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(6) ANALYSES OF CROSS-PLY RECTANGULAR PLATES OF BIMODULUS COMPOSITE MATERIAL

by

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ACKNOWLEDGMENTS

This report constitutes the thesis submitted by Vallapureddy Sudhakar Reddy in partial fulfillment of the requirements for the Master of Science degree in Mechanical Engineering. For brevity, only the first two computer programs (for static bending) of Appendix E are included here. The thesis itself also includes programs for free vibration (pp. 102-112), thermal bending (pp. 113-121), and thin-plate nonlinear bending (pp. 122-127).

Dr. C.W. Bert served as thesis adviser and other members of the thesis committee were Drs. A.S. Khan and J.N. Reddy.

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NOMENCLATURE

A_{ij}	=	stretching stiffnesses
a,b	=	plate dimensions in x and y directions
B_{ij}	=	bending-stretching coupling stiffnesses
C_{rs}	=	coefficients defined in Eqs. (2.2.4), (3.3.4), and (4.3.4)
D_{ij}	=	bending stiffnesses
d_x	=	$\partial(\ )/\partial x$
E	=	Young's modulus of isotropic ordinary material
E_c,E_t	=	compressive and tensile Young's moduli (isotropic bimodulus)
E_{11}^k,E_{22}^k	=	Young's moduli in directions x and y (orthotropic bimodulus)
G_{13},G_{23}	=	longitudinal-thickness and transverse-thickness shear moduli
G_{zc},G_{zt}	=	transversely isotropic, bimodulus-material shear moduli
h	=	total thickness of plate
I	=	rotatory inertia coefficient per unit mid-plane area
K_4^2,K_5^2	=	shear-correction coefficients
L_{rs}	=	linear differential operators defined in Eqs. (2.1.10), (3.1.10), and (4.1.11)
M_i,N_i	=	stress couples and inplane stress resultants
M_i^T,N_i^T	=	thermally induced stress couples and inplane stress resultants
P	=	normal inertia coefficient per unit mid-plane area
Q_x,Q_y	=	thickness-shear stress resultants

$Q_{ijk\ell}$ = plane-stress-reduced stiffnesses

q, q_o = normal pressure and peak value of q

R = rotatory-normal-coupling inertia coefficient per unit mid-plane area

T = temperature

$T_o, T_1, \bar{T}_o, \bar{T}_1$ = temperature coefficients defined in Eqs. (4.4.1) and (4.4.2)

t = time

U,V,W = mid-plane displacement coefficients (amplitudes of u^o, v^o, w)

u,v,w = displacements in x,y,z directions

u^o, v^o, w = mid-plane displacements in x,y,z directions

V_f = fiber volume fraction

V_m = matrix volume fraction

X,Y = bending-slope coefficients (amplitudes of ψ_x, ψ_y)

x,y,z = plate coordinates in longitudinal, transverse, and downward thickness directions

Z_x, Z_y = z_{nx}/h, z_{ny}/h

z_{nx}, z_{ny} = neutral-surface positions associated with $\epsilon_x = 0$ and $\epsilon_y = 0$

α_j = coefficient of thermal expansion

α, β = π/a, π/b

ϵ_j, ϵ_j^o = strain component at arbitrary location and at mid-plane

ν = Poisson's ratio of isotropic material

ν_f, ν_m = fiber and matrix Poisson's ratios

ν_{12}, ν_{23} = major (longitudinal-transverse) and transverse-thickness Poisson's ratios

σ_x, τ_{xy} = stress components

ψ_x, ψ_y = slope functions in x and y directions

ρ = density of composite

ρ_f, ρ_m = fiber and matrix densities

ω = natural frequency

CHAPTER I

INTRODUCTION

The rate of progress of technological innovation is dependent on the development of new and better materials. The new and rapidly developing composites made a significant impact on the engineering field and are responsible for the tremendous progress that has been achieved recently in the structural and aerospace industries.

Composites are materials made up of more than one constituent material. According to this literal definition, almost all materials used in civil and mechanical engineering are composites. Wood consists of lignin and cellulose fibers and is clearly a "natural" composite, but so too are cast iron, steel and other metallic alloys, brick, natural stone, and of course reinforced concrete. The fact that none of these materials are perfectly isotropic leads us to closer definition of a composite as understood today, especially in the advanced technology industries such as aerospace and automotive.

Composites are generally laminates in which a matrix material is reinforced in either one or more planes with filaments, fibers or fibrous material, giving the composite enhanced mechanical properties over those of either the matrix or the reinforcement when used alone. The matrix can consist of metal, ceramics, glass, concrete, gypsum, or resins, and the reinforcement can be metal rods or filaments, whiskers of silicon carbide or nitride, sapphire, carbon fiber, boron fiber, and various types of glass, asbestos, and cellulose fibers.

Glass-reinforced plastics are being used extensively and successfully in the manufacture of storage vessels. They are also beginning to be used structurally in buildings, and a good deal of thought is going into the design of G.R.P. bridges. The aerospace industry continues to lead in the use of composites for very high-performance applications with products ranging from rocket casings and major portions of fuselage, wing, and empennage assemblies to compressor blades and helicopter rotor blades.

The principal reasons for using composites in place of conventional materials are:

(1) Composites are anisotropic; so in order to get the greatest economy of material, either for cost or weight saving, the reinforcing fibers can be oriented in the plane where they will be most effective.

(2) Composites, unlike metals, can often be molded with a varying thickness at no extra cost. This gives an additional freedom to economize the material.

(3) Composites have improved strength and stiffness, especially when compared with other materials on a unit weight basis. For example, composites can be made that have the same strength and stiffness as high-strength steel, yet are 70 percent lighter! Other advanced composites are as much as three times as strong as aluminum, the common aircraft structural material, yet weigh only 60 percent as much!

(4) Composite materials can be tailored to efficiently meet design requirements of strength, stiffness, fatigue, thermal conductivity,

corrosion resistance, and other parameters all in various directions.

The advent of advanced fiber-reinforced composites has been called the biggest technical revolution since the jet engine. This claim is very striking because the tremendous impact of the jet engine on military aircraft performance is readily apparent. The impact on commercial aviation is even more striking because the airlines switched from propeller-driven planes to all-jet fleets within the span of just a few years.

Currently, almost every aerospace company is developing products made with fiber-reinforced composite materials. After passing through the different stages of usage, people are now dreaming for the final stage of an all-composite high-performance airplane.

As the applications of fiber-reinforced composites in structures become more widespread, the prediction of behavior of plates constructed of such materials become increasingly important. One of the characteristics of certain composite materials, known as bimodulus materials, is that they exhibit quite different elastic properties when loaded along the fiber direction in tension as opposed to compression [1-4] (see Fig. 1.1).

These materials are listed in Table 1.1. In the literature, this class of materials has variously been called bilinear, bimodulus, different-modulus, and multi-modulus. Here the term bimodulus is believed to be most descriptive of a material having different linear stress-strain relations in compression than in tension.

The first multi-dimensional model was proposed by Ambartsumyan [5] for isotropic material, such as a composite material with spherical particles. It was later extended to the orthotropic case [6].

The second and third models are the restricted-compliance model due to Isabekyan and Khachatrayan [7] and the first-invariant model of Shapiro [8]. A fourth model is the weighted-compliance theory originated by Jones [9].

The fifth model is the fiber-governed bimodulus symmetric compliance model originated by Bert [10].

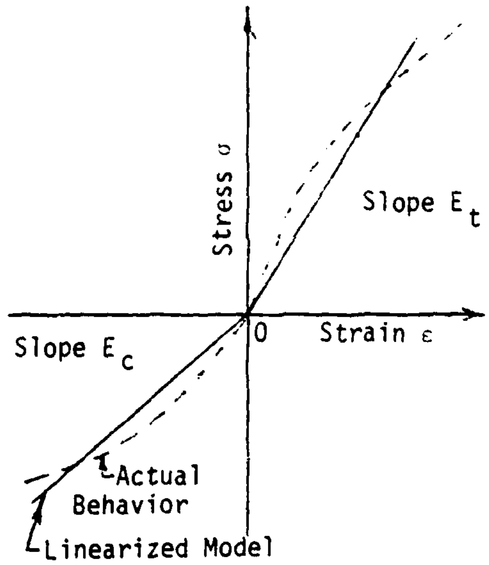

Fig. 1.1. Bimodulus idealization.

A plate subjected to a loading which produces plate bending or vibration obviously experiences both tension and compression; therefore. a more accurate analysis should take this into consideration.

Table 1.1 Some Bimodulus Materials

	Reinforcement Geometry	Ref.	Tensile Young's Modulus Divided by Compressive Young's Modulus
ATJ-S graphite	Granular	5	1.2
ZTA graphite	Granular	5	0.8
Glass-epoxy	Fibrous	5	1.25
Boron-epoxy	Fibrous	5	0.8
Graphite-epoxy	Fibrous	5	1.4
Carbon-carbon	Fibrous	5	2.0 to 5.0
Kevlar-rubber	Fibrous	6	0.77 (transverse) to 305 (longitudinal)*
Polyester-rubber	Fibrous	-	0.75 (transverse) to 16.7 (longitudinal)*

* Based on experimental results reported by Patel et al. [2].

The existing literature available in English on bending of bimodulus plates is quite sparse and, with only a few exceptions, is limited to bimodulus isotropic material [8,11-14]. Shapiro [8] considered the very simple problem of a circular plate subjected to a pure radial bending moment at its edge, but he used Love's stress-function formulation rather than plate theory. Kamiya [11] treated large deflections (geometric nonlinearity) of uniformly loaded, clamped-edge circular plates, using an iterative finite-difference technique. In [12], Kamiya applied the energy method to large deflections of simply supported rectangular plates subjected to sinusoidally distributed loading. In [13], Kamiya included the effect of thickness shear deformation, but only for the simple one-dimensional case of cylindrical bending. The only analysis applicable to anisotropic bimodulus material is the work of Jones and Morgan [14], who treated cylindrical bending of a thin, cross-ply laminate.

In the realm of plates laminated of ordinary* anisotropic materials, the theory due to Reissner and Stavsky [15], is generally recognized as the classical, linear (small-deflection) thin-plate theory. Although there have been numerous approximate solutions of this theory, only a relatively few closed-form solutions have appeared. Notable among these are the works of Whitney [16] and Whitney and Leissa [17] for both antisymmetric cross-ply and antisymmetric angle-ply rectangular plates with certain (different) kinds of simply supported edges. For an in-infinitely long strip of finite width, Padovan [18] presented a solution for the case of an arbitrary laminate.

Kamiya [19,20] considered problems of thermal stresses in a bimodulus thin plate. An annulus with axisymmetric steady temperature distribution was analyzed numerically. Ambartsumyan [21,22] presented a general theory of strains and stresses for bimodulus materials located in a temperature field. Other literature available on thermal bending [23-27] deals with plates of ordinary materials.

Vibration of plates has been treated by several authors [28-32], but the problem of bimodulus plate vibration has not been attempted previously.

Apparently, the present work is first to consider anisotropic,

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* Throughout this report, the term ordinary will be used to distinguish materials that do not exhibit bimodulus action, i.e., materials in which the tensile and compressive stiffnesses coincide.

bimodulus, thick plates finite in both directions in closed form except a few exceptions [33,34].

The problems of static bending, free vibration, thermal bending and non-linear large deflection (of thick bimodulus composite rectangular plates) have been analyzed and are presented separately in Chapters II, III, IV, and V. Numerical computations have been carried out and were compared, and good agreement was obtained with existing solutions of special cases existing in the literature.

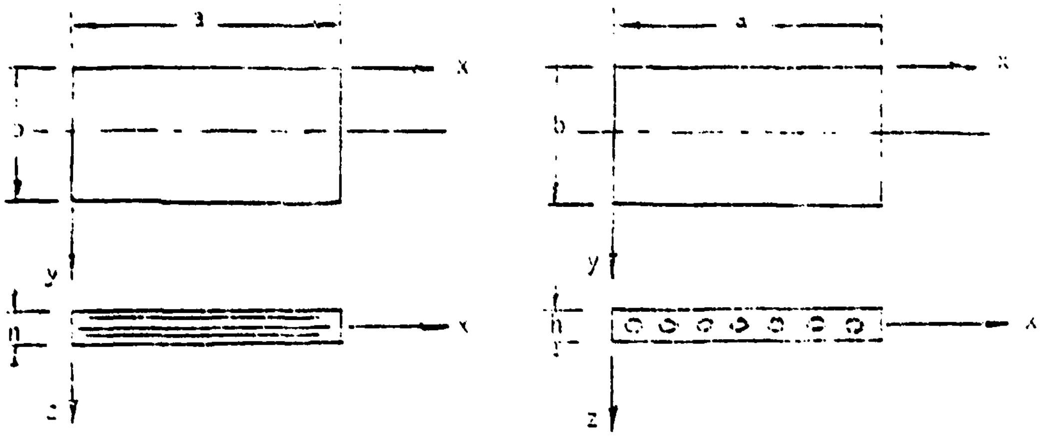
CHAPTER II

STATIC BENDING OF THICK RECTANGULAR PLATES

Consider the case of ordinary (not bimodulus) material. A single-layer plate constructed of an ordinary material that is macroscopically homogeneous is obviously symmetric about the midplane of the plate, and thus there is no coupling between bending and stretching during small-deflection bending. Likewise, a plate consisting of multiple layers of ordinary materials of various thicknesses arranged symmetrically about the midplane has no bending-stretching coupling at small deflections. However, in the case of a general laminate, i.e., one not symmetric about the midplane, bending-stretching coupling is induced.

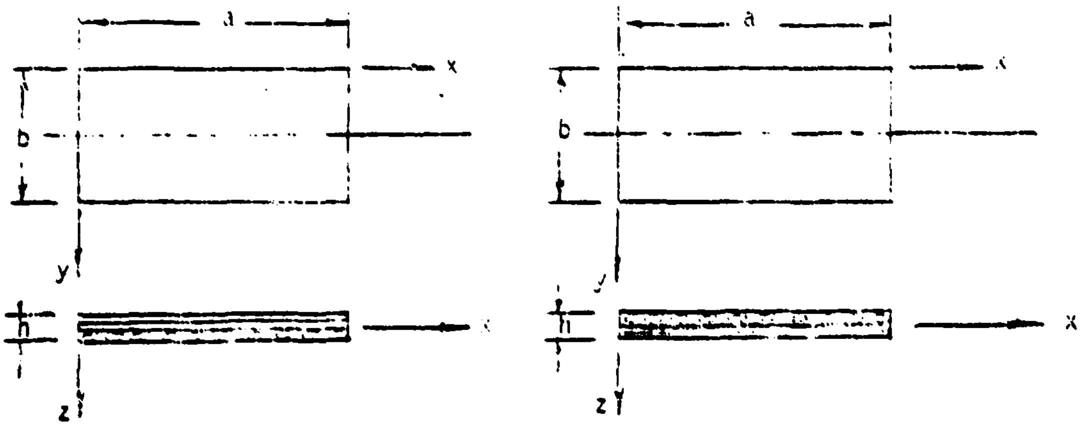
Now, consider the case of a single layer of bimodulus material. The different properties in tension and compression cause a shift in the neutral surface away from the geometric midplane, and symmetry about the midplane no longer holds. The results of this is that a single-layer bimodulus-material plate exhibits bending-stretching coupling of the orthotropic type i.e., analogous to a two-layer cross-ply plate (one layer at 0° and the other at 90°) of ordinary orthotropic material. (See Figs. 2.1, 2.2)

Using Bert's fiber-governed symmetric-matrix macroscopic material model [10], it can be assumed that there are <u>two</u> symmetric plane-



(a) Single layer with longitudinal fibers

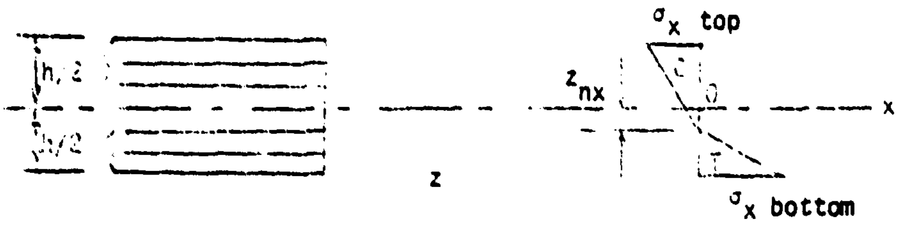
(b) Single layer with transverse fibers



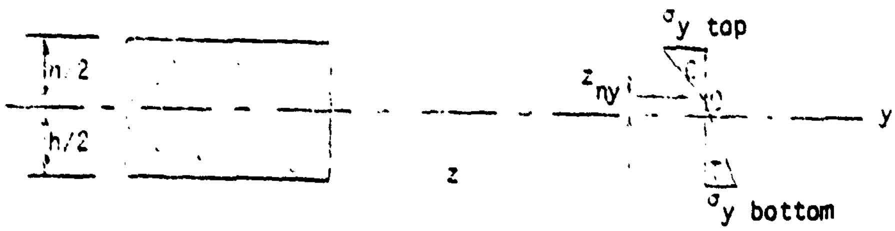
(c) Two-layer cross-ply with 90°/0° stacking sequence

(d) Two-layer cross-ply with 0°/90° stacking sequence

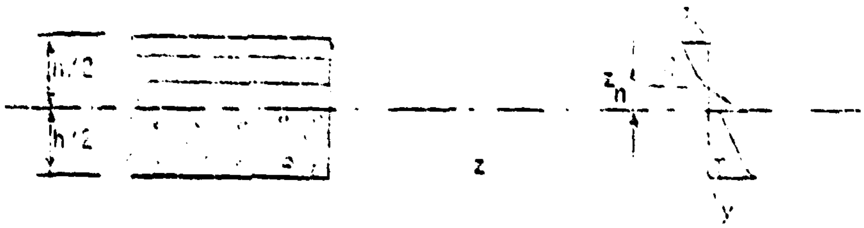
Fig. 2.1. Laminate configurations for rectangular plates.



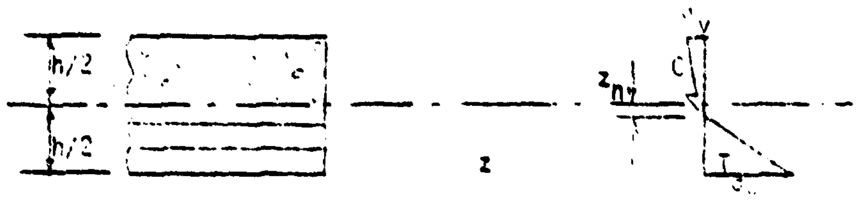
(a) Single layer with longitudinal fibers.



(b) Single layer with transverse fibers.



(c) Two-layer cross-ply laminate with 90°/0° stacking sequence.



(d) Two-layer cross-ply laminate with 0°/90° stacking sequence

Fig. 2.2. Stress distributions in bimodulus plates.

stress reduced stiffness matrices: one, when the fibers are in tension along their length, and another, when they are in compression in the same direction. Invoking the Voigt hypothesis in the fiber direction, for which it is well-established, it is assumed that the fiber-direction normal strains in the fibers and in the matrix are identical. Then the criterion for changing from tension to compression can be taken to be the fiber-direction normal strain in each layer. This is a much more convenient criterion to apply than is the fiber-direction normal-stress criterion.

2.1 Governing Equations

Consider a plate of thickness h composed of an even number of indentical orthotropic layers bonded together, arranged alternately at angles 0° and 90°. The origin of a Cartesian coordinate system is located within the central plane (x-y) with the z-axis being normal to this plane (see Fig. 2.3).

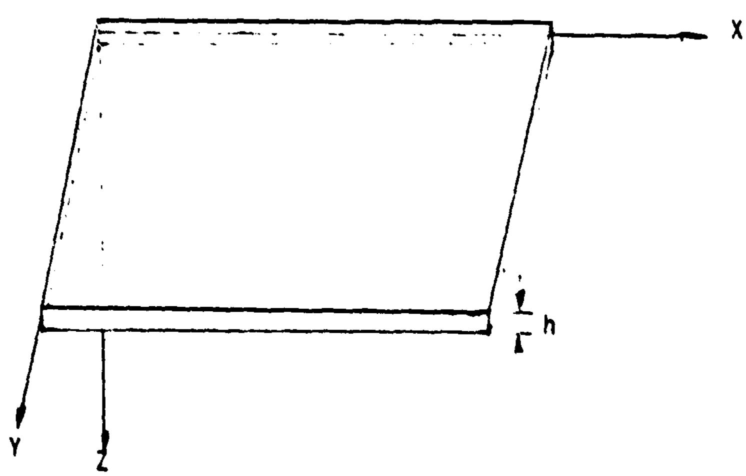


Fig. 2.3. Cartesian coordinates for rectangular plate.

The stress resultants and stress couples, each per unit length, are defined in the usual way as

$$(N_x, N_y, N_{xy}, Q_x, Q_y) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz})dz \qquad (2.1.1)$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy})z\, dz \qquad (2.1.2)$$

The theory developed by Yang, Norris, and Stavsky [35] is based on the following assumed displacement field,

$$\begin{aligned} u &= u^o(x,y) + z\psi_x(x,y) \\ v &= v^o(x,y) + z\psi_y(x,y) \\ w &= w(x,y) \end{aligned} \qquad (2.1.3)$$

where u, v, and w are the displacement components in the x, y, and z directions, respectively, and ψ_x and ψ_y are called the slope functions.

The constitutive equations for an unsymmetric cross-ply laminate can be written as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} u^o_{,x} \\ v^o_{,y} \\ v^o_{,x} + u^o_{,y} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{y,x} + \psi_{x,y} \end{Bmatrix} \qquad (2.1.4)$$

and

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} K_4^2 A_{44} & 0 \\ 0 & K_5^2 A_{55} \end{bmatrix} \begin{Bmatrix} w_{,y} + \psi_y \\ w_{,x} + \psi_x \end{Bmatrix} \qquad (2.1.5)$$

Here, differentiation is denoted by a comma, and the extensional, flexural-extensional coupling, and flexural or twisting stiffnesses for a laminate of an arbitrary number of layers are defined by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2)dz \ , \ i,j=1,2,6 \qquad (2.1.6)$$

In addition to performing the integrations in a piecewise manner from layer to layer, one also has to take into account the possibility of different properties (tension or compression) within a layer. This is worked out in detail for a two-layer cross-ply laminate in Appendix A.

The quantities K_4^2 and K_5^2 are shear correction coefficients which may be calculated by various static and dynamic methods [36].

Taking into account the shear deformation, one can write the equations of equilibrium (neglecting the body forces) as follows:

$$\begin{aligned}
&N_{x,x} + N_{xy,y} = 0 \\
&N_{xy,x} + N_{y,y} = 0 \\
&Q_{x,x} + Q_{y,y} = -q \\
&M_{x,x} + M_{xy,y} - Q_x = 0 \\
&M_{xy,x} + M_{y,y} - Q_y = 0
\end{aligned} \qquad (2.1.7)$$

Here $N_{x,x} = \partial N_x/\partial x$, $N_{y,y} = \partial N_y/\partial y$, etc.

Substituting equations (2.1.4) and (2.1.5) into equations (2.1.7), we obtain the equations of equilibrium in terms of the generalized displacements.

$$A_{11}u^o_{,xx} + A_{66}u^o_{,yy} + (A_{12} + A_{66})v^o_{,xy} + B_{11}\psi_{x,xx} + B_{66}\psi_{x,yy}$$

$$+ (B_{12} + B_{66})\psi_{y,xy} = 0$$

$$(A_{12} + A_{66})u^o_{,xy} + A_{66}v^o_{,xx} + A_{22}v^o_{,yy} + (B_{12} + B_{66})\psi_{x,xy}$$

$$+ B_{66}\psi_{y,xx} + B_{22}\psi_{y,yy} = 0$$

$$K_5^2A_{55}(\psi_{x,x} + w_{,xx}) + K_4^2A_{44}(\psi_{y,y} + w_{,yy}) = -q \qquad (2.1.8)$$

$$(B_{12} + B_{66})u^o_{,xy} + B_{66}v^o_{,xx} + B_{22}v^o_{,yy} + (D_{12} + D_{66})\psi_{x,xy}$$

$$+ D_{66}\psi_{y,xx} + D_{22}\psi_{y,yy} - K_4^2[A_{44}(\psi_y + w_{,y})] = 0$$

$$B_{11}u^o_{,xx} + B_{66}u^o_{,yy} + (B_{12} + B_{66})v^o_{,xy} + D_{11}\psi_{x,xx} + D_{66}\psi_{x,yy}$$

$$+ (D_{12} + D_{66})\psi_{y,xy} - K_5^2[A_{55}(\psi_x + w_{,x})] = 0$$

Or, in operator form

$$[L_{k\ell}] \begin{Bmatrix} u^o \\ v^o \\ w \\ h\psi_y \\ h\psi_x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q \\ 0 \\ 0 \end{Bmatrix} \qquad (2.1.9)$$

$$k,\ell = 1,2,3,4,5$$

where $[L_{k\ell}]$ is a symmetric linear differential operator matrix with the following components:

$$
\begin{aligned}
L_{11} &\equiv A_{11}d_x^2 + A_{66}d_y^2 \\
L_{12} &\equiv (A_{12} + A_{66})\, d_x d_y \\
L_{13} &\equiv 0 \\
L_{14} &\equiv [(B_{12} + B_{66})/h]\, d_x d_y \\
L_{15} &\equiv (B_{11}/h)\, d_x^2 + (B_{66}/h)\, d_y^2 \\
L_{22} &\equiv A_{66}d_x^2 + A_{22}d_y^2 \\
L_{23} &\equiv 0 \\
L_{24} &\equiv (B_{66}/h)\, d_x^2 + (B_{22}/h)\, d_y^2 \\
L_{25} &\equiv L_{14} \\
L_{33} &\equiv -\, K_5^2A_{55}d_x^2 - K_4^2A_{44}d_y^2 \\
L_{34} &\equiv -\, K_4^2(A_{44}/h)\, d_y \\
L_{35} &\equiv -\, K_5^2(A_{55}/h)\, d_x \\
L_{44} &\equiv (D_{66}/h^2)\, d_x^2 + (D_{22}/h^2)\, d_y^2 - K_4^2A_{44}/h^2 \\
L_{45} &\equiv [(D_{12} + D_{66})/h^2]\, d_x d_y \\
L_{55} &\equiv (D_{11}/h^2)\, d_x^2 + (D_{66}/h^2) d_y^2 - K_5^2A_{55}/h^2
\end{aligned}
\qquad (2.1.10)
$$

2.2. Application to Rectangular Plate Hinged on all Edges

The boundary conditions for a rectangular plate simply supported on all edges can be specified as follows:

Along the edges at $x = 0$ and $x = a$,

$$w = \psi_y = M_x = 0$$

$$v^0 = N_x = 0$$

Along the edges at $y = 0$ and $y = b$, (2.2.1)

$$w = \psi_x = M_y = 0$$

$$u^o = N_y = 0$$

Consider the loading to be sinusoidally distributed in both the x and y directions:

$$q = q_o \sin \alpha x \sin \beta y$$

Here

$$\alpha \equiv \pi/a \ , \quad \beta \equiv \pi/b$$

Furthermore, a and b are the plate dimensions along x and y axes, respectively.

The governing equations (2.1.9) and boundary conditions (2.2.1) are exactly satisfied in closed form by the following set of functions:

$$\begin{aligned}
u^o &= U \cos \alpha x \sin \beta y \\
v^o &= V \sin \alpha x \cos \beta y \\
w &= W \sin \alpha x \sin \beta y \\
h\psi_y &= Y \sin \alpha x \cos \beta y \\
h\psi_x &= X \cos \alpha x \sin \beta y
\end{aligned} \tag{2.2.2}$$

Substituting solutions (2.2.2) into the governing equations (2.1.9), one obtains

$$[C_{k\ell}] \begin{Bmatrix} U \\ V \\ W \\ Y \\ X \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_o \\ 0 \\ 0 \end{Bmatrix} \qquad k,\ell = 1,2,3,4,5 \tag{2.2.3}$$

where $[C_{k\ell}]$ is a 5x5 symmetric matrix containing the following elements:

$$C_{11} \equiv - A_{11}\alpha^2 - A_{66}\beta^2$$

$$C_{12} \equiv - (A_{12} + A_{66})\alpha\beta$$

$$C_{13} \equiv 0$$

$$C_{14} \equiv - [(B_{12} + B_{66})/h]\alpha\beta$$

$$C_{15} \equiv - (B_{11}/h)\alpha^2 - (B_{66}/h)\beta^2$$

$$C_{22} \equiv - A_{66}\alpha^2 - A_{22}\beta^2$$

$$C_{23} \equiv 0$$

$$C_{24} \equiv - (B_{66}/h)\alpha^2 - (B_{22}/h)\beta^2 \tag{2.2.4}$$

$$C_{25} \equiv C_{14}$$

$$C_{33} \equiv - (K_5^2 A_{55}\alpha^2 + K_4^2 A_{44}\beta^2)$$

$$C_{34} \equiv - K_4^2(A_{44}/h)\beta$$

$$C_{35} \equiv - K_5^2(A_{55}/h)\alpha$$

$$C_{44} \equiv - (D_{66}/h^2)\alpha^2 - (D_{22}/h^2)\beta^2 - K_4^2(A_{44})/h^2$$

$$C_{45} \equiv - [(D_{12} + D_{66})/h^2]\alpha\beta$$

$$C_{55} \equiv - (D_{11}/h^2)\alpha^2 - (D_{66}/h^2)\beta^2 - K_5^2 A_{55}/h^2$$

2.3 <u>The Positions of the Fiber-Direction Neutral Surfaces</u>

From the kinematics of the deformation

$$\begin{aligned} \varepsilon_x &= u^o_{,x} + z\ \psi_{x,x} \\ \varepsilon_y &= v^o_{,y} + z\ \psi_{y,y} \end{aligned} \tag{2.3.1}$$

Thus, the neutral-surface positions, for the longitudinal (x) and transverse (y) directions, respectively, are

$$z_{nx} = \frac{-u^o_{,x}}{\psi_{x,x}} = -\ hU/X$$

$$z_{ny} = \frac{-v^o_{,y}}{\psi_{y,y}} = -\ hV/Y \qquad (2.3.2)$$

So, in computations the values for z_{nx} and z_{ny} are first assumed to obtain the displacements. Actual displacements can then be obtained by an iterative procedure with the help of the equations (2.3.2).

2.4 Numerical Results

As the first example, we take the case of a homogeneous (single-layer) plate of transversely isotropic bimodulus material. The plane of isotropy is assumed to coincide with the midplane of the plate, and the inplane Poisson's ratio is assumed to be zero. Then the closed-form solution reduces to the simplified form [33]. Numerical results are presented in Tables 2.1 and 2.2.

Table 2.1. Comparison of Neutral-Surface Locations for Transversely Isotropic Square Plate

$E_t/E_c = G_{zt}/G_{zc}$	Neutral-Surface Location $Z^{\dagger}$		
	$G_{zc}/E_c = 0.1$	0.3	0.5
	Exact Closed-Form Solution:		
0.5	- 0.08578	- 0.08578	- 0.08578
1.0	0	0	0
2.0	+ 0.08578	+ 0.08578	+ 0.08578
	Simplified Approximate Solution [33]:		
0.5	- 0.08579	- 0.08579	- 0.08579
1.0	0	0	0
2.0	+ 0.08579	+ 0.08579	+ 0.08579
	Mixed Finite-Element Solution [33]:		
0.5	- 0.08578	- 0.08578	- 0.08578
1.0	0	0	0
2.0	+ 0.08578	+ 0.08578	+ 0.08578

† Here, $Z = z_{nx}/h = z_{ny}/h$.

Table 2.2. Comparison of Maximum Deflections for Transversely Isotropic Square Plate ($b/h=0.1$, $K^2=5/6$)

$E_t/E_c=G_{zt}/G_{zc}$	Dimensionless Deflection $WE_c h^3/q_0 b^4$		
	$G_{zc}/E_c=0.1$	0.3	0.5
	Exact Closed-Form Solution:		
0.5	0.05348	0.04774	0.04660
1.0	0.03688	0.03283	0.03201
2.0	0.02674	0.02387	0.02330
	Simplified Approximate Solution [33]:		
0.5	0.05004	0.04660	0.04591
1.0	0.03445	0.03202	0.03153
2.0	0.02530	0.02342	0.02296
	Mixed Finite-Element Solution [33]:		
0.5	0.05329	0.04743	0.04626
1.0	0.03675	0.03261	0.03178
2.0	0.02664	0.02371	0.02313

It is noted that the neutral-surface location is independent of G_{zc} and G_{zt}. The agreement among the results obtained by all three solutions is quite good.

According to classical thin-plate theory for a rectangular isotropic plate with simply supported edges, the dimensionless maximum deflection is given by

$$\frac{WE_t}{q_0 a^4} = \frac{12\,(1-\nu^2)}{\pi^4\,[1+(b/a)^{-2}]}$$

The values are computed for three aspect ratios using the above formula and are compared with the present results in Table 2.3 below.

Table 2.3. Dimensionless Deflections for Rectangular Isotropic Plate as Determined by the Thin-Plate Theory and the Present Work

Aspect Ratio a/b	$WE_{22}^{t}/q_0 a^4$					
	h/b=0.1		h/b=0.01		h/b=0.001	
	Thick*	Thin*	Thick*	Thin*	Thick*	Thin*
0.5	0.07439	0.07392	0.07401	0.07392	0.07392	0.07392
1.0	0.02908	0.02887	0.02887	0.02888	0.02887	0.02887
2.0	0.00478	0.00462	0.00462	0.00462	0.00462	0.00462

* "Thick" denotes the present thick-plate theory and "thin" denotes classical thin-plate theory.

As examples of some actual bimodulus materials, aramid-cord/rubber and polyester-cord/rubber are selected. The material properties used are listed in Table 2.4. The data are based on test results of Patel et al.[2], using the data-reduction procedure of [10], except for the thickness shear moduli, which were estimated.

Table 2.4. Elastic Properties for Two Tire-Cord/Rubber, Unidirectional, Bimodulus Composite Materials[a]

Property and Units	Aramid-Rubber k=1	Aramid-Rubber k=2	Polyester-Rubber k=1	Polyester-Rubber k=2
Longitudinal Young's modulus, GPa	3.58	0.0120	0.617	0.0369
Transverse Young's modulus, GPa	0.00909	0.0120	0.00800	0.0106
Major Poisson's ratio, dimensionless[b]	0.416	0.205	0.475	0.185
Longitudinal-transverse shear modulus, GPa[c]	0.00370	0.00370	0.00262	0.00267
Transverse-thickness shear modulus, GPa	0.00290	0.00499	0.00233	0.00475

a Fiber-direction tension is denoted by k=1, and fiber-direction compression by k=2.
b It is assumed that the minor Poisson's ratio is given by the reciprocal relation.
c It is assumed that the longitudinal-thickness shear modulus is equal to this one.

Numerical results for single-layer rectangular plates with the fibers oriented parallel to the x axis are given in Table 2.5, while those for cross-ply plates are listed in Table 2.6.

The exact closed-form solution developed here for thick, rectangular plates of single-layer and cross-ply laminates of orthotropic bimodulus material has been shown to agree well with an existing approximate solution for isotropic bimodulus plates and with a mixed finite-element solution.

To show the general trend, plots have been presented in Fig. 2.4.

Table 2.5. Neutral-Surface Positions and Dimensionless Deflections for Rectangular Plates of Single-Layer 0° Aramid-Rubber and Polyester-Rubber, as Determined by Two Different Methods (Thickness/Width, h/b=0.1)

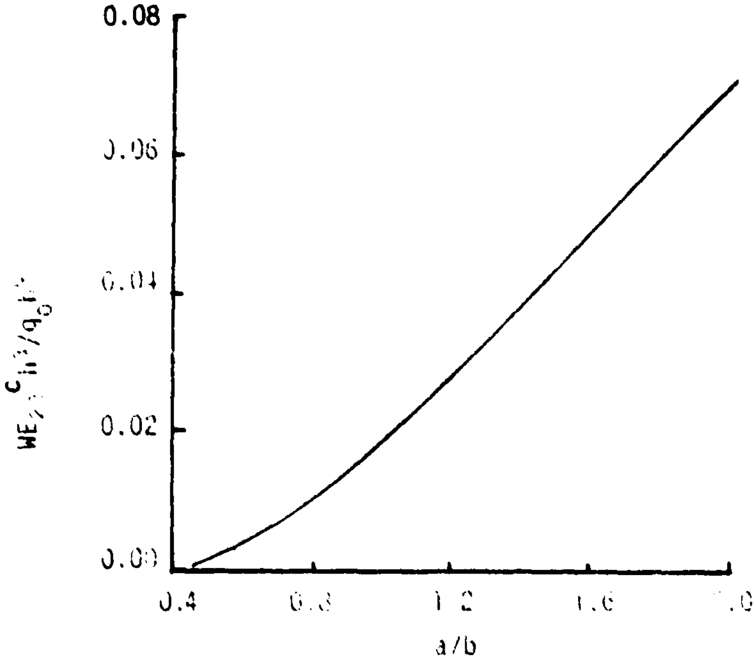
Aspect Ratio	Z_x		Z_y		$WE_{22}{}^{c}h^3/q_0b^4$	
	C.F.*	F.E.*	C.F.*	F.E.*	C.F.*	F.E.*
			Aramid-Rubber:			
0.5	0.4453	0.4454	- 0.3304	- 0.3007	0.002544	0.002750
0.6	0.4452	0.4452	- 0.2941	- 0.2734	0.004560	0.004827
0.7	0.4447	0.4447	- 0.2564	- 0.2419	0.007393	0.007712
0.8	0.4440	0.4440	- 0.2220	- 0.2117	0.01105	0.01140
0.9	0.4431	0.4431	- 0.1923	- 0.1846	0.01545	0.01582
1.0	0.4420	0.4420	- 0.1671	- 0.1614	0.02046	0.02083
1.2	0.4394	0.4394	- 0.1285	- 0.1250	0.03160	0.03193
1.4	0.4363	0.4363	- 0.1015	- 0.09919	0.04313	0.04335
1.6	0.4328	0.4329	- 0.08228	- 0.08070	0.05406	0.05416
1.8	0.4292	0.4294	- 0.06838	- 0.06724	0.06390	0.06388
2.0	0.4253	0.4254	- 0.05813	- 0.05727	0.07250	0.07236
			Polyester-Rubber:			
0.5	0.3044	0.3045	- 0.1597	- 0.1234	0.001529	0.001971
0.6	0.3044	0.3045	- 0.1538	- 0.1245	0.002652	0.003265
0.7	0.3042	0.3044	- 0.1426	- 0.1198	0.004283	0.005075
0.8	0.3039	0.3041	- 0.1299	- 0.1124	0.006517	0.007487
0.9	0.3035	0.3037	- 0.1174	- 0.1041	0.009421	0.01055
1.0	0.3029	0.3031	- 0.1061	- 0.09586	0.01303	0.01430
1.2	0.3015	0.3018	- 0.08728	- 0.08111	0.02223	0.02367
1.4	0.2999	0.3001	- 0.07329	- 0.06941	0.03348	0.03492
1.6	0.2979	0.2982	- 0.06296	- 0.06042	0.04574	0.04703
1.8	0.2957	0.2960	- 0.05528	- 0.05356	0.05793	0.05897
2.0	0.2934	0.2936	- 0.04959	- 0.04828	0.06925	0.07003

*C.F. denotes closed-form solution; F.E. signifies finite-element solution [33]. (For in-plane displacements, see Appendix D)

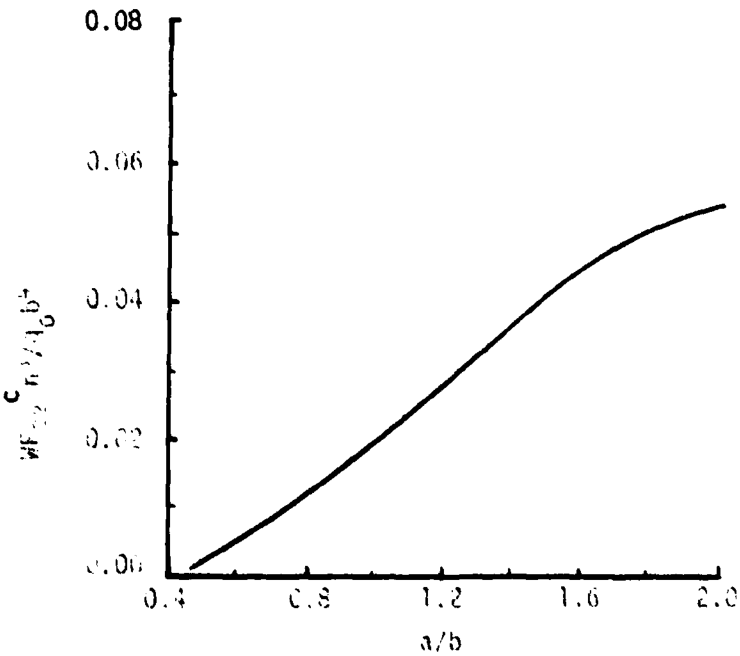
Table 2.6. Neutral-Surface Positions and Dimensionless Deflections for Rectangular Plates of Two-Layer Cross-Ply Aramid-Rubber and Polyester-Rubber, as Determined by Two Different Methods (Thickness/Width, h/b=0.1)

Aspect Ratio	Z_x		Z_y		$WE_{22}^{c}h^3/q_ob^4$	
	C.F.*	F.E.*	C.F.*	F.E.*	C.F.*	F.E.*
			Aramid-Rubber:			
0.5	0.4433	0.4431	- 0.06343	- 0.06223	0.002472	0.002576
0.6	0.4427	0.4426	- 0.05478	- 0.05443	0.004388	0.004518
0.7	0.4418	0.4418	- 0.04794	- 0.04778	0.007072	0.007220
0.8	0.4407	0.4407	- 0.04247	- 0.04237	0.01054	0.01070
0.9	0.4396	0.4396	- 0.03803	- 0.03795	0.01475	0.01490
1.0	0.4384	0.4384	- 0.03437	- 0.03430	0.01957	0.01972
1.2	0.4356	0.4356	- 0.02883	- 0.02860	0.03043	0.03054
1.4	0.4326	0.4325	- 0.02470	- 0.02477	0.04185	0.04190
1.6	0.4292	0.4292	- 0.02160	- 0.02165	0.05282	0.05280
1.8	0.4257	0.4256	- 0.01922	- 0.01923	0.06277	0.06264
2.0	0.4219	0.4219	- 0.01735	- 0.01734	0.07151	0.07137
			Polyester-Rubber:			
0.5	0.3650	0.3652	- 0.1285	- 0.1256	0.002539	0.002732
0.6	0.3644	0.3646	- 0.1178	- 0.1164	0.004527	0.004772
0.7	0.3638	0.3639	- 0.1097	- 0.1089	0.007288	0.007575
0.8	0.3631	0.3631	- 0.1036	- 0.1031	0.01078	0.01109
0.9	0.3622	0.3622	- 0.09886	- 0.09859	0.01487	0.01519
1.0	0.3613	0.3613	- 0.09526	- 0.09502	0.01933	0.01966
1.2	0.3593	0.3593	- 0.09001	- 0.09000	0.02846	0.02879
1.4	0.3571	0.3570	- 0.08660	- 0.08660	0.03674	0.03707
1.6	0.3546	0.3545	- 0.08430	- 0.08430	0.04356	0.04389
1.8	0.3519	0.3518	- 0.08267	- 0.08267	0.04890	0.04925
2.0	0.3491	0.3490	- 0.08150	- 0.08150	0.05301	0.05337

*C.F. denotes closed-form solution; F.E. signifies finite-element solution [33]. (See Appendix D for in-plane displacements.)



(a) Aramid-rubber



(b) Polyester-rubber

Fig. 2.4. Variation of dimensionless deflection with aspect ratio for two-layer cross-ply rectangular plates.

CHAPTER III

VIBRATION OF THICK CROSS-PLY LAMINATED BIMODULUS RECTANGULAR PLATES

Thin-plate theory does not take into account either the effect of transverse shear deformation or rotatory inertia, and hence it becomes inaccurate for thicker plates. Mindlin [37] considered both of these effects for homogeneous isotropic plates, by assuming that the displacement variation across the thickness is linear for u and v and constant for w. He also had to assign a value to the shearing rigidity factor on suitable physical considerations. His solution does not satisfy the governing elasticity equations exactly, but does permit the satisfaction of a set of three boundary conditions on each edge. Mindlin, Schacknow, and Deresiewicz [38] applied this method to the vibrations of thick rectangular plates with two opposite sides simply supported and the other two edges with various conditions. The present work, to the author's knowledge, is the first to consider the bimodulus property in the thick cross-ply rectangular plates.

3.1 Governing Equations

Consider a plate of thickness h composed of an even number of identical orthotropic layers bonded together, arranged alternately at angles 0° and 90° (see Fig. 2.2).

The stress and moment resultants, each per unit length, are given in the usual manner as

$$(N_x, N_y, N_{xy}, Q_x, Q_y) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz})\, dz \tag{3.1.1}$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) z\, dz \tag{3.1.2}$$

The displacement components, u, v, and w in the x, y, and z directions respectively can be expressed in terms of mid-plane displacements u^o, v^o, w^o and slope functions ψ_x and ψ_y as:

$$\begin{aligned} u &= u^o(x,y,t) + z\psi_x(x,y,t) \\ v &= v^o(x,y,t) + z\psi_y(x,y,t) \\ w &= w(x,y,t) \end{aligned} \tag{3.1.3}$$

where t is time.

Constitutive equations for an unsymmetric cross-ply laminate, as has already been mentioned in Chapter II, are:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} u^o_{,x} \\ v^o_{,y} \\ v^o_{,x} + u^o_{,y} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{y,x} + \psi_{x,y} \end{Bmatrix} \tag{3.1.4}$$

and

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} K_4^2 A_{44} & 0 \\ 0 & K_5^2 A_{55} \end{bmatrix} \begin{Bmatrix} w_{,y} + \psi_y \\ w_{,x} + \psi_x \end{Bmatrix} \tag{3.1.5}$$

Differentiation here is denoted by a comma, i.e., $(\ )_{,x} \equiv \partial(\ )/\partial x$, and the extensional, flexural-extensional coupling, and flexural stiffnesses for the laminate are defined by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (Q_{ij})(1, z, z^2)dz \qquad (3.1.6)$$

$$i,j=1,2,6$$

As usual, K_4^2 and K_5^2 are shear-correction coefficients.

Taking into account the shear deformation and the rotatory inertia, the equations of motion can be written as follows:

$$\begin{aligned}
&N_{x,x} + N_{xy,y} = P\, u^o_{,tt} + R\, \psi_{x,tt} \\
&N_{xy,x} + N_{y,y} = P\, v^o_{,tt} + R\, \psi_{y,tt} \\
&Q_{x,x} + Q_{y,y} = P\, w_{,tt} \qquad (3.1.7) \\
&M_{x,x} + M_{xy,y} - Q_x = R u^o_{,tt} + I\, \psi_{x,tt} \\
&M_{xy,x} + M_{y,y} - Q_y = R\, v^o_{,tt} + I\, \psi_{y,tt}
\end{aligned}$$

Here P, R, and I are the normal, in-plane, and rotatory inertia coefficients per unit mid-plane area and are defined by

$$(P, R, I) = \int_{-h/2}^{h/2} \rho(1, z, z^2)\, dz \qquad (3.1.8)$$

where ρ is the material density.

Substituting equations (3.1.4) and (3.1.5) into equations (3.1.7), we obtain the equations of motion. In operator form,

$$[L_{k\ell}]\begin{Bmatrix} u^o \\ v^o \\ w \\ h\psi_y \\ h\psi_x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad k,\ell = 1,2,3,4,5 \tag{3.1.9}$$

where $[L_{k\ell}]$ is a symmetric linear differential operator matrix with the following elements:

$$
\begin{aligned}
L_{11} &\equiv A_{11}d_x^2 + A_{66}d_y^2 - Pd_t^2 \\
L_{12} &\equiv (A_{12} + A_{66})\ d_xd_y \\
L_{13} &\equiv 0 \\
L_{14} &\equiv [(B_{12} + B_{66})/h]\ d_xd_y \\
L_{15} &\equiv (B_{11}/h)d_x^2 + (B_{66}/h)d_y^2 - (R/h)\ d_t^2 \\
L_{22} &\equiv A_{66}d_x^2 + A_{22}d_y^2 - Pd_t^2 \\
L_{23} &\equiv 0 \\
L_{24} &\equiv (B_{66}/h)d_x^2 + (B_{22}/h)d_y^2 - (R/h)\ d_t^2 \qquad (3.1.10) \\
L_{25} &\equiv L_{14} \\
L_{33} &\equiv -K_5^2A_{55}d_x^2 - K_4^2A_{44}d_y^2 + Pd_t^2 \\
L_{34} &\equiv -K_4^2(A_{44}/h)\ d_y \\
L_{35} &\equiv -K_4^2(A_{55}/h)\ d_x \\
L_{44} &\equiv (D_{66}/h^2)d_x^2 + (D_{22}/h^2)d_y^2 - K_4^2A_{44}/h^2 - (I/h^2)d_t^2 \\
L_{45} &\equiv [(D_{12} + D_{66})/h^2]\ d_xd_y \\
L_{55} &\equiv [D_{11}/h^2)d_x^2 + (D_{66}/h^2)d_y^2 - K_5^2A_{55}/h^2 - (I/h^2)d_t^2
\end{aligned}
$$

3.2 Application to Plate Simply Supported on all Edges

The boundary conditions are:

$$
\begin{aligned}
&\text{Along the edges at} \quad x = 0 \text{ and } x = a,\\
&\qquad w = \psi_y = M_x = 0\\
&\qquad v^o = N_x = 0\\
&\text{Along the edges at} \quad y = 0 \text{ and } y = b,\\
&\qquad w = \psi_x = M_y = 0\\
&\qquad u^o = N_y = 0
\end{aligned}
\tag{3.2.1}
$$

3.3 Closed-Form Solution

The governing equations (3.1.9) and the boundary conditions (3.2.1) are exactly satisfied in closed form by the following set of functions:

$$
\begin{aligned}
u^o &= U \cos \alpha x \sin \beta y \, e^{i\omega t}\\
v^o &= V \sin \alpha x \cos \beta y \, e^{i\omega t}\\
w &= W \sin \alpha x \sin \beta y \, e^{i\omega t}\\
h\psi_y &= Y \sin \alpha x \cos \beta y \, e^{i\omega t}\\
h\psi_x &= X \cos \alpha x \sin \beta y \, e^{i\omega t}
\end{aligned}
\tag{3.3.1}
$$

Here, ω is the natural frequency associated with the mode having axial and transverse wave numbers m and n, and

$$\alpha \equiv m\pi/a \quad , \quad \beta \equiv n\pi/b \tag{3.3.2}$$

where a and b are plate dimensions in the x and y directions, respectively.

Substituting solutions (3.3.1) into the governing equations (3.1.9), we obtain the following:

$$[C_{k\ell}] \begin{Bmatrix} U \\ V \\ W \\ \Psi \\ \chi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (3.3.3)$$

$k,\ell = 1,2,3,4,5$

where $C_{k\ell}$ is a 5x5 symmetric determinant containing the following elements:

$$C_{11} \equiv - A_{11}\alpha^2 - A_{66}\beta^2 + P\omega^2$$

$$C_{12} \equiv - (A_{12} + A_{66})\alpha\beta$$

$$C_{13} \equiv 0$$

$$C_{14} \equiv - [(B_{12} + B_{66})/h]\alpha\beta$$

$$C_{15} \equiv - (B_{11}/h)\alpha^2 - (B_{66}/h)\beta^2 + (R/h)\omega^2$$

$$C_{22} \equiv - A_{66}\alpha^2 - A_{22}\beta^2 + P\omega^2$$

$$C_{23} \equiv 0$$

$$C_{24} \equiv - (B_{66}/h)\alpha^2 - (B_{22}/h)\beta^2 + (R/h)\omega^2 \qquad (3.3.4)$$

$$C_{25} \equiv C_{14}$$

$$C_{33} \equiv - (K_5^2A_{55}\alpha^2 + K_4^2A_{44}\beta^2 - P\omega^2)$$

$$C_{34} \equiv - K_4^2(A_{44}/h)\beta$$

$$C_{35} \equiv - K_5^2(A_{55}/h)\alpha$$

$$C_{44} \equiv - (D_{66}/h^2)\alpha^2 - (D_{22}/h^2)\beta^2 - (K_4^2A_{44}/h^2) + (I/h^2)\omega^2$$

$$C_{45} \equiv - [(D_{12} + D_{66})/h^2]\alpha\beta$$

$$C_{55} \equiv - (D_{11}/h^2)\alpha^2 - (D_{66}/h^2)\beta^2 - (K_5^2A_{55}/h^2) + (I/h^2)\omega^2$$

The frequency ω can be determined by setting $|C_{k\ell}| = 0$.

3.4 Neutral-Surface Locations

This is the same as in the preceding chapter: $z_{nx} = -hU/X$; $z_{ny} = -hV/Y$. An iterative procedure is used to obtain the final displacements.

3.5 Numerical Results

Since there is no previous analysis for vibration of bimodulus plates, the present one could be compared only with rectangular plates laminated of ordinary materials.

Comparisons with Jones, and Fortier and Rossettos are presented in Tables 3.1 and 3.2 below. It can be seen that the agreement is good.

Table 3.1. Comparison of Fundamental Natural Frequencies of Rectangular Antisymmetric Cross-Ply Plates at Different Plate Aspect Ratios
($E_{11}/E_{22}=40$, $G_{12}/E_{22}=0.5$, $\nu_{12}=0.25$, $b/h=10$)

Aspect Ratio a/b	$\frac{\omega b^2}{\pi^2}\sqrt{\frac{\rho}{D_{22}}}$	
	Jones [31]	Present
0.5	2.050	1.934
1.0	0.825	0.794
1.5	0.650	0.612
2.0	0.580	0.565
2.5	0.560	0.548
3.0	0.550	0.541

Table 3.2. Comparison of Fundamental Frequency ($m=n=1$) of a Square Cross-Ply Plate
($E_{11}/E_{22}=40$, $G_{12}/E_{22}=1$, $G_{23}/E_{22}=0.5$, $\nu_{12}=0.25$)

a/h	$\omega a^2 \sqrt{\rho/E_{22}h^3}$	
	Fortier and Rossettos [28]	Present
10	10.80	10.80
50	11.65	11.65

Typical results for (bimodulus) aramid-rubber and polyester-rubber are tabulated in the following tables (Tables 3.3 and 3.4). See Table 2.4 for the elastic properties and Appendix B for the densities.

Computations, based on the closed-form solution, have been carried out for thick, rectangular plates of cross-ply laminates and compared with existing works. Close agreement was reached.

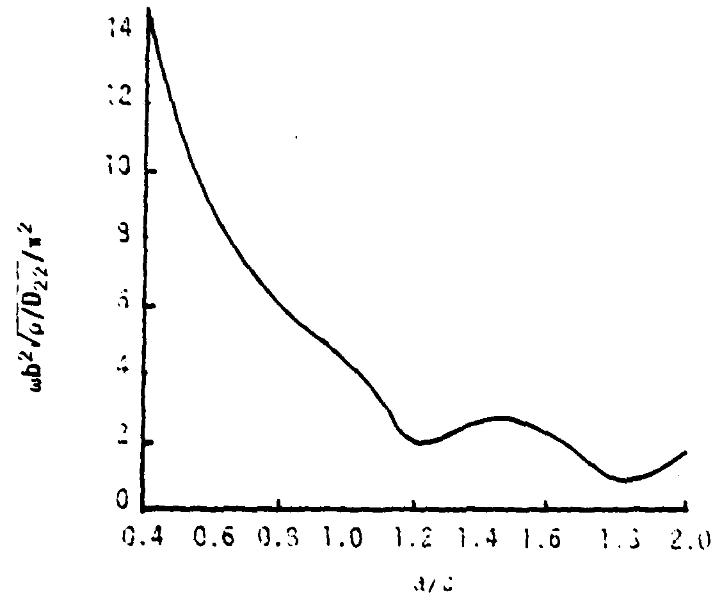
Abrupt changes in the values of Z_x and Z_y are noticeable which may be due to the bimodulus effect in combination with the eigenvalue nature of this problem (see Fig. 3.1).

Table 3.3. Values of Dimensionless Fundamental Frequencies and Neutral-Surface Locations for Single-Layer Orthotropic Rectangular Plates Having b/h=10 and Different Aspect Ratios

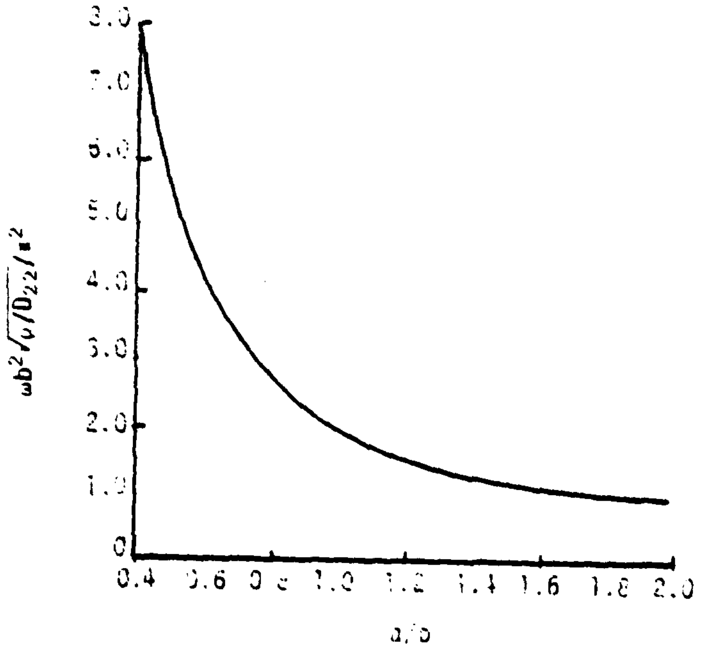
a/b	Z_x	Z_y	$\omega b^2\sqrt{\rho/D_{22}}/\pi^2$
		Aramid-Rubber:	
0.4	0.0100	0.0137	14.2370
0.6	- 0.0204	0.0296	8.9952
0.8	- 0.0346	0.0515	6.3712
1.0	0.0818	- 0.3619	4.1732
1.2	- 0.0724	- 0.2206	3.8780
1.4	0.3145	0.4442	2.9399
1.6	0.0058	- 0.5673	2.4122
1.8	0.0179	-0.3207×10^{-4}	2.1262
2.0	- 0.0257	- 0.0243	1.8935
		Polyester-Rubber:	
0.4	0.0679	- 0.2661	12.8460
0.6	0.7121×10^{-4}	0.9147×10^{-5}	7.4975
0.8	0.1314	0.0952	4.8319
1.0	0.0471	- 0.0197	3.3832
1.2	- 0.0613	0.0182	2.5949
1.4	0.0103	0.0034	2.1992
1.6	0.0903	- 0.1354	3.2867
1.8	- 0.0178	0.0474	1.9318
2.0	- 0.0178	0.0474	1.4641

Table 3.4. Values of Dimensionless Fundamental Frequencies and Neutral-Surface Locations for Cross-Ply Rectangular Plates Having b/h=10 and Different Aspect Ratios

a/b	Z_x	Z_y	$\omega b^2 \sqrt{\rho/D_{22}}/\pi^2$
		Aramid-Rubber:	
0.4	-0.2196×10^{-2}	-0.0152	14.1740
0.6	0.1929×10^{-2}	-0.0363	8.5910
0.8	-0.0205	-0.0353	6.0253
1.0	-0.0196	-0.0383	4.4521
1.2	0.0312	-0.0209	2.0092
1.4	-0.0109	0.0159	2.6748
1.6	0.0142	-0.7792×10^{-4}	2.3851
1.8	-0.1409×10^{-2}	0.1801×10^{-2}	0.8723
2.0	-0.0241	0.3046×10^{-3}	1.7419
		Polyester-Rubber:	
0.4	0.0742	0.4955×10^{-2}	7.7991
0.6	0.0283	0.1847×10^{-2}	4.3010
0.8	-0.0276	0.0219	2.7333
1.0	-0.1028	0.0449	1.9735
1.2	-0.1989	0.0667	1.6037
1.4	-0.8590×10^{-3}	0.1042	1.2888
1.6	-0.2123×10^{-4}	0.3120	1.133
1.8	-0.3218×10^{-5}	0.7538	1.002
2.0	1.2780	0.8525	0.9613



(a) Aramid-Rubber



(b) Polyester-Rubber

Fig. 3.1. Variation of fundamental vibration frequency with aspect ratio for two-layer cross-ply rectangular plates.

CHAPTER IV

THERMAL BENDING OF CROSS-PLY THICK BIMODULUS RECTANGULAR PLATES

Based upon the mathematical theory of elasticity of bimodulus materials and upon the Neumann hypothesis, Ambartsumyan [22] dealt with the development of the theory of thermoelasticity for elastic bimodulus material. Kamiya [20] developed the fundamental equations for axisymmetric plane stress problems for a bimodulus thin plate. Das and Rath [26] presented an analysis of thermal bending for a moderately thick rectangular plate subjected to a temperature distribution which is antisymmetric about the middle plane of the plate, but is arbitrary along the direction perpendicular to simply supported edges and constant along the other perpendicular direction. Bapu Rao [39] treated thermal bending of thick isotropic rectangular plates taking into consideration the shear deformation capability.

The present analysis deals with the thermal bending of unsymmetrically cross-plied bimodulus rectangular plates simply supported on all edges.

4.1 Governing Equations

The plate is subjected to a sinusoidal temperature distribution, T, Thus, temperature terms appear in the constitutive equations. The thermoelastic constitutive relations for the material are written as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}^{(s)} \begin{Bmatrix} \varepsilon_1 - \alpha_1 T \\ \varepsilon_2 - \alpha_2 T \\ \varepsilon_6 \end{Bmatrix} \tag{4.1.1}$$

The stiffness matrix [Q] takes different values in tension and compression depending upon the sign of the fiber-direction strain.

$$Q_{ij} = \begin{cases} Q_{ijc} & \text{if } \varepsilon_i < 0 \\ Q_{ijt} & \text{if } \varepsilon_i > 0 \end{cases} \tag{4.1.2}$$

The coefficients of thermal expansion α_1 and α_2 in the x and y directions, respectively, also depend upon the sign of the fiber-direction strain. Also,

$$\alpha_j = \begin{cases} \alpha_{jc} & \text{if } \varepsilon_i < 0 \\ \alpha_{jt} & \text{if } \varepsilon_i > 0 \end{cases} \tag{4.1.3}$$

The laminate constitutive relations can be represented by:

$$\begin{Bmatrix} N_x + N_x^T \\ N_y + N_y^T \\ N_{xy} \\ (M_x + M_x^T)/h \\ (M_y + M_y^T)/h \\ M_{xy}/h \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11}/h & B_{12}/h & 0 \\ A_{12} & A_{22} & 0 & B_{12}/h & B_{22}/h & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66}/h \\ B_{11}/h & B_{12}/h & 0 & D_{11}/h^2 & D_{12}/h^2 & 0 \\ B_{12}/h & B_{22}/h & 0 & D_{12}/h^2 & D_{22}/h^2 & 0 \\ 0 & 0 & B_{66}/h & 0 & 0 & D_{66}/h^2 \end{bmatrix} \begin{Bmatrix} u^o_{,x} \\ v^o_{,y} \\ v^o_{,x} + u^o_{,y} \\ h\psi_{x,x} \\ h\psi_{y,y} \\ h\psi_{y,x} + h\psi_{x,y} \end{Bmatrix} \tag{4.1.4}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} K_4^2 A_{44} & 0 \\ 0 & K_5^2 A_{55} \end{bmatrix} \begin{Bmatrix} w^o_{,y} + \psi_y \\ w^o_{,x} + \psi_x \end{Bmatrix} \tag{4.1.5}$$

where the thermally induced inplane forces are

$$\begin{Bmatrix} N_x^T \\ N_y^T \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} Q_{11}\alpha_1 + Q_{12}\alpha_2 \\ Q_{12}\alpha_1 + Q_{22}\alpha_2 \end{Bmatrix} T\, dz \tag{4.1.6}$$

and the thermally induced moments are

$$\begin{Bmatrix} M_x^T \\ M_y^T \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} Q_{11}\alpha_1 + Q_{12}\alpha_2 \\ Q_{12}\alpha_1 + Q_{22}\alpha_2 \end{Bmatrix} zT\, dz \tag{4.1.7}$$

As usual the stretching, stretching-bending, and bending stiffnesses for the laminate are defined as

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (Q_{ij})(1, z, z^2)\, dz \qquad i,j = 1,2,6 \tag{4.1.8}$$

Including shear deformation, the equations of equilibrium are:

$$\begin{aligned} &N_{x,x} + N_{xy,y} = 0 \\ &N_{xy,x} + N_{y,y} = 0 \\ &Q_{x,x} + Q_{y,y} = 0 \\ &M_{x,x} + M_{xy,y} - Q_x = 0 \\ &M_{xy,x} + M_{y,y} - Q_y = 0 \end{aligned} \tag{4.1.9}$$

Substituting equations (4.1.4) and (4.1.5) into equations (4.1.9), we get:

$$
[L_{k\ell}] \begin{Bmatrix} u^o \\ v^o \\ w^o \\ h\psi_y \\ h\psi_x \end{Bmatrix} = \begin{Bmatrix} N^T_{x,x} \\ N^T_{y,y} \\ 0 \\ M^T_{x,x} \\ M^T_{y,y} \end{Bmatrix} \qquad k,\ell = 1,2,3,4,5 \tag{4.1.10}
$$

where $[L_{k\ell}]$ is a symmetrix linear differential operator matrix with the following elements:

$$
\begin{aligned}
L_{11} &\equiv A_{11} d_x^2 + A_{66} d_y^2 \\
L_{12} &\equiv (A_{12} + A_{66})\, d_x d_y \\
L_{13} &\equiv 0 \\
L_{14} &\equiv (B_{12} + B_{66})\, d_x d_y \\
L_{15} &\equiv B_{11} d_x^2 + B_{66} d_y^2 \\
L_{22} &\equiv A_{66} d_x^2 + A_{22} d_y^2 \\
L_{23} &\equiv 0 \\
L_{24} &\equiv B_{66} d_x^2 + B_{22} d_y^2 \\
L_{25} &\equiv L_{14} \\
L_{33} &\equiv -K_5^2 A_{55} d_x^2 - K_4^2 A_{44} d_y^2 \\
L_{34} &\equiv -K_4^2 A_{44} d_y \\
L_{35} &\equiv -K_5^2 A_{55} d_x \\
L_{44} &\equiv D_{66} d_x^2 + D_{22} d_y^2 - K_4^2 A_{44} \\
L_{45} &\equiv (D_{12} + D_{66})\, d_x d_y \\
L_{55} &\equiv D_{11} d_x^2 + D_{66} d_y^2 - K_5^2 A_{55}
\end{aligned} \tag{4.1.11}
$$

4.2 Application to a Simply Supported Plate

The boundary conditions, as usual, are:

Along the edges at $x = 0$ and $x = a$,

$$w = \psi_y = M_x + M_x^T = 0$$

$$v^o = N_x + N_x^T = 0$$

Along the edges at $y = 0$ and $y = b$, (4.2.1)

$$w = \psi_x = M_y + M_y^T = 0$$

$$u^o = N_y + N_y^T = 0$$

4.3 Closed-Form Solution

The governing equations and the boundary conditions can be satisfied exactly by the following for T sinusoidally distributed along x and y:

$$
\begin{aligned}
u^o &= U \cos \alpha x \sin \beta y \\
v^o &= V \sin \alpha x \cos \beta y \\
w &= W \sin \alpha x \sin \beta y \qquad (4.3.1) \\
h\psi_y &= Y \sin \alpha x \cos \beta y \\
h\psi_x &= X \cos \alpha x \ \mathrm{con}\ \beta y
\end{aligned}
$$

where

$$\alpha = m\pi/a \ , \quad \beta = n\pi/b \qquad (4.3.2)$$

Here m and n are integers, and a and b are plate dimensions in the x and y directions.

Inserting solutions (4.3.1) into the governing equations (4.1.10), we get the following:

$$[C_{k\ell}] \begin{Bmatrix} U \\ V \\ W \\ Y \\ X \end{Bmatrix} = \begin{Bmatrix} N^T_{x,x} \\ N^T_{y,y} \\ 0 \\ M^T_{x,x} \\ M^T_{y,y} \end{Bmatrix} \qquad k,\ell=1,2,3,4,5 \tag{4.3.3}$$

where $[C_{k\ell}]$ is a symmetric matrix containing the following elements:

$$
\begin{aligned}
C_{11} &\equiv - A_{11}\alpha^2 - A_{66}\beta^2 \\
C_{12} &\equiv - (A_{12} + A_{66})\alpha\beta \\
C_{13} &\equiv 0 \\
C_{14} &\equiv - (B_{12} + B_{66})\alpha\beta \\
C_{15} &\equiv - B_{11}\alpha^2 - B_{66}\beta^2 \\
C_{22} &\equiv - A_{66}\alpha^2 - A_{22}\beta^2 \\
C_{23} &\equiv 0 \\
C_{24} &\equiv - B_{66}\alpha^2 - B_{22}\beta^2 \qquad\qquad (4.3.4) \\
C_{25} &\equiv C_{14} \\
C_{33} &\equiv - K_5^2 A_{55}\alpha^2 - K_4^2 A_{44}\beta^2 \\
C_{34} &\equiv - K_4^2 A_{44}\beta \\
C_{35} &\equiv - K_5^2 A_{55}\alpha \\
C_{44} &\equiv - D_{66}\alpha^2 - D_{22}\beta^2 - K_4^2 A_{44} \\
C_{45} &\equiv - (D_{12} + D_{66})\alpha\beta \\
C_{55} &\equiv - D_{11}\alpha^2 - D_{66}\beta^2 - K_5^2 A_{55}
\end{aligned}
$$

4.4 Mean Temperature and Temperature Gradient Sinusoidally Distributed over a Rectangular Region

Let

$$T(x,y,z) = T_o(x,y) + (z/h)\ T_1(x,y) \tag{4.4.1}$$

where

$$T_o(x,y) = \bar{T}_o \sin(m\pi x/a) \sin(n\pi y/b)$$

and

$$T_1(x,y) = \bar{T}_1 \sin(m\pi x/a) \sin(n\pi y/b) \tag{4.4.2}$$

For Case I, $z_{nx} > 0$ and $z_{ny} < 0$ with z_{nx} governing layer 1 (0°) and z_{ny} layer 2 (90°) (see Appendix C for the remaining cases).

$$N_x^T = \int_{-h/2}^{z_{ny}} (Q_{1122}\,\alpha_{122} + Q_{1222}\,\alpha_{222})\ Tdz$$

$$+ \int_{z_{ny}}^{0} (Q_{1112}\,\alpha_{112} + Q_{1212}\,\alpha_{212})\ Tdz$$

$$+ \int_{0}^{z_{nx}} (Q_{1121}\,\alpha_{121} + Q_{1221}\,\alpha_{221})\ Tdz$$

$$+ \int_{z_{nx}}^{h/2} (Q_{1111}\,\alpha_{111} + Q_{1211}\,\alpha_{211})\ Tdz \tag{4.4.3}$$

Let

$$(Q_{1122}\,\alpha_{122} + Q_{1222}\,\alpha_{222}) = \beta_{122}$$

$$(Q_{1112}\,\alpha_{112} + Q_{1212}\,\alpha_{212}) = \beta_{112}$$

$$(Q_{1121}\,\alpha_{121} + Q_{1221}\,\alpha_{221}) = \beta_{121}$$

$$(Q_{1111}\,\alpha_{111} + Q_{1211}\,\alpha_{211}) = \beta_{111} \qquad \text{etc.}$$

Then,

$$N_x^T = \beta_{122}\, T_o(z_{ny} + h/2) + \beta_{112}\, T_o(0 - z_{ny}) + \beta_{121}\, T_o(z_{nx} - 0)$$

$$+ \beta_{111}\, T_o(h/2 - z_{nx}) + \beta_{122}(T_1/2h)(z_{ny}^2 - h^2/4)$$

$$+ \beta_{112}(T_1/2h)(0 - z_{ny}^2) + \beta_{121}(T_1/2h)(z_{nx}^2 - 0)$$

$$+ \beta_{111}(T_1/2h)(h^2/4 - z_{nx}^2)$$

$$N_x^T = (\beta_{122} + \beta_{111})(T_o h/2) + (\beta_{121} - \beta_{111})\, T_o z_{nx} + (\beta_{122} - \beta_{112})$$

$$T_o z_{ny} + (\beta_{111} - \beta_{122})(T_1 h/8) + (\beta_{121} - \beta_{111})(T_1 z_{nx}^2/2h)$$

$$+ (\beta_{122} - \beta_{112})(T_1 z_{ny}^2/2h) \tag{4.4.4}$$

Similarly,

$$N_y^T = (\beta_{222} + \beta_{211})(T_o h/2) + (\beta_{221} - \beta_{211})\, T_o z_{nx} + (\beta_{222} - \beta_{212})\, T_o z_{ny}$$

$$+ (\beta_{211} - \beta_{222})(T_1 h/8) + (\beta_{221} - \beta_{211})(T_1 z_{nx}^2/2h) + (\beta_{222} - \beta_{212})$$

$$(T_1 z_{ny}^2/2h) \tag{4.4.5}$$

Now,

$$M_x^T = \int_{-h/2}^{h/2} \beta_{122}\, Tzdz + \int_{z_{ny}}^{0} \beta_{112}\, Tzdz + \int_{0}^{z_{nx}} \beta_{121}\, Tzdz + \int_{z_{nx}}^{h/2} \beta_{111}\, Tzdz$$

$$= (\beta_{111} - \beta_{122})(T_o h^2/8) + (\beta_{121} - \beta_{111})(T_o z_{nx}^2/2) + (\beta_{122} - \beta_{112})(T_o z_{ny}^2/2)$$

$$+ (\beta_{122} + \beta_{111})(T_1 h^2/24) + (\beta_{121} - \beta_{111})(T_1 z_{nx}^3/3h) + (\beta_{122} - \beta_{112})(T_1 z_{ny}^3/3h)$$

$$\tag{4.4.6}$$

And similarly,

$$M_y^T = (\beta_{211} - \beta_{222})(T_o h^2/8) + (\beta_{221} - \beta_{211})(T_o z_{nx}^2/2) + (\beta_{222} - \beta_{212})(T_o z_{ny}^2/2)$$

$$+ (\beta_{222} + \beta_{211})(T_1 h^2/24) + (\beta_{221} - \beta_{211})(T_1 z_{nx}^3/3h) + (\beta_{222} - \beta_{212})(T_1 z_{ny}^3/3h) \tag{4.4.7}$$

Using equations (4.4.4), (4.4.5), (4.4.6), and (4.4.7), we obtain the following:

$$N_{x,x}^T = \alpha\{(\beta_{122} + \beta_{111})(\bar{T}_o h/2) + (\beta_{121} - \beta_{111})\bar{T}_o z_{nx} + (\beta_{122} - \beta_{112})\bar{T}_o z_{ny}$$

$$+ (\beta_{111} - \beta_{122})(\bar{T}_1 h/8) + (\beta_{121} - \beta_{111})(\bar{T}_1 z_{nx}^2/2h)$$

$$+ (\beta_{122} - \beta_{112})(\bar{T}_1 z_{ny}^2/2h)\} \tag{4.4.8}$$

$$N_{y,y}^T = \beta\{(\beta_{222} + \beta_{211})(\bar{T}_o h/2) + (\beta_{221} - \beta_{211})\bar{T}_o z_{nx} + (\beta_{222} - \beta_{212})\bar{T}_o z_{ny}$$

$$+ (\beta_{211} - \beta_{222})(\bar{T}_1 h/8) + (\beta_{221} - \beta_{211})(\bar{T}_1 z_{nx}^2/2h)$$

$$+ (\beta_{222} - \beta_{212})(\bar{T}_1 z_{ny}^2/2h) \tag{4.4.9}$$

$$M_{x,x}^T = \alpha\{(\beta_{111} - \beta_{122})(\bar{T}_o h^2/8) + (\beta_{121} - \beta_{111})(\bar{T}_o z_{nx}^2/2) + (\beta_{122} - \beta_{112})$$

$$(\bar{T}_o z_{ny}^2/2) + (\beta_{122} + \beta_{111})(\bar{T}_1 h^2/24) + (\beta_{121} - \beta_{111})(\bar{T}_1 z_{nx}^3/3h)$$

$$+ (\beta_{122} - \beta_{112})(\bar{T}_1 z_{ny}^3/3h)\} \tag{4.4.10}$$

and

$$M^T_{y,y} = \beta\{(\beta_{211} - \beta_{222})(\bar{T}_o h^2/8) + (\beta_{221} - \beta_{211})(\bar{T}_o z^2_{nx}/2) + (\beta_{222} - \beta_{212})(\bar{T}_o z^2_{ny}/2)$$

$$+ (\beta_{222} + \beta_{211})(\bar{T}_1 h^2/24) + (\beta_{221} - \beta_{211})(\bar{T}_1 z^3_{nx}/3h)$$

$$+ (\beta_{222} - \beta_{212})(\bar{T}_1 z^3_{ny}/3h) \quad (4.4.11)$$

4.5 Neutral-Surface Locations

As explained already,

$$z_{nx} = - hU/X \quad , \quad z_{ny} = - hV/Y \quad (4.5.1)$$

Numerically, the values z_{nx} and z_{ny} are computed as follows:

The values of z_{nx} and z_{ny} are assumed in the beginning to get displacements. The above equations are used to obtain the new values of z_{nx} and z_{ny} and fed back to get one a more accurate set. This procedure is repeated until the actual deflections are obtained.

4.6 Numerical Results

Numerical results are compared with Boley and Weiner's work [40] for an isotropic-ordinary-material, single-layer, thin rectangular plate. Close agreement has been obtained (see Table 4.1 below).

Table 4.1. Comparison with Boley and Weiner's Work [40] for an Isotropic Single-Layer Thin Rectangular Plate at Different Aspect Ratios (E_{11}/E_{22} = 1.00 , $\nu_{12} = \nu_{21}$ = 0.3 , b/h = 10)

Aspect Ratio, a/b	Deflection, W/h (m=n=1)	
	Boley and Weiner [40]	Present
0.5	0.5300	0.5264
1.0	6.5858	6.5789
1.5	6.3112	6.3063
2.0	2.1104	2.1058

Typical numerical results for (bimodulus) aramid-rubber and polyester-rubber are listed in the following tables (see Table 2.4 for the properties).

The solution for an isotropic,ordinary-material,single-layer, thin rectangular plate has been specialized from the present analysis and is compared with such a solution available in the literature. Good agreement was obtained.

Sudden change in the deflection has been observed in the case of aramid-rubber. To show the general trend, graphs have been plotted (see Fig. 4.1).

Table 4.2. Values of $Wh/\alpha_2^t \bar{T}_1 b^2$ and Neutral-Surface Locations for a Single-Layer Orthotropic Rectangular Plate ($b/h=10$, $\alpha_1{}^t/\alpha_1{}^c=0.5$, $\alpha_2{}^t/\alpha_2{}^c=1.0$, $\alpha_1{}^t/\alpha_2{}^t=0.1$, $\bar{T}_o/\bar{T}_1=1.0$)

a/b	Z_x	Z_y	$Wh/\alpha_2{}^t \bar{T}_1 b^2$
		Aramid-Rubber:	
0.5	- 2.3007	- 0.1666	0.4533×10^{-1}
0.6	- 1.8521	- 0.2237	0.5547×10^{-1}
0.7	- 1.5573	- 0.2745	0.6579×10^{-1}
0.8	- 1.3482	- 0.3160	0.7636×10^{-1}
0.9	- 1.1192	- 0.3473	0.8722×10^{-1}
1.0	- 1.0708	- 0.3690	0.9839×10^{-1}
1.2	- 0.8959	- 0.3886	0.1218
1.4	- 0.7766	- 0.3861	0.1467
1.6	- 0.6914	- 0.3709	0.1732
1.8	- 0.6286	- 0.3498	0.2011
2.0	- 0.5819	- 0.3266	0.2301
		Polyester-Rubber:	
0.5	- 2.1099	- 0.6511	0.1401×10^{-1}
0.6	- 1.6790	- 0.8482	0.1764×10^{-1}
0.7	- 1.4029	- 1.0080	0.2156×10^{-1}
0.8	- 1.2112	- 1.1218	0.2583×10^{-1}
0.9	- 1.0710	- 1.1903	0.3046×10^{-1}
1.0	- 0.9649	- 1.2194	0.3546×10^{-1}
1.2	- 0.8172	- 1.1933	0.4672×10^{-1}
1.4	- 0.7233	- 1.1062	0.5940×10^{-1}
1.6	- 0.6630	- 1.0015	0.7318×10^{-1}
1.8	- 0.6258	- 0.9016	0.8757×10^{-1}
2.0	- 0.6060	- 0.8158	0.1020

(See Appendix D for the in-plane displacements)

Table 4.3. Values of $Wh/\alpha_2{}^t \bar{T}_1 b^2$ and Neutral-Surface Locations for a Cross-Ply Rectangular Plate ($b/h=10$, $\alpha_1{}^t/\alpha_2{}^t=0.1$, $\alpha_1{}^t/\alpha_1{}^c=0.5$, $\alpha_2{}^t/\alpha_2{}^c=1.0$, $\bar{T}_0/\bar{T}_1=1.0$)

a/b	Z_x	Z_y	$Wh/\alpha_2{}^t \bar{T}_1 b^2$
		Aramid-Rubber:	
0.5	0.1276	-0.1510×10^3	-0.2620×10^{-1}
0.6	0.1184	-0.1722×10^3	-0.3983×10^{-1}
0.7	0.1165	-0.2347×10^3	-0.5973×10^{-1}
0.8	0.1191	-0.5316×10^3	-0.8814×10^{-1}
0.9	0.0578	-0.3450	0.3660×10^{-2}
1.0	0.0980	-0.3498	0.4432×10^{-2}
1.2	0.1304	-0.4327	0.5848×10^{-2}
1.4	0.1399	-0.3375	0.6998×10^{-2}
1.6	0.1402	-0.3339	0.7852×10^{-2}
1.8	0.1358	-0.3316	0.8453×10^{-2}
2.0	0.1286	-0.3302	0.8862×10^{-2}
		Polyester-Rubber:	
0.5	0.7442	-0.8415	-0.5453×10^{-1}
0.6	0.7433	-0.8470	-0.7863×10^{-1}
0.7	0.7477	-0.8516	-0.9269×10^{-1}
0.8	0.7584	-0.8439	-0.8915×10^{-1}
0.9	0.7772	-0.8237	-0.6903×10^{-1}
1.0	0.8075	-0.7973	-0.3825×10^{-1}
1.2	0.9274	-0.7628	0.2514×10^{-1}
1.4	3.2603	0.4369	-0.7672
1.6	3.3521	0.4656	-0.8295
1.8	3.3178	0.4636	-0.8473
2.0	3.2126	0.4440	-0.8402

(See Appendix D for the in-plane displacements)

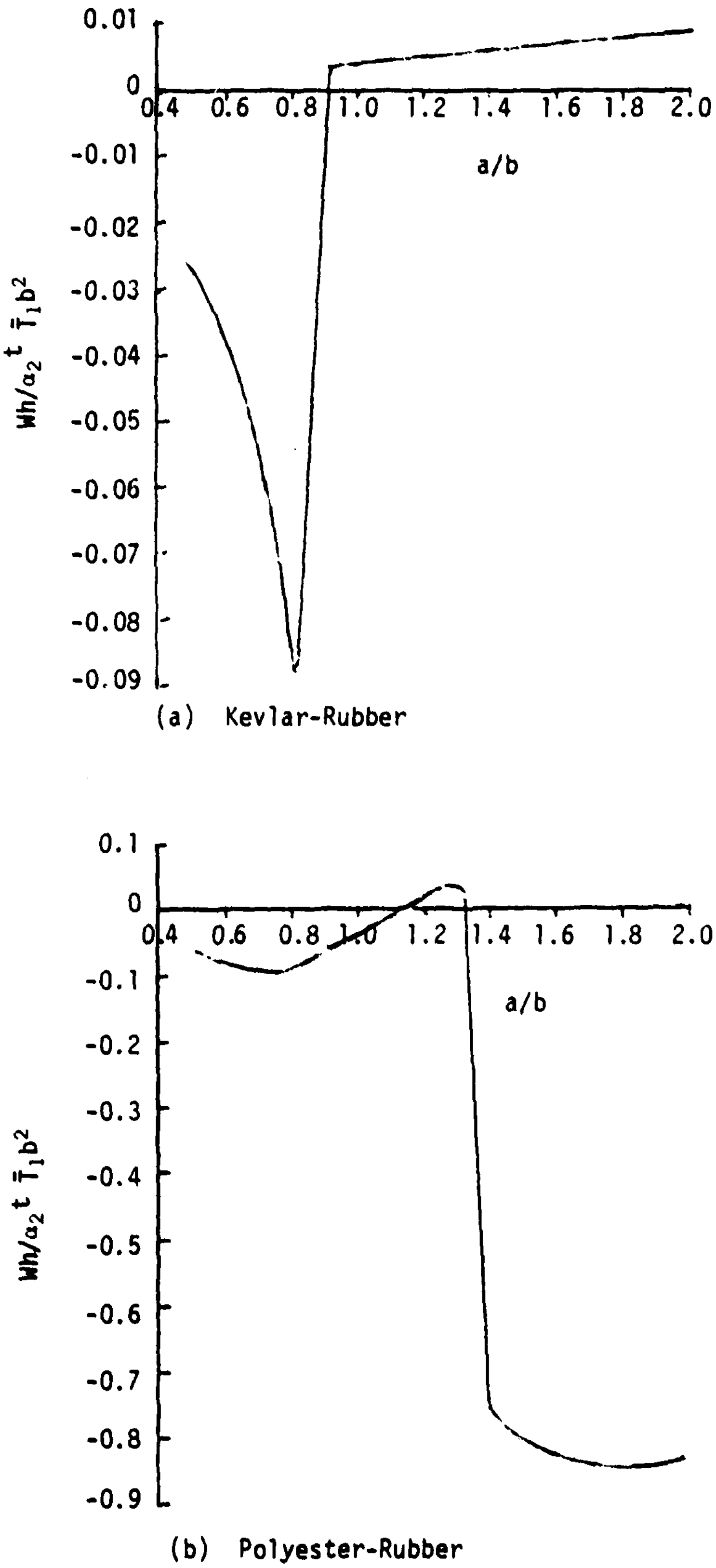


Fig. 4.1. Variation of dimensionless deflection with respect to aspect ratio for two-layer cross-ply rectangular plates.

CHAPTER V

LARGE DEFLECTIONS OF BIMODULUS CROSS-PLY THIN RECTANGULAR PLATE

This class of problem has been attempted in different ways by various people [11,12,41-49]. Nonlinear bending of ordinary orthotropic, single-layer thin plates subjected to uniform loading has been treated by Niyogi [41] among others.

Perhaps the first large deflection analysis of unsymmetrically laminated plates was due to Pister and Dong [42], who considered isotropic rectangular plates. The arbitrarily laminated fully anisotropic equations of the von Karman type were probably first presented by Whitney and Leissa [17], who did not solve them. Large-deflection analyses of unsymmetrical, laminated rectangular plates have been published in [43-49].

Large-deflection analyses of plates made of bimodulus materials have been limited to isotropic bimodulus materials. Kamiya [11,12] analyzed both the circular and the rectangular planforms.

Due to the complicated algebra involved in the nonlinear behavior of thick plates, the work in this chapter has been reduced to thin plates. An approximate solution is obtained by using the Galerkin technique.

5.1 Basic Equations

Consider a thin rectangular plate of thickness h and plate dimensions a and b (in the x and y directions, respectively) subjected to nonlinear bending.

In view of Kirchhoff's thin-plate hypothesis, the displacement components u, v, and w in the x, y, and z directions can be expressed in terms of mid-plane displacements u^o, v^o, and w^o as:

$$
\begin{aligned}
u &= u^o(x,y) - z\, w_{,x}(x,y) \\
v &= v^o(x,y) - z\, w_{,y}(x,y) \\
w &= w(x,y)
\end{aligned}
\qquad (5.1.1)
$$

where the comma denotes differentiation.

The laminate constitutive relations can be written as:

$$
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}
=
\begin{bmatrix}
A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\
A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\
0 & 0 & A_{66} & 0 & 0 & B_{66} \\
B_{11} & B_{12} & 9 & D_{11} & D_{12} & 0 \\
B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\
0 & 0 & B_{66} & 0 & 0 & D_{66}
\end{bmatrix}
\begin{Bmatrix} u^o_{,x} + w_{,x}^2/2 \\ v^o_{,y} + w_{,y}^2/2 \\ u^o_{,y} + v^o_{,x} + w_{,x}w_{,y} \\ -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix}
\qquad (5.1.2)
$$

The quantities A_{ij}, B_{ij}, and D_{ij} are the stretching, stretching-bending coupling, and bending stiffnesses defined by

$$(A_{ij},B_{ij},D_{ij}) = \int_{-h/2}^{h/2} (1,z,z^2)\, Q_{ij}\, dz \qquad i,j=1,2,6 \tag{5.1.3}$$

The equilibrium equations are (neglecting body forces and body moments):

$$\begin{aligned}
&N_{x,x} + N_{xy,y} = 0 \\
&N_{xy,x} + N_{y,y} = 0 \\
&N_x w,_{xx} + 2N_{xy} w,_{xy} + N_y w,_{yy} + M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + q = 0
\end{aligned} \tag{5.1.4}$$

where q represents the normal load.

Substituting equations (5.1.2) into equations (5.1.4), we get the following. For equilibrium in the x direction:

$$A_{11}(u^o,_{xx} + w,_x w,_{xx}) + A_{12}(v^o,_{xy} + w,_y w,_{xy}) - B_{11}w,_{xxx} - B_{12}w,_{xyy}$$

$$+ A_{66}(u^o,_{yy} + u^o,_{xy} + w,_x w,_{yy} + w,_y w,_{xy}) - 2B_{66}\, w,_{xyy} = 0$$

or

$$A_{11}u^o,_{xx} + A_{66}u^o,_{yy} + (A_{12} + A_{66})v^o,_{xy} + w,_x(A_{11}w._{xx} + A_{66}w,_{yy})$$

$$+ w,_y(A_{12} + A_{66})w,_{xy} - B_{11}w,_{xxx} - (B_{12} + 2B_{66})w,_{xyy} = 0 \tag{5.1.5}$$

For equilibrium in the y direction:

$$A_{66}(u^o,_{xy} + v^o,_{xx} + w,_x w,_{xy} + w,_y w,_{xx}) - 2B_{66}w,_{xxy}$$

$$+ A_{12}(u^o,_{xy} + w,_x w,_{xy}) + A_{22}(v^o,_{yy} + w,_y w,_{yy})$$

$$- B_{12}w,_{xxy} - B_{22}w,_{yyy} = 0$$

or

$$A_{22}v^o_{,yy} + A_{66}v^o_{,xx} + (A_{12}+A_{66})u^o_{,xy} + w_{,y}(A_{22}w_{,yy}+A_{66}w_{,xx})$$

$$+ w_{,x}(A_{12}+A_{66})w_{,xy} - B_{22}w_{,yyy} - (B_{12}+2B_{66})w_{,xxy} = 0 \qquad (5.1.6)$$

For equilibrium in the z direction:

$$[A_{11}(u^o_{,x} + \tfrac{1}{2}w_{,x}^2) + A_{12}(v^o_{,y} + \tfrac{1}{2}w_{,y}^2) + B_{11}(-w_{,xx}) + B_{12}(-w_{,yy})]w_{,xx}$$

$$+ 2[A_{66}(u^o_{,y} + v^o_{,x} + w_{,x}w_{,y}) + B_{66}(-2w_{,xy})]w_{,xy}$$

$$+ [A_{12}(u^o_{,x} + \tfrac{1}{2}w_{,x}^2) + A_{22}(v^o_{,y} + \tfrac{1}{2}w_{,y}^2) - B_{12}w_{,xx} - B_{22}w_{,yy}]w_{,yy}$$

$$+ B_{11}(u_{,xxx} + w_{,x}w_{,xxx} + w_{,xx}^2) + B_{12}(v^o_{,xxy} + w_{,y}w_{,xxy} + w_{,xy}^2)$$

$$- D_{11}w_{,xxxx} - D_{12}w_{,xxyy}$$

$$+ 2[B_{66}(u^o_{,xyy} + v^o_{,xxy} + w_{,x}w_{,xyy} + w_{,xy}^2 + w_{,y}w_{,xxy} + w_{,xx}w_{,yy})$$

$$- 2D_{66}w_{,xxyy}] + B_{12}(u^o_{,xyy} + w_{,x}w_{,xyy} + w_{,xy}^2)$$

$$+ B_{22}(v^o_{,yyy} + w_{,y}w_{,yyy} + w_{,yy}^2) - D_{12}w_{,xxyy} - D_{22}w_{,yyyy} + q = 0$$

or

$$B_{11}u^o_{,xxx} + (B_{12} + 2B_{66})(u^o_{,xyy} + v^o_{,xxy}) + B_{22}v^o_{,yyy}$$

$$+ w_{,x}[B_{11}w_{,xxx} + (B_{12} + 2B_{66})w_{,xyy}] + w_{,y}[B_{22}w_{,yyy} + (B_{12} + 2B_{66})w_{,xxy}]$$

$$+ 2w_{,xx}(B_{66} - B_{12})w_{,yy} + 2w_{,xy}w_{,xy}(B_{12} + B_{66}) - w_{,yy}B_{22}w_{,yy}$$

$$- D_{11}w_{,xxxx} - 2(D_{12} + 2D_{66})w_{,xxyy} - D_{22}w_{,yyyy}$$

$$+ \{q + w_{,xx}[A_{11}(u^o_{,x} + \tfrac{1}{2}w_{,x}^2) + A_{12}(v^o_{,y} + \tfrac{1}{2}w_{,y}^2)]$$

$$+ w_{,yy}[A_{12}(u^o_{,x} + \tfrac{1}{2}w_{,x}^2) + A_{22}(v^o_{,y} + \tfrac{1}{2}w_{,y}^2)]$$

$$+ 2w_{,xy}[A_{66}(u^o_{,y} + v^o_{,x} + w_{,x}w_{,y}) - 2B_{66}w_{,xy}]\} = 0 \qquad (5.1.7)$$

5.2 Simply-Supported Boundary Conditions

Along the edges at $x = 0,a$

$$w = w,_{yy} = M_x = 0$$
$$v^o = N_x = 0$$

Along the edges at $y = 0,b$ (5.2.1)

$$w = w,_{xx} = M_y = 0$$
$$u^o = N_y = 0$$

5.3 Solution

The set of equations (5.1.5-5.1.7) are coupled and nonlinear in nature, and an exact solution appears to be extremely difficult to obtain. Hence, an approximate solution will be obtained here. Let

$$\begin{aligned} q &= Q \sin \alpha x \sin \beta y \\ w &= W \sin \alpha x \sin \beta y \\ u^o &= U \cos \alpha x \sin \beta y \\ v^o &= V \sin \alpha x \cos \beta y \end{aligned} \qquad (5.3.1)$$

By the Galerkin method (equations (5.1.5)-(5.1.7) and equations (5.3.1) are combined):

$$\int_0^a \int_0^b \{(A_{11}u^o,_{xx} + A_{66}v^o,_{yy} + (A_{12} + A_{66})v^o,_{xy} + w,_x(A_{11}w,_{xx} + A_{66}w,_{yy})$$
$$+ w,_y(A_{12} + A_{66})w,_{xy} - B_{11}w,_{xxx} - (B_{12} + 2B_{66})w,_{xyy})\} \cos \alpha x \sin \beta y \, dxdy = 0$$

or

$$\int_0^a\int_0^b \{- A_{11} \frac{\pi^2}{a^2} U \cos^2 \alpha x \sin^2 \beta y - A_{66} \frac{\pi^2}{b} U \cos^2 \alpha x \sin^2 \beta y$$

$$- V(A_{12} + A_{66}) \frac{\pi^2}{ab} \cos^2 \alpha x \sin^2 \beta y - A_{11} \frac{\pi^3}{a^3} W^2 \cos^2 \alpha x \sin^3 \beta y \sin \alpha x$$

$$- A_{66} \frac{\pi^3}{ab^2} W^2 \cos^2 \alpha x \sin^3 \beta y \sin \alpha x$$

$$+ W^2(A_{12} + A_{66}) \frac{\pi^3}{ab^2} \cos^2 \alpha x \cos^2 \beta y \sin \alpha x \sin \beta y + B_{11} W \frac{\pi^3}{a^3} \cos^2 \alpha x \sin^2 \beta y$$

$$+ W(B_{12} + 2B_{66}) \frac{\pi^3}{ab^2} \cos^2 \alpha x \sin^2 \beta y\} \, dxdy = 0$$

or

$$- A_{11} \frac{\pi^2}{a^2} U \frac{ab}{4} - A_{66} \frac{\pi^2}{b^2} U \frac{ab}{4} - (A_{12} + A_{66})V \frac{\pi^2}{ab} \frac{ab}{4}$$

$$- A_{11} \frac{\pi^3}{a^3} W^2 \cdot \frac{8ab}{9\pi^2} - A_{66} \frac{\pi^3}{ab^2} W^2 \cdot \frac{8ab}{9\pi^2} + W^2(A_{12} + A_{66}) \frac{\pi^3}{ab^2} \cdot \frac{4ab}{9\pi^2}$$

$$+ B_{11} W \frac{\pi^3}{a^3} \cdot \frac{ab}{4} + W(B_{12} + 2B_{66}) \frac{\pi^3}{ab^2} \cdot \frac{ab}{4} = 0$$

or

$$U(+ \frac{A_{11}\pi^2 b}{4a} + \frac{A_{66}\pi^2 a}{4b}) + V[+ (A_{12} + A_{66}) \frac{\pi^2}{4}]$$

$$- W[+ \frac{B_{11}\pi^3 b}{4a} + (B_{12} + 2B_{66}) \frac{\pi^3}{4b}]$$

$$+ W^2[+ \frac{8}{9} A_{11} \frac{\pi b}{a^2} + \frac{8}{9} A_{66} \frac{\pi}{b} - \frac{4}{9} (A_{12} + A_{66}) \frac{\pi}{b}] = 0$$

or

$$U\{\frac{\pi^2}{4ab} (A_{11}b^2 + A_{66}a^2)\} + V\{\frac{\pi^2}{4} (A_{12} + A_{66})\} + W\{- \frac{\pi^3 b}{4a^2} B_{11} - \frac{\pi^3}{4b} (B_{12} + 2B_{66})\}$$

$$+ W^2 \{ \frac{8}{9} \frac{\pi}{b} (A_{11} \frac{b^2}{a^2} + A_{66} - \frac{1}{2} (A_{12} + A_{66}))\} = 0 \qquad (5.3.2)$$

Similarly,

$$\int_0^a\int_0^b \{A_{22}v^o_{,yy} + A_{66}v^o_{,xx} + (A_{12} + A_{66})u^o_{,xy} + w_{,y}(A_{22}w_{,yy} + A_{66}w_{,xx})$$

$$+ w_{,x}(A_{12} + A_{66})w_{,xy} - B_{22}w_{,yyy} - (B_{12} + 2B_{66})w_{,xxy}\}\sin \alpha x \cos \beta y \; dxdy = 0$$

or

$$\int_0^a\int_0^b \{- A_{22} \frac{\pi^2}{b^2} V \sin^2 \alpha x \cos^2 \beta y - A_{66} \frac{\pi^2}{a^2} V \sin^2 \alpha x \cos^2 \beta y$$

$$- (A_{12} + A_{66}) U \frac{\pi^2}{ab} \sin^2 \alpha x \cos^2 \beta y - A_{22}W^2 \frac{\pi^3}{b^3} \sin^3 \alpha x \cos^2 \beta y \sin \beta y$$

$$- A_{66}W^2 \frac{\pi^3}{a^2b} \sin^3 \alpha x \cos^2 \beta y \sin \beta y$$

$$+ (A_{12} + A_{66})W^2 \frac{\pi^3}{a^2b} \sin \alpha x \cos^2 \alpha x \sin \beta y \cos^2 \beta y + B_{22} \frac{\pi^3}{b^3} W \sin^2 \alpha x \cos^2 \beta y$$

$$+ (B_{12} + 2B_{66}) W \frac{\pi^3}{a^2b} \sin^2 \alpha x \cos^2 \beta y\} = 0$$

or

$$- A_{22} \frac{\pi^2}{b^2} V \frac{ab}{4} - A_{66} \frac{\pi^2}{a^2} V \frac{ab}{4} - (A_{12} + A_{66}) U \frac{\pi^2}{ab} \frac{ab}{4} - A_{22}W^2 \frac{\pi^3}{b^3} \frac{4a}{3\pi} \frac{2b}{3\pi}$$

$$- A_{66}W^2 \frac{\pi^3}{a^2b} \frac{8ab}{9\pi^2} + (A_{12} + A_{66})W^2 \frac{\pi^3}{a^2b} \frac{4ab}{9\pi^2} + B_{22}W \frac{\pi^3}{b^3} \frac{ab}{4}$$

$$+ (B_{12} + 2B_{66})W \frac{\pi^3}{a^2b} \frac{ab}{4} = 0$$

or

$$V(+ A_{22} \frac{\pi^2 a}{4b} + A_{66} \frac{\pi^2 b}{4a}) + U(A_{12} + A_{66}) \frac{\pi^4}{4}$$

$$- W[B_{22} \frac{\pi^3 a}{4b^2} + (B_{12} + 2B_{66}) \frac{\pi^3}{4a}]$$

$$+ W^2[+ A_{22} \frac{8\pi a}{9b^2} + A_{66} \frac{8\pi}{9a} - (A_{12} + A_{66}) \frac{4\pi}{9a}] = 0$$

or

$$U\{\frac{\pi^2}{4}(A_{12}+A_{66})\} + V\{\frac{\pi^2}{4}(\frac{a}{b}A_{22}+\frac{b}{a}A_{66})\}$$

$$+ W\{-B_{22}\frac{\pi^3 a}{4b^2} - \frac{\pi^3}{4a}(B_{12}+2B_{66})\}$$

$$+ W^2\{\frac{8\pi}{9a}(\frac{a^2}{b^2}A_{22}+A_{66}-\frac{1}{2}(A_{12}+A_{66})\} = 0 \tag{5.3.3}$$

Also,

$$\int_0^a\int_0^b \{B_{11}u^o_{,xxx} + (B_{12}+2B_{66})(u^o_{,xyy}+v^o_{,xxy}) + B_{22}v^o_{,yyy}$$

$$+ w_{,x}[B_{11}w_{,xxx} + (B_{12}+2B_{66})w_{,xyy}]$$

$$+ w_{,y}[B_{22}w_{,yyy} + (B_{12}+2B_{66})w_{,xxy}]$$

$$+ 2w_{,xx}(B_{66}-B_{12})w_{,yy} + 2w_{,xy}w_{,xy}(B_{12}+B_{66}) - w_{,yy}B_{22}w_{,yy}$$

$$- D_{11}w_{,xxxx} - 2(D_{12}+2D_{66})w_{,xxyy} - D_{22}w_{,yyyy}$$

$$+ q + w_{,xx}[A_{11}(u^o_{,x}+\frac{1}{2}w_{,x}^2) + A_{12}(v^o_{,y}+\frac{1}{2}w_{,y}^2)]$$

$$+ w_{,yy}[A_{12}(u^o_{,x}+\frac{1}{2}w_{,x}^2) + A_{22}(v^o_{,y}+\frac{1}{2}w_{,y}^2)]$$

$$+ 2w_{,xy}[A_{66}(u^o_{,y}+v^o_{,x}+w_{,x}w_{,y}) - 2B_{66}w_{,xy}]\} \sin\alpha x \sin\beta y\, dxdy=0$$

or

$$\int_0^a\int_0^b \{\frac{\pi^3}{a^3}B_{11}U\sin^2\alpha x\sin^2\beta y + (B_{12}+2B_{66})(\frac{\pi^3}{ab^2}U+\frac{\pi^3}{a^2b}V)\sin^2\alpha x\sin^2\beta y$$

$$+ B_{22}\frac{\pi^3}{b^3}V\sin^2\alpha x\sin^2\beta y - B_{11}W^2\frac{\pi^4}{a^4}\cos^2\alpha x\sin\alpha x\sin^3\beta y$$

$$- (B_{12}+2B_{66})\frac{\pi^4}{a^2b^2}W^2\cos^2\alpha x\sin\alpha x\sin^3\beta y$$

$$- B_{22}W^2 \frac{\pi^4}{b^4} \sin^3 \alpha x \sin \beta y \cos^2 \beta y$$

$$- (B_{12} + 2B_{66}) \frac{\pi^4}{a^2b^2} W^2 \sin^3 \alpha x \sin \beta y \cos^2 \beta y$$

$$+ 2(B_{66} - B_{12}) \frac{\pi^4}{a^2b^2} W^2 \sin^3 \alpha x \sin^3 \beta y$$

$$+ 2W^2(B_{12} + B_{66}) \frac{\pi^4}{a^2b^2} \sin \alpha x \sin \beta y \cos^2 \alpha x \cos^2 \beta y$$

$$- B_{22} \frac{\pi^4}{b^4} W^2 \sin^3 \alpha x \sin^3 \beta y - D_{11}W \frac{\pi^4}{a^4} \sin^2 \alpha x \sin^2 \beta y$$

$$- 2(D_{12} + 2D_{66})W \frac{\pi^4}{a^2b^2} \sin^2 \alpha x \sin^2 \beta y - D_{22}W \frac{\pi^4}{b^4} \sin^2 \alpha x \sin^2 \beta y$$

$$+ Q \sin^2 \alpha x \sin^2 \beta y + A_{11} \frac{\pi^3}{a^3} UW \sin^3 \alpha x \sin^3 \beta y$$

$$- A_{11} \frac{\pi^4}{2a^4} W^3 \sin^2 \alpha x \cos^2 \alpha x \sin^4 \beta y + A_{12}WV \frac{\pi^3}{a^2b} \sin^3 \alpha x \sin^3 \beta y$$

$$- A_{12} \frac{\pi^4}{2a^2b^2} W^3 \sin^4 \alpha x \cos^2 \beta y \sin^2 \beta y + A_{12} \frac{\pi^3}{ab^2} UW \sin^3 \alpha x \sin^3 \beta y$$

$$- A_{12} \frac{\pi^4}{2a^2b^2} W^3 \cos^2 \alpha x \sin^2 \alpha x \sin^4 \beta y + A_{22} \frac{\pi^3}{b^3} VW \sin^3 \alpha x \sin^3 \beta y$$

$$- A_{22} \frac{\pi^4}{2b^4} W^3 \sin^4 \alpha x \cos^2 \beta y \sin^2 \beta y$$

$$+ 2A_{66} UW \frac{\pi^3}{ab^2} \sin \alpha x \cos^2 \alpha x \sin \beta y \cos^2 \beta y$$

$$+ 2A_{66} VW \frac{\pi^3}{a^2b} \sin \alpha x \cos^2 \alpha x \sin \beta y \cos^2 \beta y$$

$$+ 2A_{66} W^3 \frac{\pi^4}{a^2b^2} \sin^2 \alpha x \cos^2 \alpha x \sin^2 \beta y \cos^2 \beta y$$

$$- 4B_{66} W^2 \frac{\pi^4}{a^2b^2} \sin \alpha x \cos^2 \alpha x \sin \beta y \cos^2 \beta y\} = 0$$

or

$$\frac{\pi^3}{a^3} B_{11}U \frac{ab}{4} + (B_{12} + 2B_{66})(\frac{\pi^3 U}{4b} + \frac{\pi^3 V}{4a}) + B_{22} \frac{\pi^3}{b^3} V \frac{ab}{4} - B_{11}W^2 \frac{\pi^4}{a^4} \frac{8ab}{9\pi^2}$$

$$- B_{11}W^2 \frac{\pi^4}{a^4} \frac{8ab}{9\pi^2} - (B_{12} + 2B_{66}) \frac{\pi^4}{a^2b^2} W^2 \frac{8ab}{9\pi^2} - B_{22}W^2 \frac{\pi^4}{b^4} \frac{8ab}{9\pi^2}$$

$$- (B_{12} + 2B_{66}) \frac{\pi^4}{a^2b^2} W^2 \frac{8ab}{9\pi^2} + 2W^2(B_{66} - B_{12}) \frac{\pi^4}{a^2b^2} \frac{16ab}{9\pi^2}$$

$$- 2W^2(B_{12} + B_{66}) \frac{\pi^4}{a^2b^2} \frac{4ab}{9\pi^2} - B_{22} \frac{\pi^4}{a^2b^2} W^2 \frac{16ab}{9\pi^2} - D_{11}W \frac{\pi^4}{a^4} \frac{ab}{4}$$

$$- 2(D_{12} + 2D_{66}) W \frac{\pi^4}{a^2b^2} \frac{ab}{4} - D_{22}W \frac{\pi^4}{b^4} \frac{ab}{4} + Q \frac{ab}{4} + A_{11} \frac{\pi^3}{a^3} UW \frac{16ab}{9\pi^2}$$

$$- A_{11} \frac{\pi^4}{2a^4} W^3 \frac{3ab}{64} + A_{12}WV \frac{\pi^3}{a^2b} \frac{16ab}{9\pi^2} - A_{12} \frac{\pi^4}{2a^2b^2} W^3 \frac{3ab}{64} + A_{12} \frac{\pi^3}{ab^2} UW \frac{16ab}{9\pi^2}$$

$$- A_{12} W^3 \frac{\pi^4}{2a^2b^2} \frac{3ab}{64} + A_{22} \frac{\pi^3}{b^3} VW \frac{16ab}{9\pi^2} - A_{22} \frac{\pi^4}{2b^4} W^3 \frac{3ab}{64} + 2A_{66} UW \frac{\pi^3}{ab^2} \frac{4ab}{9\pi^2}$$

$$+ 2A_{66}VW \frac{\pi^3}{a^2b} \frac{4ab}{9\pi^2} + 2A_{66} W^3 \frac{\pi^4}{a^2b^2} \frac{ab}{64} - 4B_{66} W^2 \frac{\pi^4}{a^2b^2} \frac{4ab}{9\pi^2} = 0$$

$$U\{B_{11} \frac{\pi^3 b}{4a^2} + (B_{12} + 2B_{66} \frac{\pi^3}{4b}\} + V\{(B_{12} + 2B_{66}) \frac{\pi^3}{4a} + B_{22} \frac{\pi^3 a}{4b^2}\}$$

$$+ W\{- D_{11} \frac{\pi^4 b}{4a^3} - 2(D_{12} + 2D_{66}) \frac{\pi^4}{4ab} - D_{22} \frac{\pi^4 a}{4b^3}\}$$

$$+ W^2\{-B_{11} \frac{\pi^2 8b}{9a^3} - (B_{12} + 2B_{66}) \frac{\pi^2 8}{9ab} - B_{22} \frac{\pi^2 8a}{9b^3} - (B_{12} + 2B_{66}) \frac{\pi^2 8}{9ab}$$

$$+ 2(B_{66} - B_{12}) \frac{\pi^2 16}{9ab} + 2(B_{12} + B_{66}) \frac{\pi^2 4}{9ab} - B_{22} \frac{\pi^2 16}{9ab} - 4B_{66} \frac{\pi^4 4}{9ab}\}$$

$$+ W^3\{-A_{11} \frac{\pi^4 3b}{128a^3} - A_{12} \frac{\pi^4 3}{128ab} - A_{12} \frac{\pi^4 3}{128ab} - A_{22} \frac{\pi^4 3a}{128b^3} + 2A_{66} \frac{\pi 4}{64ab}\}$$

$$+ UW\{A_{11} \frac{16b\pi}{9a^2} + A_{12} \frac{16\pi}{9b} + A_{66} \frac{8\pi}{9b}\}$$

$$+ VW\{A_{12} \frac{\pi 16}{9a} + A_{22} \frac{\pi 16a}{9b^2} + 2A_{66} \frac{\pi 4}{9a}\} + q_o \frac{ab}{4} = 0 \qquad (5.3.4)$$

The above three equations yield a cubic equation in W which can be solved by using a standard subroutine (such as ZRPOLY).

5.4 Neutral-Surface Positions

By Kirchhoff's hypothesis, the neutral-surface locations are defined by $\varepsilon_x = \varepsilon_y = 0$ which implies

$$\begin{aligned} z_{nx} &= u^o_{,x}/w_{,xx} = U/W\alpha \\ z_{ny} &= v^o_{,y}/w_{,yy} = V/W\beta \end{aligned} \tag{5.4.1}$$

The iterative method is applied as in the other problems to get the actual deflection.

5.5 Numerical Results

The present solution is compared with that of Kamiya [11], the only solution available for large deflections of bimodulus isotropic rectangular thin plates, simply supported on all edges. Good agreement is obtained (see Table 5.1).

Table 5.1. Comparison of Nondimensional Deflection, W/h, with Kamiya's Solution [11]. ($a/b=1$, $\nu^c=0.2$, $q_o a^4/E_{22}{}^c h^4 = 16.91$)

E^t/E^c	$16.91\, WE_{22}{}^c h^4/q_o a^4$		$Z_x=z_{nx}/h$	$Z_y=z_{ny}/h$
	Kamiya	Present	Present	
1.0	0.4000	0.4010	-0.0795	-0.0795
1.5	0.3200	0.3177	-0.0398	-0.0100
2.0	0.2700	0.2658	0.2658	-0.0167

Table 5.2. Dimensionless Deflections and Neutral-Surface Locations for a Single-Layer Orthotropic Rectangular Plate at (i) $\bar{q}_o = q_o b^4/E_{22}{}^C h^4 = 1.0$, (ii) $\bar{q}_o = q_o b^4/E_{22}{}^C h^4 = 200$

a/b	Z_x (i)	Z_x (ii)	Z_y (i)	Z_y (ii)	W/h (i)	W/h (ii)
			Aramid-Rubber:			
0.5	0.0768	0.0876	- 0.0688	- 0.0803	0.6651×10^{-4}	0.0139
0.6	0.0780	0.0828	- 0.0636	- 0.0724	0.1384×10^{-3}	0.0278
0.7	0.0780	0.0732	- 0.0598	- 0.0667	0.2567×10^{-3}	0.0489
0.8	0.0795	0.0611	- 0.0564	- 0.0644	0.4377×10^{-3}	0.0774
0.9	0.0795	0.0462	- 0.0536	- 0.0659	0.7001×10^{-3}	0.1126
1.0	0.0805	0.0290	- 0.0512	- 0.0699	0.1063×10^{-2}	0.1530
1.2	0.0805	-0.0093	- 0.0477	- 0.0885	0.2178×10^{-2}	0.2435
1.4	0.0805	-0.0498	- 0.0464	- 0.1146	0.3945×10^{-2}	0.3395
1.6	0.0805	-0.0906	- 0.0446	- 0.1447	0.6528×10^{-2}	0.4365
1.8	0.0789	-0.1370	- 0.0446	- 0.1765	0.1003×10^{-1}	0.5326
2.0	0.0773	-0.1699	- 0.0446	- 0.2086	0.1449×10^{-1}	0.6270
			Polyester-Rubber:			
0.5	0.0706	0.0565	- 0.0515	- 0.0441	0.3090×10^{-3}	0.0572
0.6	0.0706	0.0376	- 0.0491	- 0.0390	0.6361×10^{-3}	0.1058
0.7	0.0706	0.0126	- 0.0471	- 0.0390	0.1166×10^{-2}	0.1674
0.8	0.0716	0.0126	- 0.0454	- 0.0504	0.1962×10^{-2}	0.2389
0.9	0.0716	-0.0447	- 0.0441	- 0.0619	0.3087×10^{-2}	0.3094
1.0	0.0716	-0.0747	- 0.0441	- 0.0796	0.4597×10^{-2}	0.3832
1.2	0.0703	-0.1334	- 0.0427	- 0.1231	0.8974×10^{-2}	0.5305
1.4	0.0683	-0.1899	- 0.0427	- 0.1712	0.1529×10^{-1}	0.6743
1.6	0.0656	-0.2433	- 0.0437	- 0.2202	0.2344×10^{-1}	0.8130
1.8	0.0623	-0.2929	- 0.0472	- 0.2681	0.3298×10^{-1}	0.9453
2.0	0.0589	-0.3382	- 0.0503	- 0.3137	0.4337×10^{-1}	1.0700

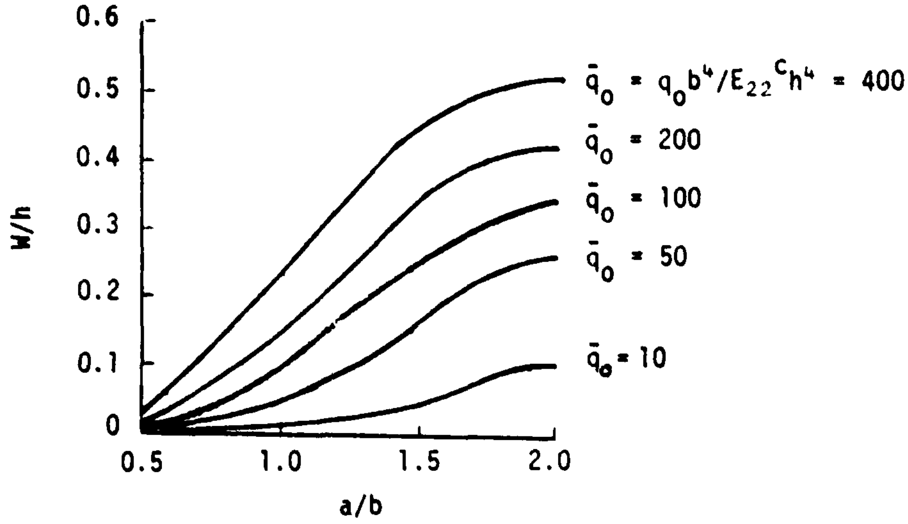
(See Appendix D for the in-plane displacements)

Table 5.3. Dimensionless Deflections and Neutral-Surface Locations for a Cross-Ply Bimodulus Rectangular Plate at (i) $\bar{q}_0 = q_0 b^4/E_{22}{}^c h^4 = 1.0$, (ii) $\bar{q}_0 = q_0 b^4/E_{22}{}^c h^4 = 200$

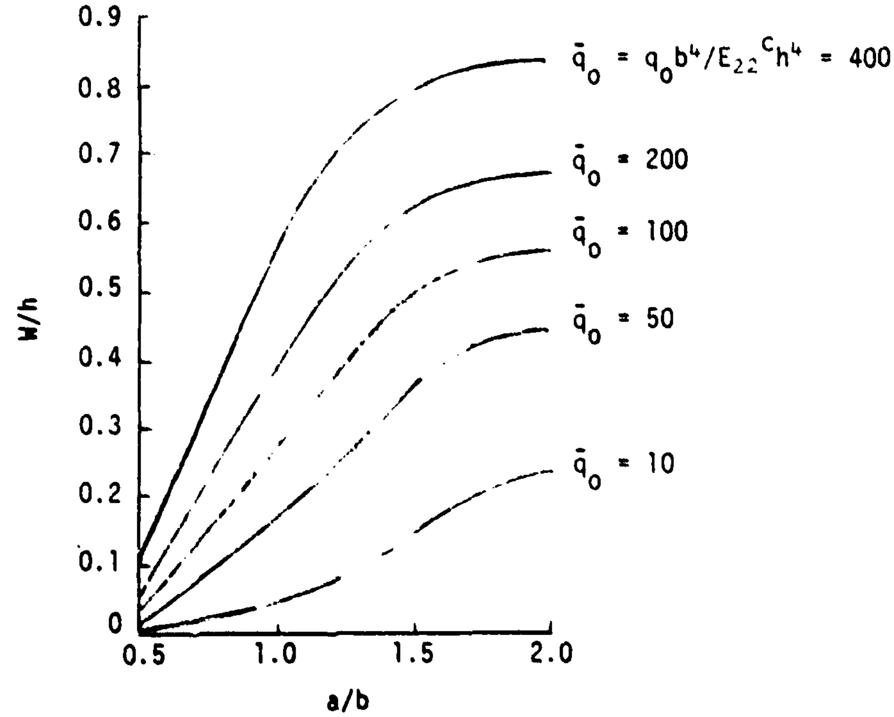
a/b	Z_x (i)	Z_x (ii)	Z_y (i)	Z_y (ii)	W/h (i)	W/h (ii)
			Aramid-Rubber:			
0.5	0.0934	0.0875	- 0.0280	- 0.0316	0.7149×10^{-4}	0.0139
0.6	0.0934	0.0827	- 0.0247	- 0.0329	0.1480×10^{-3}	0.0278
0.7	0.0934	0.0731	- 0.0221	- 0.0398	0.2734×10^{-3}	0.0488
0.8	0.0934	0.0611	- 0.0202	- 0.0544	0.4645×10^{-3}	0.0774
0.9	0.0934	0.0462	- 0.0189	- 0.0766	0.7402×10^{-3}	0.1126
1.0	0.0934	0.0292	- 0.0189	- 0.1033	0.1121×10^{-2}	0.1531
1.2	0.0934	-0.0072	- 0.0172	- 0.1633	0.2284×10^{-2}	0.2404
1.4	0.0921	-0.0397	- 0.0172	- 0.2184	0.4127×10^{-2}	0.3187
1.6	0.0909	-0.0625	- 0.0172	- 0.2580	0.6800×10^{-2}	0.3741
1.8	0.0895	-0.0752	- 0.0172	- 0.2815	0.1041×10^{-1}	0.4077
2.0	0.0874	-0.0826	- 0.0188	- 0.2950	0.1498×10^{-1}	0.4270
			Polyester-Rubber:			
0.5	0.0895	0.0634	- 0.0815	- 0.0959	0.3519×10^{-3}	0.0616
0.6	0.0895	0.0438	- 0.0805	- 0.1128	0.7206×10^{-3}	0.1124
0.7	0.0895	0.0181	- 0.0805	- 0.1413	0.1312×10^{-2}	0.1759
0.8	0.0895	-0.0104	- 0.0805	- 0.1789	0.2188×10^{-2}	0.2469
0.9	0.0895	-0.0396	- 0.0805	- 0.2216	0.3403×10^{-2}	0.3307
1.0	0.0895	-0.0678	- 0.0805	- 0.2673	0.4999×10^{-2}	0.3931
1.2	0.0869	-0.1437	- 0.0832	- 0.4042	0.1527×10^{-1}	0.5994
1.4	0.0869	-0.1437	- 0.0832	- 0.4042	0.1527×10^{-1}	0.5994
1.6	0.0854	-0.1554	- 0.0856	- 0.4356	0.2217×10^{-1}	0.6444
1.8	0.0840	-0.1576	- 0.0882	- 0.4517	0.2947×10^{-1}	0.6669
2.0	0.0828	-0.1542	- 0.0909	- 0.4605	0.3659×10^{-1}	0.6776

(For the in-plane displacements, seee Appendix D)

Comparison of the present theory with the existing solution for bimodulus isotropic thin plates shows close agreement. Typical computations are shown in the above two tables, and graphs are presented (Fig. 5.1) to observe the general trend.

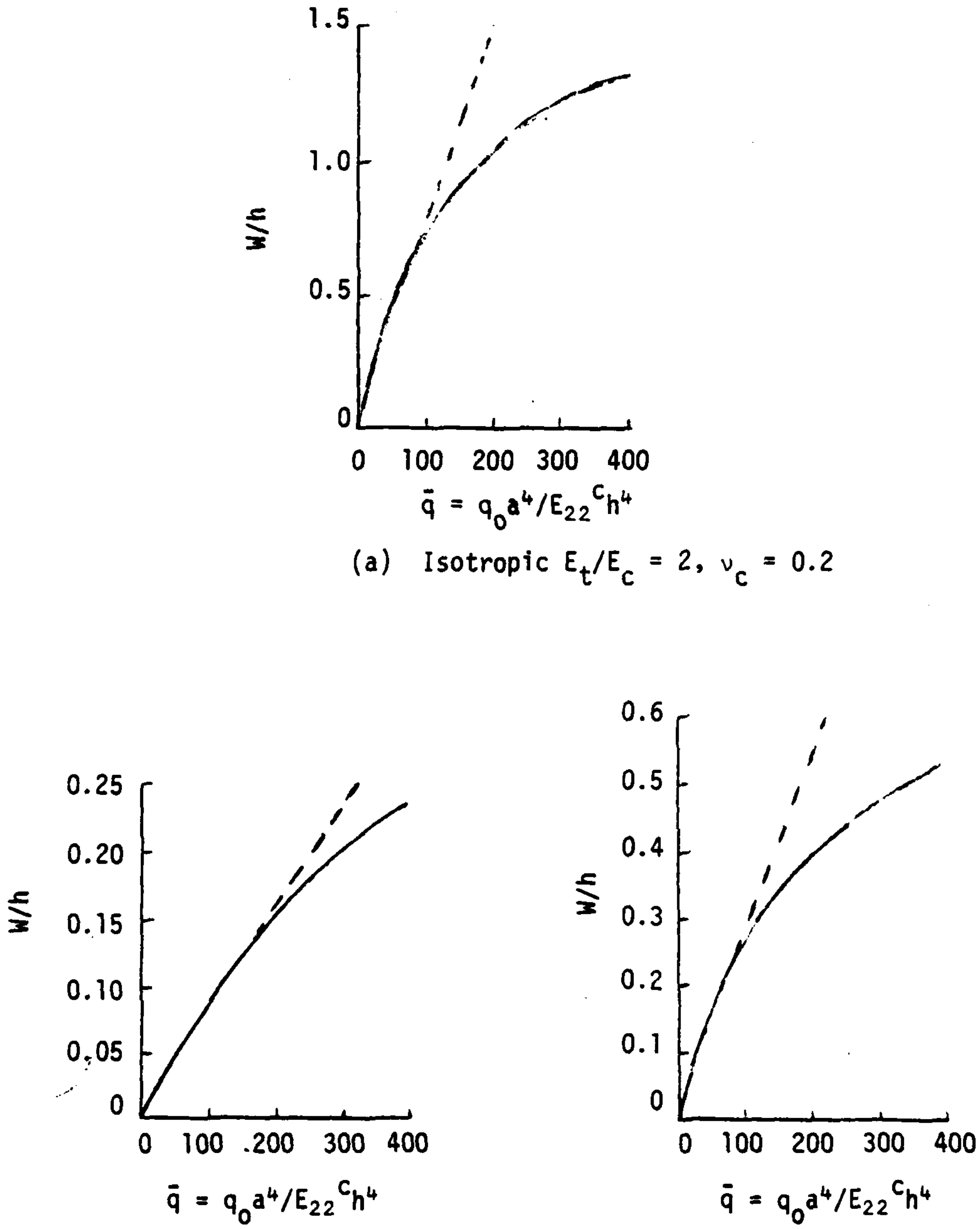


(a) Aramid-Rubber



(b) Polyester-Rubber

Fig. 5.1. Variation of dimensionless deflection with aspect ratio for two-layer cross-ply rectangular plates.



(a) Isotropic $E_t/E_c = 2$, $\nu_c = 0.2$

(b) Aramid-Rubber (Cross-Ply)

(c) Polyester-Rubber (Cross-Ply)

Fig. 5.2. Variation of dimensionaless deflection with dimensionless load for bimodulus square plate (a/b=1). (Dashed line represents linear case.)

CHAPTER VI

CONCLUSIONS

Good agreement exists between the closed-form, small-deflection solutions presented here and previous approximate analyses carried out by several authors for special cases (isotropic bimodulus material) as was shown in Chapters II-IV. It is also shown that the exact solutions developed here offer a good check for finite-element analysis available (see Chapter II). The results of Chapters III and IV can also be used for this purpose.

Good agreement was also obtained between the approximate Galerkin-type solution presented in Chapter V and an existing solution for the isotropic bimodulus case.

It has been observed that for materials with different properties in tension and compression, the location of the neutral surface may vary considerably from the geometric mid-plane. Materials with markedly different properties in tension and compression are, of course, most affected.

There is a sudden jump in neutral-surface locations with the change in aspect ratio in the case of free vibration. This is probably due to the eigenvalue nature of this problem. They also assume out-of-plate values in the case of thermal bending.

Stacking sequence for two-layer, cross-ply laminates plays an important role in both the amount of deflection and the failure mode. For some plate aspect ratios, the maximum stress can be developed at the bottom of the plate. However, if the stacking sequence is reversed, the maximum stress may occur in those fibers closest to the mid-plane.

Typical computer programs are presented in Appendix III.

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APPENDIX A

DERIVATION OF THE PLATE STIFFNESSES FOR TWO-LAYER CROSS-PLY LAMINATE OF BIMODULUS MATERIAL

In the solution of problems involving laminates comprised of bimodulus-material layers, it is necessary to evaluate the integral forms involved in the definitions of the plate stiffnesses, Eq. (2.1.6). This is accomplished here for the case of a two-layer cross-ply laminate.

Each layer is assumed to be of the same thickness, h/2, and the same orthotropic elastic properties with respect to the fiber direction. Since each layer is oriented at either 0° or 90° to the x axis, the laminate is also orthotropic, i.e., there are no stiffnesses with subscripts 16 and 26.

The bottom layer is denoted as layer 1, i.e., $\ell = 1$ in $Q_{ijk\ell}$, and occupies the thickness space from $z = 0$ to $z = h/2$, where z is measured positive downward from the midplane. The top layer is denoted as layer 2, i.e., $\ell = 2$, and occupies the thickness space from $z = -h/2$ to $z = 0$.

In the general case derived in this derivation, it is assumed that the upper portion of the top layer (ℓ=2) is in compression (k=2 in $Q_{ijk\ell}$) in the fiber direction and that the lower portion of the top layer is in tension (k=1), while the inner portion of the bottom layer (ℓ=1), from $z = 0$ to $z = z_{nx}$, is in compression (k=2), while the outer portion (from z_{nx} to h/2) of layer 1 is in tension (k=1).

Thus, the general integral expression for A_{ij}, the first of Eqs. (2.1.6), may be taken as the sum of the integrals for each of these regions:

<u>Case 1</u>
($Z_y<0$, $Z_x>0$)

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ijk\ell}\, dz$$

$$= \int_{-h/2}^{z_{ny}} Q_{ij22}\, dz + \int_{z_{ny}}^{0} Q_{ij12}\, dz + \int_{0}^{z_{nx}} Q_{ij21}\, dz + \int_{z_{nx}}^{h/2} Q_{ij11}\, dz \qquad \text{(A-1)}$$

Since the planar reduced stiffnesses $Q_{ijk\ell}$ are each respectively constant in the appropriate regions, Eq. (A-1) integrates to the following result:

$$A_{ij} = (Q_{ij22} + Q_{ij11})(h/2) + (Q_{ij21} - Q_{ij11})z_{nx} + (Q_{ij22} - Q_{ij12})z_{ny} \qquad \text{(A-2)}$$

or

$$A_{ij} = (1/2)(Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})Z_x + (Q_{ij22} - Q_{ij12})Z_y \qquad \text{(A-3)}$$

Similarly

$$B_{ij} = \int_{-h/2}^{h/2} zQ_{ijk\ell}\, dz$$

$$= \int_{-h/2}^{z_{ny}} zQ_{ij22}\, dz + \int_{z_{ny}}^{0} zQ_{ij12}\, dz + \int_{0}^{z_{nx}} zQ_{ij21}\, dz + \int_{z_{nx}}^{h/2} zQ_{ij11}\, dz \qquad \text{(A-4)}$$

$$= (-Q_{ij22} + Q_{ij11})(h^2/8) + (Q_{ij21} - Q_{ij11})(z_{nx}{}^2/2)$$
$$+ (Q_{ij22} - Q_{ij12})(z_{ny}{}^2/2) \tag{A-5}$$

or

$$B_{ij}/h^2 - (1/8)(-Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})(Z_x{}^2/2)$$
$$+ (Q_{ij22} - Q_{ij12})(Z_y{}^2/2) \tag{A-6}$$

Also

$$D_{ij} = \int_{-h/2}^{h/2} z^2 Q_{ijk\ell}\, dz$$

$$= \int_{-h/2}^{z_{ny}} z^2 Q_{ij22}\, dz + \int_{z_{ny}}^{0} z^2 Q_{ij12}\, dz + \int_{0}^{z_{nx}} z^2 Q_{ij21}\, dz + \int_{z_{nx}}^{h/2} z^2 Q_{ij11}\, dz \tag{A-7}$$

$$= (Q_{ij22} + Q_{ij11})(h^3/24) + (Q_{ij21} - Q_{ij11})(z_{nx}{}^3/3)$$
$$+ (Q_{ij22} - Q_{ij12})(z_{ny}{}^3/3) \tag{A-8}$$

or

$$D_{ij}/h^3 = (1/24)(Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})(Z_x{}^3/3)$$
$$+ (Q_{ij22} - Q_{ij12})(Z_y{}^3/3) \tag{A-9}$$

Similarly

Case 2
$(Z_y > 0,\ Z_x < 0)$

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ijk\ell}\, dz$$

$$= \int_{-h/2}^{z_{nx}} Q_{ij22}\, dz + \int_{z_{nx}}^{0} Q_{ij12}\, dz + \int_{0}^{z_{ny}} Q_{ij21}\, dz + \int_{z_{ny}}^{h/2} Q_{ij11}\, dz$$

$$= (Q_{ij11} + Q_{ij22})(h/2) + (Q_{ij22} - Q_{ij12})z_{nx} + (Q_{ij21} - Q_{ij11})z_{ny}$$

or

$$A_{ij/h} = (Q_{ij11} + Q_{ij22})/2 + (Q_{ij22} - Q_{ij12})Z_x + (Q_{ij21} - Q_{ij11})Z_y$$

and

$$B_{ij}/h^2 = (Q_{ij11} - Q_{ij22})/8 + (Q_{ij22} - Q_{ij12})(Z_x{}^2/2) + (Q_{ij21} - Q_{ij11})(Z_y{}^2/2) \tag{A-10}$$

$$D_{ij}/h^3 = (Q_{ij11} + Q_{ij22})/24 + (Q_{ij22} - Q_{ij12})(Z_x{}^3/3) + (Q_{ij21} - Q_{ij11})(Z_y{}^3/3)$$

Case 3
$(Z_x>0,\ Z_y>0)$

$$A_{ij}/h = (Q_{ij11} + Q_{ij22})/2 + (Q_{ij21} - Q_{ij11})Z_x$$

$$B_{ij}/h^2 = (Q_{ij11} - Q_{ij22})/8 + (Q_{ij21} - Q_{ij11})(Z_x{}^2/2) \tag{A-11}$$

$$D_{ij}/h^3 = (Q_{ij11} + Q_{ij22})/24 + (Q_{ij21} - Q_{ij11})(Z_x{}^3/3)$$

Case 4
$(Z_x<0,\ Z_y<0)$

$$A_{ij}/h = (Q_{ij11} + Q_{ij22})/2 + (Q_{ij22} - Q_{ij12})Z_y$$

$$B_{ij}/h^2 = (Q_{ij11} - Q_{ij22})/8 + (Q_{ij22} - Q_{ij12})(Z_y{}^2/2) \tag{A-12}$$

$$D_{ij}/h^3 = (Q_{ij11} + Q_{ij22})/24 + (Q_{ij22} - Q_{ij12})(Z_y{}^3/3)$$

For neutral surface going out of plane,

Case 5
($Z_x>0.5$, $Z_y>-0.5$)

$$\begin{aligned} A_{ij}/h &= (Q_{ij21} + Q_{ij12})/2 \\ B_{ij}/h^2 &= (Q_{ij21} - Q_{ij12})/8 \\ D_{ij}/h^3 &= (Q_{ij21} + Q_{ij12})/24 \end{aligned} \tag{A-13}$$

Case 6
($Z_x>-0.5$, $Z_y>0.5$)

$$\begin{aligned} A_{ij}/h &= (Q_{ij11} + Q_{ij22})/2 \\ B_{ij}/h^2 &= (Q_{ij11} - Q_{ij22})/8 \\ D_{ij}/h^3 &= (Q_{ij11} + Q_{ij22})/24 \end{aligned} \tag{A-14}$$

Case 7
($Z_x>0.5$, $Z_y>0.5$)

$$\begin{aligned} A_{ij}/h &= (Q_{ij21} + Q_{ij22})/2 \\ B_{ij}/h^2 &= (Q_{ij21} - Q_{ij22})/8 \\ D_{ij}/h^3 &= (Q_{ij21} + Q_{ij22})/24 \end{aligned} \tag{A-15}$$

Case 8
($Z_x>-0.5$, $Z_y>-0.5$)

$$\begin{aligned} A_{ij}/h &= (Q_{ij11} + Q_{ij12})/2 \\ B_{ij}/h^2 &= (Q_{ij11} - Q_{ij12})/8 \\ D_{ij}/h^3 &= (Q_{ij11} + Q_{ij12})/24 \end{aligned} \tag{A-16}$$

Single Layer

For a single layer (0°),

$$A_{ij}/h = (Q_{ij11} + Q_{ij21})/2 + (Q_{ij21} - Q_{ij11})Z_x$$

$$B_{ij}/h^2 = (Q_{ij11} - Q_{ij21})/8 + (Q_{ij21} - Q_{ij11})Z_x/2 \tag{A-17}$$

$$D_{ij}/h^3 = (Q_{ij11} + Q_{ij21})/24 + (Q_{ij21} - Q_{ij11})Z_x/3$$

For neutral surface out of plane,

Case 1
(Z_x>0.5)

$$A_{ij}/h = Q_{ij21}$$

$$B_{ij}/h^2 = 0 \tag{A-18}$$

$$D_{ij}/h^3 = Q_{ij21}/12$$

Case 2
(Z_x>-0.5)

$$A_{ij}/h = Q_{ij11}$$

$$B_{ij}/h^2 = 0 \tag{A-19}$$

$$D_{ij}/h^3 = Q_{ij11}/12$$

APPENDIX B

DENSITIES OF ARAMID, POLYESTER, AND RUBBER

Fibers:

According to the Kevlar Data Manual [50], the densities of the fibers of aramid and polyester are:

$$\text{Kevlar (aramid)} = 0.052\ lb_f/in^3$$

$$\text{Polyester} = 0.049\ lb_f/in^3$$

Rubber:

Reference [51] lists nautral rubber and isoprene rubber at specific gravity, 0.93, and SBR plus BR rubber at specific gravity, 0.94.

Taking 0.93, the density of rubber can be calculated as:

$$\text{Density} = 0.93 \times 62.4\ lb_f/ft^3\ (H_2O)/1728\ in^3/ft^3$$

$$= 0.034\ lb_f/in^3$$

Composite:

If ρ represents the density (f for fiber and m for matrix) and V represents the volume fraction, the density of the composite ρ can be written as:

$$\rho = \rho_f V_f + \rho_m V_m$$

$$= \rho_m + (\rho_f - \rho_m) V_f$$

The values of V_f according to Ref. [2] are:

Aramid/rubber : 0.140

Polyester/rubber: 0.149

Thus, the densities of the composites can be listed as follows:

Composite	Density $\rho(lb_f/in^3)$
Aramid/rubber	0.037
Polyester/rubber	0.036

APPENDIX C

THERMAL FORCE AND MOMENT EXPRESSIONS FOR CASES II - VIII

Expressions for $N_{1,x}^{T}$, $N_{2,y}^{T}$, $M_{1,x}^{T}$, and $M_{2,y}^{T}$ were derived for Case I in Chapter IV. In a similar way, one can obtain the expressions for the above-mentioned quantities for the remaining seven cases as follows:

<u>Case II</u>
$z_{nx}>0,\ z_{ny}>0$

$$
\begin{aligned}
N_{1,x}^{T} &= \alpha\{(\beta_{122}+\beta_{111})(\bar{T}_0 h/2) + (\beta_{121}-\beta_{111})(\bar{T}_0 z_{nx}) \\
&\quad + (\beta_{111}-\beta_{122})(\bar{T}_1 h/8) + (\beta_{121}-\beta_{111})(\bar{T}_1 z_{nx}^2/2h)\} \\
N_{2,y}^{T} &= \beta\{(\beta_{222}+\beta_{211})(\bar{T}_0 h/2) + (\beta_{221}-\beta_{211})(\bar{T}_0 z_{nx}) \\
&\quad + (\beta_{211}-\beta_{222})(\bar{T}_1 h/8) + (\beta_{221}-\beta_{211})(\bar{T}_1 z_{nx}^2/2h)\} \\
M_{1,x}^{T} &= \alpha\{(\beta_{111}-\beta_{122})(\bar{T}_0 h^2/8) + (\beta_{121}-\beta_{111})(\bar{T}_0 z_{nx}^2/2) \\
&\quad + (\beta_{122}+\beta_{111})(\bar{T}_1 h^2/24) + (\beta_{121}-\beta_{111})(\bar{T}_1 z_{nx}^3/3h)\} \\
M_{2,y}^{T} &= \beta\{(\beta_{211}-\beta_{222})(\bar{T}_0 h^2/8) + (\beta_{221}-\beta_{211})(\bar{T}_0 z_{nx}^2/2) \\
&\quad + (\beta_{222}-\beta_{211})(\bar{T}_1 h^2/24) + (\beta_{221}-\beta_{211})(\bar{T}_1 z_{nx}^3/3h)\}
\end{aligned}
\tag{C-1}
$$

Case III

$z_{nx}<0,\ z_{ny}>0$

$$
\begin{aligned}
N_{1,x}^{T} &= \alpha\{(\beta_{122}+\beta_{111})(\bar{T}_0 h/2) + (\beta_{121}-\beta_{111})(\bar{T}_0 z_{ny}) \\
&\quad + (\beta_{122}-\beta_{112})(\bar{T}_0 z_{nx}) + (\beta_{111}-\beta_{122})(\bar{T}_1 h/8) \\
&\quad + (\beta_{121}-\beta_{111})(\bar{T}_1 z_{ny}^2/2h) + (\beta_{122}-\beta_{112})(\bar{T}_1 z_{nx}^2/2h)\} \\
N_{2,y}^{T} &= \beta\{(\beta_{222}+\beta_{211})(\bar{T}_0 h/2) + (\beta_{221}-\beta_{211})(\bar{T}_0 z_{ny}) \\
&\quad + (\beta_{222}-\beta_{212})(\bar{T}_0 z_{nx}) + (\beta_{211}-\beta_{222})(\bar{T}_1 h/8) \\
&\quad + (\beta_{221}-\beta_{211})(\bar{T}_1 z_{ny}^2/2h) + (\beta_{222}-\beta_{212})(\bar{T}_1 z_{nx}^2/2h)\} \\
M_{1,x}^{T} &= \alpha\{(\beta_{111}-\beta_{122})(\bar{T}_0 h^2/8) + (\beta_{121}-\beta_{111})(\bar{T}_0 z_{ny}^2/2) \\
&\quad + (\beta_{122}-\beta_{112})(\bar{T}_0 z_{nx}^2/2) + (\beta_{122}+\beta_{111})(\bar{T}_1 h^2/24) \\
&\quad + (\beta_{121}-\beta_{111})(\bar{T}_1 z_{ny}^3/3h) + (\beta_{122}-\beta_{112})(\bar{T}_1 z_{nx}^3/3h)\} \\
M_{2,y}^{T} &= \beta\{(\beta_{211}-\beta_{222})(\bar{T}_0 h^2/8) + (\beta_{221}-\beta_{211})(\bar{T}_0 z_{ny}^2/2) \\
&\quad + (\beta_{222}-\beta_{212})(\bar{T}_0 z_{nx}^2/2) + (\beta_{222}+\beta_{211})(\bar{T}_1 h^2/24) \\
&\quad + (\beta_{221}-\beta_{211})(\bar{T}_1 z_{ny}^3/3h) + (\beta_{222}-\beta_{212})(\bar{T}_1 z_{nx}^3/3h)\}
\end{aligned}
\tag{C-2}
$$

Case IV

$z_{nx}<0,\ z_{ny}<0$

$$
\begin{aligned}
N_{1,x}^{T} &= \alpha\{(\beta_{122}+\beta_{111})(\bar{T}_0 h/2) + (\beta_{122}-\beta_{121})(\bar{T}_0 z_{ny}) \\
&\quad + (\beta_{111}-\beta_{122})(\bar{T}_1 h/8) + (\beta_{122}-\beta_{121})(\bar{T}_1 z_{ny}^2/2h)\}
\end{aligned}
$$

$$N_{2,y}^{T} = \beta\{(\beta_{222}+\beta_{211})(\bar{T}_o h/2) + (\beta_{222}-\beta_{221})(\bar{T}_o z_{ny})$$

$$+ (\beta_{211}-\beta_{222})(\bar{T}_1 h/8) + (\beta_{222}-\beta_{221})(\bar{T}_1 z_{ny}^{2}/2/h)\}$$

$$M_{1,x}^{T} = \alpha\{(\beta_{111}-\beta_{122})(\bar{T}_o h^2/8) + (\beta_{122}-\beta_{121})(\bar{T}_o z_{ny}^{2}/2) \tag{C-3}$$

$$+ (\beta_{122}+\beta_{111})(\bar{T}_1 h^2/24) + (\beta_{122}-\beta_{121})(\bar{T}_1 z_{ny}^{3}/3h)\}$$

$$M_{2,y}^{T} = \beta\{(\beta_{211}-\beta_{222})(\bar{T}_o h^2/8) + (\beta_{222}-\beta_{221})(\bar{T}_o z_{ny}^{2}/2)$$

$$+ (\beta_{222}+\beta_{211})(\bar{T}_1 h^2/24) + (\beta_{222}-\beta_{221})(\bar{T}_1 z_{ny}^{3}/3h)\}$$

For neutral surface going out of plane,

<u>Case V</u>
$Z_x > 0.5$, $Z_y < -0.5$

$$N_{1,x}^{T} = \alpha\{(\beta_{121}+\beta_{112})\bar{T}_o/2 + (\beta_{121}-\beta_{112})\bar{T}_1/8\}$$

$$N_{2,y}^{T} = \beta\{(\beta_{221}+\beta_{212})\bar{T}_o/2 + (\beta_{221}-\beta_{212})\bar{T}_1/8\}$$

$$M_{1,x}^{T} = \alpha\{(\beta_{121}-\beta_{112})\bar{T}_o/8 + (\beta_{121}+\beta_{112})\bar{T}_1/24\} \tag{C-4}$$

$$M_{2,y}^{T} = \beta\{(\beta_{221}-\beta_{212})\bar{T}_o/8 + (\beta_{221}+\beta_{212})\bar{T}_1/24\}$$

<u>Case VI</u>
$Z_x < -0.5$, $Z_y > 0.5$

$$N_{1,x}^{T} = \alpha\{(\beta_{121}+\beta_{112})\bar{T}_o/2 + (\beta_{121}-\beta_{112})\bar{T}_1/8\}$$

$$N_{2,y}^{T} = \beta\{(\beta_{221}+\beta_{212})\bar{T}_o/2 + (\beta_{221}-\beta_{212})\bar{T}_1/8\}$$

$$M_{1,x}^{T} = \alpha\{(\beta_{121}-\beta_{112})\bar{T}_o/8 + (\beta_{121}+\beta_{112})\bar{T}_1/24\} \tag{C-5}$$

$$M_{2,y}^{T} = \beta\{(\beta_{221}-\beta_{212})\bar{T}_o/8 + (\beta_{221}+\beta_{212})T_1/24\}$$

Case VII
$Z_x>0.5,\ Z_y>0.5$

$$N_{1,x}^{T} = \alpha\{(\beta_{111}+\beta_{112})\bar{T}_0/2 + (\beta_{111}-\beta_{112})\bar{T}_1/8\}$$

$$N_{2,y}^{T} = \beta\{(\beta_{211}+\beta_{212})\bar{T}_0/2 + (\beta_{211}-\beta_{212})\bar{T}_1/8\}$$

$$M_{1,x}^{T} = \alpha\{(\beta_{111}-\beta_{112})\bar{T}_0/8 + (\beta_{111}+\beta_{112})\bar{T}_1/24\} \qquad \text{(C-6)}$$

$$M_{2,y}^{T} = \beta\{(\beta_{211}-\beta_{212})\bar{T}_0/8 + (\beta_{211}+\beta_{212})\bar{T}_1/24\}$$

Case VIII
$Z_x<-0.5,\ Z_y<-0.5$

$$N_{1,x}^{T} = \alpha\{(\beta_{121}+\beta_{122})\bar{T}_0/2 + (\beta_{121}-\beta_{122})\bar{T}_1/8\}$$

$$N_{2,y}^{T} = \beta\{(\beta_{221}+\beta_{222})\bar{T}_0/2 + (\beta_{221}-\beta_{222})\bar{T}_1/8\}$$

$$M_{1,x}^{T} = \alpha\{(\beta_{121}-\beta_{122})\bar{T}_0/8 + (\beta_{121}+\beta_{122})\bar{T}_1/24\} \qquad \text{(C-7)}$$

$$M_{2,y}^{T} = \beta\{(\beta_{221}-\beta_{222})\bar{T}_0/8 + (\beta_{221}+\beta_{222})\bar{T}_1/24\}$$

For a single layer, change β_{112} to β_{111}, β_{122} to β_{121}, β_{212} to β_{211} and β_{222} to β_{221}.

APPENDIX D

IN-PLANE DISPLACEMENTS AND SLOPE COEFFICIENTS

The in-plane displacements and the slope coefficients corresponding to the middle-surface deflections presented in the preceding chapters are listed in the following tables.

Table D2.5. In-Plane Displacements and Slope Coefficients Associated with Table 2.5

a/b	$(UE_{22}^{c}h^3/q_o b^4)\times10^{-2}$	$(VE_{22}^{c}h^3/q_o b^4)\times10^{-2}$	$(XE_{22}^{c}h^3/q_o b^4)\times10^{-2}$	$(YE_{22}^{c}h^3/q_o b^4)\times10^{-2}$
		Aramid-Rubber (NL=1)		
0.5	0.0510	- 0.0238	- 0.1162	- 0.0730
0.6	0.0824	- 0.0386	- 0.1883	- 0.1333
0.7	0.1204	- 0.0552	- 0.2754	- 0.2187
0.8	0.1725	- 0.0720	- 0.3727	- 0.3295
0.9	0.2064	- 0.0877	- 0.4745	- 0.4634
1.0	0.2495	- 0.1013	- 0.5754	- 0.6161
1.2	0.3264	- 0.1209	- 0.7575	- 0.9569
1.4	0.3844	- 0.1308	- 0.8982	- 1.3101
1.6	0.4222	- 0.1313	- 1.0509	- 1.9486
1.8	0.4427	- 0.1269	- 1.0778	- 2.2132
2.0	0.4503	- 0.1269	- 1.0778	- 2.2132
		Polyester-Rubber (NL=1)		
0.5	0.0150	- 0.0073	- 0.0503	- 0.0449
0.6	0.0252	- 0.0122	- 0.0848	- 0.0787
0.7	0.0387	- 0.0184	- 0.1304	- 0.1281
0.8	0.0554	- 0.0256	- 0.1871	- 0.1960
0.9	0.0749	- 0.0335	- 0.2539	- 0.2845
1.0	0.0967	- 0.0419	- 0.3291	- 0.3945
1.2	0.1442	- 0.0588	- 0.4947	- 0.6760
1.4	0.1915	- 0.0744	- 0.6616	- 1.0211
1.6	0.2327	- 0.0873	- 0.8096	- 1.3972
1.8	0.2646	- 0.0971	- 0.9264	- 1.7716
2.0	0.2863	- 0.1041	- 1.0087	- 2.1199

Table D2.6. In-Plane Displacements and Slope Coefficients Associated with Table 2.6

a/b	$(UE_{22}{}^c h^3/q_o b^4)\times10^{-2}$	$(VE_{22}{}^c h^3/q_o b^4)\times10^{-3}$	$(XE_{22}{}^c h^3/q_o b^4)\times10^{-2}$	$(YE_{22}{}^c h^3/q_o b^4)\times10^{-2}$
		Aramid-Rubber (NL=2)		
0.5	0.5967	- 0.5570	- 0.1114	- 0.0716
0.6	0.4608	- 0.4168	- 0.1789	- 0.1290
0.7	0.3612	- 0.3197	- 0.2602	- 0.2099
0.8	0.2855	- 0.2487	- 0.3517	- 0.3151
0.9	0.2267	- 0.1951	- 0.4485	- 0.4429
1.0	0.1804	- 0.1539	- 0.5456	- 0.5899
1.0	0.1804	- 0.1539	- 0.5456	- 0.5899
1.2	0.1148	- 0.0969	- 0.7247	- 0.9220
1.4	0.0736	- 0.0618	- 0.8672	- 1.2718
1.6	0.0478	- 0.0400	- 0.9673	- 1.6084
1.8	0.0315	- 0.0265	- 1.0289	- 1.9144
2.0	0.0211	- 0.0179	- 1.0603	- 2.1832
		Polyester-Rubber (NL=2)		
0.5	0.5139	- 1.1570	- 0.1169	- 0.0739
0.6	0.3980	- 0.9203	- 0.1881	- 0.1330
0.7	0.3113	- 0.7477	- 0.2731	- 0.2152
0.8	0.2440	- 0.6133	- 0.3660	- 0.3195
0.9	0.1909	- 0.5045	- 0.4597	- 0.4417
1.0	0.1489	- 0.4146	- 0.5475	- 0.5754
1.2	0.0898	- 0.2787	- 0.6882	- 0.8497
1.4	0.0541	- 0.1871	- 0.7730	- 1.0991
1.6	0.0330	- 0.1267	- 0.8099	- 1.3047
1.8	0.0206	- 0.0872	- 0.8138	- 1.4662
2.0	0.0132	- 0.0612	- 0.7979	- 1.5905

Table D4.2. In-Plane Displacements and Slope Coefficients Associated with Table 4.2

a/b	$Uh/\alpha_2 t\,\bar{T}_1 b^2$	$(Vh/\alpha_2 t\,\bar{T}_1 b^2)\times 10^1$	$Xh/\alpha_2 t\,\bar{T}_1 b^2$	$(Yh/\alpha_2 t\,\bar{T}_1 b^2)\times 10^1$
		Aramid-Rubber (NL=1)		
0.5	- 0.1596	- 0.1163	- 6.9363	- 0.6982
0.7	- 0.2226	- 0.1669	- 0.1430	- 0.6081
0.9	- 0.2854	- 0.2034	- 0.2395	- 0.5857
1.0	- 0.3168	- 0.2172	- 0.2959	- 0.5886
1.2	- 0.3795	- 0.2383	- 0.4236	- 0.6131
1.4	- 0.4421	- 0.2531	- 0.5692	- 0.6556
1.6	- 0.5045	- 0.2638	- 0.7298	- 0.7110
1.8	- 0.5669	- 0.2716	- 0.9018	- 0.7765
2.0	- 0.6292	- 0.2775	- 1.0813	- 0.8495
		Polyester-Rubber (NL=1)		
0.5	- 0.1619	- 0.1333	- 0.0767	- 0.2048
0.7	- 0.2226	- 0.1848	- 0.1587	- 0.1833
0.9	- 0.2826	- 0.2197	- 0.2639	- 0.1846
1.0	- 0.3124	- 0.2325	- 0.3238	- 0.1906
1.2	- 0.3719	- 0.2516	- 0.4550	- 0.2108
1.4	- 0.4310	- 0.2647	- 0.5959	- 0.2393
1.6	- 0.4898	- 0.2740	- 0.7388	- 0.2736
1.8	- 0.5482	- 0.2807	- 0.8760	- 0.3114
2.0	- 0.6062	- 0.2858	- 1.0002	- 0.3503

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Table D4.3. In-Plane Displacements and Slope Coefficients Associated with Table 4.3

a/b	$(Uh/\alpha_2 t\ \bar{T}_1 b^2)\times10^{-1}$	$(Vh/\alpha_2 t\ \bar{T}_1 b^2)\times10^{-1}$	$Xh/\alpha_2 t\ \bar{T}_1 b^2$	$Yh/\alpha_2 t\ \bar{T}_1 b^2$
		Aramid-Rubber (NL=2)		
0.5	- 3.3455	$- 0.1701 \times 10^4$	2.6210	- 1.1264
0.7	- 5.4763	$- 0.2371 \times 10^4$	4.7012	- 1.0100
0.9	0.0229	- 0.7557	- 0.0397	- 1.2135
1.0	0.0547	- 0.7789	- 0.0558	- 0.2227
1.2	0.1112	- 0.8233	- 0.0853	- 0.2403
1.4	0.1513	- 0.8597	- 0.1081	- 0.2548
1.6	0.1738	- 0.8867	- 0.1240	- 0.2655
1.8	0.1821	- 0.9054	- 0.1341	- 0.2730
2.0	0.1803	- 0.9177	- 0.1401	- 0.2779
		Polyester-Rubber (NL=2)		
0.5	- 46.395	- 32.850	6.2344	- 3.9037
0.7	- 66.708	- 33.152	8.9231	- 3.8930
0.9	- 63.472	- 34.827	8.1669	- 4.2283
1.0	- 55.893	- 36.025	6.9217	- 4.5183
1.2	- 40.106	- 37.695	4.3247	- 4.942
1.4	- 37.079	$- 0.1555 \times 10^3$	1.1373	0.3559×10^3
1.6	- 42.873	$- 0.1650 \times 10^3$	1.2790	0.3544×10^3
1.8	- 47.826	$- 0.1578 \times 10^3$	1.4415	0.3404×10^3
2.0	- 52.199	$- 0.1428 \times 10^3$	1.6249	0.3218×10^3

Table D5.2. In-Plane Displacements Associated with Table 5.2

a/b	Aramid-Rubber (NL=1) U (i)	Aramid-Rubber (NL=1) V (i)	Polyester-Rubber (NL=1) U (i)	Polyester-Rubber (NL=1) V (i)
0.5	0.3214×10^{-5}	-0.1441×10^{-5}	0.1377×10^{-4}	-0.5047×10^{-5}
0.6	0.5647×10^{-5}	-0.2798×10^{-5}	0.2374×10^{-4}	-0.9365×10^{-5}
0.7	0.9079×10^{-5}	-0.4861×10^{-5}	0.3743×10^{-4}	-0.1730×10^{-4}
0.8	0.1366×10^{-4}	-0.7005×10^{-5}	0.5517×10^{-4}	-0.2604×10^{-4}
0.9	0.1958×10^{-4}	-0.1185×10^{-4}	0.7719×10^{-4}	-0.4209×10^{-4}
1.0	0.2688×10^{-4}	-0.1717×10^{-4}	0.1029×10^{-3}	-0.6247×10^{-4}
1.2	0.4615×10^{-4}	-0.3269×10^{-4}	0.1651×10^{-3}	-0.1199×10^{-3}
1.4	0.7139×10^{-4}	-0.5631×10^{-4}	0.2345×10^{-3}	-0.2068×10^{-3}
1.6	0.1027×10^{-3}	-0.9065×10^{-4}	0.3018×10^{-3}	-0.3290×10^{-3}
1.8	0.1382×10^{-3}	-0.1379×10^{-3}	0.3567×10^{-3}	-0.4887×10^{-3}
2.0	0.1758×10^{-3}	-0.2007×10^{-3}	0.3981×10^{-3}	-0.6647×10^{-3}

a/b	U (ii)	V (ii)	U (ii)	V (ii)
0.5	0.7656×10^{-3}	-0.3512×10^{-3}	0.2051×10^{-2}	-0.8036×10^{-3}
0.6	0.1195×10^{-2}	-0.6279×10^{-3}	0.2072×10^{-2}	-0.1289×10^{-2}
0.7	0.1605×10^{-2}	-0.1012×10^{-2}	0.9315×10^{-3}	-0.2085×10^{-2}
0.8	0.1857×10^{-2}	-0.1543×10^{-2}	-0.1215×10^{-2}	-0.3786×10^{-2}
0.9	0.1816×10^{-2}	-0.2295×10^{-2}	-0.4870×10^{-2}	-0.5964×10^{-2}
1.0	0.1382×10^{-2}	-0.3351×10^{-2}	-0.9015×10^{-2}	-0.9549×10^{-2}
1.2	-0.6104×10^{-3}	-0.6757×10^{-2}	-0.1856×10^{-1}	-0.2047×10^{-1}
1.4	-0.3813×10^{-2}	-0.1221×10^{-1}	-0.2876×10^{-1}	-0.3620×10^{-1}
1.6	-0.7777×10^{-2}	-0.1981×10^{-1}	-0.3886×10^{-1}	-0.5615×10^{-1}
1.8	-0.1216×10^{-1}	-0.2949×10^{-1}	-0.4835×10^{-1}	-0.7949×10^{-1}
2.0	-0.1675×10^{-1}	-0.4106×10^{-1}	-0.5686×10^{-1}	-0.1053

Table D5.3. In-Plane Displacements Associated with Table 5.3

	Aramid-Rubber (NL=2)		Polyester-Rubber (NL=2)	
a/b	$U_{(i)}$	$V_{(i)}$	$U_{(i)}$	$V_{(i)}$
0.5	0.4194×10^{-5}	-0.6301×10^{-6}	0.1982×10^{-4}	-0.9016×10^{-5}
0.6	0.7255×10^{-5}	-0.1140×10^{-5}	0.3383×10^{-4}	-0.1823×10^{-4}
0.7	0.1148×10^{-4}	-0.1889×10^{-5}	0.5280×10^{-4}	-0.3298×10^{-4}
0.8	0.1705×10^{-4}	-0.2944×10^{-5}	0.7696×10^{-4}	-0.5484×10^{-4}
0.9	0.2412×10^{-4}	-0.4382×10^{-5}	0.1062×10^{-3}	-0.8536×10^{-4}
1.0	0.3282×10^{-4}	-0.6299×10^{-5}	0.1399×10^{-3}	-0.1259×10^{-3}
1.2	0.5547×10^{-4}	-0.1200×10^{-4}	0.2173×10^{-3}	-0.2402×10^{-3}
1.4	0.8529×10^{-4}	-0.2126×10^{-4}	0.2972×10^{-3}	-0.3993×10^{-3}
1.6	0.1214×10^{-3}	-0.3567×10^{-4}	0.3708×10^{-3}	-0.5964×10^{-3}
1.8	0.1626×10^{-3}	-0.5737×10^{-4}	0.4308×10^{-3}	-0.8177×10^{-3}
2.0	0.2055×10^{-3}	-0.9012×10^{-4}	0.4750×10^{-3}	-0.1046×10^{-2}

a/b	$U_{(ii)}$	$V_{(ii)}$	$U_{(ii)}$	$V_{(ii)}$
0.5	0.2483×10^{-2}	-0.1847×10^{-2}	0.7623×10^{-3}	-0.1374×10^{-3}
0.6	0.2562×10^{-2}	-0.3998×10^{-2}	0.1194×10^{-2}	-0.2888×10^{-3}
0.7	0.1401×10^{-2}	-0.7835×10^{-2}	0.1599×10^{-2}	-0.6180×10^{-3}
0.8	-0.1053×10^{-2}	-0.1391×10^{-1}	0.1849×10^{-2}	-0.1338×10^{-2}
0.9	-0.4506×10^{-2}	-0.2240×10^{-1}	0.1807×10^{-2}	-0.2729×10^{-2}
1.0	-0.8479×10^{-2}	-0.3307×10^{-1}	0.1391×10^{-2}	-0.5014×10^{-2}
1.2	-0.1568×10^{-1}	-0.5676×10^{-1}	-0.4823×10^{-3}	-0.1241×10^{-1}
1.4	-0.1938×10^{-1}	-0.7625×10^{-1}	-0.2872×10^{-2}	-0.2195×10^{-1}
1.6	-0.1973×10^{-1}	-0.8835×10^{-1}	-0.2872×10^{-2}	-0.2195×10^{-1}
1.8	-0.1835×10^{-1}	-0.9482×10^{-1}	-0.5559×10^{-2}	-0.3967×10^{-1}
2.0	-0.1636×10^{-1}	-0.9814×10^{-1}	-0.5559×10^{-2}	-0.3967×10^{-1}

APPENDIX E

TYPICAL COMPUTER SOLUTIONS

Sample computer printouts are attached. The programs are presented separately for each problem. They are self-explanatory. The dimension statement allows required values for different quantities. For example, reduced stiffnesses take 5x5x2x2 values (2 represents 2 layers, and the other 2 indicate different properties in tension (1) and compression (2)) and THETA takes either zero or 1.5708 radians.

The ISML library subroutine LEQT2F solves N equations in N unknowns. EIGZF solves eigenvalue problem, and ZRPOLY solves cubic equation. It should be mentioned that WKAREA, WK, etc. are part of such subroutines.

The values of z_{nx} and z_{ny} are assumed in the beginning to get the displacements. An iterative procedure is then adopted until a precision of $\pm$ 0.0001 is achieved to get the actual deflections. Later on the whole procedure is repeated for different aspect ratios.

Computations are carried out for a single-layer rectangular plate by making NL=1 (number of layers) and, of course, with a proper set of equations for A_{ij}, B_{ij}, and D_{ij}. With proper A_{ij}, B_{ij}, and D_{ij} and NL=2, the program calculates the deflections and neutral-surface locations for a two-layer, cross-ply rectangular plate. THETA takes the values 0 and 1.5708 (corresponding to 0° and 90°) radians for the two-layer, cross-ply case.

Close attention is needed to use the equations for stiffnesses and neutral-surface locations.

COMPUTER PROGRAM FOR STATIC BENDING (SINGLE LAYER)

```
$JCB TIME=(1,55)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION E11(2),E22(2),G12(2),G23(2),G31(2),ANU12(2),ANU21(2),Q
     *Q(5,5,2),Q(5,5,2,2),A(3,3),B(3,3),D(3,3),K(5,5),CDASH(5,1),
     **KAREA(50),THETA(2)
      READ,E11(1),E11(2),E22(1),E22(2),G12(1),G12(2),G23(1),G23(2),
     *G31(1),G31(2),ANU12(1),ANU12(2),ANU21(1),ANU21(2),THETA(1),
     *THETA(2)
   10 DO 20 I=1,5
      DO 20 J=1,5
      DO 20 K=1,2
      DO 20 L=1,2
      QQ(I,J,K)=0.0
      Q(I,J,K,L)=0.0
   20 CONTINUE
      NL = 1.0
      DO 30 I=1,5
      DO 30 J=1,5
      DO 30 K=1,2
      DENOM = 1-ANU12(K)*ANU21(K)
      QQ(1,1,K)=E11(K)/DENOM
      QQ(2,2,K)=E22(K)/DENOM
      QQ(1,2,K)=ANU12(K)*QQ(2,2,K)
      QQ(2,1,K)=QQ(1,2,K)
      QQ(3,3,K)=G12(K)
      QQ(4,4,K)=G23(K)
      QQ(5,5,K)=G31(K)
      DO 30 L=1,NL
      X=DCOS(THETA(L))
      Y=DSIN(THETA(L))
      X2=X**2
      Y2=Y**2
      X4=X**4
      Y4=Y**4
      Q(1,1,K,L) = QQ(1,1,K)*X4+QQ(2,2,K)*Y4
      Q(1,2,K,L) = QQ(1,2,K)*(Y4+X4)
      Q(2,1,K,L)=Q(1,2,K,L)
      Q(2,2,K,L) = QQ(1,1,K)*Y4+QQ(2,2,K)*X4
      Q(3,3,K,L)=QQ(3,3,K)*(Y4+X4)
      Q(4,4,K,L)=QQ(4,4,K)*X2+QQ(5,5,K)*Y2
      Q(5,5,K,L)=QQ(4,4,K)*Y2+QQ(5,5,K)*X2
   30 CONTINUE
      AX=5.0
      BY=10.0
      ZX=0.001
      ZY=-0.001
      PRINT 10,E11,E22,G12,G23,G31,ANU12,ANU21,THETA,ALFA,TBARO,TBAR1
   40 ZX2=ZX**2
      ZY2=ZY**2
      ZX3=ZX**3
      ZY3=ZY**3
      MM=1
      NN=1
      ALPHA = MM*3.1416/AX
      BETA = NN*3.1416/BY
      DO 50 I=1,3
      DO 50 J=1,3
      A(I,J) = (Q(I,J,1,1)+Q(I,J,2,1))/2+(Q(I,J,2,1)-Q(I,J,1,1))*ZX
      B(I,J) = (Q(I,J,1,1)-Q(I,J,2,1))/8+(Q(I,J,2,1)-Q(I,J,1,1))*ZX2/2
      D(I,J)=(Q(I,J,1,1)+Q(I,J,2,1))/24+(Q(I,J,2,1)-Q(I,J,1,1))*ZX3/3
```

```
      A(J,I)=A(I,J)
      B(J,I)=B(I,J)
      D(J,I)=D(I,J)
   50 CONTINUE
      A44=(G(4,4,1,1)+G(4,4,2,2))/2+(G(4,4,2,1)-G(4,4,1,1))*ZX+
     *(G(4,4,2,2)-G(4,4,1,2))*ZY
      A55=(G(5,5,1,1)+G(5,5,2,2))/2+(G(5,5,2,1)-G(5,5,1,1))*ZX+
     *(G(5,5,2,2)-G(5,5,1,2))*ZY
      SQRK4=0.833
      SQRK5=0.833
      C(1,1)=-A(1,1)*ALPHA**2-A(3,3)*BETA**2
      C(1,2)=-(A(1,2)+A(3,3))*ALPHA*BETA
      C(1,3)=0.0
      C(1,4)=-(B(1,2)+B(3,3))*ALPHA*BETA
      C(1,5)=-B(1,1)*ALPHA**2-B(3,3)*BETA**2
      CDASH(1,1)=0.0
      C(2,1)=C(1,2)
      C(2,2)=-A(3,3)*ALPHA**2-A(2,2)*BETA**2
      C(2,3)=0.0
      C(2,4)=-B(3,3)*ALPHA**2-B(2,2)*BETA**2
      C(2,5)=C(1,4)
      CDASH(2,1)=0.0
      C(3,1)=C(1,3)
      C(3,2)=C(2,3)
      C(3,3)=-SQRK5*A55  *ALPHA**2-SQRK4* A44  *BETA**2
      C(3,4)=-SQRK4* A44  *BETA
      C(3,5)=-SQRK5* A55  *ALPHA
      CDASH(3,1)=-S22(2)/DY**4
      C(4,1)=C(1,4)
      C(4,2)=C(2,4)
      C(4,3)=C(3,4)
      C(4,4)=-D(3,3)*ALPHA**2-D(2,2)*BETA**2-SQRK4*A44
      C(4,5)=-(D(1,2)+D(3,3))*ALPHA*BETA
      CDASH(4,1)=0.0
      C(5,1)=C(1,5)
      C(5,2)=C(2,5)
      C(5,3)=C(3,5)
      C(5,4)=C(4,5)
      C(5,5)=-D(1,1)*ALPHA**2-D(3,3)*BETA**2-SQRK5*A55
      CDASH(5,1)=0.0
      M=1
      N=5
      IA=5
      IDGT=4
      CALL LEQT2F(C,M,N,IA,CDASH,IDGT,WKAREA,IER)
      ZXX=-CDASH(1,1)/CDASH(5,1)
      ZYY=-CDASH(2,1)/CDASH(4,1)
      ZXA = DABS(DABS(ZX)-DABS(ZXX))
      ZYA = DABS(DABS(ZY)-DABS(ZYY))
      IF (ZXA.GT.0.0001)ZX=ZXX
      IF (ZYA.GT.0.0001) GO TO 60
      GO TO 70
   60 ZY=ZYY
      GO TO 40
   70 U=CDASH(1,1)
      V=CDASH(2,1)
      W=CDASH(3,1)
      YY=CDASH(4,1)
      XX=CDASH(5,1)
      PRINT 10,CDASH,ZX,ZY,AX
```

COMPUTER PROGRAM FOR STATIC BENDING (CROSS-PLY)

```
$JOB TIME=(1,55)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION E11(2),E22(2),G12(2),G23(2),G31(2),ANU12(2),ANU21(2),Q
     *Q(5,5,2),Q(5,5,2,2),A(3,3),B(3,3),D(3,3),C(5,5),CDASH(5,5),
     **KAREA(50),THETA(2)
      READ,E11(1),E11(2),E22(1),E22(2),G12(1),G12(2),G23(1),G23(2),
     *G31(1),G31(2),ANU12(1),ANU12(2),ANU21(1),ANU21(2),THETA(1),
     *THETA(2)
   10 DO 20 I=1,5
      DO 20 J=1,5
      DO 20 K=1,2
      DO 20 L=1,2
      QQ(I,J,K)=0.0
      Q(I,J,K,L)=0.0
   20 CONTINUE
      NL=2
      DO 30 I=1,5
      DO 30 J=1,5
      DO 30 K=1,2
      DENOM = 1-ANU12(K)*ANU21(K)
      QQ(1,1,K)=E11(K)/DENOM
      QQ(2,2,K)=E22(K)/DENOM
      QQ(1,2,K)=ANU12(K)*QQ(2,2,K)
      QQ(2,1,K)=QQ(1,2,K)
      QQ(3,3,K)=G12(K)
      QQ(4,4,K)=G23(K)
      QQ(5,5,K)=G31(K)
      DO 30 L=1,NL
      X=DCOS(THETA(L))
      Y=DSIN(THETA(L))
      X2=X**2
      Y2=Y**2
      X4=X**4
      Y4=Y**4
      Q(1,1,K,L) = QQ(1,1,K)*X4+QQ(2,2,K)*Y4
      Q(1,2,K,L) = QQ(1,2,K)*(Y4+X4)
      Q(2,1,K,L)=Q(1,2,K,L)
      Q(2,2,K,L) = QQ(1,1,K)*Y4+QQ(2,2,K)*X4
      Q(3,3,K,L)=QQ(3,3,K)*(Y4+X4)
      Q(4,4,K,L)=QQ(4,4,K)*X2+QQ(5,5,K)*Y2
      Q(5,5,K,L)=QQ(4,4,K)*Y2+QQ(5,5,K)*X2
   30 CONTINUE
      AX=5.0
      BY=10.0
      ZX=0.001
      ZY=-0.001
      PRINT 16,E11,E22,G12,G23,G31,ANU12,ANU21,THETA,ALFA,THARO,TBAR1
   40 ZX2=ZX**2
      ZY2=ZY**2
      ZX3=ZX**3
      ZY3=ZY**3
      MM=1
      NN=1
      ALPHA = MM*3.1416/AX
      BETA = NN*3.1416/BY
      DO 50 I=1,3
      DO 50 J=1,3
      A(I,J)=(Q(I,J,1,1)+Q(I,J,2,2))/2+(Q(I,J,2,1)-Q(I,J,1,1))*ZX+
     *(Q(I,J,2,2)-Q(I,J,1,2))*ZY
      B(I,J)=(Q(I,J,1,1)-Q(I,J,2,2))/8+(Q(I,J,2,1)-Q(I,J,1,1))*ZX**2/2
```

```
      *+(Q(I,J,2,2)-Q(I,J,1,2))*ZY**2/2
       D(I,J)=(Q(I,J,1,1)+Q(I,J,2,2))/24+(Q(I,J,2,1)-Q(I,J,1,1))*ZX**3/3
      *+(Q(I,J,2,2)-Q(I,J,1,2))*ZY**3/3
       A(J,I)=A(I,J)
       B(J,I)=B(I,J)
       D(J,I)=D(I,J)
   50  CONTINUE
       A44=(Q(4,4,1,1)+Q(4,4,2,2))/2+(Q(4,4,2,1)-Q(4,4,1,1))*ZX+
      *(Q(4,4,2,2)-Q(4,4,1,2))*ZY
       A55=(Q(5,5,1,1)+Q(5,5,2,2))/2+(Q(5,5,2,1)-Q(5,5,1,1))*ZX
      *(Q(5,5,2,2)-Q(5,5,1,2))*ZY
       SQRK4=0.832
       SQRK5=0.833
       C(1,1)=-A(1,1)*ALPHA**2-A(3,3)*BETA**2
       C(1,2)=-(A(1,2)+A(3,3))*ALPHA*BETA
       C(1,3)=0.0
       C(1,4)=-(B(1,2)+B(3,3))*ALPHA*BETA
       C(1,5)=-B(1,1)*ALPHA**2-B(3,3)*BETA**2
       CDASH(1,1)=0.0
       C(2,1)=C(1,2)
       C(2,2)=-A(3,3)*ALPHA**2-A(2,2)*BETA**2
       C(2,3)=0.0
       C(2,4)=-B(3,3)*ALPHA**2-B(2,2)*BETA**2
       C(2,5)=C(1,4)
       CDASH(2,1)=0.0
       C(3,1)=C(1,3)
       C(3,2)=C(2,3)
       C(3,3)=-SQRK5*A55   *ALPHA**2-SQRK4* A44   *BETA**2
       C(3,4)=-SQRK4* A44   *BETA
       C(3,5)=-SQRK5* A55   *ALPHA
       CDASH(3,1)=-E22(2)/ZY**4
       C(4,1)=C(1,4)
       C(4,2)=C(2,4)
       C(4,3)=C(3,4)
       C(4,4)=-D(3,3)*ALPHA**2-D(2,2)*BETA**2-SQRK4*A44
       C(4,5)=-(D(1,2)+D(3,3))*ALPHA*BETA
       CDASH(4,1)=0.0
       C(5,1)=C(1,5)
       C(5,2)=C(2,5)
       C(5,3)=C(3,5)
       C(5,4)=C(4,5)
       C(5,5)=-D(1,1)*ALPHA**2-D(3,3)*BETA**2-SQRK5*A55
       CDASH(5,1)=0.0
       M=1
       N=5
       IA=5
       IDGT=4
       CALL LEQT2F(C,M,N,IA,CDASH,IDGT,WKAREA,IER)
       ZXX=-CDASH(1,1)/CDASH(5,1)
       ZYY=-CDASH(2,1)/CDASH(4,1)
       ZXA = DABS(DABS(ZX)-DABS(ZXX))
       ZYA = DABS(DABS(ZY)-DABS(ZYY))
       IF (ZXA.GT.0.0001)ZX=ZXX
       IF (ZYA.GT.0.0001) GO TO 60
       GO TO 70
   60  ZY=ZYY
       GO TO 40
   70  U=CDASH(1,1)
       V=CDASH(2,1)
       W=CDASH(3,1)
```

```
      YY=CDASH([illegible],1)
      XX=CDASH(5,1)
      PRINT 10,CDASH,ZX,ZY,AX
      AX=AX+1.0
      IF (AX.LT.21.0) GO TO 40
   10 FORMAT(1X,5E12.5)
      STOP
      END
```

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18. SUPPLEMENTARY NOTES

This report is a slight condensation of the first author's master's thesis.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Bimodulus materials, classical solutions, closed-form solutions, composite materials, fiber-reinforced materials, free vibration, geometric nonlinearity, laminated plates, large deflections, moderately thick plates, natural frequencies, rectangular plates, shear flexible plate theory, static plate bending, thermal bending.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A differential-equation formulation is presented for the equations governing the small-deflection elastic behavior of thick plates laminated of anisotropic bimodulus materials (which have different elastic stiffnesses depending upon the sign of the fiber-direction strains). This study is concerned with the problems of bending due to pressure loading, free vibrations, and thermal expansion of thick plates having finite transverse shear moduli. Large static deflections of thin plates are also analyzed by an approximate technique. A simple scheme is used for determining explicitly the location of the (over)

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20. Abstract - Cont'd

neutral surface which is defined on the basis of fiber-direction strain. This provides a basis for rational design of such plates. Exact closed-form solutions are presented for freely supported rectangular plates subjected to a sinusoidally distributed normal pressure. Based on experimentally measured bimodulus properties, some numerical computations are carried out. Good agreement is obtained when compared with numerical results existing in the literature for special cases. The results presented here can be used to validate finite-element codes being developed for analysis of thick plates laminated of bimodulus material.