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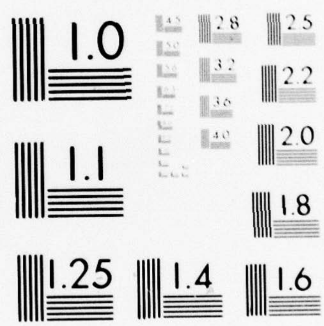

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

A PHYSICIST'S GUIDE TO J_2

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See 1473 in back

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ABSTRACT

This report presents a first-order discussion of the perturbations in the orbital motion of a satellite about an oblate primary. The approach is non-traditional in that the secular changes in the semi-major axis, eccentricity, inclination, and longitude of the ascending node are all derived directly from the translational and rotational equations of motion. The exactly solvable problem of motion in the primary's equatorial plane is used to obtain the advance of the argument of periastron and the consequent change in the time of periastron passage. In addition, a simple discussion of a very restricted three-body problem is given to illustrate the effects of third-body perturbations.

CONTENTS

I. INTRODUCTION

A. Motivation and Outline

Celestial mechanics is one of the oldest fields of study in physics and astronomy. With the advent of the space age and the rapid development of the electronic digital computer, the field has undergone a resurgence of both theoretical development and practical use. Old subjects have their own vocabulary, their own mathematical methods, their own symbolism, their own assumed knowledge base, etc. Although many scientists and engineers lack this specialized training, there is still the need for them to be cognizant of the principal results of the two-body problem and of perturbation theory. For most artificial satellites, the principal perturbation is that due to the oblateness of the earth (symbolized by a non-zero value for J_2). In this report I derive all of the first-order secular variations due to J_2 in a non-traditional (for a celestial mechanic) fashion. Thus, one can see, without the appurtenances of celestial mechanics, both the physical cause of the perturbations and their effects. Some of these results are scattered in the standard reference books. However, the treatment of the exact solution for motion in the earth's equatorial plane is new. Finally, as the satellite's distance from the earth increases, the perturbations of its motion due to the sun and the moon become more important. Hence, a very simple treatment of third-body perturbations is also included.

The report has some caveats the reader needs to know. First, because almost no new physics is involved, the equations have been kept to a minimum. Second, to keep the framework of the discussion constant, this is a geocentric report. Nowhere is any particular property of the earth used nor does the geocentric aspect limit the discussion. Lastly, except for §IID*, the entire report is for first-order perturbation theory only. Extrapolating the conclusions reached herein beyond this limitation is not necessarily correct.

We next present brief summaries of two-body elliptic motion and the geopotential. Following that is a detailed discussion of motion in the earth's equatorial plane. This illustrates various techniques of analysis and an analytic solution to such a problem. We then turn to motion in any plane and exhibit another method of analysis. This time the equations for the rates of change of the angular momentum and the energy are used to further our understanding of the motion. This section closes with the perturbation equations in the classical form. The last section deals with a primitive three-body problem so that the effects of the sun and moon can be illustrated.

B. Brief Summary of Two-Body Elliptic Motion

The classical two-body problem has the artifical satellite, of negligible mass m, revolving about the earth whose mass is $M_{\oplus}$. The coordinate system with origin at the center of mass of the earth is assumed to be an inertial one. We will take the extension of the earth's equator as the xy plane and have the x axis point in the direction of the Vernal Equinox. The z axis points in the direction of the North Celestial Pole

*Equations (46, 47) are exact but our handling of them is first-order.

and the y axis completes a right-handed triple. The force is central [the potential is given in Eq. (11c)] so the motion takes place in a plane. The plane's normal is the angular momentum vector. The form of the orbit is an ellipse which can be parametrically described by

$$r = a(1-e^2)/(1 + e\cos v) \tag{1a}$$

or

$$r = a(1 - e\cos E). \tag{1b}$$

The ellipse's semi-major axis is a, its eccentricity is e, v is the azimuthal coordinate in the orbital plane measured from the point of closest approach (i.e., perigee; v is also called the true anomaly), and E is an auxiliary angular variable named the eccentric anomaly. For an arbitrary orientation of polar coordinates in the orbital plane the distance, r, would be given by

$$r = a(1-e^2)/[1 + e\cos(u-\omega)], \tag{1c}$$

$$u = v + \omega, \tag{2}$$

where ω is called the argument of perigee. See Figure 1. The time dependence of the orbit is implicitly given by the relationships between the true anomaly, the eccentric anomaly, and the mean anomaly, M;

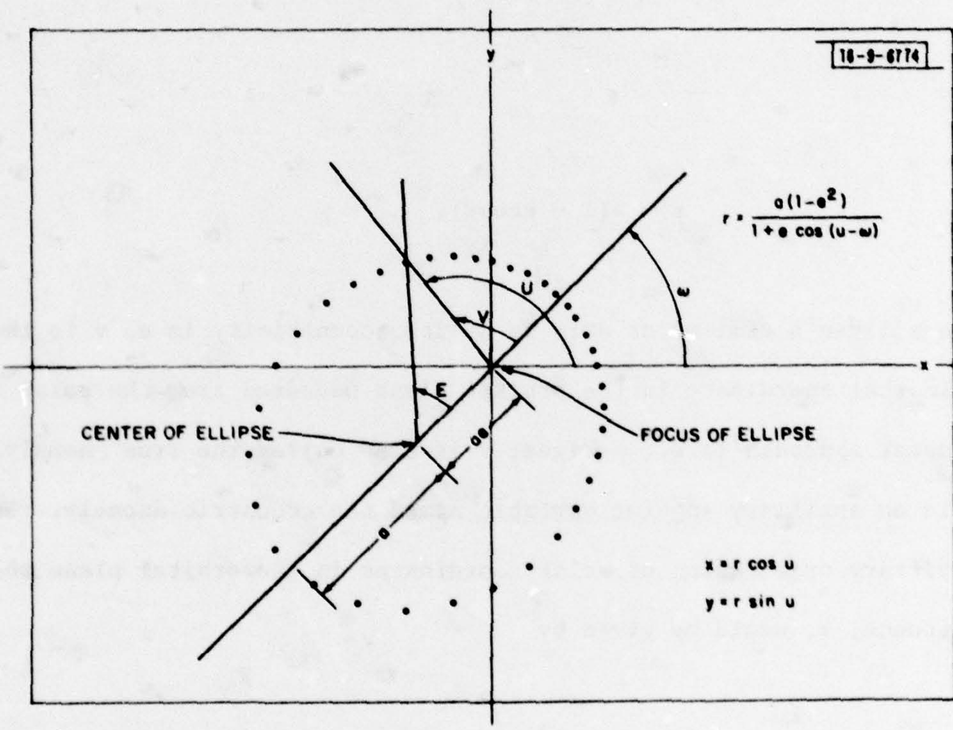

Fig. 1. Illustration of the meaning of the angles E, u, v, and ω. The diagram is drawn for a = 1, e = 0.5, ω = 45°.

$$\tan(v/2) = [(1 + e)/(1 - e)]^{1/2}\tan(E/2), \qquad (3)$$

$$E - e\sin E = M = n(t - T) = n(t - T_o) + M_o, \qquad (4)$$

where the mean motion, n, is related to the semi-major axis and period, P, of the motion by

$$n = (GM_{\oplus}/a^3)^{1/2} \equiv (\mu/a^3)^{1/2} = 2\pi/P. \qquad (5)$$

G is the universal constant of gravitation. The standard epoch is the time of perigee passage, T, when M = 0. If some other, arbitrary, epoch $t = T_o$ is used, then $M = M_o$ at that instant.

To complete the three dimensional picture of the motion, we need to know the direction of the angular momentum vector, $\hat{L}$,

$$\hat{L} = (\sin\Omega\sin i, -\cos\Omega\sin i, \cos i). \qquad (6a)$$

The inclination of the orbital plane to the equator is symbolized by $i\varepsilon[0,180°]$. That place on the equator where the plane of the orbit intersects it and the satellite is moving northward is called the ascending node. Ω is the longitude of the ascending node, $\Omega\varepsilon[0,360°)$. See Figure 2. The magnitude of the angular momentum vector is given by mL,

$$L = [\mu a(1 - e^2)]^{1/2}, \ \underline{L} = L\hat{L}. \qquad (6b)$$

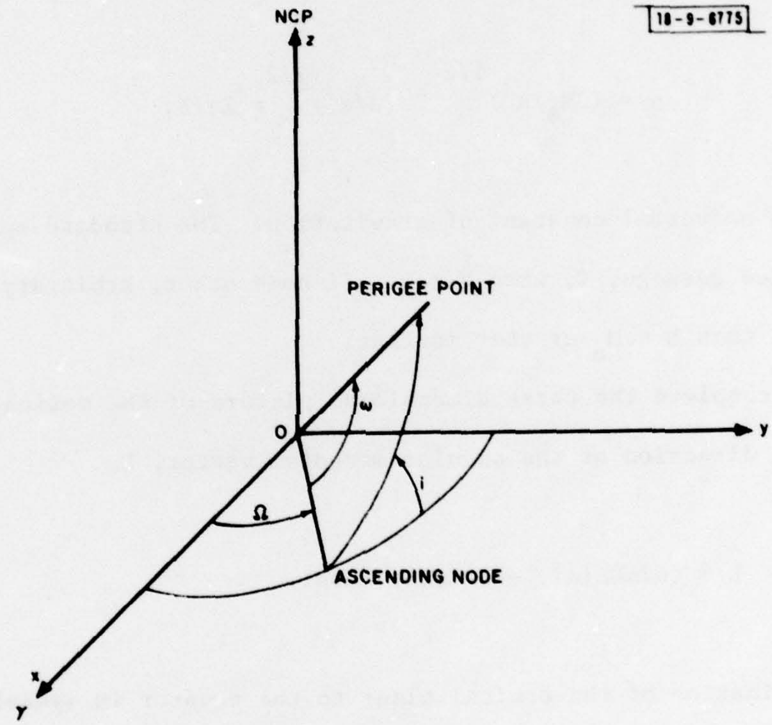

Fig. 2. Illustration of the meaning of the angles i, Ω, and ω.

The angle (for which there is no standard notation) $\omega + \Omega$ is called the longitude of perigee. Finally, to go from the cartesian orbital plane location vector

$$\underline{r} = r(\cos v, \sin v, o), \tag{7a}$$

to the cartesian geocentric location vector or spherical geocentric location vector,

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix}, \tag{7b}$$

we need to rotate by the orthogonal matrix S,

$$S = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7c}$$

The results are

$$\begin{aligned} x/r &= \cos\Omega\cos u - \sin\Omega\cos i\sin u, \\ y/r &= \sin\Omega\cos u + \cos\Omega\cos i\sin u, \\ z/r &= \sin i\sin u, \end{aligned} \tag{8}$$

or

$$\alpha = \Omega + \tan^{-1}(\cos i \tan u), \qquad (9a)$$

$$\delta = \sin^{-1}(\sin i \sin u), \qquad (9b)$$

where u was defined in Eq. (2). Alternative expressions of the invariable plane are

$$x\sin\Omega\sin i - y\cos\Omega\sin i + z\cos i = 0 \qquad (10a)$$

and

$$\tan\delta = \tan i\sin(\alpha - \Omega). \qquad (10b)$$

All of this is derived directly from the equations of motion. In the plane of the orbit their form is

$$\ddot{r} - r\dot{v}^2 = -\partial U/\partial r;\ dw/dt \equiv \dot{w}\ \forall w, \qquad (11a)$$

$$r\ddot{v} + 2\dot{r}\dot{v} = -(1/r)\partial U/\partial v, \qquad (11b)$$

$$U = -\mu/r. \qquad (11c)$$

The key to the solution of the two-body problem and the remainder of the analysis presented herein is the ability to analytically integrate Eq. (11b) once.

C. Brief Summary of the Geopotential Through J_2

The gravitational potential at a point located $\underline{r}$ outside of a body whose mass density is ρ is given by

$$U(\underline{r}) = -G \int_{\text{Volume}} \frac{\rho(\underline{r}')d\underline{r}'}{|\underline{r} - \underline{r}'|} . \tag{12}$$

If one uses the spherical harmonic expansion of the denominator of the integral one finds, through second-order terms,

$$U(\underline{r}) \simeq (G/r)\int\rho(\underline{r}')d\underline{r}' - (G/r^2)\int r'\rho(\underline{r}')\cos\beta d\underline{r}'$$

$$+ (G/2r^2)\int(r')^2\rho(\underline{r}')(1 - 3\cos^2\beta)d\underline{r}'. \tag{13}$$

The angle between $\underline{r}$ and $\underline{r}'$ is denoted by β. The first integral is the total mass ($M_\oplus$ for the earth). The second integral vanishes because the origin of the coordinate system is at the center of mass. The third integral is equal to [(three times the moment of inertia about the line joining the center of mass and $\underline{r}$) minus (the sum of the principal moments of inertia)]. If we employ the symmetry of an oblate spheroid and express the last integral in terms of the principal moments of inertia of the spheroid (A = equatorial moment of inertia, C = polar moment of inertia) then

$$U(r,\delta) = -(\mu/r)[1 + J_2R_\oplus^2(1 - 3\sin^2\delta)/(2r^2)] \tag{14a}$$

where

$$J_2 = (A - C)/(M_\oplus R_\oplus^2) = 1.0827 \times 10^{-3} \text{ for the earth,} \tag{14b}$$

and δ is the geocentric latitude of $\underline{r}$. We shall abbreviate

$$\eta = J_2 R_\oplus^2, \tag{15}$$

and note that the ratio of the forces due to the earth's total mass and that due to its oblateness on an object a distance $R > R_\oplus$ away is

$$\varepsilon = \eta/R^2. \tag{16}$$

For a stationary satellite, $\varepsilon \approx 2.5 \times 10^{-5} \approx 5$ arc seconds.

II. MOTION IN THE EARTH'S EQUATORIAL PLANE

With $\delta = 0$ the potential, from Eq. (14a), is

$$U(r,0) = -(\mu/r)(1 + J_2R_\oplus^2/2r^2) = -(\mu/r)(1 + \eta/2r^2). \tag{17}$$

The equations of motion, Eqs. (11a, 11b), have the two integrals

$$(\dot{r}^2 + r^2\dot{v}^2)/2 + U(r,0) = E(\leq 0 \text{ here}), \tag{18a}$$

$$r^2\dot{v} = L\ (\geq 0 \text{ always}). \tag{18b}$$

The existence of the angular momentum integral allows a discussion of the orbit, $r(v)$, instead of the time dependence of the motion. Since,

$$r' \equiv dr/dv = \dot{r}/\dot{v},\ \dot{r} = Lr'/r^2. \tag{19}$$

The conservation of energy equation, Eq. (18a), can be written as

$$(r')^2 + r^2 = 2r^4[E - U(r,0)]/L^2. \tag{20}$$

When $J_2 = 0$ the semi-major axis and eccentricity were defined by

$$a = -\mu/2E,\ e^2 = 1 + 2EL^2/\mu^2. \tag{21}$$

As long as J_2 is small, these quantities will have a meaningful interpretation as a "semi-major axis" and "eccentricity". In any case, they too are constants of the motion. Eliminating $\mathcal{E}$ and L^2 from Eq. (20), the equation describing the orbit may be reexpressed as

$$(r')^2 = - rF(r)/[a^2(1 - e^2)]^{1/2}, \tag{22a}$$

with

$$F(r) = r(r - r_+)(r - r_-) - a\eta, \tag{22b}$$

and

$$r_{\pm} = a(1 \pm e). \tag{22c}$$

This will be the starting point for the remainder of the analysis presented in §II. We note that if J_2 were zero, then the roots of F would be 0, r_- (perigee distance), and r_+ (apogee distance).

A. Circular Orbits

A circular orbit is always possible for non-zero J_2 and always stable for small J_2. Since $r' = 0$, the radius of the orbit is a root of F. It's actually simpler to derive it from Eq. (20). The result is ($\varepsilon = \eta/a^2$)

$$\text{radius} = (a/2)[1 + (1 - 6\varepsilon)^{1/2}]<a. \tag{23}$$

Thus, the only effect is to reduce the size of the orbit. But from Eq. (17) we see that if r is constant, then a positive J_2 merely augments the central mass; hence for the same energy the satellite would be more tightly bound.

B. "Elliptical Orbits": Apogee and Perigee

It is clear from Eq. (22a) that the roots of F represent the turning points of the orbit. It is a simple matter to show that F has three real roots (two of which are equal for a circular orbit) and we know that as $J_2 \to 0$ they must approach 0, r_-, and r_+. The one near zero is not a physical turning point. Let us call the three true roots of F (for $J_2 \neq 0$), R_o, R_p (for perigee), and R_a (for apogee). Their values can be approximated through the use of Newton's method. One finds ($\varepsilon = \eta/a^2$ still)

$$R_o \simeq a\varepsilon/(1 - e^2), \tag{24a}$$

$$R_p \simeq r_- - a\varepsilon/[2e(1 - e)] < r_-, \tag{24b}$$

$$R_a \simeq r_+ + a\varepsilon/[2e(1 + e)] > r_+. \tag{24c}$$

Thus, perigee is closer in and apogee further out for the perturbed "ellipse" than for the unperturbed ellipse. The "semi-major axis", A, and "eccentricity", E, can be derived by using the formulas for elliptical motion, namely

$$2A = R_a + R_p, \tag{25a}$$

$$2AE = R_a - R_p, \tag{25b}$$

or

$$A/a = 1 - \varepsilon/[2(1 - e^2)] < 1, \tag{26a}$$

$$E/e = 1 + \varepsilon/(1 + e^2)/[2e^2(1 - e^2)] > 1. \tag{26b}$$

We already knew that the "focal distance" was smaller and we now see that the reason is that the "eccentricity" is larger and the "semi-major axis" is smaller.

It now seems logical to ask about the utility of

$$r = A(1 - E^2)/(1 + E\cos v), \tag{27}$$

as a representation of the orbit. If we express this through first order in η we find

$$r = r_o(v) - \frac{a\varepsilon(\cos v + 3e + 3e^2\cos v + e^3)}{2e(1 - e^2)(1 + e\cos v)^2}, \tag{28a}$$

where

$$r_o(v) = a(1 - e^2)/(1 + e\cos v). \tag{28b}$$

Equation (27) can only represent the orbit except for those terms in v which vanish at $v = 0$ and $v = \pi$. However, the value of r at the turning points is independent of v, so that there can be no secular change in A or E due to J_2.

C. "Elliptical Orbits": The Orbit

Assuming that Eqs. (22) can not be exactly solved, we try a physicist's perturbation approach. We write

$$r(v) = r_o(v) + \varepsilon r_1(v) + \varepsilon^2 r_2(v) + \dots, \tag{29}$$

substitute this into the equation, separate powers of ε, and solve the resulting system of equations. If we do this [but we actually use $r(v) = r_o(v) + \varepsilon(1 - e^2)r_1(v)$ for simplicity] the coefficient of ε (which must vanish) is

$$r_1' + p(v)r_1 = q(v), \tag{30a}$$

$$p(v) = (e\cos^2 v - \cos v - 2e)\cos v/(1 + e\cos v), \tag{30b}$$

$$q(v) = a(1 + e\cos v)\csc v/[2e(1 - e^2)^2]. \tag{30c}$$

The solution of Eqs. (30) for $r_1(v)$ is

$$r_1(v) = \frac{C\sin v}{(1 + e\cos v)^2} - \frac{a[\cos v + 3e + 3e^2(\cos v + v\sin v) + e^3(1 + \sin^2 v)]}{2e(1 - e^2)(1 + e\cos v)^2}. \tag{30d}$$

The arbitrary constant of integration is C and the initial condition is $r(0) = R_p$ with the expression (24b) used for R_p. Although the initial condition does not determine the value of C, the existence and uniqueness theorem for the differential equation (30a-c) allows us to set C = 0. Hence,

$$r(v) = r_o(v) - \frac{a\varepsilon[\cos v + 3e + 3e^2(\cos v + v\sin v) + e^3(1 + \sin^2 v)]}{2e(1 - e^2)(1 + e\cos v)^2}. \tag{31}$$

Compare with Eqs. (28).

We can now extend our analysis to compute the advance of perigee. The presence of the secular term, $\propto v\sin v$, means that although $r(0) = r(2\pi) = r(4\pi) = \ldots$ and $r(\pi) = r(3\pi) = r(5\pi) \ldots$ (cf. page 14), it is not necessarily true that the minimum and maximum values of r occur at these true anomalies. By construction, the first perigee occurs for v = 0. We know the first apogee must occur near $v = \pi$ [cf. Eq. (1a) or (28b)]. By computing $r'(v)$ from Eq. (31), setting $v = v_a = \pi + \varepsilon V_a$ in the result, and solving for V_a by expanding all terms through first order in ε we find

$$v_a \simeq \pi\{1 + 3\varepsilon/[2(1 - e^2)^2]\} > \pi. \tag{32a}$$

Similarly, the second perigee must occur near $v = 2\pi$, and an analogous procedure yields

$$v_p = 2v_a > 2\pi. \tag{32b}$$

Hence, the advance of the argument of perigee is symmetrically distributed over each half of the orbit. Clearly the time of perigee passage does not increase by only P. Its change is given by $(v_p - 2\pi)/\langle\dot{v}\rangle$ where the angular bracket denotes an average angular rate. A simple average for $\dot{v}$ is obtained from Eq. (18b) at the perigee and apogee points, whence

$$T_p - P \simeq 3\pi\varepsilon/[n(1 - e^2)^{3/2}] > 0. \tag{33}$$

The reader may notice that $r(\pi) = R_a$ with R_a given by Eq. (24c) and then wonder why v_a of Eq. (32a) is greater than π. The reason is that the Taylor series for the cosine contains no first-order terms.

Finally, to illustrate these effects in an exaggerated fashion Eq. (31) has been graphed for $a = 1$, $e = 0.25$, $\varepsilon = 0.05$ and $\varepsilon = 0.10$ at 10° intervals in the true anomaly. The solution of Eqs. (22) in the form given by Eq. (29) can not be reliably carried out beyond $\varepsilon v \sim 1$ except by piecewise continuation.

D. The Exact Solution

The exact solution of Eqs. (22a, b) involves elliptic functions. We need, at minimum, the sine amplitude, sn, the cosine amplitude, cn, and the delta amplitude, dn. We can define the sine amplitude via

$$\mu = \int_o^\phi (1 - \kappa^2\sin^2\theta)^{1/2}d\theta, \quad \mathrm{sn}(\mu, \kappa) \equiv \sin\phi, \tag{34b}$$

and the others by

$$\mathrm{sn}^2(\mu, \kappa) + \mathrm{cn}^2(\mu, \kappa) = 1, \tag{34a}$$

$$\kappa^2\mathrm{sn}^2(\mu, \kappa) + \mathrm{dn}^2(\mu, \kappa) = 1. \tag{34c}$$

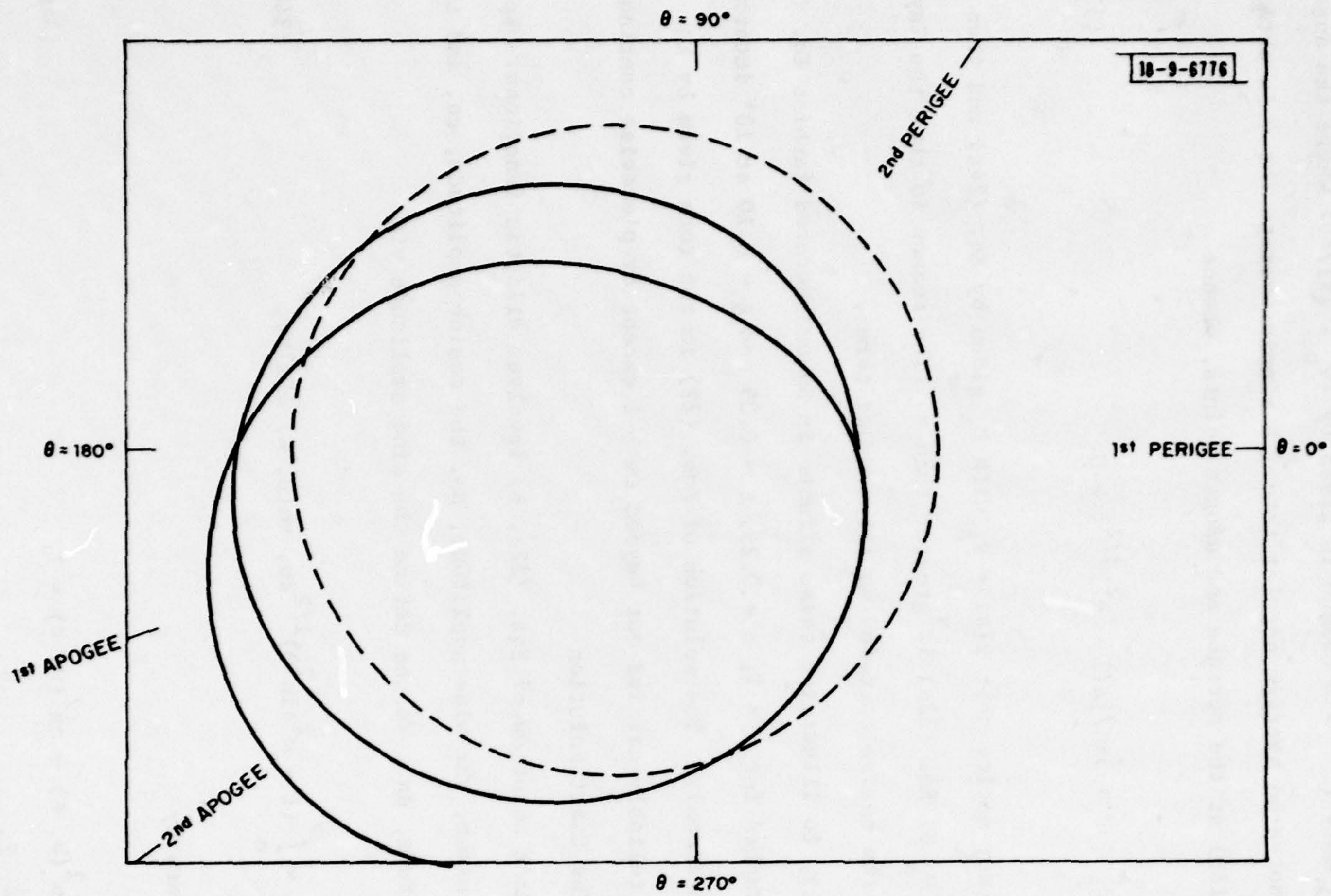

Fig. 3. Perturbed motion (solid curve) and unperturbed motion (dashed curve) for $a = 1$, $e = 0.25$, $\varepsilon = 0.1$ and $a = 1$, $e = 0.25$, $\varepsilon = 0$.

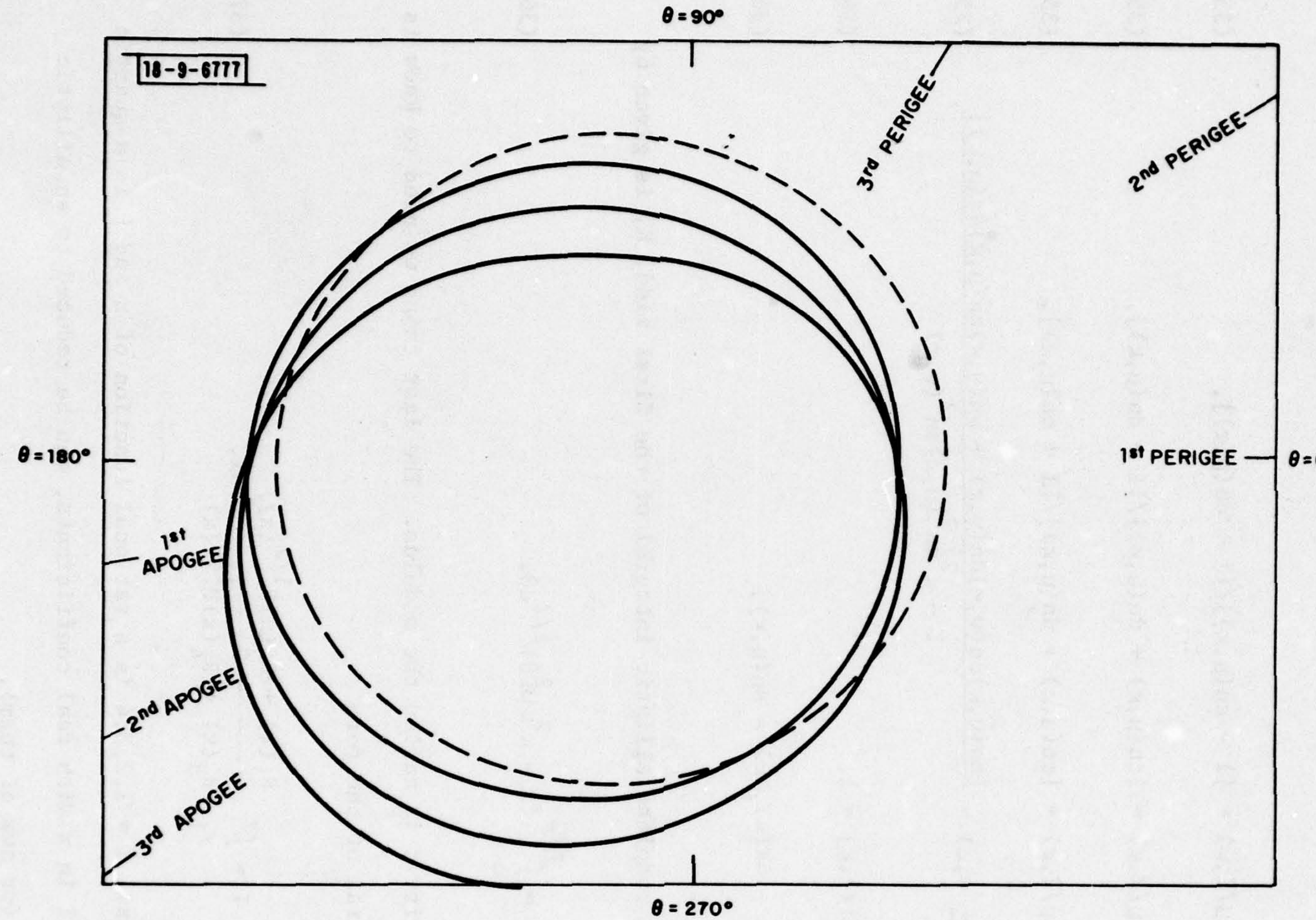

Fig. 4. Perturbed motion (solid curve) and unperturbed motion (dashed curve) for $a = 1$, $e = 0.25$, $\varepsilon = 0.05$ and $a = 1$, $e = 0.25$, $\varepsilon = 0$.

While these functions have various addition theorems, half-angle formulas, periodicities, etc. we shall need few of them here. In particular though,

$$\mathrm{sn}^2(\mu/2,\kappa) = [1 - \mathrm{cn}(\mu,\kappa)]/[1 + \mathrm{dn}(\mu,\kappa)], \tag{35a}$$

$$\mathrm{cn}^2(\mu/2,\kappa) = [\mathrm{cn}(\mu,\kappa) + \mathrm{dn}(\mu,\kappa)]/[1 + \mathrm{dn}(\mu,\kappa)], \tag{35b}$$

$$\mathrm{dn}^2(\mu/2,\kappa) = [\mathrm{cn}(\mu,\kappa) + \mathrm{dn}(\mu,\kappa)]/[1 + \mathrm{cn}(\mu,\kappa)], \tag{35c}$$

$$\mathrm{sn}(\mu \pm \nu,\kappa) = \frac{[\mathrm{sn}(\mu,\kappa)\mathrm{cn}(\nu,\kappa)\mathrm{dn}(\nu,\kappa) + \mathrm{sn}(\nu,\kappa)\mathrm{cn}(\mu,\kappa)\mathrm{dn}(\mu,\kappa)]}{1 - \kappa^2\mathrm{sn}^2(\mu,\kappa)\mathrm{sn}^2(\nu,\kappa)}, \tag{35d}$$

$$\mathrm{sn}[K(\kappa),\kappa] = 1, \tag{36a}$$

$$\mathrm{sn}[\mu + 4K(\kappa), \kappa] = \mathrm{sn}(\mu,\kappa), \tag{36b}$$

where the complete elliptic integral of the first kind, K, is given by

$$K(\kappa) = \int_0^{\pi/2} (1 - \kappa^2\sin^2\theta)^{1/2} d\theta. \tag{36c}$$

The quantity κ is called the modulus. The last thing we need to know is that any integral of the form

$$I = \int_{y_1}^{y} \frac{R_1(x) + R_2(x)P^{1/2}(x)}{R_3(x) + R_4(x)P^{1/2}(x)} dx, \tag{37}$$

where $R_i(x)$, $i = 1,2,3,4$ is a rational function of x and P is a quartic polynomial in x with real coefficients, can be reduced to an elliptic function (or sum of them).

Now, Eqs. (22a, 22b) are of the form (37) if we rewrite them as

$$dr/dv = \pm \{-rF(r)/[a^2(1 - e^2)]\}^{1/2}. \tag{38}$$

We also know that $r' \geq 0$ for $v \varepsilon [0, v_a]$ and $r' \leq 0$ for $v \varepsilon [v_a, v_p]$. The true anomalies of first apogee passage and second perigee passages are approximately given in Eqs. (32). Thus,

$$\int_{R_p}^{r} \frac{ds}{[-sF(s)]^{1/2}} = \frac{+1}{a(1 - e^2)^{1/2}} \int_{o}^{v} dw; \; R_p \leq r \leq R_a, \; 0 \leq v \leq v_a, \tag{39a}$$

$$\int_{R_a}^{r} \frac{ds}{[-sF(s)]^{1/2}} = \frac{-1}{a(1 - e^2)^{1/2}} \int_{v_a}^{v} dw; \; R_p \leq r \leq R_a, \; v_a \leq v \leq v_p. \tag{39b}$$

There exists a standard technique for reducing integrals of the form (37) to the standard forms [of which (34a) is one example]. The results are for our cases

$$\gamma v/2 = sn^{-1}\left(\left[\frac{R_a - R_o}{R_a - R_p} \cdot \frac{r - R_p}{r - R_o}\right]^{1/2}, k\right); \; v \varepsilon [0, v_a], \tag{40a}$$

$$\gamma(v - v_a)/2 = sn^{-1}\left(\left[\frac{R_p}{R_a - R_p} \cdot \frac{R_a - r}{r}\right]^{1/2}, k\right); \; v \varepsilon [v_a, v_p], \tag{40b}$$

where the modulus, k, is given by

$$k^2 = \frac{R_o(R_a - R_p)}{R_p(R_a - R_o)}, \; k^2 \to 0 \text{ as } \varepsilon \to 0, \tag{40c}$$

and the quantity γ is given by

$$\gamma^2 = \frac{R_p(R_a - R_o)}{a^2(1 - e^2)}, \; \gamma^2 \to 1 \text{ as } \varepsilon \to 0. \tag{40d}$$

If we use Eqs. (34, 35, and 36) then <u>both</u> of Eqs. (40a, b) can be transformed into

$$r(v) = \frac{R_oR_p\{[1 + dn(\gamma v,k)] -k^2 [1 - cn(\gamma v,k)]\}}{R_o[1 + dn(\gamma v,k)] - R_pk^2[1 - cn(\gamma v,k)]} . \tag{41}$$

We can now rederive all of our earlier work as a check. From Eq. (36c) we see that if κ is small,

$$K(\kappa) \simeq (\pi/2)(1 + \kappa^2/4). \tag{36d}$$

Then from Eq. (36a) it follows that

$$v_a \simeq \pi \{1 + 3\varepsilon/[2(1 - e^2)^2]\}. \tag{32a}$$

From Eqs. (39) and this result Eq. (32b) follows immediately. Also, from Eqs. (34) it follows that if κ is small,

$$cn(\mu,\kappa) \simeq \cos\mu + (\kappa^2/4)(\mu - \sin\mu\cos\mu)\sin\mu, \tag{34d}$$

$$dn(\mu,\kappa) \simeq 1 - (\kappa^2/2)\sin^2\mu, \tag{34e}$$

so, from these approximations, Eqs. (40c, d and 41) the result in Eq. (31) can be derived. The periodicity of the motion is assured because cn and dn share 4K as a period. The only remaining point is the time dependence of r and v which can be computed directly from Eqs. (18) and involves other elliptic functions.

III. GENERAL MOTION

In this section the rates of change of the energy and the angular momentum are used to derive the first-order secular perturbations in four (a, e, i, and Ω) of the six orbital elements. The analysis of the preceding section has determined the changes in the other two except for a multiplicative dependence on inclination (which is non-zero for $i = 0$) and an additive dependence on inclination (which vanishes for $i = 0$). At the end of this section one version of the classical perturbation equations is given and solved.

The potential was given in Eqs.(14, 15).

A. The Work Done by J_2

The work done by J_2, dE/dt, is given by

$$dE/dt = - \nabla U \cdot \underline{\dot{r}}. \tag{42a}$$

Over the course of one revolution the change in E is

$$\Delta E = \S \dot{E} dt. \tag{42b}$$

One can't simply note that $\nabla \times \nabla U = \underline{0}$ and use Green's theorem to evaluate this though. The sign, §, means integrate around one complete circuit of the perturbed orbit. We can not assume that this leads to the same point in space. If we split U into $U_o + \eta U$, where $U_o = - \mu/r$ then,

$$\Delta E = \Delta E_o + \Delta E_1 = \oint [- \nabla U_o - \eta \nabla U_1] \cdot \underline{\dot{r}} dt. \tag{42c}$$

Now, to first-order in η we can deform the contour for the U_o integration to be the unperturbed orbit. Now we exploit the fact that the force due to U_o is conservative so $\Delta E_o = 0$. Since ηU_1 is already first-order, we can again deform the contour and $\Delta E_1 = 0$. But,

$$E = - \mu/2a, \tag{21}$$

so

$$\Delta a = 0 \text{ or } da/dt)_{sec} = 0, \tag{43a}$$

where $da/dt)_{sec}$ is the first-order secular rate of change of the semi-major axis due to J_2.

B. The Torque Due to J_2

The rate of change of the angular momentum due to J_2 is given by

$$d\underline{L}/dt = \underline{r} \times \underline{F} = - \underline{r} \times \nabla U = - \eta \underline{r} \times \nabla U_1, \tag{44a}$$

since U_o represents a central force. By direct computation we find

$$\underline{\dot{L}} = - 3(\mu \eta z/r^5)(y, -x, 0), \tag{44b}$$

and

$$\Delta \underline{L} = \int_0^P \dot{\underline{L}}dt = \frac{-3\mu\pi\eta\sin 2i}{2na^2(1-e^2)^{3/2}} (\cos\Omega, \sin\Omega, 0), \tag{44c}$$

$$= (\Delta L_x, \Delta L_y, \Delta L_z). \tag{45a}$$

From Eqs. (6)

$$\begin{aligned}
\Delta \underline{L} &= (L + \Delta L)(\sin[\Omega + \Delta\Omega]\sin[i + \Delta i], \\
&\quad - \cos[\Omega + \Delta\Omega]\sin[i + \Delta i], \cos[i + \Delta i]) - \underline{L} \\
&\simeq \Delta L(\sin\Omega\sin i, - \cos\Omega\sin i, \cos i) \\
&\quad + L\Delta\Omega(\cos\Omega\sin i, \sin\Omega\sin i, 0) \\
&\quad + L\Delta i(\sin\Omega\cos i, - \cos\Omega\cos i, - \sin i).
\end{aligned} \tag{45b}$$

From Eqs. (45) we find

$$\begin{aligned}
\Delta L &= \Delta L_x \sin\Omega\sin i - \Delta L_y \cos\Omega\sin i + \Delta L_z \cos i, \\
L\Delta\Omega\sin i &= \Delta L_x \cos\Omega + \Delta L_y \sin\Omega, \\
L\Delta i &= \Delta L_x \sin\Omega\cos i - \Delta L_y \cos\Omega\cos i - \Delta L_z \sin i,
\end{aligned} \tag{45c}$$

so, after dividing by P and setting $\Delta i/P = di/dt)_{sec}$ etc.,

$$di/dt)_{sec} = 0, \tag{43b}$$

$$d\Omega/dt)_{sec} = - 3\varepsilon n\cos i/[2(1 - e^2)^2], \qquad (43c)$$

$$da/dt)_{sec} = [2ea/(1 - e^2)]de/dt)_{sec}, \qquad (46)$$

where Eq. (6b) has been used to derive Eq. (46) from $dL/dt)_{sec} = 0$. Equation (46) and Eq. (43a) lead to

$$de/dt)_{sec} = 0. \qquad (43d)$$

Equations (43) then summarize our results. We see, therefore, that the application of the equations for the time rates of change and E and $\underline{L}$ have simply and directly enabled us to solve two-thirds of the perturbation problem. This approach also gives a direct physical insight into the cause of the perturbations as well as their effects.

C. Classical Perturbation Theory

The two-body problem with the potential given in Eq. (11c) forms the cornerstone of celestial mechanics. It represents the situation exactly and can be solved exactly. Hence, it is treated in all mechanics texts and is fully developed analytically. The coordinate system represented by the six orbital elements has become familiar and useful. Intuitively, if the total real force in a problem only departs slightly from that given by $-\nabla U_o$, the orbital elements themselves can only change slowly. This is the foundation of classical perturbation theory. The three, second-order, differential equations given by $\underline{F} = m\underline{a}$ are replaced by six, first-order differential equations for de/dt, di/dt, etc. This can be performed in a variety of ways for both non-conservative and conservative forces. The reduction is purely a computational problem and can be simplified by using the Lagrangian formalism. The important restriction is the condition of osculation. This mathematical statement's content is that both $\underline{r}$ and $\underline{\dot{r}}$ can be computed from the osculating values of the orbital elements via the ordinary, two-body, Keplerian formulas.

I shall give one form of the conservative equations here. Others, to be found in the standard reference texts, will vary in the choice of orbital parameters. Let the perturbing force be given by

$$\underline{F}_{pert} = + \nabla U'. \tag{46}$$

Then,

$$da/dt = [2/(na)]\partial U'/\partial M, \tag{47a}$$

$$de/dt = \{(1 - e^2)/[na^2 e]\}\partial U'/\partial M - \{(1 - e^2)^{1/2}/[na^2 e]\}\partial U'/\partial\omega, \tag{47b}$$

$$d\omega/dt = - \{\cot i/[na^2(1 - e^2)^{1/2}]\}\partial U'/\partial i + \{(1 - e^2)^{1/2}/[na^2 e]\}\partial U'/\partial e, \tag{47c}$$

$$di/dt = \{\cot i/[na^2(1 - e^2)^{1/2}]\}\partial U'/\partial\omega - \{\csc i/[na^2(1 - e^2)^{1/2}]\}\partial U'/\partial\Omega, \tag{47d}$$

$$d\Omega/dt = \{\csc i/[na^2(1 - e^2)^{1/2}]\}\partial U'/\partial i, \tag{47e}$$

$$dM/dt = -\{(1-e^2)^2/[na^2e]\}\partial U'/\partial e - [2/(na)]\partial U'/\partial a. \quad (47f)$$

As an illustration of how to use these, we take

$$U' = (\eta\mu)(1 - 3\sin^2\delta)/(2r^3), \quad (48)$$

from Eq. (14a). Since U' is already of first order, we can replace δ and r via Eqs. (9b and 1) in U', compute the derivatives on the right-hand sides of Eqs. (47), regard a, e, ..., as constants, and integrate. This will give us knowledge we have not yet derived, the explicit time dependence of a, e, ..., as opposed to their secular rates of change. Of course, if we integrate over a period we reproduce Eqs. (43a - d) and can derive the inclination dependence for $d\omega/dt$ and dT/dt. The latter results are

$$d\omega/dt)_{sec} = 3\varepsilon n(5\cos^2 i - 1)/[4(1-e^2)^2], \quad (43e)$$

$$dT/dt)_{sec} = -3\varepsilon(1 - 3\sin^2 i\sin^2\omega)/[2(1-e)^3]. \quad (43f)$$

Note that for $\cos^2 i = 1/5$ ($i = 63° 26' 5.''82$ or $i = 116° 33' 54.''18$) there is no perigee advance due to J_2 (in first-order theory!).

Equations (47), or their non-conservative generalization, give us a very powerful tool for studying long-term perturbing effects analytically since short-period perturbations can be left out (they'll be averaged over anyway).

IV. A VERY RESTRICTED THREE-BODY PROBLEM

We could, of course, discuss third body perturbations within the framework of classical perturbation theory. However, in its simplest formulation, this problem admits an analytical first order solution so we'll follow a physicist's approach. Moreover, this time we will solve directly for the time dependence of r and v instead of for r(v). To set the stage the satellite's unperturbed orbit is circular, viz

$$r = a, \; v = 2\pi t/P \text{ with } 4\pi^2 a^3/P^2 = \mu. \tag{49a}$$

In the plane of the orbit lies another body, of mass M_p, also revolving about the earth in a circular orbit,

$$r_p = a_p, \; v_p = 2\pi t/P_p + \psi, \; 4\pi^2 a_p^3/P_p^2 = \mu. \tag{49b}$$

However, $a_p >> a$ or $P >> P_p$. Thus, we can regard the perturbing body to be fixed for a few revolutions of the satellite (a poor approximation for the Moon but a reasonable one for the Sun). The potential for the problem is

$$U = -\mu/r - GM_p/|\underline{r}_p - \underline{r}|, \tag{50}$$

or

$$U = -\mu/r - GM_p/[a_p^2 + r^2 - 2ra_p\cos(v - v_p]^{1/2}. \tag{51}$$

The equations of motion of the satellite are still Eqs. (11a, b). If we use the approximations mentioned above they become

$$\ddot{r} - r\dot{v}^2 = -\mu/r^2 + \mu\varepsilon\cos(v - \psi), \tag{52a}$$

$$r\ddot{v} + 2\dot{r}\dot{v} = -\varepsilon\sin(v - \psi), \tag{52b}$$

$$\varepsilon = (GM_p/a_p^2)/GM_\oplus/a^2). \tag{52c}$$

For the Sun $\varepsilon = 0.026$, for the Moon $\varepsilon = 0.00015$. We set

$$r = a + \varepsilon r_1,\ v = 2\pi t/P + \varepsilon v_1, \tag{53}$$

substitute into Eqs. (52), linearize, and find

$$\ddot{r}_1 - 12\pi^2 r_1/P^2 - 4\pi a\dot{v}_1/P = (\mu/a^2)\cos(2\pi t/P - \psi), \tag{54a}$$

$$a\ddot{v}_1 + 4\pi\dot{r}_1/P = -(\mu/a^2)\sin(2\pi t/P - \psi). \tag{54b}$$

As initial conditions we use

$$r(0) = 0,\ v(0) = 0, \tag{55a}$$

$$\dot{r}(0) = 0,\ \dot{v}(0) = 2\pi/P. \tag{55b}$$

We can integrate the v_1 equation once, substitute that into the r_1 equation and then integrate it twice. We then return to the v_1 equation to find it. The answer is

$$r/a = 1 + \varepsilon[B\cos(2\pi t/P + \phi) + (3\pi t/P)\sin(2\pi t/P - \psi) - 2\cos\psi], \tag{56a}$$

$$v = 2\pi t/P - \varepsilon\{2B\sin(2\pi t/P + \phi) - (3\pi t/P)[\cos(2\pi t/P - \psi) + \cos\psi] + 2\sin(2\pi t/P - \psi) + 5\sin\psi\}, \tag{56b}$$

where

$$B\cos\phi = 2\cos\psi, \tag{56c}$$

$$B\sin\phi = -(3/2)\sin\psi, \tag{56d}$$

and this solution is valid for times $|t| \lesssim P/(2\pi\varepsilon)$.

Given the artificial nature of the problem, it seems best just to illustrate the motion for a few values of ε and ψ. Figures 5 - 8 show the unperturbed circular orbit as a full curve and the perturbed orbit as a series of equally spaced dots [Δt = (1/36)'th of the unperturbed period] for $\psi = 0(90)360°$, $\varepsilon = 0.05$ from $t = 0$ until $t = P$. From the location of M_p and the direction of the initial velocity vector, we can simply interpret the results. It is a numerical accident that for $\psi = 90°$ or $270°$ $v(P) = 0$.

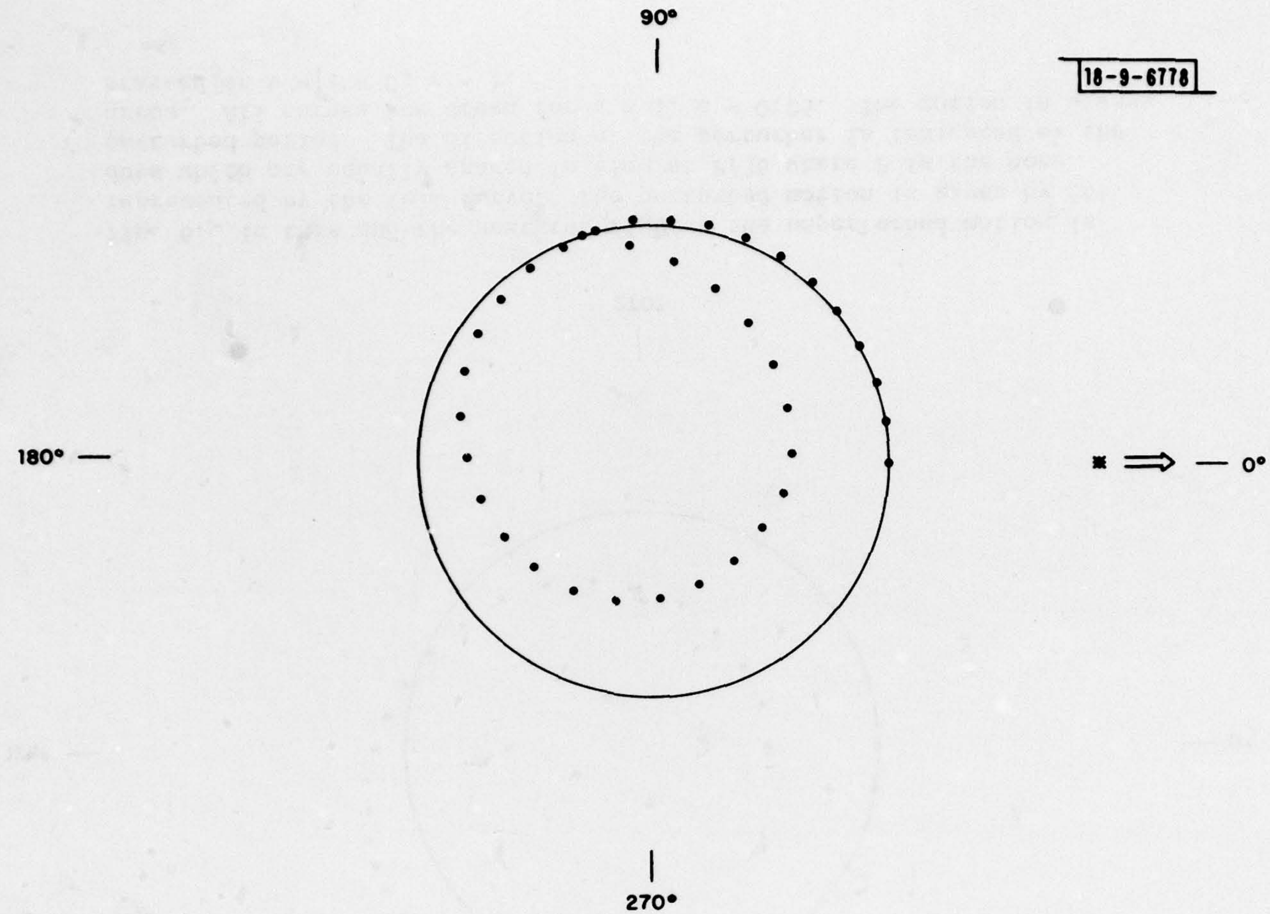

Fig. 5. In this and the next three figures the unperturbed motion is represented by the full curve. The perturbed motion is given by the dots which are equally spaced in time at $P/36$ where P is the non-perturbed period. The direction of the perturber is indicated by the arrow. All curves are drawn for $a = 1$, $\varepsilon = 0.05$. The motion is always started at $v = t = 0$, $r = 1$.

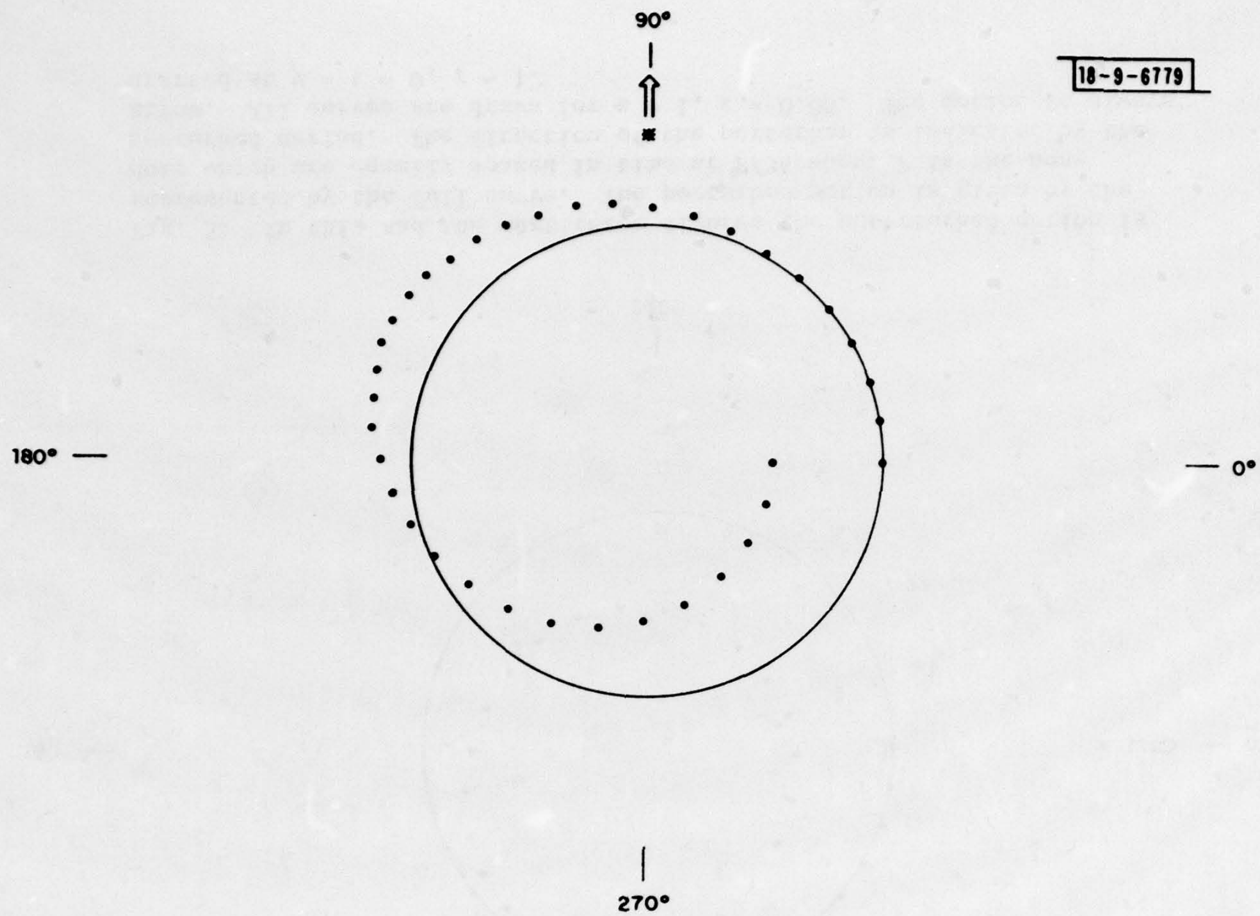

Fig. 6. In this and the next two figures the unperturbed motion is represented by the full curve. The perturbed motion is given by the dots which are equally spaced in time at P/36 where P is the non-perturbed period. The direction of the perturber is indicated by the arrow. All curves are drawn for $a = 1$, $\varepsilon = 0.05$. The motion is always started at $v = t = 0$, $r = 1$.

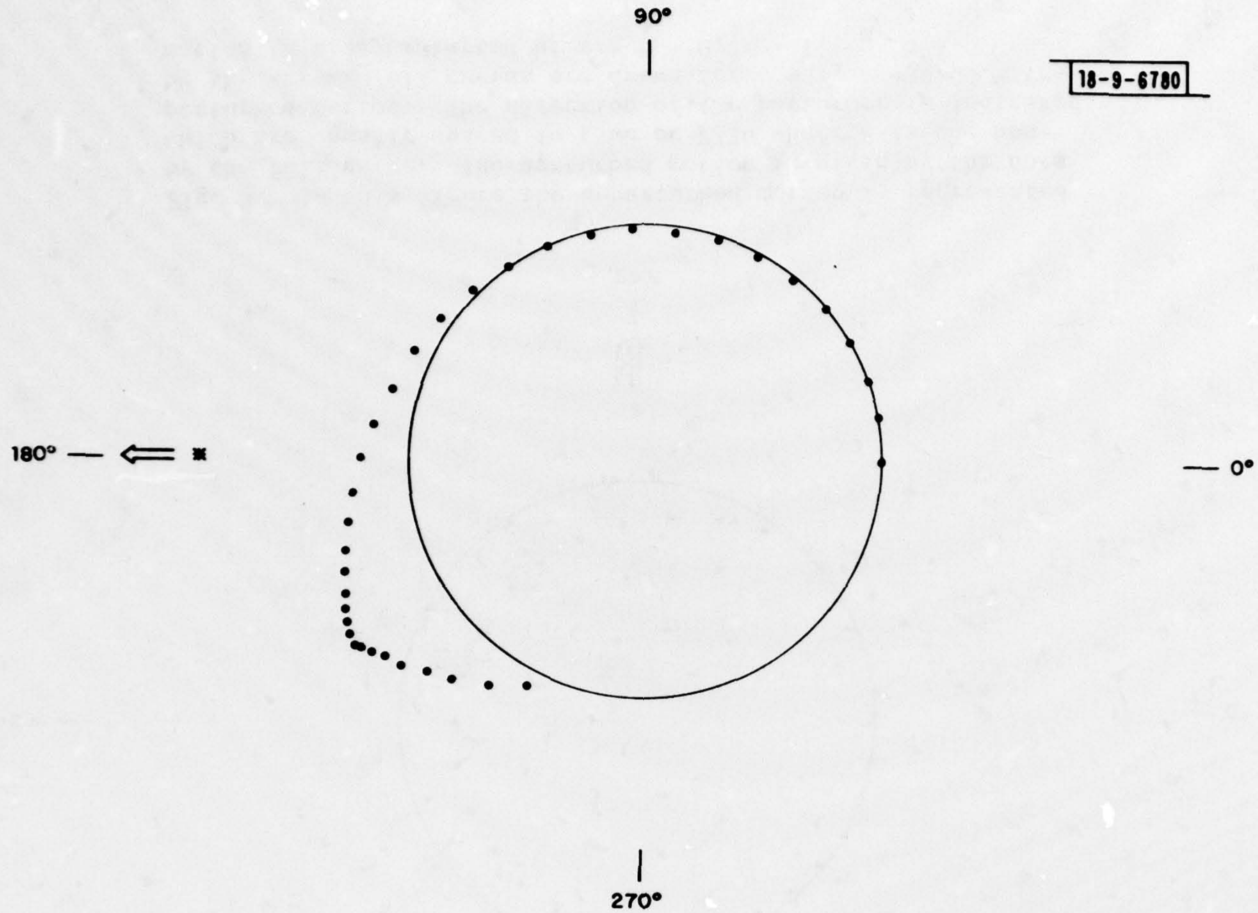

Fig. 7. In this and the next figure the unperturbed motion is represented by the full curve. The perturbed motion is given by the dots which are equally spaced in time at $P/36$ where P is the non-perturbed period. The direction of the perturber is indicated by the arrow. All curves are drawn for $a = 1$, $\varepsilon = 0.05$. The motion is always started at $v = t = 0$, $r = 1$.

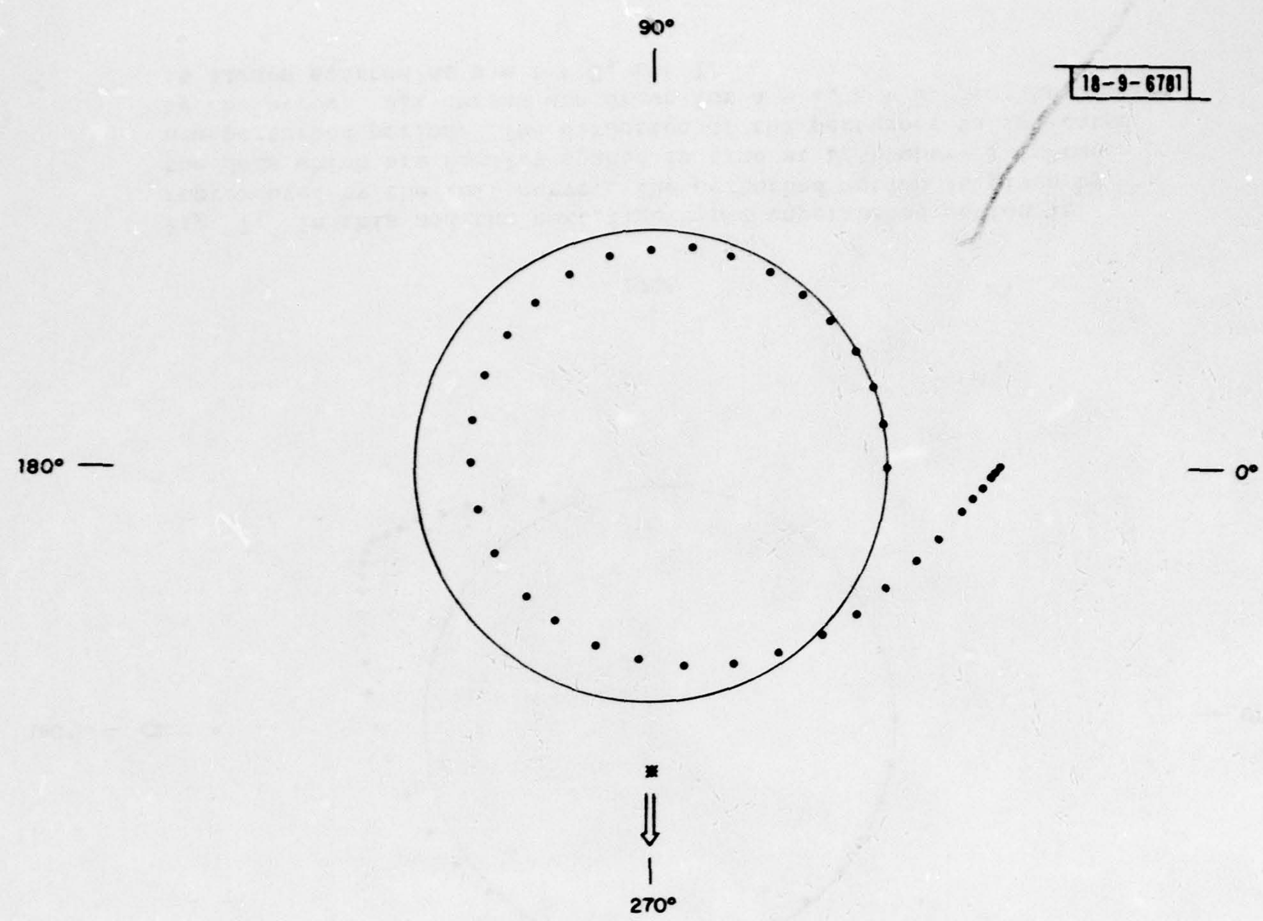

Fig. 8. In this figure the unperturbed motion is represented by the full curve. The perturbed motion is given by the dots which are equally spaced in time at P/36 where P is the non-perturbed period. The direction of the perturber is indicated by the arrow. All curves are drawn for $a = 1$, $\varepsilon = 0.05$. The motion is always started at $v = t = 0$, $r = 1$.

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This report presents a first-order discussion of the perturbations in the orbital motion of a satellite about an oblate primary. The approach is non-traditional in that the secular changes in the semi-major axis, eccentricity, inclination, and longitude of the ascending node are all derived directly from the translational and rotational equations of motion. The exactly solvable problem of motion in the primary's equatorial plane is used to obtain the advance of the argument of periastron and the consequent change in the time of periastron passage. In addition, a simple discussion of a very restricted three-body problem is given to illustrate the effects of third-body perturbations.

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