overline{\nabla} \times \overline{A}) = \mu \overline{J} \qquad (8)$$

Using the vector identity

$$\overline{\nabla} \times \overline{\nabla} \times \overline{A} = \overline{\nabla}(\overline{\nabla} \cdot \overline{A}) - \bigstar \overline{A} \qquad (9)*$$

Equation 8 becomes

$$\overline{\nabla}(\overline{\nabla} \cdot \overline{A}) - \bigstar \overline{A} = \mu \overline{J} \qquad (10)$$

The magnetostatic vector potential is characterized by the important property that its divergence can be conveniently chosen to be zero.

$$\overline{\nabla} \cdot \overline{A} = 0 \qquad (11)$$

Equation (10) reduces to the vector Poisson's differential equation.

$$\bigstar \overline{A} = - \mu \overline{J} \qquad (12)$$

This is the governing equation for our calculations.

The general boundary conditions to be satisfied at the interfaces of stationary dissimilar media may be derived from the limiting integral forms of Maxwell's equations and are given by

---

*The vector Laplacian operator is designated by $\bigstar$.

6

$$\overline{n}_{12} \cdot (\overline{B}_2 - \overline{B}_1) = 0 \text{ or } B_{n1} = B_{n2} \tag{13a}$$

$$\overline{n}_{12} \times (\overline{H}_2 - \overline{H}_1) = \overline{J}_s \text{ or } H_{t2} - H_{t1} = J_s \tag{13b}$$

where the subscripts 1 and 2 indicate the media under consideration, and $\overline{n}_{12}$ denotes the unit normal vector to the interface and is directed from medium 1 into medium 2. In the case where the materials are linear and isotropic equations (13a) and (13b) become

$$\overline{n}_{12} \cdot (\mu_2 \overline{H}_2 - \mu_1 \overline{H}_1) = 0 \tag{13c}$$

$$\overline{n}_{12} \times \left( \frac{\overline{B}_2}{\mu_2} - \frac{\overline{B}_1}{\mu_1} \right) = \overline{J}_s \tag{13d}$$

$\overline{J}_s$ is a true surface current density that may exist at the interface. At an interface where $\overline{J}_s$ is 0, Equations (13b) and (13d) need to be modified accordingly.

## THIN COIL SURROUNDING A FERROMAGNETIC
## PROLATE SPHEROIDAL SHELL

GENERAL SOLUTION

We now proceed to solve the boundary value problem of a ferromagnetic prolate spheroidal shell of homogeneous permeability $\mu_2$ surrounded by an infinitesimally thin prolate spheroidal current band of constant current density $\overline{J}$. The geometry of the problem suggests that a prolate spheroidal coordinate system as shown in Figure 1 be used in the solution. Figure 2, a cross section of the problem geometry, identifies the four regions of interest. The boundaries of the prolate spheroidal shell are
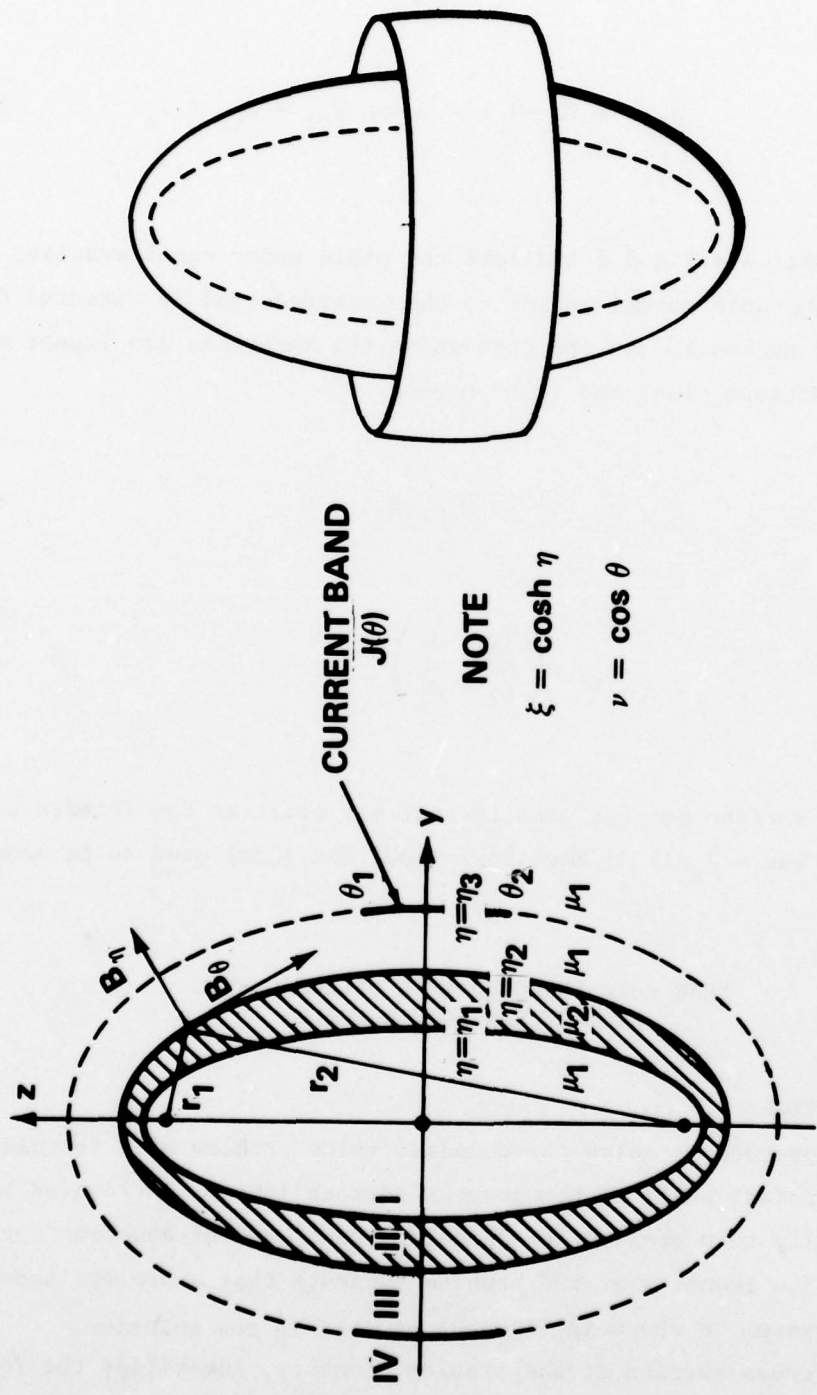
Figure 2 - Cross Section of Ferromagnetic Spheroidal Shell
Surrounded by Infinitesimally Thin Current Band

determined by $\eta = \eta_1$ = constant and $\eta = \eta_2$ = constant. The dc current lies in the boundary $\eta = \eta_3$ constant. Regions I, III, and IV have a permeability equal to that of free space $\mu_o$, which for convenience will be labelled $\mu_1$. Ampere's law states

$$\overline{\nabla} \times \overline{H} = \overline{J} \tag{14}$$

and since $\overline{\nabla} \cdot \overline{B} = 0$, the induction $\overline{B}$ must be the curl of some vector field $\overline{A}$. The governing differential equation for $\overline{A}$ when homogeneous and linear materials are considered is from Equation (12)

$$\bigstar \overline{A} = -\mu \overline{J} \tag{15}$$

The general expression in prolate spheroidal coordinates for a current density is

$$\overline{J} = J_\eta \overline{e}_\eta + J_\theta \overline{e}_\theta + J_\psi \overline{e}_\psi \tag{16}$$

In the problem presented herein, the current density has only a psi ($\psi$) component $\left[ J_\psi(\theta)\overline{e}_\psi \right]$ which means that the vector potential has only a psi component $A_\psi \overline{e}_\psi$. The vector potential $\overline{A} = A_\psi \overline{e}_\psi$ is a function of the prolate spheroidal coordinates $\eta$, $\theta$ $\left[ \text{i.e. } A_\psi = A_\psi(\eta,\theta) \right]$. The constant current density, which lies on the boundary between regions III and IV, can be expressed by the function

$$\overline{J} = \begin{cases} 0 & \text{, if } \theta < \theta_1 \text{ or } \theta > \theta_2 \\ J_\psi(\theta)\overline{e}_\psi, & \text{if } \theta_1 \leq \theta \leq \theta_2 \end{cases} \tag{17}$$

9

where $J_\psi(\theta)$ is equal to a constant J along $\eta = \eta_3$ for $\theta_1 \leq \theta \leq \theta_2$ .

Therefore, Equation (15) has only an azimuthal component and can be expressed as

$$\bigstar\overline{A}_\psi = \bigstar\overline{A}_\psi(\eta,\theta) = 0 \quad \begin{pmatrix} \text{in regions I} \\ \text{through IV} \end{pmatrix} \tag{18}$$

When the psi component of vector Laplacian $\bigstar A_\psi$ is expanded in prolate spheroidal coordinates, Equation (18) can be expressed as (see Appendix A)

$$\frac{\partial}{\partial \eta}\left[\frac{1}{\sinh\eta}\frac{\partial(\sinh\eta\ A_\psi)}{\partial\eta}\right] + \frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial(\sin\theta\ A_\psi)}{\partial\theta}\right] = 0 \tag{19}$$

(in regions I-IV)

Applying the method of separation of variables, let us assume that $A_\psi$ can be expressed as the product of two functions

$$A_\psi = H(\cosh\eta)\ G(\cos\theta) \tag{20}$$

where $H(\cosh\eta)$ is a function of $\cosh\eta$ only and $G(\cos\theta)$ is a function of $\cos\theta$ only. Substituting this form of the component of the vector potential $\overline{A}$ into Equation (19), we have after separation of variables

$$\frac{d^2H}{d\eta^2} + \coth\eta\ \frac{dH}{d\eta} - \left(p(p+1) + \frac{1}{\sinh^2\eta}\right)H = 0 \tag{21a}$$

$$\frac{d^2G}{d\theta^2} + \cot\theta\ \frac{dG}{d\theta} + \left(p(p+1) - \frac{1}{\sin^2\theta}\right)G = 0 \tag{21b}$$

10

where the separation constant is $p(p+1)$ and $p$ is an integer from one to infinity. It is well known that differential equations of the form

$$\frac{d^2H'}{d\eta^2} + \coth\eta \; \frac{dH'}{d\eta} - \left( p(p+1) + \frac{m^2}{\sinh^2\eta} \right)H' = 0 \tag{22a}$$

have the general solution of the form

$$H' = C_1 P_p^m (\cosh\eta) + C_2 Q_p^m (\cosh\eta) \tag{22b}$$

where $C_1$ and $C_2$ are constants, and it is known that a differential equation of the form

$$\frac{d^2G'}{d\theta^2} + \cot\theta \; \frac{dG'}{d\theta} + \left[ p(p+1) - \frac{m^2}{\sin^2\theta} \right]G' = 0 \tag{23a}$$

has the general solution of the type

$$G' = C_3 P_p^m (\cos\theta) + C_4 Q_p^m (\cos\theta) \tag{23b}$$

where $C_3$ and $C_4$ are constants. $P_p^m$ and $Q_p^m$ are the associated Legrendre functions of the first and second kind, respectively. Comparison of Equations (21), (22), and (23) shows that in Equations (22) and (23), $m^2$ is equal to 1. This requires that m always equal unity. The solutions of Equations (21a) and (21b) are expressed as

$$H(\cosh\eta) = A \; P_p^1 (\cosh\eta) + B \; Q_p^1 (\cosh\eta) \tag{24a}$$

11

$$G(\cos\theta) = A' \, P_p^1 (\cos\theta) + B' \, Q_p^1 (\cos\theta) \tag{24b}$$

The general solution of Equation (19) may be formed from the product of solutions in Equations (24a) and (24b) which yield

$$A_\psi = H \, (\cosh\eta) \, G \, (\cos\theta) = \sum_{p=1}^{\infty} H_p \, (\cosh\eta) \, G_p \, (\cos\theta) \tag{25}$$

$$A_\psi = \sum_{p=1}^{\infty} \left[ A \, P_p^1 \, (\cosh\eta) + B \, Q_p^1 \, (\cosh\eta) \right] \\ \times \left[ A' \, P_p^1 \, (\cos\theta) + B' \, Q_p^1 \, (\cos\theta) \right] \tag{26}$$

For the prolate spheroidal system, the associated Legendre functions of the second kind are infinite at $\cos\theta = \pm 1$ and as such cannot be included in a general solution for a given region which includes $\theta = 0$ or $\theta = \pi$. Therefore, in our case the constant $B'$ is set equal to zero. Equation (26) reduces to

$$A_\psi = \sum_{p=1}^{\infty} \left[ K_1 P_p^1 \, (\cosh\eta) + K_2 Q_p^1 \, (\cosh\eta) \right] P_p^1 \, (\cos\theta) \tag{27}$$

where $K_1$ and $K_2$ are constants ($K_1 = AA'$, $K_2 = AB'$). When the substitutions $\xi = \cosh\eta$ and $\nu = \cos\theta$ are made in Equation (27), $A_\psi$ can be expressed as

$$A_\psi = \sum_{p=1}^{\infty} \left[ K_1 P_p^1 (\xi) + K_2 Q_p^1 (\xi) \right] P_p^1 (\nu) \tag{28}$$

This is the general form of the psi component of the vector potential that will be used to determine the potentials $A_\psi$ in each region.

12

## BOUNDARY CONDITIONS

The form of the component of the vector potential $A_\psi$ in regions (I-IV) is determined from Equation (28). These magnetostatic vector potentials in regions I-IV are:

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[ A_p P_p^1(\xi) \right] P_p^1(\nu)$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[ B_p P_p^1(\xi) + C_p Q_p^1(\xi) \right] P_p^1(\nu)$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[ D_p P_p^1(\xi) + E_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{29}$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[ F_p Q_p^1(\xi) \right] P_p^1(\nu)$$

Because the potential must be finite in each of the regions I, II, and III and approach zero as $\xi \to \infty$ in region IV, the following constants were set equal to zero.

a. For $A_{\psi I}$ the constant associated with $Q_p^1(\xi)\ P_p^1(\nu)$ was set equal to zero because

$$Q_p^1(\xi) \to \infty \text{ at } \xi = +1$$

b. For $A_{\psi IV}$ the constant associated with $P_p^1(\xi)\ P_p^1(\nu)$ was set equal to zero because $P_p^1(\xi) \to \infty$ as $\xi \to \infty$.

$$\text{(We note } Q_p^1(\xi) \to 0 \text{ as } \xi \to \infty)$$

13

The constants $A_p$, $B_p$, $C_p$, $D_p$, $E_p$, and $F_p$ are to be determined from the boundary conditions. At each interface, the basic laws of magnetostatics (Equations (2a) and (2b)) reduce to boundary conditions on $\overline{B}$ and $\overline{H}$ that can be used to evaluate these six constants. The normal component of $\overline{B}$ across each boundary must be continuous, i.e., $(\overline{B}_2 - \overline{B}_1) \cdot \overline{n}_{12} = 0$ where the quantity $\overline{n}_{12}$ is the unit outward normal to the surface. This provides the following boundary conditions which must be satisfied by the solutions given in Equation (29) for each region.

$$B_{\eta I} = B_{\eta II} \text{ at } \eta = \eta_1 \tag{30a}$$

$$B_{\eta II} = B_{\eta III} \text{ at } \eta = \eta_3 \tag{30b}$$

$$B_{\eta III} = B_{\eta IV} \text{ at } \eta = \eta_3 \tag{30c}$$

The eta or normal component of the magnetic field ($B_\eta$) is expressed in terms of the vector potential as

$$B_\eta = (\overline{\nabla} x \overline{A}_\psi)_\eta = \frac{1}{e_2 e_3} \frac{\partial (e_3 A_\psi)}{\partial \theta}$$

$$\tag{31}$$

$$= - \frac{1}{a\left(\xi^2 - \nu^2\right)^{1/2}} \frac{\partial}{\partial \nu} \left[ (1-\nu^2)^{1/2} A_\psi \right]$$

14

where

$$\overline{B} = \overline{\nabla} \times \overline{A}$$

$$= \frac{1}{a(\sinh^2\eta + \sin^2\theta)(\sinh\eta \sin\theta)} \times$$

$$\begin{vmatrix} \overline{e}_\eta(\sinh^2\eta + \sin^2\theta)^{1/2} & \overline{e}_\theta(\sinh^2\eta + \sin^2\theta)^{1/2} & \overline{e}_\psi \sinh\eta \sin\theta \\ \frac{\partial}{\partial\eta} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\psi} \\ 0 & 0 & A_\psi \sinh\eta \sin\theta \end{vmatrix}$$

and

$$\xi = \cosh\eta, \quad e_1 = e_2 = a(\sinh^2\eta + \sin^2\theta)^{1/2} = a(\xi^2 - \nu^2)^{1/2}$$

$$\nu = \cos\theta, \quad e_3 = a \sinh\eta \sin\theta$$

However, since the vector potentials in each region are functions of $P_p^1(\nu)$, we can simplify Equation (30) to constraints on $A_\psi$ at the interfaces:

$$A_{\psi I} = A_{\psi II} \text{ at } \eta = \eta_1 \tag{32a}$$

$$A_{\psi II} = A_{\psi III} \text{ at } \eta = \eta_2 \tag{32b}$$

$$A_{\psi III} = A_{\psi IV} \text{ at } \eta = \eta_3 \tag{32c}$$

15

The second set of boundary conditions states that the theta or tangential component of $\overline{H}$ across each boundary must satisfy the relationship

$$\overline{n}_{12} \times (\overline{H}_2 - \overline{H}_1) = \overline{J}_s \tag{33}$$

where $J_s$ (which equals $J_\psi(\theta)\overline{e}_\psi$) is the real surface current density in the limit of vanishing width between the two regions. Using the relationship $\overline{B} = \mu\overline{H}$, Equation (33) can be expressed as

$$\frac{B_{\theta 2}}{\mu_2} - \frac{B_{\theta 1}}{\mu_1} = J_\psi(\theta) \tag{34}$$

Referring to the curl in Equation (31), we can write $B_\theta$ in the form

$$B_\theta = (\overline{\nabla} \times \overline{A}_\psi)_\theta \tag{35}$$

$$= \frac{-1}{e_1 e_3} \frac{\partial (e_3 A_\psi)}{\partial \eta} = -\frac{1}{a(\xi^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[ (\xi^2 - 1)^{1/2} A_\psi \right]$$

From Equations (34) and (35) the tangential components of $\overline{B}$ in regions I-IV must satisfy the relationships:

$$\left( \frac{1}{\mu_2} \right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[ (\xi^2 - 1)^{1/2} A_{\psi II} \right] \Bigg|_{\xi = \xi_1} =$$

$$\left( \frac{1}{\mu_1} \right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[ (\xi^2 - 1)^{1/2} A_{\psi I} \right] \Bigg|_{\xi = \xi_1} \tag{36a}$$

16

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi III}\right]\Bigg|_{\xi=\xi_2} =$$

(36b)

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi II}\right]\Bigg|_{\xi=\xi_2}$$

$$-\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi IV}\right]\Bigg|_{\xi=\xi_3}$$

$$+\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi III}\right]\Bigg|_{\xi=\xi_3} =$$

(36c)

$$\sum_p J_p(\theta) = \sum_p \frac{KG_p P_p^1(\nu)}{a\left(\xi_3^2-\nu^2\right)^{1/2}}$$

It is understood that the summation runs from $p = 1$ to $p = \infty$.
The general expressions for the potentials in each region (Equation (29))
are then substituted into the boundary conditions (Equations (32) and (36))
are solved for the six constants ($A_p$, $B_p$, $C_p$, $D_p$, $E_p$, and $F_p$). Since there
are six equations with six unknowns, the potential in each region can be
specified. The six boundary value equations are presented below. The
index p in the summation sign has <u>both</u> even and odd values and takes on

17

values from 1 to $\infty$. It is noted at this point that the current density $J_\psi(\theta)$ must be expanded into a set of associated Legendre functions in order to evaluate the constants in the vector potential (Equation (29)). The detailed expansion is presented in Appendix B. The six expressions for the boundary conditions are:

$$A_p P_p^1(\xi_1) P_p^1(\nu) = \left[ B_p P_p^1(\xi_1) + C_p Q_p^1(\xi_1) \right] P_p^1(\nu) \tag{37a}$$

$$\left(\frac{1}{\mu_2}\right) \frac{1}{a\left(\xi_1^2 - \nu^2\right)^{1/2}} \frac{\partial}{\partial \xi} \left[ \left(\xi^2 - 1\right)^{1/2} \left( B_p P_p^1(\xi) + C_p Q_p^1(\xi) \right) P_p^1(\nu) \right] \Bigg|_{\xi = \xi_1} =$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_1^2 - \nu^2\right)^{1/2}} \frac{\partial}{\partial \xi} \left[ \left(\xi^2 - 1\right)^{1/2} \left( A_p P_p^1(\xi) \right) P_p^1(\nu) \right] \Bigg|_{\xi = \xi_1} \tag{37b}$$

$$\left[ B_p P_p^1(\xi_2) + C_p Q_p^1(\xi_2) \right] P_p^1(\nu) = \left[ D_p P_p^1(\xi_2) + E_p Q_p^1(\xi_2) \right] P_p^1(\nu) \tag{37c}$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_2^2 - \nu^2\right)^{1/2}} \frac{\partial}{\partial \xi} \left[ \left(\xi^2 - 1\right)^{1/2} \left( D_p P_p^1(\xi) + E_p Q_p^1(\xi) \right) P_p^1(\nu) \right] \Bigg|_{\xi = \xi_2} =$$

$$\left(\frac{1}{\mu_2}\right) \frac{1}{a\left(\xi_2^2 - \nu^2\right)^{1/2}} \frac{\partial}{\partial \xi} \left[ \left(\xi^2 - 1\right)^{1/2} \left( B_p P_p^1(\xi) + C_p Q_p^1(\xi) \right) P_p^1(\nu) \right] \Bigg|_{\xi = \xi_2} \tag{37d}$$

18

$$\left[ D_p P_p^1(\xi_3) + E_p Q_p^1(\xi_3) \right] P_p^1(\nu) = F_p Q_p^1(\xi_3) P_p^1(\nu) \tag{37e}$$

$$- \left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_3^2 - \nu^2\right)^{1/2}} \frac{\partial}{\partial \xi} \left[ \left(\xi^2 - 1\right)^{1/2} \left( F_p Q_p^1(\xi) \right) P_p^1(\nu) \right] \Bigg|_{\xi = \xi_3} +$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_3^2 - \nu^2\right)^{1/2}} \frac{\partial}{\partial \xi} \left[ \left(\xi^2 - 1\right)^{1/2} \left( D_p P_p^1(\xi) + E_p Q_p^1(\xi) \right) P_p^1(\nu) \right] \Bigg|_{\xi = \xi_3} = \tag{37f}$$

$$J_p(\theta) = \frac{KG_p P_p^1(\nu)}{a\left(\xi_3^2 - \nu^2\right)^{1/2}}$$

If we make the following substitution

$$P_p^\Delta(\xi) = \frac{d}{d\xi} \left[ \left(\xi^2 - 1\right)^{1/2} P_p^1(\xi) \right] \tag{38}$$

$$Q_p^\Delta(\xi) = \frac{d}{d\xi} \left[ \left(\xi^2 - 1\right)^{1/2} Q_p^1(\xi) \right] \tag{39}$$

and perform simple algebric manipulations, the six boundary conditions can be simplified to:

$$A_p P_p^1(\xi_1) = B_p P_p^1(\xi_1) + C_p Q_p^1(\xi_1) \tag{40a}$$

19

$$\left(\frac{1}{\mu_2}\right)\left[B_\rho P_p^\Delta(\xi_1) + C_p Q_p^\Delta(\xi_1)\right] = \left(\frac{1}{\mu_1}\right) A_p P_p^\Delta(\xi_1) \tag{40b}$$

$$B_p P_{\bar{p}}^1(\xi_2) + C_p Q_p^1(\xi_2) = D_p P_p^1(\xi_2) + E_p Q_p^1(\xi_2) \tag{40c}$$

$$\left(\frac{1}{\mu_1}\right)\left[D_p P_p^\Delta(\xi_2) + E_p Q_p^\Delta(\xi_2)\right] = \left(\frac{1}{\mu_2}\right)\left[B_p P_p^\Delta(\xi_2) + C_p Q_p^\Delta(\xi_2)\right] \tag{40d}$$

$$D_p P_p^1(\xi_3) + E_p Q_p^1(\xi_3) = F_p Q_p^1(\xi_3) \tag{40e}$$

$$-\left(\frac{1}{\mu_1}\right) F_p Q_p^\Delta(\xi_3) + \left(\frac{1}{\mu_1}\right) D_p P_p^\Delta(\xi_3) + \left(\frac{1}{\mu_1}\right) E_p Q_p^\Delta(\xi_3) = \frac{J_p(\theta)\ a\left(\xi_3^2 - \nu^2\right)^{1/2}}{P_p^1(\nu)} \tag{40f}$$

where

$$J_p(\theta) = \frac{K\ G_p P_p^1(\nu)}{a\left(\xi_3^2 - \nu^2\right)^{1/2}}$$

The solution of these six simultaneous equations to obtain $E_p$ in terms of known quantities gives:

$$E_p = \frac{-\frac{1}{\mu_2}\left([x]J_p^{II}\ P_p^\Delta(\xi_2)\right) - \frac{1}{\mu_2}\left(J_p^{II}\right)Q_p^\Delta(\xi_2) + \frac{1}{\mu_1}\left(J_p^I\right)P_p^1(\xi_2)}{\frac{1}{\mu_2}\left([x][z]P_p^\Delta(\xi_2)\right) + \frac{1}{\mu_2}\left([z]Q_p^\Delta(\xi_2)\right) - \left(\frac{1}{\mu_1}\right)Q_p^\Delta(\xi_2)} \tag{41a}$$

20

where

$$[x] = \frac{\left(\left(\dfrac{\mu_1}{\mu_2}\right)\dfrac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} - \dfrac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}\right)}{\left(1 - \dfrac{\mu_1}{\mu_2}\right)} \tag{41b}$$

$$[z] = \frac{Q_p^1(\xi_2)}{[x]P_p^1(\xi_2) + Q_p^1(\xi_2)} \tag{41c}$$

$$J_p(\theta) = \frac{K\,G_p P_p^1(\nu)}{a\left(\xi_3^2 - \nu^2\right)^{1/2}} \quad , \quad (\nu = \cos\theta) \tag{41d}$$

$$J_p^I = \left\{\frac{-\mu_1 J_p(\theta)\,a\left(\xi_3^2 - \nu^2\right)^{1/2}}{P_p^1(\nu)Q_p^{\Delta}(\xi_3)}\right\} \bigg/ \left(\frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^{\Delta}(\xi_3)}{Q_p^{\Delta}(\xi_3)}\right) \tag{41e}$$

$$J_p^{II} = \frac{J_p^I P_p^1(\xi_2)}{[x]P_p^1(\xi_2) + Q_p^1(\xi_2)} \tag{41f}$$

The numerical values for the other five coefficients can be obtained from the following equations:

$$C_p = J_p^{II} + E_p[z] \tag{42a}$$

$$B_p = [x]C_p \tag{42b}$$

21

$$D_p = J_p^I \tag{42c}$$

$$A_p = B_p + C_p \, \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \tag{42d}$$

$$F_p = \frac{D_p P_p^1(\xi_3)}{Q_p^1(\xi_3)} + E_p \tag{42e}$$

Since the six coefficients can be determined for a specified problem from Equations (41) and (42), the potentials $A_{\psi I}$, $A_{\psi II}$, $A_{\psi III}$, and $A_{\psi IV}$, in regions I through IV can be completely determined. The normal $(B_\eta)$ and tangential $(B_\theta)$ to the surface $\eta = $ constant (or $\xi = $ constant) components of the magnetic induction in each region I through IV can be determined by using Equations (31) and (35).

<div align="center">

THIN COIL INTERNAL TO A FERROMAGNETIC
PROLATE SPHEROIDAL SHELL

</div>

GENERAL SOLUTION

We now proceed to solve the boundary value problem of a ferromagnetic spheroidal shell of homogeneous permeability $\mu_2$, surrounding an infinitesimally thin prolate spheroidal current band having a constant current density $\overline{J}$. Figure 3 shows the cross section of the problem geometry. The coordinate system shown previously in Figure 1 will be used in the solution. The boundaries of the prolate spheroidal shell are determined by $\eta = \eta_3$ and $\eta = \eta_2$. The steady state current lies in the boundary $\eta = \eta_1$ and between $\theta_1 \leq \theta \leq \theta_2$. As in the previous problem, the dc current density has only a psi component $J_\psi(\theta)\overline{e}_\psi$, and thus the vector potential has only a psi component $A_\psi \overline{e}_\psi$. The vector potential is a function of the prolate spheroidal coordinates $\eta$ and $\theta$. The constant current density is expressed by Equation (17) when the boundary $\eta$ is changed to $\eta = \eta_1$.

<div align="center">22</div>
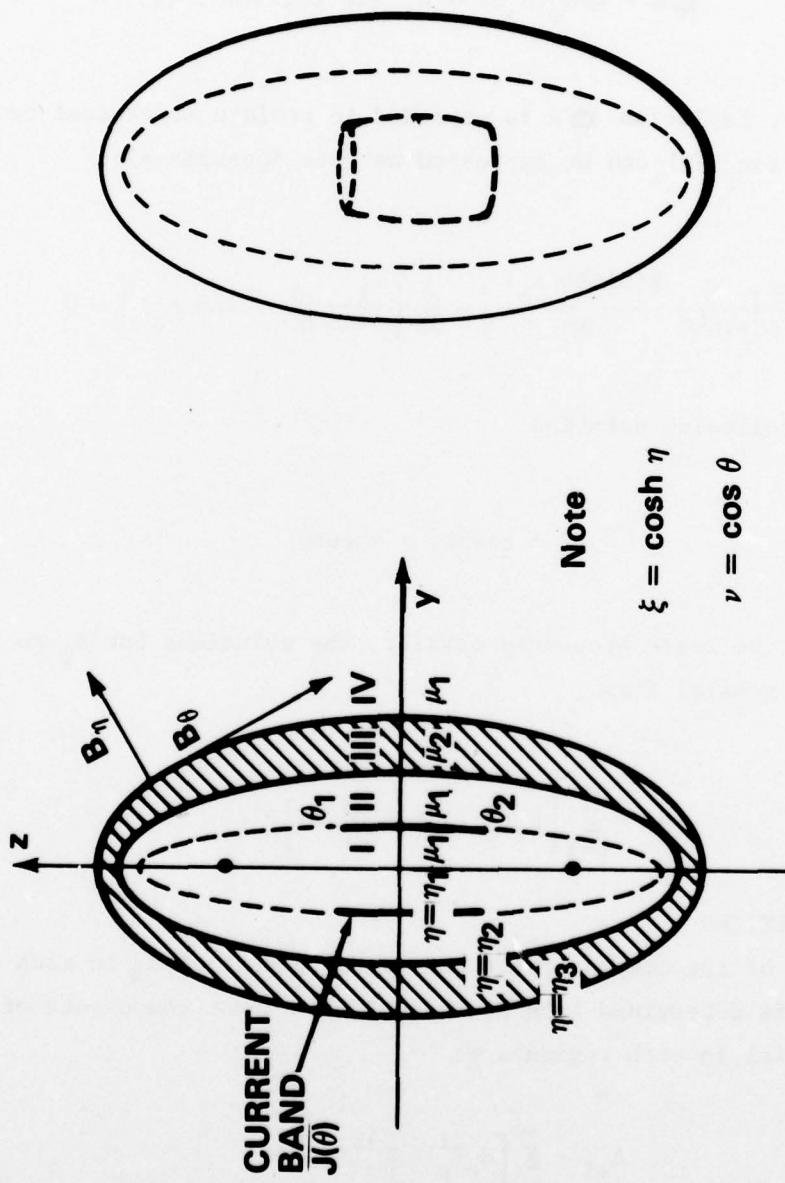
Figure 3 – Cross Section of Infinitesimally Thin Current Band Surrounded by Ferromagnetic Spheroidal Shell

The governing partial differential equation has only a psi component and is given by

$$\clubsuit \overline{A} = \clubsuit \overline{A}_\psi(\eta, \theta) = 0 \quad \text{(in regions I-IV)} \tag{43}$$

When the vector Laplacian $\clubsuit A$ is expanded in prolate spheroidal coordinates, Equation (43) can be expressed as (see Appendix A):

$$\frac{\partial}{\partial\eta}\left[\frac{1}{\sinh\eta}\frac{\partial(\sinh\eta\, A_\psi)}{\partial\eta}\right] + \frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\psi)\right] = 0$$

Adopting the following notation

$$\xi = \cosh\eta, \quad \nu = \cos\theta \tag{45}$$

and following the logic presented earlier, the solutions for $A_\psi$ in regions I-IV have the general form

$$A = \sum_{p=1}^{\infty}\left[K_1 P_p^1(\xi) + K_2 Q_p^1(\xi)\right] P_p^1(\nu) \tag{46}$$

BOUNDARY CONDITIONS

The form of the components of the vector potential $A_\psi$ in each of the regions I-IV is determined from Equation (46). These components of the vector potential in each region are:

$$A_{\psi I} = \sum_{p=1}^{\infty}\left[H_p P_p^1(\xi)\right] P_p^1(\nu) \tag{47a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty}\left[I_p P_p^1(\xi) + K_p Q_p^1(\xi)\right] P_p^1(\nu) \tag{47b}$$

24

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[ L_p P_p^1(\xi) + M_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{47c}$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[ N_p Q_{\bar{p}}^1(\xi) \right] P_p^1(\nu) \tag{47d}$$

The $P_p^1$ functions are the associated Legendre functions of the first kind of degree 1 and order p, and the $Q_p^1$ functions are associated Legendre functions of the second kind.

At each interface the basic laws of magnetostatics reduce to boundary conditions on $\bar{B}$ and $\bar{H}$ (see Equations (30) and (33)) that can be used to determine related boundary conditions on $\bar{A}$:

$$A_{\psi I} = A_{\psi II} \qquad \eta = \eta_1 \tag{48a}$$

$$A_{\psi II} = A_{\psi III} \qquad \eta = \eta_2 \tag{48b}$$

$$A_{\psi III} = A_{\psi IV} \qquad \eta = \eta_3 \tag{48c}$$

$$-\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_1^2 - \nu^2\right)^{1/2}} \frac{\partial}{\partial \xi}\left[\left(\xi^2 - 1\right)^{1/2} A_{\psi II}\right]\Bigg|_{\xi=\xi_1} +$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_1^2 - \nu^2\right)^{1/2}} \frac{\partial}{\partial \xi}\left[\left(\xi^2 - 1\right)^{1/2} A_{\psi I}\right]\Bigg|_{\xi=\xi_1} = \sum_p J_p(\theta) \tag{48d}$$

25

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi II}\right]\Bigg|_{\xi=\xi_2}=$$

(48e)

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi III}\right]\Bigg|_{\xi=\xi_2}$$

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi III}\right]\Bigg|_{\xi=\xi_3}=$$

(48f)

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi IV}\right]\Bigg|_{\xi=\xi_3}$$

These boundary conditions are then used to evaluate the constants in Equation (47). Using Equations (47) and (48) to solve for the coefficients (where the index p takes on all values from 1 to ∞) we get:

$$H_p P_p^1(\xi_1)P_p^1(\nu) = \left[I_p P_p^1(\xi_1) + K_p Q_p^1(\xi_1)\right]P_p^1(\nu)$$

(49a)

$$-\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}\left(I_p P_p^1(\xi) + K_p Q_p(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_1}+$$

(49b)

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{1/2}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}\left(H_p P_p^1(\xi)P_p^1(\nu)\right)\right]\Bigg|_{\xi=\xi_1}= J_p(\theta) = \frac{K\,G_p P_p^1(\nu)}{a\left(\xi_1^2-\nu^2\right)^{1/2}}$$

26

$$\left[I_p P_p^1(\xi_2) + K_p Q_p^1(\xi_2)\right] P_p^1(\nu) = \left[L_p P_p^1(\xi_2) + M_p Q_p^1(\xi_2)\right] P_p^1(\nu) \tag{49c}$$

$$\left(\frac{1}{\mu_2}\right) \frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{1/2}\left(L_p P_p^1(\xi) + M_p Q_p^1(\xi)\right) P_p^1(\nu)\right]\Bigg|_{\xi=\xi_2} = \tag{49d}$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{1/2}\left(I_p P_p^1(\xi) + K_p Q_p^1(\xi)\right) P_p^1(\nu)\right]\Bigg|_{\xi=\xi_2}$$

$$\left[L_p P_p^1(\xi_3) + M_p Q_p^1(\xi_3)\right] P_p^1(\nu) = N_p Q_p^1(\xi_3) P_p^1(\nu) \tag{49e}$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{1/2}\left(N_p Q_p^1(\xi)\right) P_p^1(\nu)\right]\Bigg|_{\xi=\xi_3} =$$

$$\left(\frac{1}{\mu_2}\right) \frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{1/2}\left(L_p P_p^1(\xi) + M_p Q_p^1(\xi)\right) P_p^1(\nu)\right]\Bigg|_{\xi=\xi_3}$$

If we make the following substitutions

$$P_p^\Delta(\xi) = \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{1/2} P_p^1(\xi)\right] \tag{50}$$

$$Q_p^\Delta(\xi) = \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{1/2} Q_p^1(\xi)\right] \tag{51}$$

27

and perform simple algebric manipulations, the six boundary conditions reduce to:

$$H_p P_p^1(\xi_1) = I_p P_p^1(\xi_1) + K_p Q_p^1(\xi_1) \tag{52a}$$

$$-\left(\frac{1}{\mu_1}\right)\left(I_p P_p^\Delta(\xi_1) + K_p Q_p^\Delta(\xi_1)\right) + \left(\frac{1}{\mu_1}\right) H_p P_p^\Delta(\xi_1) = \frac{J_p(\theta) a \left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^1(\nu)} \tag{52b}$$

$$I_p P_p^1(\xi_2) + K_p Q_p^1(\xi_2) = L_p P_p^1(\xi_2) + M_p Q_p^1(\xi_2) \tag{52c}$$

$$\left(\frac{1}{\mu_2}\right)\left(L_p P_p^\Delta(\xi_2) + M_p Q_p^\Delta(\xi_2)\right) = \left(\frac{1}{\mu_1}\right)\left(I_p P_p^\Delta(\xi_2) + K_p Q_p^\Delta(\xi_2)\right) \tag{52c}$$

$$L_p P_p^1(\xi_3) + M_p Q_p^1(\xi_3) = N_p Q_p^1(\xi_3) \tag{52e}$$

$$\left(\frac{1}{\mu_1}\right) N_p Q_p^\Delta(\xi_3) = \left(\frac{1}{\mu_2}\right)\left(L_p P_p^\Delta(\xi_3) + M_p Q_p^\Delta(\xi_3)\right) \tag{52f}$$

It should be noted in the above equations that the current density $J_\psi(\theta)$ was expanded into a set of associated Legendre functions in order to evaluate the constants in the vector potential components (see Appendix B).

28

The solution of these six simultaneous equations (52a through 52f) to obtain $L_p$ in terms of known quantities is:

$$L_p = \frac{\left(\frac{1}{\mu_1}\right)\left[J_p^{II}P_p^{\Delta}(\xi_2) + J_p^{I}Q_p^{\Delta}(\xi_2)\right]}{\left(-\frac{1}{\mu_1}\right)[v]\,P_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_2}\right)P_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_2}\right)[u]Q_p^{\Delta}(\xi_2)} \tag{53}$$

where

$$[u] = \frac{\left[\dfrac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \dfrac{P_p^{\Delta}(\xi_3)}{Q_p^{\Delta}(\xi_3)}\left(\dfrac{\mu_1}{\mu_2}\right)\right]}{\left[\dfrac{\mu_1}{\mu_2} - 1\right]} \tag{54}$$

$$[v] = \frac{P_p^1(\xi_2) + [u]\,Q_p^1(\xi_2)}{P_p^1(\xi_2)} \tag{55}$$

$$J_p(\theta) = \frac{K\,G_p P_p^1(\nu)}{a\left(\xi_1^2 - \nu^2\right)^{1/2}} \tag{56}$$

$$J_p^I = \frac{\mu_1 J_p(\theta)a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^{\Delta}(\xi_1)P_p^1(\nu)} \bigg/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)}\right] \tag{57}$$

$$J_p^{II} = -J_p^I\,\frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} \tag{58}$$

29

The numerical values for the other five coefficients can be obtained from the following equations:

$$K_p = J_p^I \tag{59}$$

$$M_p = L_p[u] \tag{60}$$

$$I_p = J_p^{II} + L_p[v] \tag{61}$$

$$N_p = L_p \; \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + M_p \tag{62}$$

$$H_p = I_p + \frac{K_p Q_p^1(\xi_1)}{P_p^1(\xi_1)} \tag{63}$$

The components of the potential $A_\psi$ in regions I-IV can be determined since the coefficients $H_p$, $I_p$, $K_p$, $L_p$, $M_p$, and $N_p$ can be calculated for a specific problem. The normal ($B_\eta$) and tangential ($B_\theta$) components (to the surface $\eta = $ constant or $\xi = $ constant) of the magnetic induction in each region I-IV can be determined by using Equations (31) and (35).

30

# APPENDIX A

## DERIVATION OF THE VECTOR LAPLACIAN $\bigstar\overline{A}_\psi(\eta,\theta)$

We note that a distinction is made between the Laplacian operating on a scalar $\nabla^2\phi$ and the Laplacian operating on a vector $\bigstar\overline{A}(\eta,\theta)$ [i.e., $\nabla^2\overline{A}(\eta,\theta)$]. The vector Poisson's equation in rectangular coordinates can be treated as three uncoupled scalar equations where $\nabla^2 A_i = J_i$ for $i = x,y,z$. However, if the vector Poisson's equation is resolved into orthogonal components in other coordinate systems the differential operation mixes the components together giving coupled equations.

The vector Laplacian $\bigstar\overline{A}$ can be derived by using the well-known vector identity

$$\overline{\nabla} \times (\overline{\nabla}\times\overline{A}) = \overline{\nabla}(\overline{\nabla}\cdot\overline{A}) - \bigstar\overline{A} \tag{A-1}$$

In the case herein the Coulomb gage was chosen ($\overline{\nabla}\cdot\overline{A}=0$); therefore, the following identity applied in our magnetostatic work

$$\bigstar\overline{A} = -\overline{\nabla} \times (\overline{\nabla}\times\overline{A}) \tag{A-2}$$

The expression in prolate spheroidal coordinates for $\bigstar\overline{A}_\psi(\eta,\theta)$ will now be derived since only the psi component of the vector potential $\overline{A}(\eta,\theta)$ exists in this problem.

Taking the curl of $\overline{A}$ results in the expression

$$\overline{\nabla} \times \overline{A}_\psi(\eta,\theta) =$$

$$\frac{1}{a(\sinh^2\eta+\sin^2\theta)\sinh\eta\,\sin\theta} \times$$

$$\begin{vmatrix} \overline{e}_\eta(\sinh^2\eta+\sin^2\theta)^{1/2} & \overline{e}_\theta(\sinh^2\eta+\sin^2\theta)^{1/2} & \overline{e}_\psi\sinh\eta\,\sin\theta \\ \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\psi} \\ 0 & 0 & A_\psi\sinh\eta\,\sin\theta \end{vmatrix} \tag{A-3}$$

31

$$= \left[\frac{\left(\sinh^2\eta + \sin^2\theta\right)^{-1/2}}{a\ \sinh\eta\ \sin\theta}\right] \times \left[\overline{e}_\eta \frac{\partial}{\partial\theta}(\sinh\eta\ \sin\theta\ A_\psi)\right.$$

$$\left. - \overline{e}_\theta \frac{\partial}{\partial\eta}(\sinh\eta\ \sin\theta\ A_\psi)\right] \tag{A-4}$$

The negative of the curl of $\overline{\nabla}\mathrm{x}\overline{A}$ $(\eta,\theta)$ results in the equation (where $\bigstar^2\overline{A}_\psi = -\ \overline{\nabla}\ \mathrm{x}\ \overline{\nabla}\ \mathrm{x}\ \overline{A}_\psi$)

$$- \overline{\nabla}\ \mathrm{x}\ \overline{\nabla}\ \mathrm{x}\ \overline{A}_\psi = \bigstar\ \overline{A}_\psi = -\left[\frac{1}{a(\sinh^2\eta + \sin^2\theta)\sinh\eta\ \sin\theta}\right]\ \mathrm{x}$$

$$\begin{vmatrix} \overline{e}_\eta(\sinh^2\eta + \sin^2\theta)^{1/2} & \overline{e}_\theta(\sinh^2\eta + \sin^2\theta)^{1/2} & \overline{e}_\psi(\sinh\eta\ \sin\theta) \\[2mm] \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\psi} \\[2mm] \dfrac{1}{a(\sinh\eta\ \sin\theta)}\dfrac{\partial}{\partial\theta}(\sinh\eta\ \sin\theta\ A_\psi), & \dfrac{-1}{(\sinh\eta\ \sin\theta)}\dfrac{\partial}{\partial\eta}(\sinh\eta\ \sin\theta\ A_\psi), & 0 \end{vmatrix} \tag{A-5}$$

$$= \frac{(\sinh^2\eta + \sin^2\theta)^{-1/2}}{a(\sinh\eta\ \sin\theta)} \times \overline{e}_\psi\left[\frac{\partial}{\partial\eta}\ \frac{1}{a\ \sinh\eta\ \sin\theta}\ \frac{\partial}{\partial\eta}(\sinh\eta\ \sin\theta\ A_\psi)\right.$$

$$\left. + \frac{\partial}{\partial\theta}\ \frac{1}{a\ \sinh\eta\ \sin\theta}\ \frac{\partial}{\partial\theta}(\sinh\eta\ \sin\theta\ A_\psi)\right] \tag{A-6}$$

$$= \frac{\left(\sinh^2\eta + \sin^2\theta\right)^{-1/2}}{a^2(\sinh\eta \, \sin\theta)} \, \overline{e}_\psi \left( \frac{\partial}{\partial\eta} \, \frac{1}{\sinh\eta} \, \frac{\partial}{d\eta} \, \sinh\eta \, A_\psi \right)$$

(A-7)

$$+ \frac{\partial}{\partial\theta} \, \frac{1}{\sin\theta} \, \frac{\partial}{\partial\theta} \, (\sin\theta \, A_\psi)$$

Now, if $\maltese \overline{A}_\psi$ is set equal to zero, then the vector potential $A_\psi(\eta,\theta)$ will satisfy the equation (see Equation (19) in text of report).

$$\frac{\partial}{\partial\eta} \left[ \frac{1}{\sinh\eta} \, \frac{\partial}{\partial\eta} (\sinh\eta \, A_\psi) \right] + \frac{\partial}{\partial\theta} \left[ \frac{1}{\sin\theta} \, \frac{\partial}{\partial\theta} (\sin\theta \, A_\psi) \right] = 0 \qquad \text{(A-8)}$$

It is also interesting to note at this point that $\overline{\nabla} \cdot A_\psi(\eta,\theta) = 0$ as it must, since the Coulomb gage was chosen. The divergence of the vector $\overline{A}$ in prolate spheroidal coordinates is

$$\overline{\nabla} \cdot \overline{A} =$$

$$\frac{1}{a(\sinh^2\eta + \sin^2\theta)} \left\{ \frac{1}{\sinh\eta} \, \frac{\partial}{\partial\eta} \left[ \left(\sinh^2\eta + \sin^2\theta\right)^{1/2} \sinh\eta \, A_\eta \right] \right.$$

(A-9)

$$\left. + \frac{1}{\sin\theta} \, \frac{\partial}{\partial\theta} \left[ \left(\sinh^2\eta + \sin^2\theta\right)^{1/2} \sin\theta \, A_\theta \right] \right\} + \frac{1}{a \, \sinh\eta \, \sin\theta} \, \frac{\partial A_\psi}{\partial\psi}$$

Since the two components $A_\eta$ and $A_\theta$ are zero in our case, and $A_\psi$ is a function of $\eta$ and $\theta$ only, then $\overline{\nabla} \cdot \overline{A}_\psi(\eta,\theta)$ is equal to zero as required.

33

# APPENDIX B

## EXPANSION OF CURRENT $J_\psi(\theta)$ IN ASSOCIATED
## LEGENDRE FUNCTIONS $P_p^1(\cos\theta)$

Let us assume the expansion is similar to Purczynski's[4] (see Equation (56)).

$$J_\psi(\theta) = \frac{J \sum_{p=1}^{\infty} G_p P_p^1(\cos\theta)}{a\left(\sinh^2\eta_1 + \sin^2\theta\right)^{1/2}} \tag{B-1}$$

In this case the current band was chosen to be on boundary $\eta=\eta_1$.

The coefficients $G_p$ are determined from Equation (B-1) by multiplying both sides by $\sin\theta\, P_{p'}^1(\cos\theta)$ and $a(\sinh^2\eta_1 + \sin^2\theta)^{1/2}$ and integrating

$$\int_0^\pi J_\psi(\theta)a\left(\sinh^2\eta_1 + \sin^2\theta\right)^{1/2} P_{p'}^1(\cos\theta)\sin\theta d\theta =$$

$$\int_0^\pi J\, G_p P_{p'}^1(\cos\theta)P_p^1(\cos\theta)\,\frac{a\left(\sinh^2\eta_1+\sin^2\theta\right)^{1/2}}{a\left(\sinh^2\eta_1+\sin^2\theta\right)^{1/2}}\,\sin\theta d\theta = \tag{B-2}$$

$$J\, G_p\,\frac{2p(p+1)}{2p+1}$$

where the identity

$$\int_0^\pi P_{p'}^m(\cos\theta)P_p^m(\cos\theta)\sin\theta d\theta = \frac{2(p+m)!\,\delta_{p,p'}}{(2p+1)(p-m)!}$$

was used to simplify Equation (B-2).

When $J_\psi(\theta)$ is set conveniently to a constant J, $G_p$ can be expressed as

$$G_p = \frac{(2p+1)a}{2p(p+1)} \int_o^\pi \left(\sinh^2\eta_1 + \sin^2\theta\right)^{1/2} P_p^1(\cos\theta)\sin\theta d\theta \qquad (B-3)$$

Since the current in this problem extends from $\theta_1$ to $\theta_2$ the expression for $G_p$ can be further modified to read

$$G_p = \frac{(2p+1)a}{2p(p+1)} \int_{\theta_1}^{\theta_2} \left(\sinh^2\eta_1 + \sin^2\theta\right)^{1/2} P_p^1(\cos\theta)\sin\theta d\theta \qquad (B-4)$$

By making the following substitutions

$$\xi = \cosh\eta \ , \ \nu = \cos\theta \qquad (B-5)$$

$G_p$ can be conveniently written as

$$G_p = - \frac{(2p+1)a}{2p(p+1)} \int_{\nu_1}^{\nu_2} \left(\xi_1^2 - \nu^2\right)^{1/2} P_p^1(\nu)d\nu \qquad (B-6)$$

This integral can be evaluated with much difficulty in closed mathematical form. Purczynski[4] evaluated these integrals for $C_{2p}$ (even terms) and $C_{2p+1}$ (odd terms) in terms of very complicated series expansions.

In the computer work to be performed, the integrals for $G_p$ will be evaluated numerically, the reason being it has been found that a large number of coefficients were necessary to obtain a good fit for the current expanded in terms of associated Legendre functions for the analogous spherical shell problem.

36

## APPENDIX C

### DERIVATION OF THE COEFFICIENTS FOR THE VECTOR.
### POTENTIALS FOR A THIN COIL SURROUNDING
### A FERROMAGNETIC SPHEROIDAL SHELL

In this appendix the coefficients are derived for the vector potential in regions I-IV for a ferromagnetic spheroidal shell surrounded by an infinitely thin current band. For a detailed discussion of the ferromagnetic problem see the section in the text of this report entitled "Thin Coil Surrounding a Ferromagnetic Prolate Spheroidal Shell." The magnetic vector potentials in each region are given by:

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[ A_p P_p^1(\xi) \right] P_p^1(\nu) \tag{C-1a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[ B_p P_p^1(\xi) + C_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{C-1b}$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[ D_p P_p^1(\xi) + E_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{C-1c}$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[ F_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{C-1d}$$

The coefficients in Equations (C-1a) to (C-1d) are obtained by substituting these equations into Equations (C-2a) to C-2f).

$$A_{\psi I} = A_{\psi II} \quad \text{at } \eta = \eta_1 \tag{C-2a}$$

$$A_{\psi II} = A_{\psi III} \quad \text{at } \eta = \eta_2 \tag{C-2b}$$

$$A_{\psi III} = A_{\psi IV} \quad \text{at } \eta = \eta_3 \tag{C-2c}$$

37

$$\left(\frac{1}{\mu_2}\right) \frac{1}{a\left(\xi_1^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1^2\right)^{1/2}A_{\psi II}\right]\Bigg|_{\xi=\xi_1} =$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_1^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi I}\right]\Bigg|_{\xi=\xi_1}$$

(C-2d)

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi III}\right]\Bigg|_{\xi=\xi_2} =$$

$$\left(\frac{1}{\mu_2}\right) \frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi II}\right]\Bigg|_{\xi=\xi_2}$$

(C-2e)

$$-\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi IV}\right]\Bigg|_{\xi=\xi_3}$$

$$+\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi III}\right]\Bigg|_{\xi=\xi_3}$$

(C-2f)

$$= \sum_p J_p = \sum_p \frac{K \, G_p P_p^1(\nu)}{a\left(\xi_3^2-\nu^2\right)^{1/2}}$$

38

Let us define the following functions as

$$P_p^\Delta(\xi) = \frac{d}{d\xi}\left[\left(\xi^2-1\right)^{1/2} P_p^1(\xi)\right] \tag{C-2g}$$

$$Q_p^\Delta(\xi) = \frac{d}{d\xi}\left[\left(\xi^2-1\right)^{1/2} Q_p^1(\xi)\right] \tag{C-2h}$$

After appropriate substitution of Equations (C-1a) to (C-1d) into Equations (C-2a) to (C-2f) and using (C-2g) and (C-2h), the following boundary value equations are obtained.

$$A_p P_p^1(\xi_1) = B_p P_p^1(\xi_1) + C_p Q_p^1(\xi_1) \tag{C-3a}$$

$$\left(\frac{1}{\mu_2}\right)\left[B_p P_p^\Delta(\xi_1) + C_p Q_p^\Delta(\xi_1)\right] = \left(\frac{1}{\mu_1}\right) A_p P_p^\Delta(\xi_1) \tag{C-3b}$$

$$B_p P_p^1(\xi_2) + C_p Q_p^1(\xi_2) = D_p P_p^1(\xi_2) + E_p Q_p^1(\xi_2) \tag{C-3c}$$

$$\left(\frac{1}{\mu_1}\right)\left[D_p P_p^\Delta(\xi_2) + E_p Q_p^\Delta(\xi_2)\right] = \left(\frac{1}{\mu_2}\right)\left[B_p P_p^\Delta(\xi_2) + C_p Q_p^\Delta(\xi_2)\right] \tag{C-3d}$$

$$D_p P_p^1(\xi_3) + E_p Q_p^1(\xi_3) = F_p Q_p^1(\xi_3) \tag{C-3e}$$

39

$$-\left(\frac{1}{\mu_1}\right)F_p Q_p^{\Delta}(\xi_3) + \left(\frac{1}{\mu_1}\right)D_p P_p^{\Delta}(\xi_3) + \left(\frac{1}{\mu_1}\right)E_p Q_p^{\Delta}(\xi_3) = \frac{J_p \, a\left(\xi_3^2 - \nu^2\right)^{1/2}}{P_p^1(\nu)} \qquad \text{(C-3f)}$$

where

$$J_p = \frac{k \, G_p P_p^1(\nu)}{a\left(\xi^2 - \nu^2\right)^{1/2}}$$

These algebraic equations provide six simultaneous equations with six unknowns, and they can be solved for the coefficients $A_p$, $B_p$, $C_p$, $D_p$, $E_p$, and $F_p$.

After solving Equation (C-3a) for $A_p$, we obtain

$$A_p = B_p + C_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \qquad \text{(C-4)}$$

The solution of $A_p$ from Equation (C-3b) is

$$A_p = B_p\left(\frac{\mu_1}{\mu_2}\right) + C_p \frac{\mu_1}{\mu_2}\left(\frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)}\right) \qquad \text{(C-5)}$$

By equating Equations (C-4) and (C-5) and solving for $B_p$, one obtains

$$B_p = \frac{C_p\left[\frac{\mu_1}{\mu_2}\left(\frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)}\right) - \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}\right]}{\left[1 - \frac{\mu_1}{\mu_2}\right]}$$

40

where

$$B_p = [x] \, C_p \qquad \qquad (C-7)$$

and

$$[x] = \frac{\left[ \dfrac{\mu_1}{\mu_2} \left( \dfrac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} \right) - \dfrac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right]}{\left[ 1 - \dfrac{\mu_1}{\mu_2} \right]}$$

Solving Equation (C-3e) for $F_p$, we derive the result

$$F_p = D_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + E_p \qquad \qquad (C-8)$$

In addition, the solution for $F_p$ from Equation (C-3f) is

$$F_p = \frac{-\mu_1 J_p \, a \left( \xi_3^2 - \nu^2 \right)^{1/2}}{Q_p^{\Delta}(\xi_3) P_p^1(\nu)} + \frac{D_p P_p^{\Delta}(\xi_3)}{Q_p^{\Delta}(\xi_3)} + E_p \qquad \qquad (C-9)$$

Equating (C-8) and (C-9) and solving for $D_p$, one derives the result

$$D_p = J_p^I \qquad \qquad (C-10)$$

where

$$J_p^I = \left[ \frac{-\mu_1 J_p \, a\left(\xi_3^2 - \nu^2\right)^{1/2}}{P_p^1(\nu) Q_p^\Delta(\xi_3)} \right] \Bigg/ \left[ \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)} \right]$$

Solving Equation (C-3c) for $C_p$ and using Equations (C-7) and (C-10), we arrive at the result

$$C_p = \frac{J_p^I P_p^1(\xi_2)}{\left( [x] P_p^1(\xi_2) + Q_p^1(\xi_2) \right)} + \frac{E_p Q_p^1(\xi_2)}{\left( [x] P_p^1(\xi_2) + Q_p^1(\xi_2) \right)} \qquad (C-11)$$

This can be expressed as

$$C_p = J_p^{II} + E_p \left[ z \right] \qquad (C-12)$$

where

$$J_p^{II} = \frac{J_p^I P_p^1(\xi_2)}{\left[ [x] P_p^1(\xi_2) + Q_p^1(\xi_2) \right]}$$

$$\left[ z \right] = \frac{Q_p^1(\xi_2)}{\left[ [x] P_p^1(\xi_2) + Q_p^1(\xi_2) \right]}$$

42

Using Equation (C-3d) to solve for $E_p$ and substituting Equations (C-7), (C-10), and (C-12), we arrive at the expression

$$E_p = \frac{-\frac{1}{\mu_2}[x]J_p^{II}\, P_p^\Delta(\xi_2) - \frac{1}{\mu_2}\left(J_p^{II}\right)Q_p^\Delta(\xi_2) + \frac{1}{\mu_1}J_p^I P_p^\Delta(\xi_2)}{\frac{1}{\mu_2}\left([x][z]\,P_p^\Delta(\xi_2)\right) + \frac{1}{\mu_2}\left([z]\,Q_p^\Delta(\xi_2)\right) - \frac{1}{\mu_1}Q_p^\Delta(\xi_2)} \tag{C-13}$$

The constants are thus determined by using Equations (C-4), (C-7), (C-8), (C-10), (C-12), and (C-13). For convenience, (C-4), (C-7), (C-8), (C-10), and (C-12) are listed below

$$C_p = J_p^{II} + E_p[z]$$

$$B_p = [x]C_p$$

$$D_p = J_p^I$$

$$\hspace{10cm} \tag{C-14}$$

$$A_p = B_p + C_p\,\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}$$

$$F_p = D_p\,\frac{P_p^1(\xi_3)}{Q_p^1(\xi_2)} + E_p$$

## APPENDIX D

### DERIVATION OF THE COEFFICIENTS FOR THE VECTOR
### POTENTIAL FOR A THIN COIL INTERNAL TO
### A FERROMAGNETIC SPHEROIDAL SHELL

In this appendix the coefficients are derived for the vector potentials in regions I-IV for an infinitely thin current band surrounded by a ferromagnetic spheroidal shell. For a detailed discussion of the ferromagnetic problem see the section in the text of this report entitled "Thin Coil Internal to a Ferromagnetic Prolate Spheroidal Shell." The magnetic vector potential in each region is given by:

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[ H_p P_p^1(\xi) \right] P_p^1(\nu) \tag{D-1a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[ I_p P_p^1(\xi) + K_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{D-1b}$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[ L_p P_p^1(\xi) + M_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{D-1c}$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[ N_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{D-1d}$$

The coefficients in Equations (D-1a) to (D-1d) are obtained by substituting these equations into the boundary conditions (Equations (D-2a) to (D-2f)).

$$A_{\psi I} = A_{\psi II} \quad \text{at } \eta = \eta_1 \tag{D-2a}$$

$$A_{\psi II} = A_{\psi III} \quad \text{at } \eta = \eta_2 \tag{D-2b}$$

$$A_{\psi III} = A_{\psi IV} \quad \text{at } \eta = \eta_3 \tag{D-2c}$$

45

$$-\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi II}\right]\Bigg|_{\xi=\xi_1} \quad + \qquad\qquad \text{(D-2d)}$$

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi I}\right]\Bigg|_{\xi=\xi_1} = \sum_p J_p(\theta)$$

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi II}\right]\Bigg|_{\xi=\xi_2} =$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(D-2e)}$$

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi III}\right]\Bigg|_{\xi=\xi_2}$$

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi III}\right]\Bigg|_{\xi=\xi_3} =$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(D-2f)}$$

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{1/2}A_{\psi IV}\right]\Bigg|_{\xi=\xi_3}$$

Let us define the following functions as

$$P_p^\Delta(\xi) = \frac{d}{d\xi}\left[\left(\xi^2-1\right)^{1/2}P_p^1(\xi)\right]$$

$$Q_p^\Delta(\xi) = \frac{d}{d\xi}\left[\left(\xi^2-1\right)^{1/2}Q_p^1(\xi)\right]$$

46

After appropriate substitution of Equations (D-1a) to (D-1d) into Equations (D-2a) to (D-2f), the following boundary value equations are obtained.

$$H_p P_p^1(\xi_1) = I_p P_p^1(\xi_1) + K_p Q_p^1(\xi_1) \tag{D-3a}$$

$$-\left(\frac{1}{\mu_1}\right)\left(I_p P_p^\Delta(\xi) + K_p Q_p^\Delta(\xi_1)\right) + \left(\frac{1}{\mu_1}\right) H_p P_p^\Delta(\xi_1) = \frac{J_p \, a\left(\xi^2 - \nu^2\right)^{1/2}}{P_p^1(\nu)} \tag{D-3b}$$

$$I_p P_p^1(\xi_2) + K_p Q_p^1(\xi_2) = L_p P_p^1(\xi_2) + M_p Q_p^1(\xi_2) \tag{D-3c}$$

$$\left(\frac{1}{\mu_2}\right)\left(L_p P_p^\Delta(\xi_2) + M_p Q_p^\Delta(\xi_2)\right) = \left(\frac{1}{\mu_1}\right)\left(I_p P_p^\Delta(\xi_2) + K_p Q_p^\Delta(\xi_2)\right) \tag{D-3d}$$

$$L_p P_p^1(\xi_3) + M_p Q_p^1(\xi_3) = N_p Q_p^1(\xi_3) \tag{D-3e}$$

$$\left(\frac{1}{\mu_1}\right) N_p Q_p^\Delta(\xi_3) = \left(\frac{1}{\mu_2}\right)\left(L_p P_p^\Delta(\xi_3) + M_p Q_p^\Delta(\xi_3)\right) \tag{D-3f}$$

These algebraic equations provide six simultaneous equations with six unknowns, and they can be solved for the coefficients $H_p$, $I_p$, $K_p$, $M_p$, and $N_p$.

Solving Equation (D-3a) for $H_p$ in terms of $I_p$ and $K_p$, we arrive at the equation

$$H_p = I_p + K_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \tag{D-4}$$

47

Solving Equation (D-3b) for $H_p$, we obtain the equation

$$H_p = I_p + K_p \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} + \frac{\mu_1 J_p a \left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)} \tag{D-5}$$

Equating Equations (D-4) and (D-5) and solving for $K_p$ yields

$$K_p = J_p^I \tag{D-6}$$

where

$$J_p^I = \left[\frac{\mu_1 J_p a \left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)}\right] \Big/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)}\right]$$

Now solving Equation (D-3e) for $N_p$, we obtain

$$N_p = L_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + M_p \tag{D-7}$$

Additionally, by solving Equation (D-3f) for $N_p$, one obtains the expression

$$N_p = \frac{\left(\frac{\mu_1}{\mu_2}\right)\left(L_p P_p^\Delta(\xi_3) + M_p Q_p^\Delta(\xi_3)\right)}{Q_p^\Delta(\xi_3)} \tag{D-8}$$

48

By equating Equations (D-7) and (D-8), we may obtain $M_p$ in terms of $L_p$

$$M_p = L_p[u] \tag{D-9}$$

where

$$[u] = \frac{\left[\dfrac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \dfrac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)}\left(\dfrac{\mu_1}{\mu_2}\right)\right]}{\left[\dfrac{\mu_1}{\mu_2} - 1\right]}$$

Solving (D-3c) for $I_p$ and using Equations (D-6) and (D-9), we derive the result

$$I_p = J_p^{II} + L_p[v] \tag{D-10}$$

where

$$J_p^{II} = - J_p^I \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)}$$

$$[v] = \frac{P_p^1(\xi_2) + [u]Q_p^1(\xi_2)}{P_p^1(\xi_2)}$$

Now if we use Equation (D-3d) and substitute from Equations (D-6), (D-9), and (D-10) the constant $L_p$ can be determined to be

$$L_p = \frac{\left(\dfrac{1}{\mu_1}\right)\left[J_p^{II}P_p^\Delta(\xi_2) + J_p^I Q_p^\Delta(\xi_2)\right]}{-\left(\dfrac{1}{\mu_1}\right)[v]P_p^\Delta(\xi_2) + \left(\dfrac{1}{\mu_2}\right)P_p^\Delta(\xi_2) + \left(\dfrac{1}{\mu_2}\right)[u]Q_p^\Delta(\xi_2)} \tag{D-11}$$

The constants have now been found. After the numerical value for $L_p$ is calculated on the computer for a specific problem, the numerical values for the other coeffieients can be obtained from the following equations:

$$K_p = J_p^I$$

$$M_p = L_p[u]$$

$$I_p = J_p^{II} + L_p[v] \tag{D-12}$$

$$N_p = L_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + M_p$$

$$H_p = I_p + K_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}$$

## APPENDIX E

### DETERMINATION OF THE MAGNETIC VECTOR
### POTENTIAL FOR AN INFINITESIMALLY THIN
### PROLATE SPHEROIDAL CURRENT BAND

In this appendix the Potentials $A_{\psi I}$ in the inner region and $A_{\psi II}$ in outer region are derived for the infinitesimally thin current band in a homogeneous medium of permeability $\mu_1$ (see Figure E-1).

The potential in the inner region $A_{\psi I}$ and the outer region $A_{\psi II}$ of the infinitesimally thin current band problem are solutions to the vector Laplace's equation $\clubsuit \overline{A} = 0$. These solutions can be expressed as:

$$A_{\psi I} = \sum_{p=1}^{\infty} A_p P_p^1(\xi) \, P_p^1(\nu) \tag{E-1a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty} F_p Q_p^1(\xi) \, P_p^1(\nu) \tag{E-1b}$$

The coefficients $A_p$ and $F_p$ are determined from the boundary conditions of the problem. After algebraic manipulation such as with Equations (32) and (36) in the text, the boundary conditions for the normal component of $\overline{B}$ and the tangential component of $\overline{H}$ becomes:

$$A_{\psi I} = A_{\psi II} \text{ at } \eta = \eta_1 \tag{E-2a}$$

$$-\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_1^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{1/2} A_{\psi II}\right]\Bigg|_{\xi=\xi_1} +$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_1^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{1/2} A_{\psi I}\right]\Bigg|_{\xi=\xi_1} = \sum_p J_p(\theta) \tag{E-2b}$$
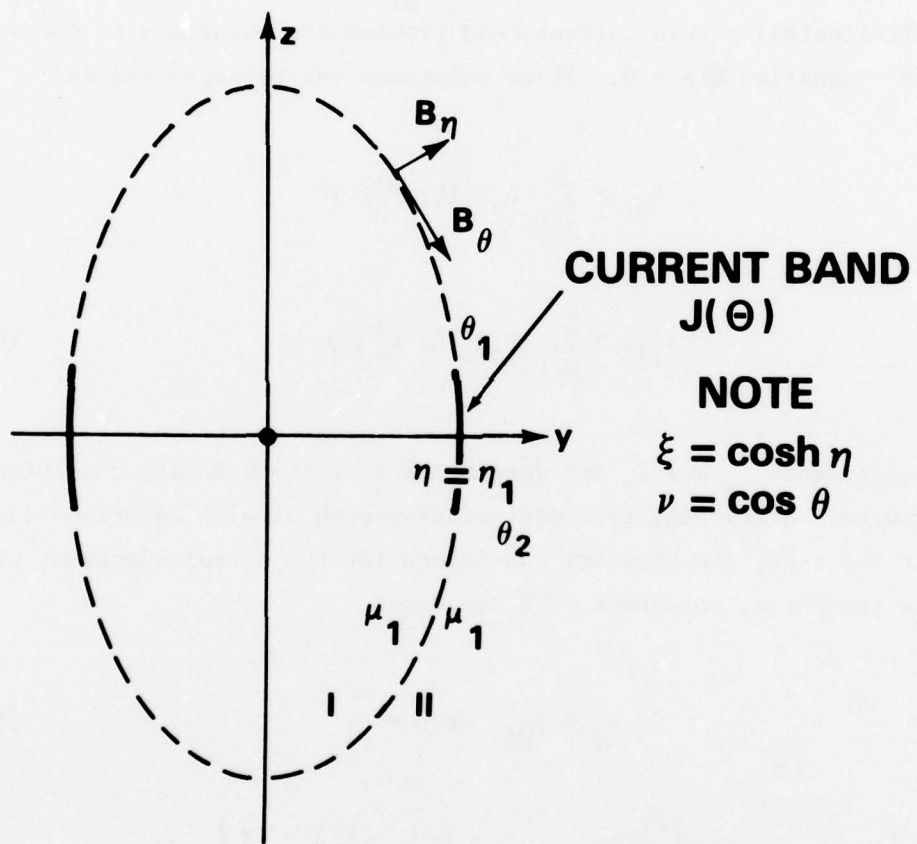
51

Figure E-1 - Infinitesimally Thin Spheroidal
Current Band

Using the relationship $\xi = \cosh\eta$ and $\nu = \cos\theta$ and substituting the expressions for $A_{\psi I}$ and $A_{\psi II}$ (Equations (E-1a) and (E-1b)) into the boundary value equations (Equations (E-2a) and (E-2b)) provides us with the following algebraic equations for the coefficients:

$$A_p P_p^1(\xi_1) = F_p Q_p^1(\xi_1) \tag{E-3a}$$

$$-\frac{1}{\mu_1}\left[F_p Q_p^\Delta(\xi_1)\right] + \frac{1}{\mu_1}\left[\left(A_p P_p^\Delta(\xi_1)\right)\right] = \frac{J_p(\theta)\; a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^1(\nu)}$$

The following substitutions were used to simplify the above expressions.

$$P_p^\Delta(\xi) = \frac{d}{d\xi}\left[\left(\xi^2 - 1\right)^{1/2} P_p^1(\xi)\right] \tag{E-4}$$

$$Q_p^\Delta(\xi) = \frac{d}{d\xi}\left[\left(\xi^2 - 1\right)^{1/2} Q_p^1(\xi)\right] \tag{E-5}$$

These equations are solved for $A_p$ and $F_p$ by simple algebraic manipulation.

$$A_p = F_p\, \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \tag{E-6a}$$

$$F_p = \left[\frac{\mu_1 J_p(\theta)\; a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)}\right] \Bigg/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)}\right] \tag{E-6b}$$

53

The potential $A_{\psi I}$ and $A_{\psi II}$ are determined by substituting the expression for $A_p$ and $F_p$ into Equations (E-1a) and (E-1b)

$$A_{\psi I} = \sum_{p=1}^{\infty} \left\{ \left[ \frac{\mu_1 J_p(\theta) a \left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^{\Delta}(\xi_1) P_p^1(\nu)} \left( \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right) \right] \middle/ \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} \right] \right\} P_p^1(\xi) P_p^1(\nu)$$

(E-7a)

$$A_{\psi II} = \sum_{p=1}^{\infty} \left\{ \frac{\mu_1 J_p(\theta) a \left(\xi_1 - \nu^2\right)^{1/2}}{P_p^{\Delta}(\xi_1) P_p^1(\nu)} \middle/ \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} \right] \right\} Q_p^1(\xi) P_p^1(\nu)$$

(E-7b)

where

$$J_p(\theta) = \frac{K \, G_p \, P_p^1(\nu)}{a \left(\xi_1^2 - \nu^2\right)^{1/2}}$$

(E-8)

54

## APPENDIX F

### REDUCTION OF THE MAGNETIC VECTOR POTENTIAL FOR A THIN COIL SURROUNDING A FERROMAGNETIC SPHEROIDAL SHELL TO THAT OF A THIN COIL IN FREE SPACE WHEN IN THE LIMIT $\mu_2$ EQUALS $\mu_1$

In this appendix, the coefficients $A_p$, $B_p$, $C_p$, $D_p$, $E_p$, and $F_p$ for the potentials are evaluated for the system consisting of a ferromagnetic shell with permeability $\mu_2$ surrounded by an infinitesimally thin current band in a homogeneous medium with permeability $\mu_1$ in the limit as $\mu_2 = \mu_1$. These coefficients are utilized in Equation (29) in the section of this report entitled "Thin Coil Surrounding a Ferromagnetic Prolate Spheroidal Shell." The variables are defined in Figure 2 located in the text. When $\mu_2$ us set equal to $\mu_1$ the problem reduces to that of finding the potentials in the two regions of a simple current band (see Figure 1-E in Appendix E , since the ferromagnetic shell will now have a permeability $\mu_1$ equal to that of the homogeneous medium with permeability $\mu_1$.

In this limit the coefficients should assume the following form:

$$A_p = B_p = D_p \tag{F-1a}$$

$$C_p = E_p = 0 \tag{F-1b}$$

and where $A_p$ and $F_p$ should reduce to the coefficients for the potentials in the two regions for the spheroidal band problem (see Appendix E). If the coefficients assume this mathematical form it will prove that the mathematical form of the coefficients for the spheroidal shell surrounded by a thin current band are mathematically correct.

The mathematical solution for $E_p$ in terms of known quantities was derived in Appendix C and was reported in the text of this report (see Equation (41a)).

55

$$E_p = \frac{-\frac{1}{\mu_2}\left([x]J_p^{II}P_p^{\Delta}(\xi_2)\right) - \frac{1}{\mu_2}J_p^{II}Q_p^{\Delta}(\xi_2) + \frac{1}{\mu_1}(J_p^{I})P_p^{\Delta}(\xi_2)}{\frac{1}{\mu_2}\left([x][z]P_p^{\Delta}(\xi_2)\right) + \frac{1}{\mu_2}\left([z]Q_p^{\Delta}(\xi_2)\right) - \left(\frac{1}{\mu_1}\right)Q_p^{\Delta}(\xi_2)} \tag{F-2a}$$

$$[x] = \frac{\left(\frac{\mu_1}{\mu_2}\right)\frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} - \frac{Q_p^{1}(\xi_1)}{P_p^{1}(\xi_1)}}{\left(1 - \frac{\mu_1}{\mu_2}\right)} \tag{F-2b}$$

$$[z] = \frac{Q_p^{1}(\xi_2)}{[x]P_p^{1}(\xi_2) + Q_p^{1}(\xi_2)} \tag{F-2c}$$

$$J_p^{I} = \left\{ \frac{-\mu_1 J_p(\theta)a\left(\xi_3^2 - \nu^2\right)^{1/2}}{P_p^{1}(\nu)Q_p^{\Delta}(\xi_3)} \right\} \Bigg/ \left[ \frac{P_p^{1}(\xi_3)}{Q_p^{1}(\xi_3)} - \frac{P_p^{\Delta}(\xi_3)}{Q_p^{\Delta}(\xi_3)} \right] \tag{F-2d}$$

$$J_p^{II} = \frac{J_p^{I}P_p^{1}(\xi_2)}{[x]P_p^{1}(\xi_2) + Q_p^{1}(\xi_2)} \tag{F-2e}$$

The coefficient $E_p$ will now be evaluated when the limit is taken with $\mu_2 = \mu_1$ which cause $[x]$ to approach infinity ($\infty$). Also the expression for $J_p^{II}$ is substituted into Equation (F-2a).

56

$$E_p \bigg|_{\mu_2=\mu_1} = \underset{[x]\to\infty}{\text{limit}}\ E_p$$

$$E_p \bigg|_{\mu_2=\mu_1} = \underset{[x]\to\infty}{\text{limit}}\ \frac{-\dfrac{1}{\mu_1}[x]\left\{\dfrac{J^I_p P^I_p(\xi_2)}{[x]P^1_p(\xi_2)+Q^1_p(\xi_2)}\right\}P^\Delta_p(\xi_2) - \dfrac{1}{\mu_1}\left\{\dfrac{J^I_p P^1_p(\xi_2)}{[x]P^1_p(\xi_2)+Q^1_p(\xi_2)}\right\}Q^\Delta_p(\xi_2) + \dfrac{1}{\mu_1}\left(J^I_p\right)P^\Delta_p(\xi_2)}{-\dfrac{1}{\mu_1}[x]\left(\dfrac{Q^1_p(\xi_2)}{[x]P^1_p(\xi_2)+Q^1_p(\xi_2)}\right)P^\Delta_p(\xi_2) + \dfrac{1}{\mu_1}\left(\dfrac{Q^1_p(\xi_2)Q^\Delta_p(\xi_2)}{[x]P^1_p(\xi_2)+Q^1_p(\xi_2)}\right) - \dfrac{1}{\mu_1}Q^\Delta_p(\xi_2)}$$

$$= \frac{-\dfrac{1}{\mu_1}J^I_p P^\Delta_p(\xi_2) + \dfrac{1}{\mu_1}J^I_p P^\Delta_p(\xi_2)}{\dfrac{1}{\mu_1}\left[\dfrac{Q^1_p(\xi_2)P^\Delta_p(\xi_2) - P^1_p(\xi_2)Q^\Delta_p(\xi_2)}{P^1_p(\xi_2)}\right]}$$

$$E_p \bigg|_{\mu_2=\mu_1} = 0$$

$$(F\text{-}3)$$

The expression for $B_p$ (see Equations (42b) and (41b)) is

$$B_p = [x] \, C_p = [x] J_p^{II} + E_p [x][z] \qquad (F-4a)$$

where

$$C_p = J_p^{II} + E_p [z] \qquad (F-4b)$$

The expression for $B_p$ when $\mu_2$ equals $\mu_1$ can be expressed as

$$\left. B_p \right|_{\mu_2 = \mu_1} = \lim_{\mu_2 \to \mu_1} \left\{ [x] \left[ \left. E_p \right|_{\mu_2 = \mu_1} \right] [z] \right\} + \lim_{\mu_2 \to \mu_2} \left( [x] J_p^{II} \right) \qquad (F-5)$$

$$\left. B_p \right|_{\mu_2 \to \mu_1} = J_p^{I}$$

where

$$\lim_{\mu_2 \to \mu_1} \Big[ [x][z] \Big] = Q_p^1(\xi_2) \Big/ P_p^1(\xi_2)$$

and

$$\lim_{\mu_2 \to \mu_1} \Big[ [x] J_p^{II} \Big] = \lim_{[x] \to \infty} \left[ \frac{[x] J_p^{I} P_p^1(\xi_2)}{[x] P_p^1(\xi_2) + Q_p^1(\xi_2)} \right] = J_p^{I}$$

58

and

$$E_p \bigg|_{\mu_2 = \mu_1} = 0 \quad \text{from Equation (F-3)}$$

The expression for $D_p \bigg|_{\mu_2 = \mu_1}$ equals $J_p^I$

(See Equation (42c) in the text of this report). The expression for $C_p$ (see Equation (42a)) in the text of this report is:

$$C_p = J_p^{II} + E_p[z] \tag{F-6}$$

The expression for $C_p$ when $\mu_2 = \mu_1$ can be expressed as:

$$C_p \bigg|_{\mu_2 = \mu_1} = \lim_{[x] \to \infty} \left[ E_p \bigg|_{\mu_2 = \mu_1} \right] [z] + \lim_{[x] \to \infty} \left[ \frac{J_p^I P_p^1(\xi_2)}{[x] P_p^1(\xi_2) + Q_p^1(\xi_2)} \right]$$

$$C_p \bigg|_{\mu_2 = \mu_1} = 0 \tag{F-7}$$

where

$$\lim_{[x] \to \infty} [z] = 0$$

and

$$\lim_{[x] \to \infty} J_p^{II} = 0$$

59

The expression for $A_p$ is (see Equation (42d) in the text)

$$A_p = B_p + C_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \qquad (F-8)$$

The expressions for $A_p$ when $\mu_2=\mu_1$ is

$$A_p\bigg|_{\mu_2=\mu_1} = B_p\bigg|_{\mu_2=\mu_1} + \left[C_p\bigg|_{\mu_2=\mu_1}\right] \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \qquad (F-9)$$

$$A_p\bigg|_{\mu_2=\mu_1} = B_p\bigg|_{\mu_2=\mu_1} = J_p^I$$

where $C_p\bigg|_{\mu_2=\mu_1} = 0$

The expression for $F_p$ (see Equation (42e) in the text) is

$$F_p = \frac{D_p P_p^1(\xi_3)}{Q_p^1(\xi_3)} + E_p \qquad (F-10)$$

The expression for $F_p$ when $\mu_2=\mu_1$ is

$$F_p\bigg|_{\mu_2=\mu_1} = \left[D_p\bigg|_{\mu_2=\mu_1}\right] \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + E_p\bigg|_{\mu_2=\mu_1}$$

60

$$F_p \Bigg|_{\mu_2 = \mu_1} = \left[ \frac{\mu_1 J_p \ a \left( \xi_3^2 - \nu^2 \right)^{1/2}}{P_p^\Delta(\xi_3) P_p^1(\nu)} \right] \Bigg/ \left[ \frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)} - \frac{Q_p^\Delta(\xi_3)}{P_p^\Delta(\xi_3)} \right] \qquad \text{(F-11)}$$

where $\quad E_p \Bigg|_{\mu_2 = \mu_1} = 0$ from Equation (F-3)

This means that in the four regions, the potential used in Equation (29) of the main text of this report reduce when $\mu_2 = \mu_1$ to the form

$$A_{\psi I, \ II, \ III} = \sum_{p=1}^\infty \left[ A_p \Bigg|_{\mu_2 = \mu_1} P_p^1(\xi) P_p^1(\nu) \right] \qquad \text{(F-12a)}$$

$$A_{\psi IV} = \sum_{p=1}^\infty \left[ F_p \Bigg|_{\mu_2 = \mu_1} Q_p^1(\xi) P_p(\nu) \right]$$

These are the solutions for the potentials of the current band in a region of space with homogeneous permeability $\mu_1$ (see Equations (E-1a) and (E-2a)). We now have the solution for the two potentials in regions I and II ($A_{\psi I, \ II, \ III}$, and $A_{\psi IV}$, respectively) for the simple current band problem. This indicates that the form of the coefficients $A_p$ through $F_p$ is correct.

The mathematical expressions for $A_p$ and $F_p$ which has been evaluated in the limit as $\mu_2 = \mu_2$, will be compared with the coefficients $A_p$ and $F_p$, respectively, for the two regions of the current band problem (see Appendix E).

$A_p$ from Equation (F-9) is

$$A_p \Bigg|_{\mu_2 = \mu_1} = J_p^I \qquad \text{(F-13)}$$

61

$A_p$ from Equation (E-6a) which we will denote here by $A_p'$ is

$$A_p' = \left\{ \left[ \frac{\mu_1 J_p\, a \left( \xi_1^2 - \nu^2 \right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)} \right] \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right] \right\} \Bigg/ \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \qquad \text{(F-14)}$$

By simple algebraic manipulation, the identity

$$\frac{1}{Q_p^\Delta(\xi_1)} \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] = \frac{1}{P_p^\Delta(\xi_1)} \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right] \left[ \frac{P_p^\Delta(\xi_1)}{Q_p^\Delta(\xi_1)} - \frac{P_p^1(\xi_1)}{Q_p^1(\xi_1)} \right]$$

$$\text{(F-15)}$$

can be changed to read

$$\left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \left[ \frac{P_p^1(\xi_1)}{Q_p^1(\xi_1)} \right] = - \left[ \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \left[ \frac{P_p^1(\xi_1)}{Q_p^1(\xi_1)} - \frac{P_p^\Delta(\xi_1)}{Q_p^\Delta(\xi_1)} \right]$$

$$\text{(F-16)}$$

substituting Equation (F-16) into (F-14) we have

$$A_p' = \left\{ \frac{-\mu_1 J_p a \left( \xi_1^2 - \nu^2 \right)^{1/2}}{Q_p^\Delta(\xi_1) P_p^1(\nu)} \right\} \Bigg/ \left[ \frac{P_p^1(\xi_1)}{Q_p^1(\xi_1)} - \frac{P_p^\Delta(\xi_1)}{Q_p^\Delta(\xi_1)} \right] \qquad \text{(F-17)}$$

comparing Equation (F-17) to (41e) shows that

$$A_p' = J_p^I$$

Now $F_p$ from Equation (E-6b) which will be denoted by $F_p'$ is

$$F_p' = \left[ \frac{\mu_1 J_p \, a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)} \right] \bigg/ \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \qquad \text{(F-18)}$$

which is identical to $F_p$ as defined in Equation (F-11) so that $F_p = F_p'$

Thus, the mathematical expressions for $A_p$ and $F_p$ (Equations (F-13) and (F-11), respectively) for the ferromagnetic spheroidal shell surrounded by a thin current band in the limit as $\mu_2 = \mu_1$ are the same as the coefficients $A_p$ and $F_p$ (see Equations (E-6a) and (E-6b), respectively) for the vector potentials in the regions of the current band in free space (see Appendix E).

It is noted that when making the comparison, $\xi_3$ must be set equal to $\xi_1$. For comparison, the coefficients for the current band problem are

$$A_p = F_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} = J_p^I \qquad \text{(F-19)}$$

$$F_p = \left[ \frac{\mu_1 J_p \, a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)} \right] \bigg/ \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \qquad \text{(F-20)}$$

and the coefficients for the ferromagnetic shell problem with $\mu_2 = \mu_1$ are

$$A_p = J_p^I = F_p \frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)} \qquad \text{(F-21)}$$

$$F_p = \left[\frac{\mu_1 J_p a\left(\xi_3^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_3) P_p^1(\nu)}\right] \Bigg/ \left[\frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)} - \frac{Q_p^\Delta(\xi_3)}{P_p^\Delta(\xi_3)}\right] \qquad \text{(F-22)}$$

## APPENDIX G

### REDUCTION OF THE MAGNETIC VECTOR POTENTIAL FOR A THIN COIL INTERNAL TO A FERROMAGNETIC PROLATE SPHEROIDAL SHELL TO THAT OF A THIN COIL IN FREE SPACE WHEN IN THE LIMIT $\mu_2$ EQUALS $\mu_1$

In this appendix, the coefficients $H_p$, $I_p$, $K_p$, $L_p$, $M_p$, and $N_p$ for the potentials are evaluated for the system consisting of an infinite-simally thin current band surrounded by a ferromagnetic shell with permeability $\mu_2$ in a homogeneous medium with permeability $\mu_1$ in the limit as $\mu_2 = \mu_1$. These coefficients are utilized in Equation (47) in the section of this report entitled "Thin Coil Internal to a Ferromagnetic Prolate Spheroidal Shell." The variables are defined in Figure 3 located on page 23 of this report. When $\mu_2$ is set equal to $\mu_1$ the problem reduces to that of finding the potential in the two regions of a simple current band (see Figure 1-E in Appendix E), since the ferromagnetic shell will now have a permeability $\mu_1$ equal to that of the homogeneous medium with permeability $\mu_1$.

In this limit the coefficients should assume the following form

$$K_p = M_p = N_p \tag{G-1a}$$

$$I_p = L_p = 0 \tag{G-1b}$$

and where $H_p$ and $N_p$ should reduce to the coefficients for the potentials in the two regions for the spheroidal band problem (see Appendix E). If the coefficients assume this mathematical form it will prove that the coefficients for the current band surrounded by a spheroidal shell are correct.

The mathematical solution for $L_p$ in terms of known quantities was derived in Appendix D and was reported in the text of this report (see Equation (53)).

$$L_p = \frac{\left(\dfrac{1}{\mu_1}\right)\left[J_p^{II} P_p^{\Delta}(\xi_2) + J_p^{I} Q_p^{\Delta}(\xi_2)\right]}{\left(-\dfrac{1}{\mu_1}\right)[v]P_p^{\Delta}(\xi_2) + \left(\dfrac{1}{\mu_2}\right)P_p^{\Delta}(\xi_2) + \left(\dfrac{1}{\mu_2}\right)[u]Q_p^{\Delta}(\xi_2)} \tag{G-2a}$$

65

where

$$[u] = \frac{\left[\dfrac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \dfrac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)}\left(\dfrac{\mu_1}{\mu_2}\right)\right]}{\left[\dfrac{\mu_1}{\mu_2} - 1\right]} \tag{G-2b}$$

$$[v] = \frac{\left[P_p^1(\xi_2) + [u]\,Q_p^1(\xi_2)\right]}{P_p^1(\xi_2)} \tag{G-2c}$$

$$J_p = \frac{k\,G_p\,P_p^1(\nu)}{a\left(\xi_1^2 - \nu^2\right)^{1/2}} \tag{G-2d}$$

$$J_p^I = \left[\frac{\mu_1 J_p\, a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)}\right] \Bigg/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)}\right] \tag{G-2e}$$

$$J_p^{II} = -\,J_p^I\,\frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} \tag{G-2f}$$

The coefficient $L_p$ will now be evaluated when the limit is taken with $\mu_2 = \mu_1$ which causes $[u]$ and $[v]$ to approach infinity ($\infty$).

$$L_p\Bigg|_{\mu_2 = \mu_1} = \lim_{\substack{[u] \to \infty \\ [v] \to \infty}} \left\{\frac{\dfrac{1}{\mu_1}\left[J_p^{II} P_p^\Delta(\xi_2) + J_p^I Q_p^\Delta(\xi_2)\right]}{\left(-\dfrac{1}{\mu_1}\right)[v] P_p^\Delta(\xi_2) + \left(\dfrac{1}{\mu_1}\right) P_p^\Delta(\xi_2) + \left(\dfrac{1}{\mu_1}\right)[u] Q_p^\Delta(\xi_2)}\right\}$$

$$\tag{G-3}$$

$$L_p\Bigg|_{\mu_2 = \mu_1} = 0$$

where

$$\lim_{\mu_2=\mu_1}[u] = \frac{\left[\dfrac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \dfrac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)}\left(\dfrac{\mu_1}{\mu_1}\right)\right]}{\left[\dfrac{\mu_1}{\mu_1} - 1\right]} = \infty \qquad (G-4)$$

$$\lim_{\mu_2=\mu_1}[v] = \frac{\left[P_p^1(\xi_2) + [u]Q_p^1(\xi_2)\right]}{P_p^1(\xi_2)} = \infty \qquad (G-5)$$

The expression for $I_p$ (see Equations (61) and (55))

$$I_p = J_p^{II} + L_p[v] \qquad (G-6)$$

$$\left. I_p \right|_{\mu_2=\mu_1} = J_p^{II} + \lim_{\mu_2=\mu_1}[L_p][v] \qquad G-7)$$

where $\lim_{\mu_2=\mu_1} L_p[v] =$

$$\lim_{[u]\to\infty} \frac{\left(\dfrac{1}{\mu_1}\right)\left[- J_p^I \dfrac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} P_p^\Delta(\xi_2) + J_p^I Q_p^\Delta(\xi_2)\right]\left[\dfrac{P_p^1(\xi_2)}{P_p^1(\xi_2)} + \dfrac{[u]Q_p^1(\xi_2)}{P_p^1(\xi_2)}\right]}{\left(-\dfrac{1}{\mu_1}\right)\left[\dfrac{P_p^1(\xi_2)}{P_p^1(\xi_2)} + \dfrac{[u]Q_p^1(\xi_2)}{P_p^1(\xi_2)}\right] P_p^\Delta(\xi_2) + \dfrac{1}{\mu_1} P_p^\Delta(\xi_2) + \left(\dfrac{1}{\mu_1}\right)[u]Q_p^\Delta(\xi_2)} = - J_p^{II}$$

$$(G-8)$$

67

Thus

$$I_p \bigg|_{\mu_2=\mu_1} = J_p^{II} - J_p^{II} \tag{G-9}$$

$$I_p \bigg|_{\mu_2=\mu_1} = 0 \tag{G-10}$$

Now, the expression for $M_p$ when $\mu_2$ equals $\mu_1$ can be written as (see Equation 60)

$$M_p \bigg|_{\mu_2=\mu_1} = \underset{\mu_2=\mu_1}{\text{limit }} L_p[u] \tag{G-11a}$$

where $\underset{\mu_2=\mu_1}{\text{limit }} L_p[u] =$

$$\underset{[u]\to\infty}{\text{limit}} \frac{\left(\frac{1}{\mu_1}\right)\left[-J_p^I \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} P_p^\Delta(\xi_2) + J_p^I Q_p(\xi_2)\right][u]}{\left\{\left(-\frac{1}{\mu_1}\right)\left[\frac{P_p^1(\xi_2)}{P_p^1(\xi_2)}P_p^\Delta(\xi_2) + \frac{[u]Q_p^1(\xi_2)}{P_p^1(\xi_2)}P_p^\Delta(\xi_2)\right] + \left(\frac{1}{\mu_1}\right)P_p^\Delta(\xi_2) + \frac{1}{\mu_1}[u]Q_p^\Delta(\xi_2)\right\}} = \tag{G-11b}$$

$$= \frac{\dfrac{\mu_1 J_p(\theta)\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1)P_p^1(\nu)}}{\left[\dfrac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \dfrac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)}\right]} = J_p^I$$

68

$$M_p \bigg|_{\mu_2=\mu_1} = J_p^I \qquad (G-12)$$

It is noted that this expression is identical to $F_p$ of Equation (E-6b). The coefficient $N_p$ is expressed as (see Equation 62)

$$N_p = L_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + M_p \qquad (G-13)$$

and since $L_p$ was shown to be zero in Equation (G-3), $N_p$ is equal to $M_p$. The expression for $K_p$ is defined in Equation (59) to be

$$K_p = J_p^I \qquad (G-14)$$

which is equal to $N_p$ and $M_p$.

The last coefficient is $H_p$ which can be expressed as (see Equation 63)

$$H_p = I_p + K_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \qquad (G-14)$$

$$H_p \bigg|_{\mu_2=\mu_1} = \left[ \frac{\mu_1 J_p(\theta) \, a \left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)} \right] \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right] \bigg/ \left[ \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \qquad (G-16)$$

This is identical with the expression for $A_p$ in Equation (E-6a) of Appendix E.

Thus, the mathematical expressions for $H_p$ and $N_p$ (Equations (G-16) and (G-13), respectively) for the thin current band surrounded by a spheroidal shell in the limit as $\mu_2 = \mu_1$ are the same as the coefficients $A_p$ and $F_p$ (see Equations (E-6a) and (E-6b), respectively) for the vector potential in the regions of the current band in free space (see Appendix E).

For comparison, the coefficients for the ferromagnetic shell problem with $\mu_2 = \mu_1$ are

$$M_p \bigg|_{\mu_2 = \mu_1} = J_p^I = \left[\frac{\mu_1 J_p(\theta)\, a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p(\nu)}\right] \Bigg/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)}\right] \qquad \text{(G-17)}$$

$$H_p \bigg|_{\mu_2 = \mu_1} = J_p^I \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} = \left[\frac{\mu_1 J_p(\theta)\, a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)}\right]\left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}\right] \Bigg/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)}\right]$$

$$\text{(G-18)}$$

and the coefficients for the current band problem are

$$A_p = F_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} = \left[\frac{\mu_1 J_p(\theta)\, a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)}\right]\left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}\right] \Bigg/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)}\right]$$

$$\text{(G-19)}$$

$$F_p = J_p^I = \left[\frac{\mu_1 J_p(\theta)\, a\left(\xi_1^2 - \nu^2\right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(\nu)}\right] \Bigg/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)}\right] \qquad \text{(G-20)}$$

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73