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ON REVERSIBLE TRANSFORMATIONS OF SPACE ELEMENTS.(U)

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BASEL MATHEMATICAL NOTES

6 On reversible transformations of space elements

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I. Introduction.

1.1. Consider, for $n > 1$, the coordinates, $x_1, \dots, x_n$, of a general point of the n-dimensional space, depending on and arbitrarily often differentiable with respect to m parameters $T_1, \dots, T_m$. Denote generally the derivatives $\frac{\partial x_\nu}{\partial T_\mu}$ by $p_{\nu\mu}$ $(\nu = 1, \dots, n; \mu = 1, \dots, m)$.

In this paper we are going to consider the transformation

$$(I.1) \qquad y_\nu = Y^*_\nu(x_\nu, p_{\nu\mu}) \quad (\nu = 1, \dots, n) \quad , \qquad ^{1)}$$

sub nu   sub nu   nu   sub nu mu

where the Y^*_ν are homogeneous of dimension 0 in the $p_{\nu\mu}$ and have the further property:

sub nu

Differentiating y_ν in (I.1) with respect to the T_μ and putting

$$q_{\nu\mu} := \frac{\partial y_\nu}{\partial T_\mu}$$

we can, eliminating the $p_{\nu\mu}$ and their derivatives, express the x_ν in function of y_ν and $q_{\nu\mu}$,

$$(I.2) \qquad x_\nu = X^*_\nu(y_\nu, q_{\nu\mu}) \quad (\nu = 1, \dots, n) \quad ,$$

where the X^*_ν are homogeneous of dimension 0 in the $q_{\nu\mu}$; and inversely (I.1) can be deduced differentiating (I.2) and eliminating the $q_{\nu\mu}$. The functions X^*_ν, Y^*_ν are assumed arbitrarily often differentiable in their arguments. We will denote the transformation, described by (I.1) and (I.2), with T^*.

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1) Here and everywhere later in this paper, if expressions like u_ν, v_μ, $w_{\nu\mu}$, $t_\varkappa$ occur inside parentheses, $(u_\nu, v_\mu, w_{\nu\mu}, t_\varkappa)$, this stands for

$$(u_1, \dots, u_n; v_1, \dots, v_m; w_{11}, \dots, w_{nm}; t_1, \dots, t_k)$$

independently of the same greek indices occurring outside of these parentheses.

Such transformations will be called <u>reversible transformations</u>.[2]

1.2. We prove in chapter II that the matrices

$$(I.3) \qquad \left(\frac{\partial(Y_\nu^*)}{\partial(p_{\nu\mu})}\right) \;,\quad \left(\frac{\partial(X_\nu^*)}{\partial(q_{\nu\mu})}\right) \qquad (\nu=1,\dots,n;\mu=1,\dots,m) \qquad ^{3)}$$

have the same maximal rank which is denoted throughout the whole paper with k. We obtain then in the same chapter the existence of two sets of k functions

$$(I.4) \qquad r_\varkappa = r_\varkappa^*(x_\nu, p_{\nu\mu}) \quad , \quad s_\varkappa = s_\varkappa^*(y_\nu, q_{\nu\mu}) \quad (\varkappa=1,\dots,k) \;,$$

where each set is independent, and which have the property that the expressions Y_ν^* in (I.1) and X_ν^* in (I.2) can be written as

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[2]) These transformations for n=2, m=1 were discussed in the author's paper, Sur une classe des transformations différentielles dans l'espace à trois dimensions, Commentarii mathematici helvetici, vol.13, pp.156-194, vol.14, pp.23-60 (1942), and for arbitrary n and m=1 in a second paper by the author, Sur les transformations réversibles d'éléments de ligne, Acta mathematica, vol.16, pp.151-182 (1942). See also G. Stohler's doctoral dissertation, Ueber eine Klasse von einparametrigen Differential-Transformationsgruppen, Commentarii mathematici helvetici, vol.18, pp.76-121 (1945)

[3]) The expressions used here and in what follows serve to denote the rectangular differential matrix formed of all derivatives of the expressions in the "numerator" with respect to all variables ocurring in the "denominator".

$$(I.5)\qquad Y_\nu^* =: Y_\nu(x_\nu, r_\varkappa) \quad , \quad X_\nu^* =: X_\nu(y_\nu, s_\varkappa) \quad (\nu=1,\dots,n) \quad ,$$

where the matrices

$$\left(\frac{\partial(Y_\nu)}{\partial(r_\varkappa)}\right) \quad , \quad \left(\frac{\partial(X_\nu)}{\partial(s_\varkappa)}\right)$$

have the same rank k.

Hence, there exists a one to one transformation between two (n+k)-dimensional spaces $(x_\nu, r_\varkappa)$ and $(y_\nu, s_\varkappa)$,

$$(I.6)\qquad T \begin{cases} y_\nu = Y_\nu(x_\nu, r_\varkappa) \quad , & s_\varkappa = S_\varkappa(x_\nu, r_\varkappa) \\ x_\nu = X_\nu(y_\nu, s_\varkappa) \quad , & r_\varkappa = R_\varkappa(y_\nu, s_\varkappa) \end{cases}$$
$$(\nu=1,\dots,n; \varkappa=1,\dots,k) \quad .$$

Now we can formulate the main problem with which we deal in this paper. <u>If a one to one transformation</u> T, <u>(I.6), is given, to describe necessary and sufficient conditions which must be satisfied in order that there exists a reversible transformation</u> T* <u>leading to the transformation</u> T (chapter II).

1.3. In order to deal with this problem we introduce in chapter III the so called <u>property</u> U. <u>An expression</u> $U(x_\nu, r_\varkappa, p_{\nu\mu})$ <u>is said to possess the property</u> U, if, using the relation (I.6) and the relations obtained by differentiation of these equations with respect to the T_μ, <u>it can be expressed in the form</u>,

$$(I.7)\qquad U = V(y_\nu, s_\varkappa, q_{\nu\mu}) \quad .$$

It turns out that the following partial differential equations are characteristic for the functions U with the property U:

$$(I.8)\qquad J_{\mu,\varkappa}\, U := \sum_{\nu=1}^{n} X'_{\nu s_\varkappa}\, U'_{p_{\nu\mu}} = 0 \quad (\mu=1,\dots,m; \varkappa=1,\dots,k) \quad ,$$

$$(I.9)\qquad \Delta_{\mu,\lambda}\, U := \sum_{\nu=1}^{n} p_{\nu\lambda}\, U'_{p_{\nu\mu}} = 0 \quad (\mu,\lambda=1,\dots,m) \quad .$$

The meaning of the system (I.9) is discussed in the Appendix B. The partial differential equations (I.8) are independent and their system is complete. The same holds for the partial differential equations (I.9). The system consisting of (I.8) and (I.9) taken together is also complete but in exceptional cases it could happen that linear relations exist between the equations (I.8) and (I.9):

$$\text{(I.10)} \qquad \sum_{\mu,\varkappa} \alpha_\varkappa J_{\mu,\varkappa} \equiv \sum_{\nu=1}^{n} \beta_\lambda \Delta_{\mu,\lambda} \qquad (\mu=1,\ldots,m;\varkappa=1,\ldots,k) \ ,$$

where the $\alpha_\varkappa$ and β_λ do not depend on μ. If there are exactly d such independent relations, the total number of independent integrals of the equations (I.8) and (I.9) is

$$\text{(I.11)} \qquad N := mn - m(m+k-d) = m(n-m-k+d)$$

where mn is the total number of the variables $p_{\nu\mu}$.

1.4. The above problem with d=0 is treated in chapter VII. We construct here a system of N functions $U^{(\sigma)}$ $(\sigma=k+1,\ldots,k+N)$ which are independent, as long as the $r_\varkappa$ are considered as independent variables, and form the total system of N independent integrals of the equations (I.8) and (I.9). We can therefore write

$$\text{(I.12)} \qquad r_\varkappa = \varphi_\varkappa(U^{(k+1)},\ldots,U^{(k+N)}) \quad (\varkappa=1,\ldots,k) \quad .$$

These equations can be solved with respect to the $r_\varkappa$ and give the corresponding expressions (II.6) of $r_\varkappa^*$, provided that the equations (I.12) are solvable,

$$\text{(I.13)} \qquad \frac{\partial(\varphi_1-r_1,\ldots,\varphi_k-r_k)}{\partial(r_1,\ldots,r_k)} \neq 0 \quad ,$$

where the $\varphi_{\varkappa}$ are differentiated "through the" $U^{(\sigma)}$. We have to add to (I.13) the additional condition

$$\text{(I.14)} \qquad \left. \frac{\partial(U^{(k+1)},\ldots,U^{(k+N)})}{\partial(r_1,\ldots,r_k)} \right|_{r_\varkappa = r_\varkappa^*} = N \quad .$$

The functions φ in (I.12) are indefinitely often differentiable arbitrary functions.

As soon as the expressions (I.4) of the $r_\varkappa^*$ are found we can obtain, using (I.6) for $s_\varkappa$, the expressions (I.4) of the $s_\varkappa$ in the y_ν and $q_{\nu\mu}$.

At the end of the chapter VII we discuss the method on an example.

1.5. As to the exceptional cases, d=1,...,m, we give in the chapters IV and V the complete discussion for the case d=m. As to the cases $1 \leqslant d \leqslant m$, we derive in chapter VIII, section 9, the inequality

$$\text{(I.15)} \qquad k \leqslant \frac{n-m}{d+1} + \theta \quad , \quad 0 < \theta \leqslant \frac{d}{d+1} \quad .$$

Further, using a method leading to (I.15), we solve in the sections 7.10-7.16 our problem completely for d=1.

The method used in the chapter VIII consists, in principle, in adding to the equations (I.6) d additional equations of the type

$$x_{n+\varsigma} = y_{n+\varsigma} \quad (\varsigma=1,\ldots,d) \quad .$$

In this way we make d to 0 for the enlarged system without changing the $r_\varkappa$ and $s_\varkappa$. This allows to obtain (I.15). However, the method of chapter VIII can apparently be only extended to our new enlarged system for $d=1$, since for $d > 1$ the condition corresponding to (VII.16) is no longer satisfied.

The discussion given in the chapter VI ought to become useful for the cases $d=2,\dots,m-1$.

The author hopes to discuss in another communication applications of the results of this paper to the theory of differential equations solvable without integration (integrallos auflösbare Differentialgleichungen).

II. Main definitions. Rank.

2.1. We consider in what follows 2n arbitrarily often differentiable functions

$$x_1, \ldots, x_n \; ; \; y_1, \ldots, y_n$$

depending on $m < n$ variables $T_1, \ldots, T_m$. We will use in particular the indices $\nu, \nu'; \varkappa, \varkappa'; \mu, \mu'; \lambda, \lambda'$, which run through the corresponding ranges: $1, \ldots, n$; $1, \ldots, k$; $1, \ldots, m$; $k+1, \ldots, n$. These ranges hold always if the corresponding letters are summation indices or in arguments so that for instance $f(x_{\nu'})$ means $f(x_1, \ldots x_n)$.

Put

$$\text{(II.1)} \qquad \frac{\partial x_\nu}{\partial T_\mu} =: p_{\nu\mu} \quad ; \quad \frac{\partial y_\nu}{\partial T_\mu} =: q_{\nu\mu} \quad (\nu=1,\ldots,n; \mu=1,\ldots,m)$$

and consider the three following (open) domains:

1) G_T an m-dimensional domain in the space of the $T_1, \ldots, T_m$;
2) G_p an (m+1)-dimensional domain in the space of the (m+1)n variables x_ν, $p_{\nu\mu}$;
3) G_q an (m+1)n-dimensional domain in the space of the (m+1)n variables y_ν, $q_{\nu\mu}$.

Assume that to the points of G_T correspond always points lying in G_p and G_q.

We choose an inner point A_o in G_T, to which correspond points in G_p, G_q and $G_p \times G_q$. These three points in G_p, G_q and $G_p \times G_q$ will be also denoted by A_o.

2.2. <u>A reversible transformation</u>, T, <u>of the</u> x_ν <u>into the</u> y_ν <u>is defined by two systems of equations</u>:

(II.2a) $$y_\nu = Y^*_\nu(x_1,\ldots,x_n;p_{11},\ldots,p_{nm}) \quad (\nu=1,\ldots,n) ,$$

(II.2b) $$x_\nu = X^*_\nu(y_1,\ldots,y_n;q_{11},\ldots,q_{nm}) \quad (\nu=1,\ldots,n) ,$$

if the Y^*_ν, X^*_ν have derivatives of any order in G_p, G_q and possess the following four properties , A, B, C and D:

A. The Jacobians of order n,

(II.3) $$\left|\frac{\partial(Y^*_\nu)}{\partial(x_\nu)}\right| \quad , \quad \left|\frac{\partial(X^*_\nu)}{\partial(y_\nu)}\right|$$

remain $\neq 0$ in G_p, G_q.

B. The functions X^*_ν, Y^*_ν remain invariant for any non-singular arbitrarily often differentiable transformation of the variables $T_1,\ldots,T_m$.

C. The relations (II.2b) follow from the relations (II.2a) by differentiation and elimination and the equations (II.2a) follow from the equations (II.2b), again by differentiation and elimination.

The content of the assumption B will be investigated in the section III.

We denote the maximal rank of the $n \times nm$-matrix

(II.4) $$\left(\frac{\partial(Y^*_\nu)}{\partial(p_{\rho\mu})}\right) \quad (\nu,\rho=1,\ldots,n;\mu=1,\ldots,m)$$

in G_p by k and that of the matrix

(II.5) $$\left(\frac{\partial(X^*_\nu)}{\partial(q_{\rho\mu})}\right) \quad (\nu,\rho=1,\ldots,n;\mu=1,\ldots,m)$$

in G_q by k'. Then our fourth property is:

D. A_o can be chosen in such a way that the ranks of the matrices (II.4) and (II.5) have in A_o their maximal values, k, k'. Obviously we can assume, restricting if necessary the domains G_T, G_p and G_q around A_o, that the rank of (II.4) is k <u>everywhere</u> in G_p and that a fixed subdeterminant of order k of (II.4) remains $\neq 0$ in G_p and that the analogous property subsists for (II.5) in G_q.

2.3. Then there exists a set of k functions

$$\text{(II.6)}\quad r_\varsigma = r^*_\varsigma(x_1,\ldots,x_n;p_{11},\ldots,p_{nm}) =: R_\varsigma(x_1,\ldots,x_n;Y^*_1,\ldots,Y^*_n) \quad (\varsigma=1,\ldots,k)$$

which are independent in G_p as functions of the $p_{\nu\mu}$, and which have derivatives of all orders and are such that all n expressions Y^*_ν can be written in the form

$$\text{(II.7)}\qquad y_\nu = Y^*_\nu =: Y_\nu(x_1,\ldots,x_n;r_1,\ldots,r_k) \qquad (\nu=1,\ldots,n)$$

and the rank of the $n\times k$-matrix

$$\text{(II.8)}\qquad \left(\frac{\partial(Y_1,\ldots,Y_n)}{\partial(r_1,\ldots,r_k)}\right)$$

has exactly the value k. The (n+k)-dimensional domain $[x_\nu, r_\varkappa]$ which is a proper part of G_p, will be denoted by G_r. For instance we can choose as the r_ς a subset of k among the n functions Y^*_ν, corresponding to a non-vanishing subdeterminant of order k of the matrix (II.4).

Similarly there exists a set of k' functions

$$\text{(II.9)}\quad s_\sigma = s^*_\sigma(y_1,\ldots,y_n;q_{11},\ldots,q_{nm}) =: S_\sigma(y_1,\ldots,y_n;X^*_1,\ldots,X^*_n) \quad (\sigma=1,\ldots,k')$$

which are independent in G_q as functions of the $q_{\nu\mu}$, and which have derivatives of all orders and are such that all n expressions X^*_ν can be written in the form

$$(\text{II.10}) \qquad x_\nu = X^*_\nu =: X_\nu(y_1,\dots,y_n;s_1,\dots,s_{k'}) \quad (\nu=1,\dots,n)$$

where the rank of the n×k'-matrix

$$(\text{II.11}) \qquad \left(\frac{\partial(X_1,\dots,X_n)}{\partial(s_1,\dots,s_{k'})}\right)$$

has exactly the value k'. The domain $[y_\nu, s_\varkappa]$ which is a part of G_q will be called G_s.

2.4. As the r^*_ς are independent as functions of the $p_{\nu\mu}$, the n+k variables

$$(\text{II.12}) \qquad x_1,\dots,x_n \;;\; r_1,\dots,r_k$$

are independent in G_p, since any relation between these variables would give a differential equation satisfied by the x_ν. Denote the space of all arbitrarily often differentiable functions of the variables (II.12) in G_r by Γ_x.

Similarly the n+k' variables

$$y_1,\dots,y_n \;;\; s_1,\dots,s_{k'}$$

are independent in G_q, and we denote the space of all arbitrarily often differentiable functions of these variables in G_s by Γ_y.

Replacing now in the formula (II.6) the Y^*_ν by y_ν and the x_ν by their expressions X_ν in the y_ν and s_σ, we obtain

(II.13) $$r_\rho = R_\rho(y_\nu, s_\sigma) \quad (\rho=1,\dots,k)$$

and similarly

(II.14) $$s_\sigma = S_\sigma(x_\nu, r_\rho) \quad (\sigma=1,\dots,k') \quad .$$

But the formulas (II.10) and (II.13) give a continuous transformation of G_s into G_r and the formulas (II.7) and (II.14) a continuous transformation of G_r into G_s. It follows that the dimensions $n+k$, $n+k'$ of G_r and G_s are equal and therefore

(II.15) $$k = k' \quad .$$

2.5. Consider the values of the r_κ corresponding to A_o and those of the s_κ equally corresponding to A_o. The corresponding points of G_r and G_s will be again denoted by A_o as well as their projections into the spaces of the r_κ and of the s_κ.

The point-to-point reversible transformation between the regions G_r and G_s given by the formulas (II.7), (II.10), (II.13) and (II.14) will be called characteristic transformation, T^*, belonging to T. This transformation is of course not uniquely determined by (II.2a), (II.2b), as the choice of the expressions r^*_κ and s^*_κ is highly arbitrary. The main problem of this paper is: Given a point-to-point transformation, T^*, between G_r and G_s, how to find suitable expressions r_ρ and s_σ such that, introducing the values of r_ρ and s_σ from (II.13) and (II.14) into the equations (II.7) and (II.10), we obtain formulas (II.2a) and (II.2b) defining a reversible transformation T.

2.6. As the rank of (II.8) is k, we can assume, after a convenient reordering of the Y_ν, that

$$\frac{\partial(Y_1,\dots,Y_k)}{\partial(r_\varkappa)} \neq 0 \quad . \tag{II.16}$$

Therefore the following equations

$$Y_\varkappa(x_\nu,r_\varkappa) - y_\varkappa = 0 \quad (\varkappa=1,\dots,k) \tag{II.17}$$

can be solved with respect to the $r_\varkappa$ in a neighbourhood of A_o so that we can write

$$r_\varkappa = \bar{R}_\varkappa(x_\nu,y_1,\dots,y_k) \quad (\varkappa=1,\dots,k) \quad . \tag{II.18}$$

Introducing these values into the expressions of Y_ν, (II.7) ($\nu=1,\dots,n$), we obtain in a neighbourhood of A_o

$$\Omega_\nu := Y_{\nu+k}(x_\nu,\bar{R}_\varkappa) - y_{\nu+k} = \Omega_\nu(x_1,\dots,x_n;y_1,\dots,y_n) = 0 \quad (\nu=1,\dots,n-k) \quad . \tag{II.19}$$

Obviously the rank of the matrix

$$\left(\frac{\partial(\Omega_\mu)}{\partial(y_\nu)}\right) \qquad (\nu=1,\dots,n;\mu=1,\dots,n-k) \tag{II.20}$$

is n-k as the last n-k variables y_ν are isolated in the Ω_ν.

2.7. We are now going to show that <u>the rank of the matrix</u>

$$\left(\frac{\partial(\Omega_\mu)}{\partial(x_\nu)}\right) \qquad (\nu=1,\dots,n;\mu=1,\dots,n-k) \tag{II.21}$$

<u>is also</u> n-k.

This follows easily from the lemma A1 of Appendix A making the following identifications: Replace the $r_\varkappa$ by $z_\varkappa$, n-k by $m=m_o$, the $Y_{\nu+k}-y_{\nu+k}$ ($\nu=1,\ldots,n-k$) by α_ν ($\nu=1,\ldots,n-k$), the $Y_\varkappa$ ($\varkappa=1,\ldots,k$) by $\beta_\varkappa$ and $y_\varkappa$ ($\varkappa=1,\ldots,k$) by $U_\varkappa$. Then the assumption (A 2) is satisfied by (II.16) while the matrix (A 3) with m+k columns has the rank m+k. The $\bar{z}_\varkappa$ becomes $\bar{R}_\varkappa$ and it follows that the rank of (II.21) is $\geq$ n-k and therefore =n-k as the matrix (II.1) has n-k columns.

2.8. Denote now the 2n-dimensional space of $[x_1,\ldots,x_n;y_1,y_2,\ldots,y_n]$ by Γ^*. Then the n-k relations (II.19) cut from Γ^* a region, Γ, of n+k dimensions. We can therefore say that those points of Γ^* belong to Γ whose coordinates are related by the relations (II.7) for convenient $r_\varkappa$. But these relations are equivalent to the relations (II.9) for convenient $s_\varkappa$ and this signifies that we obtain the same region Γ starting from the formulas (II.9) and eliminating the $s_\varkappa$. We will therefore generalize the system (II.19) of the Ω_ν admitting each system of equations

$$\text{(II.22)} \qquad \Omega_\nu(x_1,\ldots,x_n;y_1,\ldots,y_n) = 0 \quad (\nu=1,\ldots,n-k),$$

defining Γ in Γ^* and such that the ranks of the corresponding matrices (II.20) and (II.21) are exactly n-k, while the Ω_ν are arbitrarily often differentiable.

In so far we could use the characterization of points in Γ the 2n+2k variables

$$\left[x_\nu,\ y_\nu,\ r_\varkappa,\ s_\varkappa\right] \qquad ,$$

or any subset of these 2n+2k variables containing at least n+k variables independent with respect to the relations (II.7), (II.10), (II.13) and (II.14). For instance we could characterize a point of Γ by the 2n variables (x_ν, y_ν) satisfying the relations (II.22).

2.9. In the expressions $F(r_\varkappa, s_\varkappa)$ containing $r_\varkappa$ and $s_\varkappa$ these quantities are usually assumed to be "free variables" subject only to the relations (II.7), (II.10), (II.13) and (II.14). The corresponding expressions are then said to be "undevelopped". If however $r_\varkappa$ and $s_\varkappa$ are assumed to have the meaning (II.6) and (II.9), we speak of "developped" expressions and denote them by $F^*(r_\varkappa, s_\varkappa)$.

Further we denote by W^* the set of all arbitrarily often differentiable functions in convenient neighbourhoods in variables x_ν, $r_\varkappa$, y_ν, $s_\varkappa$.

Observe that, if a characteristic transformation T^* is fixed, then any function of W^* can be expressed as a function of Γ_x as well as a function of Γ_y.

III. Functions U,V.

3.1. We return now to the "invariancy" condition $\underline{\underline{B}}$. By the lemma B1 in the Appendix B, this condition amounts to the fact that the functions $Y^*_\mathfrak{z}(x_1,\ldots,x_n;p_{11},\ldots,p_{nm})$, as functions of the $p_{\nu\mu}$, depend only on quotients of determinants of order m of the matrix

$$(\text{III.1}) \qquad \begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_{n1} & \cdots & p_{nm} \end{pmatrix} ,$$

and to the fact that the functions X^*_ν, as functions of the $q_{\nu\mu}$, depend on the quotients of the subdeterminant of order m of (B 1) in Appendix B. Therefore the expressions $r_\mathfrak{S}$ in (II.6) and s_σ in (II.9) have also the corresponding properties.

Replacing in the lemma C1 in Appendix C the Y_ν by the X^*_ν, respectively the $q_{\nu\mu}$ and Y_ν by the $p_{\nu\mu}$ and Y^*_ν, we obtain the relations

$$(\text{III.2a}) \qquad \sum_{\nu=1}^{n} p_{\nu\mu} Y^{*\prime}_{\tau p_{\nu\mu'}} = 0 \quad (\mu,\mu'=1,\ldots,m;\tau=1,\ldots,n) ,$$

$$(\text{III.2b}) \qquad \sum_{\nu=1}^{n} q_{\nu\mu} X^{*\prime}_{\tau q_{\nu\mu'}} = 0 \quad (\mu,\mu'=1,\ldots,m;\tau=1,\ldots,n) ,$$

where the $Y^{*\prime}_{\tau p_{\nu\mu'}}$ and $X^{*\prime}_{\tau q_{\nu\mu'}}$ are assumed to be developped.

3.2. We consider arbitrarily often differentiable functions $U(x_\nu, r_\varkappa, p_{\nu\mu})$ of the x_ν, $r_\varkappa$ and $p_{\nu\mu}$ with the U property consisting in that that they can be represented, using (II.7), (II.10), (II.13) and (II.14), as arbitrarily often differentiable functions of the y_ν, $q_{\nu\mu}$, $s_\varkappa$,

(III.3) $$U(x_\nu, r_\varkappa, p_{\nu\mu}) = V(y_\nu, s_\varkappa, q_{\nu\mu}) \quad .$$

The functions $V(y_\nu, s_\varkappa, q_{\nu\mu})$ in (III.3) are then said to possess the V property. Obviously the Y^*_ν in (II.2a), the Y_ν in (II.7) and the r^*_s in (II.6) have the U property, while the X^*_ν in (II.2b), the X_ν in (II.10) and the $s^*_\varkappa$ in (II.9) have the V property.

3.3. Differentiating the relation (II.10),

(III.4) $$x_\nu = X_\nu(y_\nu, s_\varkappa) \quad ,$$

we obtain

(III.5) $$p_{\nu\mu} = \sum_{\nu'=1}^{n} X'_{\nu y_{\nu'}} q_{\nu'\mu} + \sum_{\varkappa=1}^{k} X'_{\nu s_\varkappa} s'_{\varkappa\mu} \quad .$$

Introducing the values (III.4) and (III.5) of the x_ν and $p_{\nu\mu}$ and (II.13) of the $r_\varkappa$ in $U(x_\nu, r_\varkappa, p_{\nu\mu})$ we obtain an expression

$$U^*(y_\nu, s_\varkappa, q_{\nu\mu}, s'_{\varkappa\mu})$$

and we have to obtain conditions under which the U^* is independent of the $s'_{\varkappa\mu}$. But in virtue of (III.4) and (III.5) we obtain, since the $s'_{\varkappa\mu}$ can be considered as arbitrary variables,

$$\frac{\partial U^*}{\partial s'_{\varkappa\mu}} = \sum_{\nu=1}^{n} U'_{p_{\nu\mu}} X'_{\nu s_\varkappa} = 0 \quad (\varkappa=1,\dots,k;\mu=1,\dots,m) \quad ,$$

and U becomes

$$(\text{III.6}) \qquad V(y_\sigma, s_\varkappa, q_{\nu\mu}) = U\left[X_\nu(y_\sigma, s_\varkappa), R_\varkappa(y_\sigma, s_\varkappa), \sum_{\nu'=1}^{n} X'_{\nu y_{\nu'}} q_{\nu'\mu}\right].$$

We see that the km relations, with developped $X'_{\nu s_\varkappa}$,

$$(\text{III.7}) \qquad \sum_{\nu=1}^{n} X'_{\nu s_\varkappa} U'_{p_{\nu\mu}} = 0 \qquad (\varkappa=1,\ldots,k;\mu=1,\ldots,m)$$

are necessary and sufficient in order that the $s'_{\varkappa\mu}$ fall out from U^*, that is that U satisfies (III.3).

3.4. The system of km linear homogeneous partial differential equations (III.7) consists of km linearly independent equations, as follows from the fact that the rank of (II.11) is k.

It follows immediately that the system of the equations (III.7) is <u>complete</u>, that is that, putting

$$(\text{III.8}) \qquad J_{\mu,\varkappa} := \sum_{\nu=1}^{n} X'_{\nu s_\varkappa} \frac{\partial}{\partial p_{\nu\mu}} \qquad (\mu=1,\ldots,m;\varkappa=1,\ldots,k) \quad ,$$

the "parentheses expressions"

$$(J_{\mu,\varkappa}, J_{\lambda,\sigma}) := J_{\mu,\varkappa} J_{\lambda,\sigma} - J_{\lambda,\sigma} J_{\mu,\varkappa} \quad (\mu,\lambda=1,\ldots,m;\varkappa,\sigma=1,\ldots,k)$$

are linearly expressible through the set of the $J_{\mu,\varkappa}$.

For obviously

$$(\text{III.9}) \quad (J_{\mu,\varkappa}, J_{\lambda,\sigma}) = \sum_{\nu=1}^{n} (J_{\mu,\varkappa} X'_{\nu s_\sigma}) \frac{\partial}{\partial p_{\nu\sigma}} - \sum_{\nu=1}^{n} (J_{\lambda,\sigma} X'_{\nu s_\varkappa}) \frac{\partial}{\partial p_{\nu\varkappa}} = 0 \,,$$

since the functions

$$J_{\mu,\varkappa} X'_{\nu s_\sigma} \quad , \quad J_{\lambda,\sigma} X'_{\nu s_\varkappa}$$

vanish. For the $X'_{\nu s_\varkappa}$ satisfy the equations (III.7), since the $X'_{\nu s_\varkappa}$ are expressible both in G_p as in G_q.

3.5. As the system (III.7) is complete it follows that this system implies exactly mk independent relations and possesses exactly nm-mk independent integrals as functions of the $p_{\nu\mu}$.

But the k functions $r^*_\varkappa(x_\nu, p_{\nu\mu})$ satisfy (III.3) in virtue of (II.13) and (II.14). It follows, as the $r^*_\varkappa$ are independent in the $p_{\nu\mu}$, using (III.8):

(III.10) $$k \leq m(n-k) \quad , \quad k \leq \frac{mn}{m+1} = n - \frac{n}{m+1} \quad .$$

In particular it follows that

(III.11) $$k < n \quad .$$

In (III.10) the equality sign holds in particular for k=m=n-1. Then we have the <u>contact transformations in $\mathbb{R}^n$</u> (see Ostrowski [1]).

3.6. In the above discussion the invariancy of the U with respect to a transformation of the T_μ was not assumed. If we now assume that the functions U are <u>invariant with respect to a transformation of the</u> T_μ, then we must add (see Appendix C) to the equations (III.7) the equations

(III.12) $$\sum_{\nu=1}^{n} p_{\nu\mu} U'_{p_{\nu\mu'}} = 0 \quad (\mu, \mu' = 1, \ldots, m) \quad .$$

and assume that all $r^*_\varkappa$ satisfy these equations.

The equations (III.12) could completely or partly be contained in the system (III.7). For instance in the case k=n-1 the set (III.12) completely depends on the equations (III.7).

Denote by N* the total number of linearly independent among the equations (III.7) and (III.12). Then we can choose among the equations (III.12) exactly N*-mk,

(III.13) $$\Delta^{(1)}U = 0, \ldots, \Delta^{(N^*-mk)}U = 0 \quad ,$$

which imply, taken together with (III.7), both systems (III.7) and (III.12). It is easy to show that the system consisting of (III.7) and (III.13) is <u>complete</u>.

3.7. Indeed, put generally

(III.14) $$\Delta_{\mu,\lambda} := \sum_{\nu=1}^{n} p_{\nu\lambda} \frac{\partial}{\partial p_{\nu\mu}} \qquad (\mu,\lambda=1,\ldots,m) \quad .$$

Then we have for

$$(\Delta_{\mu,\lambda}, J_{\mu',\varkappa}) := \Delta_{\mu,\lambda} J_{\mu',\varkappa} - J_{\mu',\varkappa}\Delta_{\mu,\lambda}$$

the expressions

$$(\Delta_{\mu,\lambda}, J_{\mu',\varkappa}) = \sum_{\nu=1}^{n} (\Delta_{\mu,\lambda} X'_{\nu s_\varkappa}) \frac{\partial}{\partial p_{\nu\mu'}} - \sum_{\nu=1}^{n} (J_{\mu',\varkappa} p_{\nu\lambda}) \frac{\partial}{\partial p_{\nu\mu}} \quad .$$

But the terms $\Delta_{\mu,\lambda} X'_{\nu s_\varkappa}$ vanish, as the $X'_{\nu s_\varkappa}$ are homogeneous of dimension 0, while

$$J_{\mu',\varkappa} p_{\nu\lambda} = \delta_{\lambda\mu'} X'_{\nu s_\varkappa} \quad .$$

Therefore

(III.15) $$(\Delta_{\mu,\lambda}, J_{\mu';\varkappa}) = -\delta_{\lambda\mu'} \sum_{\nu=1}^{n} X'_{\nu s_\varkappa} \frac{\partial}{\partial p_{\nu\mu}} = -\delta_{\lambda\mu'} J_{\mu,\varkappa} .$$

Finally we obtain easily

$$(\Delta_{\mu,\lambda}, \Delta_{\mu';\lambda'}) = \delta_{\mu\lambda'} \sum_{\nu=1}^{n} \left(p_{\nu\lambda} \frac{\partial}{\partial p_{\nu\mu'}}\right) - \delta_{\mu'\lambda} \sum_{\nu=1}^{n} \left(p_{\nu\lambda'} \frac{\partial}{\partial p_{\nu\mu}}\right) ,$$

(III.16) $$(\Delta_{\mu,\lambda}, \Delta_{\mu';\lambda'}) = \delta_{\mu\lambda'} \Delta_{\mu';\lambda} - \delta_{\mu'\lambda} \Delta_{\mu,\lambda'} ,$$

and we see that the system of operators generated by (III.7) and (III.12) is complete.

In particular it follows that the linear system of operators generated by the $\Delta_{\mu,\nu}$ for a fixed μ is complete and the same holds for the linear system of operators generated for a fixed μ by the operators $J_{\mu,\varkappa}$ ($\varkappa=1,\ldots,k;\nu=1,\ldots,n$).

On the other hand, the system of the m^2 equations (III.12) is complete and has therefore $m(n-m)$ independent integrals in the $p_{\nu\mu}$. Since there are k integrals $r_\varkappa$ it follows

(III.17) $$m(n-m) \geqslant k \quad , \quad m \leqslant n-1 \quad .$$

3.8. Assume now generally that there exists a non-trivial linear relation between the $J_{\mu,\varkappa}$ and $\Delta_{\mu,\lambda}$ and assume the $J_{\mu,\varkappa}$ as developped:

(III.18) $$\sum_{\mu,\varkappa} \alpha_{\mu\varkappa} J_{\mu,\varkappa} = \sum_{\mu,\lambda} A_{\mu\lambda} \Delta_{\mu,\lambda} ,$$

where not all $\alpha_{\mu\varkappa}$ and not all $A_{\mu\lambda}$ vanish. Then, equating on the right and on the left the parts corresponding to a general fixed μ, we obtain the relations

$$\text{(III.19)} \qquad \sum_{\varkappa=1}^{k} \alpha_{\mu\varkappa} J_{\mu,\varkappa} = \sum_{\lambda=1}^{m} A_{\mu\lambda} \Delta_{\mu,\lambda} \quad (\mu=1,\dots,m) \; .$$

Assume that for a fixed μ the relation (III.19) is not trivial and write it as

$$\text{(III.20)} \qquad \sum_{\varkappa=1}^{k} \alpha_{\varkappa} J_{\mu,\varkappa} = \sum_{\lambda=1}^{m} A_{\lambda} \Delta_{\mu,\lambda} \; .$$

Then, introducing from (III.8) and (III.14) the expressions of $J_{\mu\varkappa}$ and $\Delta_{\mu,\lambda}$ it follows, if we equate on both sides the coefficients of the single differential operators $D_{p_{\nu\mu}}$, the system of n relations equivalent with (III.20):

$$\text{(III.21)} \qquad \sum_{\varkappa=1}^{k} \alpha_{\varkappa} X'_{\nu s_{\varkappa}} = \sum_{\lambda=1}^{m} A_{\lambda} p_{\nu\lambda} \quad (\nu=1,\dots,n) \; .$$

But the relations (III.21) do not contain μ. We see that <u>if a non-trivial relation</u>

$$\text{(III.22)} \qquad \sum_{\varkappa=1}^{k} \alpha_{\varkappa} J_{\mu,\varkappa} = \sum_{\lambda=1}^{m} A_{\lambda} \Delta_{\mu,\lambda}$$

<u>holds for a certain μ, the same relation holds for any</u> $\mu=1,\dots,m$.

It follows then that if there exist <u>for a fixed</u> μ exactly

(III.23) $$d \leqslant \mathrm{Min}(m,k)$$

linearly independent relations of the type (III.22), then the number of independent equations among the equations (III.7) and (III.12) is exactly $mk+m^2-dm$ and therefore the number, N, of independent integrals of these equations is precisely

(III.24) $$N \; := \; m(n-k-m+d) \geqslant k \quad ,$$

so that the coefficient of m is $\geqslant 0$,

(III.25) $$n \geqslant k+m-d \quad , \quad n-k \geqslant m-d \quad .$$

3.9. Observe that the $n\times(k+m)$-matrix

(III.26) $$K^*_x \; := \begin{pmatrix} X'_{1s_1} & \cdots & X'_{1s_k} & p_{11} & \cdots & p_{1m} \\ X'_{2s_1} & \cdots & X'_{2s_k} & p_{21} & \cdots & p_{2m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{ns_1} & \cdots & X'_{ns_k} & p_{n1} & \cdots & p_{nm} \end{pmatrix} \quad ,$$

where all $X'_{\nu s_\varkappa}$ are assumed as developped, has the rank $\geqslant k$. Denoting this rank by k+m-d, we have therefore $d \leqslant m$.

On the other hand, by the above definition of d, d is the number of columns of the matrix (III.26) which linearly depend on the other columns.

Observe that, by (III.24),

(III.27) $$m(n-m+d) \geqslant (m+1)k \quad .$$

3.10. Observe finally that, putting

$$\text{(III.28)} \quad \hat{p}_{\nu\mu} = \sum_{\sigma=1}^{n} X'_{\nu y_\sigma} q_{\sigma\mu} \ , \quad \hat{q}_{\nu\mu} = \sum_{\sigma=1}^{n} Y'_{\nu x_\sigma} p_{\sigma\mu} \ ,$$

the formula (III.6) and the corresponding formula for U become

$$\text{(III.29)} \qquad U(x_\nu, r_\varkappa, p_{\nu\mu}) = V(y_\nu, s_\varkappa, q_{\nu\mu}) = U(X_\nu, R_\varkappa, \hat{p}_{\nu\mu}) \quad ,$$

$$\text{(III.30)} \qquad V(y_\nu, s_\varkappa, q_{\nu\mu}) = U(x_\nu, r_\varkappa, p_{\nu\mu}) = V(Y_\nu, S_\varkappa, \hat{q}_{\nu\mu}) \quad .$$

The expressions (III.28) can be obviously considered as the corresponding derivatives of the X_ν and the Y_ν computed in the <u>assumption that the</u> $s_\varkappa$, $r_\varkappa$ <u>are constants</u>.

IV. The forms u and v.

4.1. A u-form is by definition an expression

(IV.1) $$u = \sum_{\nu,\mu} U_{\nu\mu} p_{\nu\mu} \quad , \quad U_{\nu\mu} \in W^* \quad ,$$

where the $U_{\nu\mu}$ belong to W^*, with the property that, using the characteristic transformation T^*, u can be transformed into a v-form,

(IV.2) $$v = \sum_{\nu,\mu} V_{\nu\mu} q_{\nu\mu} \quad , \quad V_{\nu\mu} \in W^* \quad .$$

The coefficients $U_{\nu\mu}$ and $V_{\nu\mu}$ can be expressed both as functions of the x_ν, $r_\varkappa$ and as functions of the y_ν, $s_\varkappa$.

If we introduce into (IV.1) the expressions (III.5) of the $p_{\nu\mu}$ we obtain

(IV.3) $$u = \sum_{\nu,\mu,\lambda} U_{\nu\mu} X'_{\nu y_\lambda} q_{\lambda\mu} + \sum_{\varkappa,\mu} \Big(\sum_{\nu} U_{\nu\mu} X'_{\nu s_\varkappa} \Big) s'_{\varkappa\mu} \quad ,$$

where the indices ν and λ run from 1 to n, the index μ runs from 1 to m and the index $\varkappa$ from 1 to k.

Since here the $s'_{\varkappa\mu}$ can be considered as independent variables, we obtain as necessary and sufficient for the u-form (IV.1):

(IV.4) $$\sum_{\nu=1}^{n} U_{\nu\mu} X'_{\nu s_\varkappa} = 0 \quad (\mu=1,\ldots,m;\varkappa=1,\ldots,k) \ .$$

4.2. If we make in (IV.3) all $U_{\nu\mu}$ with $\mu \neq \mu_o$ to zero, we obtain a <u>single u-form</u> corresponding to μ_o:

$$(IV.5) \qquad u^{(\mu_o)} := \sum_{\nu=1} U_{\nu\mu_o} p_{\nu\mu_o} = \sum_{\nu,\lambda=1} U_{\nu\mu_o} X'_{\nu y_\lambda} q_{\lambda\mu_o} =: v^{(\mu_o)}$$

and we see that, taking in (IV.1) together the groups of terms belonging to the same index μ, u is decomposed into a sum,

$$u = \sum_{\mu=1}^{m} u^{(\mu)} \qquad ,$$

of single u-forms $u^{(\mu)}$ belonging each to another μ.

It follows that we can restrict ourselves to the consideration of the single u-forms and the single v-forms.

Observe that by definition the u-forms form a <u>linear system</u> if we admit as coefficients all functions from W*. And the same holds also for the system of all single u-forms corresponding to a fixed value of μ.

But the k equations (IV.4) corresponding to a fixed μ are linearly independent with respect to the $U_{\nu\mu}$ since (II.11) has the rank k. Thence there are exactly ∞^{n-k} single u-forms for each μ.

4.3. Therefore the question arises to find a convenient <u>basis</u> for all <u>single u-forms</u> corresponding to a fixed μ.

We obtain a system of n-k single u-forms and single v-forms, differentiating the Ω_ν in (II.22) with respect to T_μ (μ=1,...,m),

$$(IV.6) \qquad \bar{u}_\sigma^{(\mu)} := \sum_\nu \Omega_{\sigma x_\nu} p_{\nu\mu} = -\sum_\nu \Omega_{\sigma y_\nu} q_{\nu\mu} =: -\bar{v}_\sigma^{(\mu)} \quad .$$

And it follows from the rank condition for the matrices

(II.20) and (II.21) that the n-k forms $\bar{u}_\sigma^{(\mu)}$ are linearly independent as well as the $\bar{v}_\sigma^{(\mu)}$.

Therefore the u-forms (IV.6) form a basis for the single u-forms corresponding to a fixed μ and the same holds for the v-forms $v_\sigma^{(\mu)}$ defined by (IV.6).

4.4. Another basis for the single u-forms can be obtained using the functions X_ν in (II.10) defining the characteristic transformation T*. Since the rank of (II.11) is k we can and will assume, changing if necessary the numbering of the X_ν and Y_ν, the non-vanishing of the developped determinants

$$\text{(IV.7)}\quad J := \begin{vmatrix} X'_{1s_1} & \cdots & X'_{ks_1} \\ \cdots & \cdots & \cdots \\ X'_{1s_k} & \cdots & X'_{ks_k} \end{vmatrix}, \quad K := \begin{vmatrix} Y'_{1r_1} & \cdots & Y'_{kr_1} \\ \cdots & \cdots & \cdots \\ Y'_{1r_k} & \cdots & Y'_{kr_k} \end{vmatrix}.$$

For the derivation of our basis for the single u-forms, using the X_ν, it is not even necessary to assume that the X_ν belong to T*. It is sufficient to require that the n functions $X_\nu(y_1,\ldots,y_n;s_1,\ldots,s_k)$ have with respect to the $s_\varkappa$ the Jacobian rank $=k$, that is that one of the determinants of order k from the Jacobian matrix $\left(\frac{\partial(X_1,\ldots,X_n)}{\partial(s_1,\ldots,s_k)}\right)$ does not vanish, where in particular we can assume that the determinant $J=\left(\frac{\partial(X_1,\ldots,X_k)}{\partial(s_1,\ldots,s_k)}\right)$ does not vanish. We can then define the u-form (IV.1) by the mk relations (IV.4). In order to distinguish our generalized assumptions from the original ones based on the relation U=V, we will denote the u-forms defined solely by (IV.4) as unilateral u-forms. Then it is easy to see that a basis for the single unilateral u-forms corresponding to a μ is given by

$$(\text{IV.8})\qquad u_\lambda^{(\mu)} := \begin{vmatrix} p_{\lambda\mu} & p_{1\mu} & \cdots & p_{k\mu} \\ X'_{\lambda s_1} & X'_{1s_1} & \cdots & X'_{ks_1} \\ \cdots & \cdots & \cdots & \cdots \\ X'_{\lambda s_k} & X'_{1s_k} & \cdots & X'_{ks_k} \end{vmatrix} \qquad (\lambda=k+1,\ldots,n)\quad,$$

with respect to a system of coefficients consisting of all indefinitely often differentiable functions belonging to Γ_y.

Indeed, each $u_\lambda^{(\mu)}$ satisfies the equation (IV.4) since replacing in $u_\lambda^{(\mu)}$ the $p_{\nu\mu}$ with $X'_{\nu s_\varkappa}$ amounts to making in $u_\lambda^{(\mu)}$ the first line identical to the $(\varkappa+1)$st line. The independence of the $u_\lambda^{(\mu)}$ follows from the fact that to each $u_\lambda^{(\mu)}$ corresponds a $p_{\lambda\mu}$ occuring with the coefficient J in this $u_\lambda^{(\mu)}$ only.

4.5. It follows now that a unilateral u-form written as

$$u = \sum_{\nu=1}^{n} f_\nu p_{\nu\mu}$$

with f_ν from Γ_y contains at least one of the $p_{k+1\mu},\ldots,p_{n\mu}$ with a non-vanishing coefficient unless it vanishes identically. For, representing u linearly through the basis $u_\lambda^{(\mu)}$, none of the $p_{\lambda\mu}(\lambda>k)$ is destroyed if the corresponding $u_\lambda^{(\mu)}$ has a non-vanishing coefficient in the representation. We obtain now the rule: <u>If a unilateral u-form is written for a fixed</u> μ, as

$$(\text{IV.9})\qquad u = \sum_{\nu=1}^{n} f_\nu p_{\nu\mu}\quad,\ f_\nu \in \Gamma_y\ ,$$

<u>it follows</u>, <u>using the</u> $u_\nu^{(\mu)}$ <u>from (IV.8)</u>,

$$(\text{IV.10})\qquad u = \frac{1}{J}\sum_{\nu=k+1}^{n} f_\nu u_\nu^{(\mu)}\quad.$$

Indeed, as

$$p_{\nu\mu} = \frac{1}{J} u_\nu^{(\mu)} + \{p_{1\mu}, \ldots, p_{k\mu}\} \qquad (\nu = k+1, \ldots, n)$$

we obtain from (IV.9)

$$u = \frac{1}{J} \sum_{\nu=k+1}^{n} f_\nu u_\nu^{(\mu)} + \{p_{1\mu}, \ldots, p_{k\mu}\} \qquad ,$$

denoting generally by $\{p_{1\mu}, \ldots, p_{k\mu}\}$ a linear form in the $p_{1\mu}, \ldots, p_{k\mu}$ with coefficients from Γ_y. And this $\{p_{1\mu}, \ldots, p_{k\mu}\}$, being a unilateral u-form, vanishes identically.

4.6. Similarly there exists a basis of all single v-forms belonging to a μ, consisting of the following n-k v-forms:

$$\text{(IV.11)} \qquad v_\lambda^{(\mu)} = \begin{vmatrix} q_{\lambda\mu} & q_{1\mu} & \cdots & q_{k\mu} \\ Y'_{\lambda r_1} & Y'_{1r_1} & \cdots & Y'_{kr_1} \\ \cdots & \cdots & \cdots & \cdots \\ Y'_{\lambda r_k} & Y'_{1r_k} & \cdots & Y'_{kr_k} \end{vmatrix} \qquad (\lambda = k+1, \ldots, n) .$$

To obtain a representation of $u_\lambda^{(\mu)}$ in terms of the $v_\lambda^{(\mu)}$ observe that, by (III.5),

$$p_{\nu\mu} = \sum_{t=1}^{n} X'_{\nu y_t} q_{t\mu} + \sum_{\sigma=1}^{k} X'_{\nu s_\sigma} s'_{\sigma\mu} \qquad (\nu = 1, \ldots, n; \mu = 1, \ldots, m).$$

Introducing this into (IV.8) we obtain for $\lambda = k+1, \ldots, n$:

$$u_\lambda^{(\mu)} = \begin{vmatrix} \sum_t X'_{\lambda y_t} q_{t\mu} & \sum_t X'_{1y_t} q_{t\mu} & \cdots & \sum_t X'_{ky_t} q_{t\mu} \\ X'_{\lambda s_1} & X'_{1s_1} & \cdots & X'_{ks_1} \\ \cdots & \cdots & \cdots & \cdots \\ X'_{\lambda s_k} & X'_{1s_k} & \cdots & X'_{ks_k} \end{vmatrix} +$$

$$+ \begin{vmatrix} \sum_\sigma X'_{\lambda s_\sigma} s'_{\sigma\mu} & \sum_\sigma X'_{1s_\sigma} s'_{\sigma\mu} & \cdots & \sum_\sigma X'_{ks_\sigma} s'_{\sigma\mu} \\ X'_{\lambda s_1} & X'_{1s_1} & \cdots & X'_{ks_1} \\ \cdots & \cdots & \cdots & \cdots \\ X'_{\lambda s_k} & X'_{1s_k} & \cdots & X'_{ks_k} \end{vmatrix} .$$

Here the second determinant vanishes as its first line is a linear combination of the following lines. Taking in the first determinant the summation $\sum_t q_{t\mu}$ out, it follows further

$$u_\lambda^{(\mu)} = \sum_{t=1}^{n} q_{t\mu} \begin{vmatrix} X'_{\lambda y_t} & X'_{1y_t} & \cdots & X'_{ky_t} \\ X'_{\lambda s_1} & X'_{1s_1} & \cdots & X'_{ks_1} \\ \cdots & \cdots & \cdots & \cdots \\ X'_{\lambda s_k} & X'_{1s_k} & \cdots & X'_{ks_k} \end{vmatrix} .$$

But here the terms corresponding to t=1,...,k vanish and we obtain

$$u_\lambda^{(\mu)} = \sum_{t=k+1}^{n} q_{t\mu} \begin{vmatrix} X'_{\lambda y_t} & X'_{1y_t} & \cdots & X'_{ky_t} \\ X'_{\lambda s_1} & X'_{1s_1} & \cdots & X'_{ks_1} \\ \cdots & \cdots & \cdots & \cdots \\ X'_{\lambda s_k} & X'_{1s_k} & \cdots & X'_{ks_k} \end{vmatrix} ,$$

(IV.12) $$u_\lambda^{(\mu)} = \sum_{t=k+1}^{n} A_{\lambda t}\, q_{t\mu} ,$$

(IV.13) $$A_{\lambda t} := \begin{vmatrix} X'_{\lambda y_t} & X'_{1y_t} & \cdots & X'_{ky_t} \\ X'_{\lambda s_1} & X'_{1s_1} & \cdots & X'_{ks_1} \\ \cdots & \cdots & \cdots & \cdots \\ X'_{\lambda s_k} & X'_{1s_k} & \cdots & X'_{ks_k} \end{vmatrix} = \frac{\partial(X_\lambda, X_1, \ldots, X_k)}{\partial(y_t, s_1, \ldots, s_k)} .$$

Applying to the form on the right in (IV.12) the analogue of the rule concerning (IV.9) and (IV.10), we obtain

(IV.14) $$u_\lambda^{(\mu)} = \frac{1}{K} \sum_{t=k+1}^{n} A_{\lambda t}\, v_t^{(\mu)} \qquad (\lambda = k+1, \ldots, n) .$$

Similarly, it follows

(IV.15) $$v_\lambda^{(\mu)} = \frac{1}{J} \sum_{t=k+1}^{n} B_{\lambda t}\, u_t^{(\mu)} ,$$

(IV.16) $$B_{\lambda t} = \frac{\partial(Y_\lambda, Y_1, \ldots, Y_k)}{\partial(x_t, r_1, \ldots, r_k)} \qquad (\lambda = k+1, \ldots, n) .$$

V. Transformation with d = m.

5.1. It follows obviously from the relation (III.24): <u>If</u>

$$k = \frac{mn}{m+1} \tag{V.1}$$

<u>then</u> d <u>must be</u> =m. It will be seen from the following discussion that the relation (V.1) follows, from d=m.

Assume d=m. Each of the last k columns of K^*_x in (III.26) must be a linear combination of the first k columns, that is

$$p_{\nu\mu} - \sum_{\varkappa=1}^{k} m_{\varkappa}^{(\mu)} X'_{\nu s_\varkappa} = 0 \quad (\nu=1,\ldots,n;\mu=1,\ldots,m) . \tag{V.2}$$

This signifies that in each of the determinants (IV.8) the first line is a combination of the following lines, therefore all m(n-k) forms $u_\lambda^{(\mu)}$ vanish and we can write, developping the $u_\lambda^{(\mu)}$ in (IV.8),

$$u_\lambda^{(\mu)} = Jp_{\lambda\mu} - \sum_{\varkappa=1}^{k} f_{\lambda\varkappa}^{(\mu)} p_{\varkappa\mu} = 0 \quad (\lambda=k+1,\ldots,n;\mu=1,\ldots,m) , \tag{V.3}$$

where the $f_{\lambda\varkappa}^{(\mu)}$ belong to W*.

In the equations (V.3) we can express, in virtue of the characteristic transformation T*, all coefficients $f_{\lambda\varkappa}^{(\mu)}/J$ in G_r, that is through the variables

$$x_1,\ldots,x_n;r_1,\ldots,r_k \quad .$$

Denote the rank of the Jacobian matrix of the $m(n-k) \geqslant k$ expressions in (V.3) with respect to the $r_{\varkappa}$,

$$\left(\frac{\partial (u_{\lambda}^{(\mu)})}{\partial (r_{\varkappa})} \right) \qquad (V.4)$$

by $\varrho^* \leqslant k$.

5.2. We are going to show that the number $m(n-k)$ of the equations (V.3) cannot exceed ϱ^*,

$$m(n-k) \leqslant \varrho^* \quad . \qquad (V.5)$$

For otherwise the $r_{\varkappa}$ could be eliminated using certain ϱ^* different equations

$$u_{\lambda_\sigma}^{(\mu_\sigma)} = 0 \quad (\sigma=1,\dots,\varrho^*) \quad . \qquad (V.6)$$

A $u_{\lambda_0}^{(\mu_0)}$ different from all $u_{\lambda_\sigma}^{(\mu_\sigma)}$ in (V.6) becomes then a not identically satisfied differential equation, as the $u_{\lambda_\sigma}^{(\mu_\sigma)}$ in (V.6) <u>do not depend</u> on $p_{\lambda_0 \mu_0}$. Since this is impossible, (V.5) is proved and it follows, by (III.40), $m(n-k) \leqslant \varrho^* \leqslant k \leqslant m(n-k)$ and thence

$$m(n-k) = \varrho^* = k \quad .$$

We see that the matrix (V.4) is a square, $k \times k$, non-singular matrix, and from $m(n-k)=k$ follows (V.1).

5.3. It follows that the expressions of $r_{\varkappa}$ in the x_{ν} and $p_{\nu\mu}$ can be obtained solving the equations (V.3) with respect to r_1, $r_2,\dots,r_k$. Obviously a completely analogous result holds for the expressions of the $s_\varkappa$ in terms of the y_ν and $q_{\varkappa\mu}$, as in virtue of (IV.14) and (IV.15) all $v_\lambda^{(\mu)}$ vanish then and only then if all $u_\lambda^{(\mu)}$ vanish, and then the rank of the k$\times$k-matrix

$$(V.7) \qquad K_y^* = \begin{pmatrix} Y'_{1r_1} & \cdots & Y'_{1r_k} & q_{11} & \cdots & q_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ Y'_{nr_1} & \cdots & Y'_{nr_k} & q_{n1} & \cdots & q_{nm} \end{pmatrix}$$

is k.

We denote the determinant corresponding to the square matrix (V.4) by Δ_u.

5.4. Assume on the other hand that we have given a priori the transformation T* by the relations (II.7), (II.10), (II.13) and (II.14) where <u>all functions</u> X_ν, Y_ν, $R_\varkappa$, $S_\varkappa$ <u>are indefinitely often differentiable</u>.

Then, <u>assuming that the Jacobian</u>, Δ_u, <u>of the</u> $u_\lambda^{(\mu)}$ <u>with respect to the</u> r_ϱ <u>does not vanish</u>, we can solve the equations (V.3) in the form

$$(V.8) \quad \bar{u}_\lambda^{(\mu)}(p_{\nu\mu}, x_\nu, r_\nu) := p_{\lambda\mu} - \frac{1}{J}\sum_{\varkappa=1}^{k} f_{\lambda\varkappa}^{(\mu)} p_{\varkappa\mu} = 0 \quad (\mu=1,\dots,m; \lambda=k+1,\dots,n)$$

with respect to the $r_\varkappa$ in a neighbourhood of a point B_o and obtain the expressions

$$(V.9) \qquad r_\varkappa^* = \bar{R}_\varkappa(x_\nu, p_{\nu\mu}) \qquad (\varkappa=1,\dots,k)$$

of the $r_\varkappa$ in terms of the x_ν and $p_{\nu\mu}$. Putting these expressions into (II.7) we obtain expressions for the y_ν in function of the x_ν, $p_{\nu\mu}$,

$$Y_\nu(x_\nu,\bar{R}_\varrho(x_\nu,p_{\nu\mu})) \rightarrow y_\nu \; = Y^*_\nu(x_\nu,p_{\nu\mu})$$

corresponding to (II.2a).

Further, putting the $\bar{R}_\varrho$ for the r_ϱ in (II.14) we obtain the expressions

$$\text{(V.10)} \qquad s^*_\sigma(x_\nu,r_\varrho) = \bar{S}_\sigma(x_\nu,p_{\nu\mu}) \quad (\sigma=1,\ldots,k)$$

where the functions $s^*_\sigma(x_\nu, r_\varrho)$, $\bar{S}_\sigma(x_\nu,p_{\nu\mu})$ have in a neighbourhood of B_o the values of the s_σ corresponding to the transformation T*.

5.5. We have now to show the existence of the representations of the s_σ as functions of the $q_{11},\ldots,q_{nm}$. Expressing in (V.8) the quotients $f^{(\mu)}_{\lambda\varkappa}/J$ in terms of the y_ν and s_σ we obtain

$$\text{(V.11)} \qquad \bar{u}^{(\mu)}_\lambda(p_{\nu\mu};y_\nu,s_\sigma) := p_{\nu\mu} - \frac{1}{J(y_\nu,s_\varkappa)}\sum_{\varkappa=1}^{k} f^{(\mu)}_{\lambda\varkappa}(y_\nu,s_\sigma)\, p_{\varkappa\mu} \;. \qquad (\lambda=k+1,\ldots,n) \;.$$

And all these forms vanish in the neighbourhood of B_o.

But now it follows from (IV.14) that all $v^{(\mu)}_\lambda$,

$$\text{(V.12)} \qquad v_\lambda := Kq_{\lambda\mu} - \sum_{\varkappa=1}^{k} g^{(\mu)}_{\lambda\varkappa}(y_\nu,s_\sigma)q_{\varkappa\mu} = \;0 \quad (\lambda=k+1,\ldots,n)$$

vanish for $s_\sigma = \bar{S}_\sigma$ in a neighbourhood of B_o. If we now assume that <u>the Jacobian</u>,

$$\text{(V.13)} \qquad \Delta_u := \frac{\partial(v^{(\mu)}_\lambda)}{\partial(s_\varkappa)} \;,$$

of the $v_{\lambda}^{(\mu)}$ with respect to the s_{σ} does not vanish in the neighbourhood of B_o, it follows that the $\bar{S}_{\sigma}$ are unique solutions of the equations (V.12) in a convenient neighbourhood and can therefore be represented in terms of the $q_{11},\dots,q_{nm}$.

Introducing these expressions into (II.9) we obtain (II.2b), and the inversibility of the transformation T obtained in this way follows from the assumed inversibility of T*.

5.6. We have still to prove that the $r_{\varkappa}^{*}$ are independent as functions of the $p_{\nu\mu}$ and that the $s_{\varkappa}^{*}$ are independent as functions of the $q_{\nu\mu}$.

But it follows from (V.8) that with $\lambda=k+1,\dots,n$ and $\mu=1,\dots,m$,

$$\text{(V.14)} \qquad \frac{\partial(\bar{u}_{\lambda}^{(\mu)})}{\partial(p_{\lambda\mu})} = \pm 1 \quad ,$$

where the determinant is for variable λ and μ of the order $k=m(n-k)$. On the other hand, if we put with $\lambda=k+1,\dots,n$ and $\mu=1,\dots,m$,

$$\text{(V.15)} \qquad \Delta_1 := \frac{\partial(r_{\varkappa})}{\partial(p_{\lambda\mu})} \quad ; \quad \Delta := \frac{\partial(\bar{u}_{\lambda}^{(\mu)})}{\partial(r_{\varkappa})} = \Delta_u/J^n \neq 0 \ ,$$

both determinants are of the order k and the inequality $\Delta \neq 0$ follows from the assumption. But by (V.14) and (V.15) $\Delta\Delta_1=\pm 1$, $\Delta_1 \neq 0$. The independence of the $r_{\varkappa}$ is proved and the independence of the $s_{\varkappa}$ follows by symmetry.

5.7. We have finally to prove that the $r_{\varkappa}$ and $s_{\varkappa}$ are absolutely invariant with respect to the linear transformations of the T_{μ}, that is to say that for the $r_{\varkappa}$ and $s_{\varkappa}$ the relations

$$\sum_{\nu=1}^{n} p_{\nu\mu}\frac{\partial w}{\partial p_{\nu\mu'}} = 0 \quad (\mu,\mu'=1,\dots,m)$$

are linear combinations of the relations

$$\sum_{\nu=1}^{n} X'_{\nu r_{\varkappa}} \frac{\partial w}{\partial p_{\nu\mu'}} = 0 \quad .$$

This signifies that the relations hold:

$$p_{\nu\mu} = \sum_{\varkappa=1}^{k} m_{\varkappa}^{(\mu)} X'_{\nu r_{\varkappa}} \qquad (\mu=1,\ldots,m) \qquad .$$

But these relations follow from the fact that K^*_x has the rank k in virtue of the relations (V.8).

5.8. We observe finally that the special choice of the basis forms $u_\lambda^{(\mu)}$ and $v_\lambda^{(\mu)}$ is not essential. Indeed, if an arbitrary basis for the u-forms is given, obviously their Jacobian with respect to the $r_\varkappa$ does not vanish then and only then when this is true for the $u_\lambda^{(\mu)}$, and similar situation prevails for the v-forms and $s_\varkappa$. We can therefore obtain the $r_\varkappa$, equating to 0 a complete set of the basis elements of the u-forms, and similarly for the $s_\varkappa$ and the v-forms.

5.9. We can summarize our results in the following statement:

<u>Assume given a transformation</u> T* <u>with (II.7), (II.10), (II.13) and (II.14), where all functions occurring in these formulas have derivatives of all orders in certain domains corresponding by</u> T*. <u>Assume that</u> d=m <u>and that</u> $JK \neq 0$.

1) <u>If</u> T* <u>is a characteristic transformation of a reversible</u> T, <u>given by (II.2a), (II.2b), then both Jacobians</u> Δ_u, Δ_v <u>do not vanish with indeterminants</u> $p_{\nu\mu}$, $q_{\nu\mu}$ <u>and the expressions of the</u> $r_\varkappa^*(x_\nu, p_{\nu\mu})$, $s_\varkappa^*(y_\nu, q_{\nu\mu})$ <u>satisfy (V.8) and (V.12)</u>.

2) <u>If the functions</u> X_ν, Y_ν, $R_\varkappa$, $S_\varkappa$ <u>defining</u> T* <u>satisfy (V.4) and (V.13) then</u> T* <u>is a characteristic transformation of a reversible transformation</u> T, <u>and the expressions of the</u> $r_\varkappa^*$, $s_\varkappa^*$ <u>in</u> $p_{\nu\mu}$, $q_{\nu\mu}$ <u>are obtained, uniquely in convenient neighbourhoods, from the equations (V.8) and (V.13)</u>.

5.10. Example.

Assume

$$n = 6 \ , \ k = 4 \ , \ m = 2 \qquad \text{(V.16)}$$

and put for T^*:

$$\left\{\begin{array}{l} X_{\varkappa} = Y_{\varkappa} = r_{\varkappa} = s_{\varkappa} \quad (\varkappa=1,\dots,4) \ , \\ X_5 = y_5 + \frac{1}{2}(s_1^2+s_2^2) \ , \ X_6 = y_6 + \frac{1}{2}(s_3^2+s_4^2) \ , \\ Y_5 = x_5 - \frac{1}{2}(r_1^2+r_2^2) \ , \ Y_6 = x_6 + \frac{1}{2}(r_3^2+r_4^2) \ . \end{array}\right. \qquad \text{(V.17)}$$

Then

$$u_{\lambda}^{(\mu)} = \left| \begin{array}{ccccc} p_{\lambda\mu} & p_{1\mu} & \cdots & & p_{4\mu} \\ X'_{\lambda s_1} & & & & \\ \cdot & & & & \\ \cdot & & U & & \\ \cdot & & & & \\ X'_{\lambda s_4} & & & & \end{array} \right| \qquad (\lambda=5,6;\mu=1,2) \qquad \text{(V.18)}$$

where U is the Unity Matrix of order 4, and the $v_{\lambda}^{(\mu)}$ are obtained replacing in the $u_{\lambda}^{(\mu)}$ the $s_{\varkappa}$ with the $r_{\varkappa}$ and the $p_{\lambda\mu}$ with the $q_{\lambda\mu}$. Developping we obtain

$$\begin{aligned} u_5^{(\mu)} &= p_{5\mu} - p_{1\mu}X'_{5s_1} - p_{2\mu}X'_{5s_2} = p_{5\mu} - p_{1\mu}s_1 - p_{2\mu}s_2 \ , \\ u_6^{(\mu)} &= p_{6\mu} - p_{3\mu}X'_{6s_3} - p_{4\mu}X'_{6s_4} = p_{6\mu} - p_{3\mu}s_3 - p_{4\mu}s_4 \ , \\ v_5^{(\mu)} &= q_{5\mu} + q_{1\mu}r_1 + q_{2\mu}r_2 \ , \\ v_6^{(\mu)} &= q_{6\mu} + q_{3\mu}r_3 + q_{4\mu}r_4 \end{aligned} \qquad \text{(V.19)}$$

and, solving the equations $u_{\lambda}^{(\mu)} = 0$, $v_{\lambda}^{(\mu)} = 0$,

$$r_1 = s_1 = \frac{p_{51}p_{22} - p_{52}p_{21}}{p_{11}p_{22} - p_{12}p_{21}} = - \frac{q_{51}q_{22} - q_{52}q_{21}}{q_{11}q_{22} - q_{12}q_{21}} ,$$

$$r_2 = s_2 = \frac{p_{11}p_{52} - p_{12}p_{51}}{p_{11}p_{22} - p_{12}p_{21}} = - \frac{q_{11}q_{52} - q_{12}q_{51}}{q_{11}q_{22} - q_{12}q_{21}} ,$$

(V.20)

$$r_3 = s_3 = \frac{p_{61}p_{43} - p_{62}p_{41}}{p_{31}p_{42} - p_{32}p_{41}} = - \frac{q_{61}q_{42} - q_{62}q_{41}}{q_{31}q_{42} - q_{32}q_{41}} ,$$

$$r_4 = s_4 = \frac{p_{31}p_{62} - p_{32}p_{61}}{p_{31}p_{42} - p_{32}p_{41}} = - \frac{q_{31}q_{62} - q_{32}q_{61}}{q_{31}q_{42} - q_{32}q_{41}} .$$

Eliminating $r_{\varkappa}$ and $s_{\varkappa}$ the invertible transformation T belonging to T* is immediately obtained.

VI. Determinantal Forms.

6.1. We define multiple indices $\gamma, \delta, \varepsilon$ of order i as

$$\text{(VI.1)}\qquad \begin{aligned} \gamma &:= \{\nu_1, \nu_2, \ldots, \nu_i\} && (1 \leq \nu_1 < \nu_2 < \ldots < \nu_i \leq n)\ , \\ \delta &:= \{\mu_1, \mu_2, \ldots, \mu_i\} && (1 \leq \mu_1 < \mu_2 < \ldots < \mu_i \leq m)\ , \\ \varepsilon &:= \{\lambda_1, \lambda_2, \ldots, \lambda_i\} && (k+1 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_i \leq n)\ , \end{aligned}$$

and put

$$\text{(VI.2)}\qquad \left(\frac{\partial \gamma}{\partial \delta}\right)_p := \begin{vmatrix} p_{\nu_1 \mu_1} & \cdots & p_{\nu_1 \mu_i} \\ \vdots & & \vdots \\ p_{\nu_i \mu_1} & \cdots & p_{\nu_i \mu_i} \end{vmatrix} .$$

We write further $\left(\frac{\partial \gamma}{\partial \delta}\right)_q$ for the determinant formed with the $q_{\nu\mu}$ correspondingly to (VI.2).

6.2. Assume now a fixed characteristic transformation T* and consider the general expression

$$\text{(VI.3)}\qquad \sum_{\gamma,\delta} T_{\gamma\delta} \left(\frac{\partial \gamma}{\partial \delta}\right)_p ,$$

where the $T_{\gamma\delta}$ are functions from W* and the summation extends over all γ and δ as defined in (VI.1).

If the expression can be represented in terms of the x_ν, $r_\varkappa$ and $q_{\nu\mu}$, we call it a determinantal form of order i. We have then

$$\text{(VI.4)}\qquad \sum_{\gamma,\delta} T_{\gamma\delta} \left(\frac{\partial \gamma}{\partial \delta}\right)_p = \sum_{\gamma,\delta} \hat{T}_{\gamma\delta} \left(\frac{\partial \gamma}{\partial \delta}\right)_q ,$$

where the $\hat{T}_{\gamma\delta}$ belong to W*.

If in such a form only the $T_{\gamma\delta}$ corresponding to a fixed δ are different from zero, it will be called a <u>single determinantal form</u>.

In exactly the same way we define the determinantal forms and single determinantal forms belonging to the $q_{\nu\mu}$. Obviously in (VI.4) the right-handed sum is a determinantal form of order i belonging to the $q_{\nu\mu}$.

6.3. Observe that the relation (VI.4) reduces to the requirement that the left-handed expression in it has a U property in the sense of chapter 3. Indeed the determinants (VI.2), if expressed through the $q_{\nu\mu}$, becomes a linear combination of the $\left(\frac{\partial\chi}{\partial\xi}\right)_q$ with coefficients from W*. Therefore, for a determinantal form we obtain the differential equations (III.7) belonging to $\mu=\mu_1,\dots,\mu_i$.

As in the case of u-forms the differential equations (III.7) depend only on the functions x_ν in (II.10), therefore it is reasonable to define an expression of the type (VI.3) as a <u>unilateral determinantal form of order</u> i, if it satisfies all equations (III.7).

6.4. Our first problem is to find a <u>linear basis</u> for the unilateral determinantal forms (VI.3). In particular, if we consider in (VI.4) on the left the aggregate of the terms depending on a fixed $\delta=\delta_1$, this aggregate depends on the right only on the $\left(\frac{\partial\chi}{\partial\delta}\right)_q$ corresponding to the same δ_1 and represents therefore a single determinantal form with a fixed $\delta=\delta_1$. Obviously we have only to consider, for an arbitrary δ,

$$\text{(VI.5)} \qquad D_\delta := \sum_\gamma T_{\gamma\delta}\left(\frac{\partial\chi}{\partial\delta}\right)_p .$$

In order to define convenient elements of such a basis, we return to the expression $u_\lambda^{(\mu)}$ in (IV.8) and rewrite it here:

$$(\mathrm{VI.6})\qquad u_{\lambda}^{(\mu)} = \begin{vmatrix} p_{\lambda\mu} & p_{1\mu} & \cdots & p_{k\mu} \\ X'_{\lambda s_1} & X'_{1s_1} & \cdots & X'_{ks_1} \\ \vdots & \vdots & & \vdots \\ X'_{\lambda s_k} & X'_{1s_k} & \cdots & X'_{ks_k} \end{vmatrix} \quad (\lambda=k+1,\dots,n)\ .$$

Choosing then multiple indices δ,ε of order i, as given by (VI.1), consider the expression

$$(\mathrm{VI.7})\qquad P_{\delta}^{(\varepsilon)} := \begin{vmatrix} u_{\lambda_1}^{(\mu_1)} & u_{\lambda_1}^{(\mu_2)} & \cdots & u_{\lambda_1}^{(\mu_i)} \\ u_{\lambda_2}^{(\mu_1)} & u_{\lambda_2}^{(\mu_2)} & \cdots & u_{\lambda_2}^{(\mu_i)} \\ \vdots & \vdots & & \vdots \\ u_{\lambda_i}^{(\mu_1)} & u_{\lambda_i}^{(\mu_2)} & \cdots & u_{\lambda_i}^{(\mu_i)} \end{vmatrix} \quad \begin{pmatrix} \varepsilon := (\lambda_1,\dots,\lambda_i) \\ \delta := (\mu_1,\dots,\mu_i) \end{pmatrix}.$$

We are going to show that these expressions are single unilateral determinantal forms of order i belonging to δ.

Form the determinant of order k+i:

$$(\mathrm{VI.8})\qquad G_{\delta}^{(\varepsilon)} := \begin{vmatrix} X'_{1s_1} & \cdots & X'_{1s_k} & p_{1\mu_1} & \cdots & p_{1\mu_i} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{ks_1} & \cdots & X'_{ks_k} & p_{k\mu_1} & \cdots & p_{k\mu_i} \\ X'_{\lambda_1 s_1} & \cdots & X'_{\lambda_1 s_k} & p_{\lambda_1\mu_1} & \cdots & p_{\lambda_1\mu_i} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{\lambda_i s_1} & \cdots & X'_{\lambda_i s_k} & p_{\lambda_i\mu_1} & \cdots & p_{\lambda_i\mu_i} \end{vmatrix} \ .$$

6.5. The relation between $G_{\delta}^{(\varepsilon)}$ and $P_{\delta}^{(\varepsilon)}$ as given by Sylvester's theorem is

$$J^{i-1}G_{\delta}^{(\varepsilon)} = P_{\delta}^{(\varepsilon)} \qquad \text{(VI.9)}$$

Since J does not depend explicitly on the $p_{\nu\mu}$, we obtain, developping the determinant $G_{\delta}^{(\varepsilon)}$ in subdeterminants of order i taken from the last i columns, a representation of $G_{\delta}^{(\varepsilon)}$ in the form (VI.5) for a fixed δ and thence a similar representation of $P_{\delta}^{(\varepsilon)}$.

On the other hand each of the elements $u_{\lambda\varkappa}^{(\mu\sigma)}$ of $P_{\delta}^{(\varepsilon)}$ satisfies the relations (III.7). Therefore the determinant $P_{\delta}^{(\varepsilon)}$ is also a single unilateral determinantal form belonging to δ.

Further it follows that $P_{\delta}^{(\varepsilon)}$, if expressed through the y_{ν}, $s_{\varkappa}$ and the $q_{\nu\mu}$, is equal to a single determinantal form in the $q_{\nu\mu}$ belonging to the same δ.

The determinants $G_{\delta}^{(\varepsilon)}$ in (VI.8) are subdeterminants of the fixed matrix (III.6), containing the fixed k×k-subdeterminant J. The rank $k+\varrho$ of the matrix (III.6) has been computed in (III.7). Using this value it follows that all determinants $G_{\delta}^{(\varepsilon)}$ corresponding to an $i>\varrho$ vanish, while for each $i\leq\varrho$ there exist non-vanishing $P_{\delta}^{(\varepsilon)}$.

We are going to show that the $P_{\delta}^{(\varepsilon)}$ are a basis for single determinantal forms belonging to δ.

6.6. We begin by deriving a convenient representation for the determinant (VI.2). This will be the formula (VI.14).

Solving the relation (VI.6) for $\lambda>k$ with respect to $p_{\lambda\mu}$ we obtain

$$Jp_{\lambda\mu} = u_{\lambda}^{(\mu)} + S_{\lambda}^{(\mu)} \quad ,$$

with

$$S_\lambda^{(\mu)} \quad := \quad - \begin{vmatrix} 0 & p_{1\mu} & \dots & p_{k\mu} \\ X'_{\lambda s_1} & X'_{1s_1} & \dots & X'_{ks_1} \\ \vdots & \vdots & & \vdots \\ X'_{\lambda s_k} & X'_{1s_k} & \dots & X'_{ks_k} \end{vmatrix} ,$$

$$\text{(VI.10)} \qquad S_\lambda^{(\mu)} = -\sum_{\varkappa=1}^{k} A_{\lambda\varkappa} p_{\varkappa\mu} \qquad (\lambda > k) \quad , \quad A_{\lambda\varkappa} \in W^* \; .$$

On the other hand, if $\nu \leq k$ we can write

$$S_\nu^{(\mu)} \quad := \quad Jp_{\nu\mu} \qquad ,$$

so that these $S_\nu^{(\mu)}$ are also linear forms in the $p_{\varkappa\mu}$ ($\varkappa=1,\dots,k$). Therefore, we can write generally

$$\text{(VI.11)} \qquad Jp_{\nu\mu} = \begin{cases} S_\nu^{(\mu)} & (\nu \leq k) \\ u_\nu^{(\mu)} + S_\nu^{(\mu)} & (\nu > k) \end{cases} ,$$

where the expressions $S_\nu^{(\mu)}$ are in both cases linear forms in the $p_{1\mu},\dots,p_{k\mu}$ with coefficients from W^* and can be written in the form (VI.10).

6.7. In the following part of this chapter the $i\times i$-determinants are usually represented by writing out the general column with the index μ_ψ, where $\psi=1,\dots,i$.

For the indices sequence γ in (VI.1) an $h=0,1,\dots,i$ is uniquely determined by the inequality

$$\nu_h \leq k < \nu_{h+1} \qquad , \qquad h=0,\dots,i \quad ,$$

where h=0 corresponds to $\nu_1 > k$. We denote then the elements of the partial sequence of γ, $\{\nu_{h+1}, \nu_{h+2}, \dots, \nu_i\}$, in the same order by $\lambda_1, \lambda_2, \dots, \lambda_{i-h}$, as long as $\nu_i > k$.

Then multiplying the determinant

$$\text{(VI.12)} \qquad \left(\frac{\partial\gamma}{\partial\delta}\right)_\rho := \begin{vmatrix} p_{\nu_1\mu_\psi} \\ \vdots \\ p_{\nu_i\mu_\psi} \end{vmatrix} \qquad (\psi=1,\dots,i)$$

by J^i we can write, using (VI.11),

$$J^i\left(\frac{\partial\gamma}{\partial\delta}\right)_\rho = \begin{vmatrix} s^{(\mu_\psi)}_{\nu_1} \\ \vdots \\ s^{(\mu_\psi)}_{\nu_h} \\ u^{(\mu_\psi)}_{\lambda_1} + s^{(\mu_\psi)}_{\lambda_1} \\ \vdots \\ u^{(\mu_\psi)}_{\lambda_{i-h}} + s^{(\mu_\psi)}_{\lambda_{i-h}} \end{vmatrix} \begin{pmatrix} \nu_1 < \dots < \nu_h \leq k \ ; \ h \geq 0 \ ; \\ k+1 \leq \lambda_1 < \dots < \lambda_{i-h} \leq n \end{pmatrix}.$$

Observe that, for fixed δ and γ, both the sequence of the $\nu, \nu_1, \dots, \nu_h, \lambda_1, \dots, \lambda_{i-h}$ and the sequence of the μ_ψ corresponding to δ are fixed.

6.8. Decompose here the determinant according to its rows and reorder the rows so as to bring all rows containing the $s^{(\mu_\psi)}_\nu$ first. We obtain

$$\text{(VI.13)} \qquad J^i\left(\frac{\partial\gamma}{\partial\delta}\right)_\rho = \sum \pm \begin{vmatrix} s^{(\mu\nu)}_{\sigma_1} \\ \vdots \\ s^{(\mu\nu)}_{\sigma_g} \\ u^{(\mu\nu)}_{\lambda_1} \\ \vdots \\ u^{(\mu\nu)}_{\lambda_{i-g}} \end{vmatrix} \quad \begin{pmatrix} \sigma_1 < \ldots < \sigma_g \leq n \; ; \\ k+1 \leq \lambda_1 < \ldots < \lambda_{i-g} \leq n \end{pmatrix},$$

where the right-hand algebraic sum consists of 2^{i-h} terms and, of course, g is $\geq h$. Observe that in (VI.13) the σ- and λ-sequences vary from one of the 2^{i-h} determinants to another.

Observe that in the right-hand sum of (VI.13) the term consisting only of the $u^{(\mu\nu)}_\lambda$ occurs then and only then if $h=0$, that is $\nu_1 > k$, and then this term has in (VI.13) the plus sign. Introducing

$$\varepsilon_0 = \begin{cases} 1 & (\nu_1 \geq k+1) \\ 0 & (\nu_1 \leq k) \end{cases},$$

and using (VI.7) we can therefore rewrite (VI.13) as

$$\text{(VI.14)} \qquad J^i\left(\frac{\partial\gamma}{\partial\delta}\right)_\rho = \varepsilon_0 P^{(\gamma)}_\delta + \sum_{g \geq 1} \pm \begin{vmatrix} s^{(\mu\nu)}_{\sigma_1} \\ \vdots \\ s^{(\mu\nu)}_{\sigma_g} \\ u^{(\mu\nu)}_{\lambda_1} \\ \vdots \\ u^{(\mu\nu)}_{\lambda_{i-g}} \end{vmatrix},$$

where $\sigma_1,\dots,\sigma_h$ coincide with $\nu_1,\dots,\nu_h$, while all further $\sigma_{h+1},\dots,\sigma_g$ are $>k$.

If we now multiply (VI.14) by $T_{\gamma\delta}$ and sum over all γ, we obtain on the left J^iD_δ. As to the right-hand expression, obviously, the first right-hand terms in (VI.14) only occur if γ is an ε so that we obtain here the sum $\sum_\varepsilon T_{\varepsilon\delta}P_\delta^{(\varepsilon)}$ taken over all multiple indices ε of order i. We can therefore write

$$\text{(VI.15)}\qquad J^iD_\delta = \sum_\varepsilon T_{\varepsilon\delta}P_\delta^{(\varepsilon)} + \sum_{\substack{\gamma\\ g\geq 1}} \pm T_{\gamma\delta}\begin{vmatrix} S_{\sigma_1}^{(\mu_\psi)} \\ \vdots \\ S_{\sigma_g}^{(\mu_\psi)} \\ u_{\lambda_1}^{(\mu_\psi)} \\ \vdots \\ u_{\lambda_{i-g}}^{(\mu_\psi)} \end{vmatrix} ,$$

where the right-hand expression is a polynomial in the $p_{\varkappa\mu_\psi}$ ($\psi=1,\dots,i;\varkappa=1,\dots,k$) and $u_\lambda^{(\mu_\psi)}$ ($\psi=1,\dots,i;\lambda=k+1,\dots,n$), linear for each $\psi=1,\dots,i$.

6.9. We consider the expression in (VI.15) as function of $p_{1\mu_1},p_{2\mu_1},\dots,p_{k\mu_1}$ and of the $u_\lambda^{(\mu_1)}$. Obviously we can write

$$\text{(VI.16)}\qquad J^iD_\delta = \sum_{\varkappa=1}^{k} B_\varkappa p_{\varkappa\mu_1} + \sum_{\lambda=k+1}^{n} C_\lambda u_\lambda^{(\mu_1)} + U \quad ,$$

where $B_\varkappa$, C_λ and U no longer contain $p_{1\mu_1},\dots,p_{k\mu_1}$, but are polynomials in the $p_{\varkappa\mu_\psi}$ ($\psi\neq 1,\varkappa=1,\dots,k$) and in the $u_\lambda^{(\mu_\psi)}$ ($\psi\neq 1$) with coefficients from W^*, linear for each fixed $\nu \neq 1$.

Now, observe that in (VI.16) the differential equations (III.7) <u>for</u> $\mu = \mu_1$ are satisfied for $J^i D_\delta$, U and the sum $\sum_{\lambda=k+1}^{r} C_\lambda u_\lambda^{(\mu_1)}$. Thence, they are also satisfied for the sum

$$\text{(VI.17)} \qquad \sum_{\varkappa=1}^{k} B_\varkappa p_{\varkappa\mu_1} \quad .$$

Reordering (VI.17) in products of the $p_{\varkappa\mu_\nu}$ ($\nu \neq 1$), we can write

$$\text{(VI.18)} \qquad \sum_{\varkappa=1}^{k} B_\varkappa p_{\varkappa\mu_1} = \sum_{\sigma} P_\sigma \sum_{\varkappa=1}^{k} B_\varkappa^{(\sigma)} p_{\varkappa\mu_1} \quad ,$$

where P_σ are different products of the $p_{\varkappa\mu}$ ($\mu \neq 1$) ordered in some way, and the coefficients $B_\varkappa^{(\sigma)}$ belong to W^*. Therefore for each P_σ which actually occurs in (VI.18) the corresponding sum

$$\sum_{\varkappa=1}^{k} B_\varkappa^{(\sigma)} p_{\varkappa\mu_1} \quad , \qquad B_\varkappa^{(\sigma)} \in W^* \quad ,$$

satisfies for $\mu = \mu_1$ the equations (III.7) and is therefore, being linear, a single u-form containing only $p_{1\mu_1}, \ldots, p_{k\mu_1}$. Such a form, as was proved in chapter IV, must vanish identically. We see that the sum (VI.17) identically vanishes. But $p_{1\mu_1}, \ldots, p_{k\mu_1}$ in (VI.16) occur only in the sum (VI.17). We see that D_δ is independent of $p_{1\mu_1}, \ldots, p_{k\mu_1}$.

6.10. Proceeding in the same way, for each μ_ν, we see that the right-hand expression in (VI.15) is independent of all $p_{\varkappa\mu_\nu}$ ($\varkappa = 1, \ldots, k$). Putting then all these $p_{\varkappa\mu_\nu} = 0$, we obtain from (VI.15),

$$J^i D_\delta = \sum_\varepsilon T_{\varepsilon\delta} P_\delta^{(\varepsilon)} \quad ,$$

(VI.19)
$$D_\delta = J^{-i} \sum_\varepsilon T_{\varepsilon\delta} P_\delta^{(\varepsilon)}$$

and we see that D_δ can indeed be written as a linear expression in the $P_\delta^{(\varepsilon)}$ with coefficients from Γ_y. Further, we find in (VI.19) an explicit rule for the representation of D_δ through the $P_\delta^{(\varepsilon)}$:

Throw away in (VI.5) all terms corresponding to γ with $\gamma_1 \leq k$ and replace, since the remaining sequences γ are also sequences ε, each $\left(\frac{\partial\varepsilon}{\partial\delta}\right)_\varphi$ by $P_\delta^{(\varepsilon)} J^{-i}$.

6.11. We show now that it does not exist a linear homogeneous relation between the $P_\delta^{(\varepsilon)}$ for the order i with coefficients depending only on the y_ν and $s_\varkappa$ for independent variables y_ν and $s_\varkappa$,

(VI.20)
$$\sum_{\varepsilon,\delta} T_{\varepsilon\delta} P_\delta^{(\varepsilon)} = 0 \quad .$$

Indeed, if we make all $p_{\varkappa\mu}$ ($\varkappa=1,\ldots,k;\mu=1,\ldots,m$) equal to zero, we obtain from (VI.20)

(VI.21)
$$J^i \sum_{\varepsilon,\delta} T_{\varepsilon\delta} \begin{vmatrix} p_{\lambda_1\mu_1} & \cdots & p_{\lambda_1\mu_i} \\ \vdots & & \vdots \\ p_{\lambda_i\mu_1} & \cdots & p_{\lambda_i\mu_i} \end{vmatrix} = 0 \quad .$$

For an arbitrary $\varepsilon_o = \{\lambda_1 < \ldots < \lambda_i\}$ and $\delta_o = \{\mu_1 < \ldots < \mu_i\}$ attribute to the corresponding elements $p_{\lambda_1\mu_1}, \ldots, p_{\lambda_1\mu_i}, \ldots, p_{\lambda_i\mu_1}, \ldots, p_{\lambda_i\mu_i}$ the weight 1 and to all other $p_{\nu\mu}$ the weight 0. Then the terms of the weight i occur only in the term of (VI.21) corresponding to $T_{\varepsilon_0\delta_0}$, while all other terms of (VI.21) have weights $<$i. Therefore it follows $T_{\varepsilon_0\delta_0}=0$ and since ε_0 and δ_0 were arbitrarily chosen, we see that all coefficients $T_{\varepsilon\delta}$ in (VI.21) vanish.

6.12. We assume now that the relations (II.10) and (II.13) hold together with (II.7) and (II.11). We define similarly as in (VI.7) for $P_\delta^{(\varepsilon)}$,

$$\text{(VI.22)} \qquad Q_\delta^{(\varepsilon)} := \begin{vmatrix} v_{\lambda_1}^{(\mu_1)} & \cdots & v_{\lambda_1}^{(\mu_i)} \\ \vdots & & \vdots \\ v_{\lambda_i}^{(\mu_1)} & & v_{\lambda_i}^{(\mu_i)} \end{vmatrix} .$$

It has been proved with the formulas (IV.14) and (IV.16) that the $u_\lambda^{(\mu)}$ and the $v_\lambda^{(\mu)}$ are connected by a non-singular linear transformation of order m(n-k). It is then obvious that the determinants of the order i, $P_\delta^{(\varepsilon)}$ and $Q_\delta^{(\varepsilon)}$, are also connected by non-singular linear transformations the coefficients of which are expressible through the determinants formed by the A_λ and the B_λ in (IV.13) and (IV.16).

Therefore, all $P_\delta^{(\varepsilon)}$ of the order i vanish then and only then when all $Q_\delta^{(\varepsilon)}$ of the same order vanish. This signifies that both matrices K_x^* and K_y^* have the same rank $k+\varrho$.

The expressions $Q_\delta^{(\varepsilon)}$ correspond to the subdeterminants of the matrix K_y^* in (V.7),

$$
(VI.23) \qquad H_{\delta}^{(\varepsilon)} := \begin{vmatrix} Y'_{1r_1} & \cdots & Y'_{1r_k} & q_{1\mu_1} & \cdots & q_{1\mu_i} \\ \vdots & & \vdots & \vdots & & \vdots \\ Y'_{kr_1} & \cdots & Y'_{kr_k} & q_{k\mu_1} & \cdots & q_{k\mu_i} \\ Y'_{\lambda_1 r_1} & \cdots & Y'_{\lambda_1 r_k} & q_{\lambda_1\mu_1} & \cdots & q_{\lambda_1\mu_i} \\ \vdots & & \vdots & \vdots & & \vdots \\ Y'_{\lambda_i r_1} & \cdots & Y'_{\lambda_i r_k} & q_{\lambda_i\mu_1} & \cdots & q_{\lambda_i\mu_i} \end{vmatrix}
$$

and are connected with them by the relation corresponding to (VI.9),

$$
(VI.24) \qquad K^{i-1} H_{\delta}^{(\varepsilon)} = Q_{\delta}^{(\varepsilon)} \quad .
$$

6.13. By the relations (VI.9) and (VI.24) it follows further that the $G_{\delta}^{(\varepsilon)}$ and the $H_{\delta}^{(\varepsilon)}$, again, are connected by non-singular linear transformations the coefficients of which belong to W*, for a fixed i:

$$
(VI.25) \qquad G_{\delta}^{(\varepsilon)} = \sum_{\varepsilon',\delta'} \Omega_{\delta,\delta'}^{(\varepsilon,\varepsilon')} H_{\delta'}^{(\varepsilon')} \quad .
$$

It follows further from the relations (VI.9) and (VI.24) that the relation (VI.25) holds also between the $P_{\delta}^{(\varepsilon)}$ and the $Q_{\delta}^{(\varepsilon)}$,

$$
(VI.26) \qquad P_{\delta}^{(\varepsilon)} = \sum_{\varepsilon',\delta'} \Omega_{\delta,\delta'}^{(\varepsilon,\varepsilon')} Q_{\delta'}^{(\varepsilon')} \quad .
$$

6.14. We will have in particular to do with the case i=m. In this case δ becomes

$$
\delta_0 := \{1,2,\ldots,m\} \quad ,
$$

and we put

$$P_{\delta_0}^{(\varepsilon)} =: P_\varepsilon \quad , \quad Q_{\delta_0}^{(\varepsilon)} =: Q_\varepsilon \quad .$$

Then the relation (VI.26) can be written as

$$\text{(VI.27)} \qquad P_\varepsilon = \sum_{\varepsilon'} \Omega_{\varepsilon,\varepsilon'} Q_{\varepsilon'} \qquad (i=n) \quad .$$

If we now consider an expression, $A(s_\varkappa;y_\nu;G_\delta^{(\varepsilon)})$, depending on the $s_\varkappa$, the y_ν and the $G_\delta^{(\varepsilon)}$, where all ε and δ belong to the same i, we can express the $G_\delta^{(\varepsilon)}$ linearly through the $H_\delta^{(\varepsilon)}$ and then eliminate the $s_\varkappa$ and y_γ replacing them with functions of $r_\varkappa, x_\nu$. We obtain thus an expression

$$\text{(VI.28)} \qquad B(r_\varkappa;x_\nu;H_\delta^{(\varepsilon)}) = A(s_\varkappa;y_\nu;G_\delta^{(\varepsilon)}) \quad .$$

VII. Transformations with d = 0.

7.1. In the case that $\varrho=m$ it follows from (III.30):

$$n > k + m \quad . \tag{VII.1}$$

Interchanging in K_x^*, if necessary, the rows with the indices k+1,...,n we can assume that

$$D := \begin{vmatrix} X'_{1s_1} & \cdots & X'_{1s_k} & p_{11} & \cdots & p_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{ks_1} & \cdots & X'_{ks_k} & p_{k1} & \cdots & p_{km} \\ X'_{k+1\ s_1} & & X'_{k+1\ s_k} & p_{k+1\ 1} & & p_{k+1\ m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{k+m\ s_1} & \cdots & X'_{k+m\ s_k} & p_{k+m\ 1} & \cdots & p_{k+m\ m} \end{vmatrix} \neq 0. \tag{VII.2}$$

We consider further the determinants $D_{\mu\tau}$ which are obtained from D if, for μ with $k+m \geq \mu > k$, the row in D with the index μ is deleted and the row of K_x^* with the index τ, where τ is one of the indices k+m+1,...,n, is added at the bottom,

$$D_{\mu\tau} := \begin{vmatrix} X'_{1s_1} & \cdots & X'_{1s_k} & p_{11} & \cdots & p_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{ks_1} & \cdots & X'_{ks_k} & p_{k1} & \cdots & p_{km} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{\mu-1\ s_1} & \cdots & X'_{\mu-1\ s_k} & p_{\mu-1\ 1} & \cdots & p_{\mu-1\ m} \\ X'_{\mu+1\ s_1} & \cdots & X'_{\mu+1\ s_k} & p_{\mu+1\ 1} & \cdots & p_{\mu+1\ m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{k+m\ s_1} & \cdots & X'_{k+m\ s_k} & p_{k+m\ 1} & \cdots & p_{k+m\ m} \\ X'_{\tau\ s_1} & \cdots & X'_{\tau\ s_k} & p_{\tau 1} & \cdots & p_{\tau m} \end{vmatrix} . \tag{VII.3}$$

The number of the determinants $D_{\mu\tau}$ is obviously

(VII.4) $$m\,(n-k-m) = N \quad .$$

7.2. More generally, put $\delta := \{1,\dots,m\}$ and, for an i with $1 \leq i \leq m$, denote by ε', ε'' two combinations of i indices, ε' from the sequence $\{1,\dots,k+m\}$ and ε'' from the sequence $\{k+m+1,\dots,n\}$,

(VII.5) $$\begin{aligned} \varepsilon' &= \{\mu_1 < \dots < \mu_i\} \ , \quad \mu_i \leq k+m \ , \\ \varepsilon'' &= \{\tau_1 < \dots < \tau_i\} \ , \quad \tau_1 \geq k+m+1 \ , \quad \tau_i \leq n \ . \end{aligned}$$

Denote further by ε the sequence obtained from $\{1,2,\dots,k+m\}$ by deleting the elements of ε' and adding at the end the elements of ε''. The determinant obtained from D by deleting the rows corresponding to ε' and adding at the bottom the rows corresponding to ε'' will be denoted by $D_{\varepsilon',\varepsilon''}$. It follows comparing with the determinants $G_{\delta}^{(\varepsilon)}$ (VI.8) of order m:

(VII.6) $$D_{\varepsilon',\varepsilon''} = G_{\delta}^{(\varepsilon)} \quad .$$

In particular, the determinants $D_{\mu\tau}$ corresponds to $\varepsilon' = \{\mu\}$, $\varepsilon'' = \{\tau\}$.

The number of the $D_{\varepsilon',\varepsilon''}$ corresponding to a certain i is obviously $\binom{k+m}{i}\binom{n-k-m}{i}$ and therefore the total number of all $D_{\varepsilon',\varepsilon''}$ is

(VII.7) $$M := \sum_{i=1}^{\infty} \binom{k+m}{i}\binom{n-k-m}{i} \quad .$$

where of course the series breaks up as soon as $i > k+m$ or $i > n-k-m$.

7.3. We are first going to show that the M+1 functions

(VII.8) $$D \ , \ D_{\varepsilon',\varepsilon''}$$

are V functions in the sense of chapter III, that is satisfy (III.3), if they are expressed, using (II.7) and (II.14), through the x_ν, $r_\varkappa$ and $p_{\nu\mu}$. Indeed, applying the operator $J_{\mu,\varkappa}$ in (III.8) to one of these determinants we are simply replacing the μ-th column with the $\varkappa$-th column and obtain a determinant with two identical columns. Therefore the equations (III.7) which are necessary and sufficient for the U property are satisfied.

7.4. Further, applying the operator

$$\Delta_{\mu'',\mu'} := \sum_{\nu=1}^{n} p_{\nu\mu'} \frac{\partial}{\partial p_{\nu\mu''}} \tag{VII.9}$$

to D and $D_{\varepsilon',\varepsilon''}$ we obtain again two identical columns if $\mu''=\mu'$, while if $\mu''=\mu'$ the corresponding determinant vanishes or is reproduced. But then, if $D_{\varepsilon',\varepsilon''}$ is reproduced, applying for $\mu'=\mu''$ the operator $\Delta_{\mu',\mu'}$, we have

$$\Delta_{\mu',\mu'} D_{\varepsilon',\varepsilon''}/D = \frac{D(\Delta_{\mu',\mu'} D_{\varepsilon',\varepsilon''}) - D_{\varepsilon',\varepsilon''}(\Delta_{\mu',\mu'} D)}{D^2} = 0 .$$

We see that all M quotients

$$U^{(\sigma)} := \frac{D_{\varepsilon',\varepsilon''}}{D} \quad (\sigma=k+1,\dots,k+M) \tag{VII.10}$$

ordered conveniently, beginning with $U^{(k+1)}$, satisfy as well the equations (III.7) as (III.12) and therefore are U functions invariant with respect to the choice of the $T_1,\dots,T_m$. We choose the ordering of $U^{(\sigma)}$ in such a way that the first N of them, that is $U^{(k+1)},\dots,U^{(k+N)}$ correspond to the $D_{\mu\tau}$ in (VII.3). The values of τ in (VII.3) corresponding to a σ in the first N of the $U^{(\sigma)}$ will be denoted by τ_σ.

7.5. Consider now, for a fixed τ, the m determinants $D_{\mu\tau}$ ($\mu=k+1,\ldots,k+m$) and develop them each time in the elements of the row with the index $k+\mu$. Then we obtain

$$\text{(VII.11)} \qquad D_{\mu\tau} = \sum_{\lambda=1}^{m} D_{\mu\tau}^{(\lambda)} p_{\tau\lambda} + D_{\mu\tau}^{(o)} \qquad (\mu=k+1,\ldots,k+m) \; ,$$

where the terms of the developments corresponding to the first k terms of the $k+\mu$-th row are taken together in $D_{\mu\tau}^{(o)}$.

Here the coefficients $D_{\mu\tau}^{(\lambda)}$ are subdeterminants of D and are therefore <u>independent</u> of τ. Thence, we can write (VII.11) as

$$\text{(VII.12)} \qquad D_{\mu\tau} = \sum_{\lambda=1}^{m} D_{\mu}^{(\lambda)} p_{\tau\lambda} + D_{\mu\tau}^{(o)} \qquad (\mu=k+1,\ldots,k+m) \; .$$

The coefficients $D_{\mu}^{(\lambda)}$ are obviously obtained deleting in D the μ-th row and the $k+\lambda$-th column. By the generalized Sylvester's Theorem we have

$$\text{(VII.13)} \qquad \left| D_{\mu}^{(\lambda)} \right| = JD^{m-1} \neq 0 \; . \,^{*)}$$

The $p_{\tau\lambda}$ for our fixed value of τ can be therefore expressed through the $D_{k+1\,\tau},\ldots,D_{k+m\,\tau}$,

---

*) <u>Kowalewski</u>, Einführung in die Determinantentheorie, 3rd ed., 1942. Observe that in Kowalewski's treatise the exponent of B in the last formula on page 100, $\binom{n-h-1}{m-1}$, is false and must be replaced with $\binom{n-h-1}{m-h}$.

<u>Muir-Metzler</u>, A treatise on the theory of determinants, Dover 1960, p. 190, Nr. 197.

(VII.14) $$p_{\tau\lambda} = Q_{\tau\lambda}(D_{k+1\,\tau},\dots,D_{k+m\,\tau}) \quad ,$$

where the functions $Q_{\tau\lambda}$ do not contain any $p_{\nu\mu}$ with $\nu > k+m$.

7.6. But writing then (VII.14) out for all $\tau=k+m+1,\dots,n$ and $\lambda=1,\dots,m$ we obtain the representation of the N derivatives $p_{\tau\lambda}$ through the N quotients (VII.10) corresponding to the $D_{\mu\tau}$. It follows that the first N quotients (VII.10) considered as undevelopped, are independent functions with respect to the $p_{\tau\lambda}$. Thence, denoting generally the rank of a matrix A by Rk A, we can write

(VII.15) $$\mathrm{Rk}\left(\frac{\partial(U^{(k+1)},\dots,U^{(k+N)})}{\partial(p_{\nu\mu})}\right) = N \quad ,$$

where N, given by (VII.4), is the total number of independent integrals of the joint system consisting of (III.7) and (III.12). But the following $U^{(\sigma)}$ with $\sigma > k+N$ are also integrals of this system and are therefore functions of $U^{(k+1)},\dots,U^{(k+N)}$. It follows thence

(VII.15a) $$U^{(k+N+\sigma)} = A_{\sigma}(r_{\varkappa};U^{(k+1)},\dots,U^{(k+\ )}) \quad (\sigma=1,\dots,M-N)$$

where the functions A_{σ} depend only on T* (but not on T).

Using (II.7) and (II.14) we assume from now on that the functions (VII.8) are functions of the x_{ν}, $r_{\varkappa}$ and $p_{\nu\mu}$.

7.7. We make a further assumption going beyond (VII.15), namely that (VII.15) remains true if the $U^{(\sigma)}$ are replaced with the $U^{*(\sigma)}$,

(VII.16) $$\mathrm{Rk}\left(\frac{\partial(U^{*(k+1)},\dots,U^{*(k+N)})}{\partial(p_{\nu\mu})}\right) = N \quad .$$

Then, the $r^*_{\varkappa}$, satisfying also the property U, are expressible through the $U^{*(\sigma)}$ and the equations

(VII.17) $$r_\varkappa = \varphi_\varkappa(U^{(k+1)},\dots,U^{(k+N)}) \quad (\varkappa=1,\dots,k)$$

can be solved with respect to the $r_1,\dots,r_k$ if

(VII.18) $$\frac{\partial(\varphi_\varkappa - r_\varkappa)}{\partial(r_\varkappa)} \neq 0$$

where the $\varphi_\varkappa$ are k arbitrary, indefinitely often differentiable functions. Thus the $r_\varkappa$ can represented as functions of the x_ν, $p_{\nu\mu}$,

(VII.19) $$r_\varkappa = r^*_\varkappa(x_\nu, p_{\nu\mu}) \quad (\varkappa=1,\dots,k).$$

Therefore, the $s_\varkappa$ defined, in virtue of (II.14), by

(VII.20) $$s_\varkappa = S_\varkappa(x_\nu, r^*_\varkappa) \quad (\varkappa=1,\dots,k)$$

have also the property U and can be expressed in function of y_ν, $s_\varkappa$, $q_{\nu\mu}$,

(VII.21) $$S_\varkappa = \Psi_\varkappa(y_\nu, s_\varkappa, q_{\nu\mu}) \quad (\varkappa=1,\dots,k) \quad .$$

Thus we obtain k equations

(VII.22) $$s_\varkappa = \Psi_\varkappa(y_\nu, s_\varkappa, q_{\nu\mu}) \quad (\varkappa=1,\dots,k)$$

which can be solved with respect to $s_\varkappa$ if

(VII.23) $$\frac{\partial(\Psi_\varkappa - s_\varkappa)}{\partial(s_\varkappa)} \neq 0 \quad .$$

In this way we obtain the expressions

(VII.24) $$s_\varkappa = s^*_\varkappa(y_\nu, p_{\nu\mu})$$

satisfying together with the $r^*_\varkappa$ the equations (II.7), (II.11), (II.13) and (II.14), and our problem is solved.

Example for d=0, m=2

7.8. Take

(VII.25) $$n=4 \quad , \quad m=2 \quad , \quad d=0 \quad , \quad k=1 \quad .$$

Then, from (VII.4) it follows N=2 and for $D_{\mu\tau}$ we have τ=k+m+1=n=4, while μ can assume the values 2 or 3. We obtain from (VII.2) and (VII.3) more generally

(VII.26) $$D = \begin{vmatrix} X'_{1s} & p_{11} & p_{12} \\ X'_{2s} & p_{21} & p_{22} \\ X'_{3s} & p_{31} & p_{32} \end{vmatrix} ,$$

(VII.27) $$D_{14} = \begin{vmatrix} X'_{2s} & p_{21} & p_{22} \\ X'_{3s} & p_{31} & p_{32} \\ X'_{4s} & p_{41} & p_{42} \end{vmatrix} , \quad D_{24} = \begin{vmatrix} X'_{1s} & p_{11} & p_{12} \\ X'_{3s} & p_{31} & p_{32} \\ X'_{4s} & p_{41} & p_{42} \end{vmatrix} ,$$
$$D_{34} = \begin{vmatrix} X'_{1s} & p_{11} & p_{12} \\ X'_{2s} & p_{21} & p_{22} \\ X'_{4s} & p_{41} & p_{42} \end{vmatrix} .$$

We can therefore write, by (VII.10),

(VII.28) $$U^{(2)} = D_{24}/D \quad , \quad U^{(3)} = D_{34}/D$$

and obtain with an arbitrary function φ of two variables (VII.17), if $U^{(2)}$, $U^{(3)}$ are independent,

$$r^* = \varphi(U^{*(2)}, U^{*(3)}) \qquad \text{(VII.29)}$$

7.9. We specialize now our transformation to

$$\begin{aligned} x_1 &= X_1(y_\nu, s) = y_1 + s \\ x_2 &= X_2(y_\nu, s) = y_2 - s \\ x_3 &= X_3(y_\nu, s) = y_3 + s \\ x_4 &= X_4(y_\nu, s) = y_4 - s \end{aligned} \quad , \quad \begin{aligned} y_1 &= x_1 + r \\ y_2 &= x_2 - r \\ y_3 &= x_3 + r \\ y_4 &= x_4 - r \end{aligned} \qquad \text{(VII.30)}$$

and take $r=-s$.

We obtain from (IV.7) $K=J=1$ and further

$$D = \begin{vmatrix} 1 & p_{11} & p_{12} \\ -1 & p_{21} & p_{22} \\ 1 & p_{31} & p_{32} \end{vmatrix} = \begin{vmatrix} p_{21}+p_{11} & p_{22}+p_{12} \\ p_{31}-p_{11} & p_{32}-p_{12} \end{vmatrix} , \qquad \text{(VII.31)}$$

$$D_{14} = \begin{vmatrix} -1 & p_{21} & p_{22} \\ 1 & p_{31} & p_{32} \\ -1 & p_{41} & p_{42} \end{vmatrix} = \begin{vmatrix} p_{31}+p_{21} & p_{32}+p_{22} \\ p_{41}-p_{21} & p_{42}-p_{22} \end{vmatrix} , \qquad \text{(VII.32)}$$

$$D_{24} = \begin{vmatrix} 1 & p_{11} & p_{12} \\ 1 & p_{31} & p_{32} \\ -1 & p_{41} & p_{42} \end{vmatrix} = \begin{vmatrix} p_{31}-p_{11} & p_{32}-p_{12} \\ p_{41}+p_{11} & p_{42}+p_{12} \end{vmatrix} , \qquad \text{(VII.33)}$$

$$D_{34} = \begin{vmatrix} 1 & p_{11} & p_{12} \\ -1 & p_{21} & p_{22} \\ -1 & p_{41} & p_{42} \end{vmatrix} = \begin{vmatrix} p_{21}+p_{11} & p_{22}+p_{12} \\ p_{41}+p_{11} & p_{42}+p_{12} \end{vmatrix} , \qquad \text{(VII.34)}$$

where (VII.33) and (VII.34) are assumed as independent.

Thence

$$-s^{*} = r^{*} = \varphi(D_{24}/D, D_{34}/D) \quad , \tag{VII.35}$$

where the right-hand expression is easy to be transformed into a function depending only on the $q_{\nu\mu}$.

VIII. $1 \leq d < m$

8.1. Put

$$\text{(VIII.1)} \qquad n' \ := \ n + d \quad .$$

We change the notation of chapter II in so far that the orderings of the X_ν's and Y_ν's have a gap from k+1 to k+d, where in particular the

$$X'_{1s_\varkappa}\ ,\ X'_{2s_\varkappa}\ ,\ .\ .\ .\ .\ ,\ X'_{ks_\varkappa}\ ,\ X'_{k+d+1\ s_\varkappa}\ ,\ .\ .\ .\ .\ ,\ X'_{n's_\varkappa}$$

are expressed in terms of the x_ν and $r_\varkappa$. We further introduce d auxiliary equations

$$\text{(VIII.2)} \qquad x_\nu = X_\nu = y_\nu \ ,\ y_\nu = Y_\nu = x_\nu \quad (\nu=k+1,\ldots,k+d) \quad .$$

Consider n vectors of order k+m,

$$\text{(VIII.3)} \qquad L_\nu = (X'_{\nu s_1},\ldots,X'_{\nu s_k},p_{\nu 1},\ldots,p_{\nu m})$$

where ν runs through 1,...,k,k+d+1,...,n' so that there is a gap from k+1 to k+d.

Consider further a matrix

$$\text{(VIII.4)} \qquad \tilde{K}^*_x = (L_1\ .\ .\ .\ L_k\ L_{k+d+1}\ .\ .\ .\ L_{n'})'$$

where as also in the following the accent denotes that the rows are to be written from above to below.

8.2. Assume now that the rank of $\tilde{K}^*_x$ is k+m-d,

$$\text{(VIII.5)} \qquad \mathrm{Rk}(\tilde{K}^*_x) = k+m-d \quad , \quad 1 \leq d < m \quad .$$

Then there exist exactly d independent linear relations between the columns of $\tilde{K}^*_x$

(VIII.6) $\beta_1^{(\delta)} X'_{\nu s_1} + \ldots + \beta_k^{(\delta)} X'_{\nu s_k} = \alpha_1^{(\delta)} p_{\nu 1} + \ldots + \alpha_m^{(\delta)} p_{\nu m}$ $(\delta = 1, \ldots, d; \nu = 1, \ldots, k, k+d+1, \ldots, n')$.

Obviously the coefficients $\beta_\varkappa^{(\delta)}$ and $\alpha_\mu^{(\delta)}$ are independent of the $p_{d+\delta\mu}$ $(\delta=1,\ldots,d)$. It is easy to see that in (VIII.6)

(VIII.7) $$\mathrm{Rk}(\alpha_\mu^{(\delta)}) = d \quad (\delta=1,\ldots,d;\mu=1,\ldots,m) \quad .$$

Indeed, otherwise we could obtain, eliminating the $p_{\nu\mu}$, a non-trivial relation,

$$\beta_1 X'_{\nu s_1} + \ldots + \beta_k X'_{\nu s_k} = 0 \quad (\nu=1,\ldots,k,k+d+1,\ldots,n') \quad ,$$

in contradiction to the formula (II.11), where we have to replace k' with k.

8.3. From (VIII.7), it follows that there exists a non-vanishing determinant of order d with $\alpha_\mu^{(\delta)}$ and we can assume without loss of generality that this is the determinant

(VIII.8) $$\left|\alpha_\varepsilon^{(\delta)}\right| \neq 0 \qquad (\varepsilon,\delta=1,\ldots,d) \quad ,$$

changing conveniently the ordering of the $p_{\nu\mu}$. Further, changing conveniently the order of the columns in (VIII.8), we can assume that its diagonal product does not vanish,

$$\alpha_1^{(1)} \alpha_2^{(2)} \ldots \alpha_d^{(d)} \neq 0 \quad .$$

But then, dividing all relations (VIII.6) by the corresponding

$\alpha_{\varsigma}^{(\varsigma)}$, we can finally assume without loss of generality that

$$\text{(VIII.9)} \qquad \alpha_1^{(1)} = \alpha_2^{(2)} = \ldots = \alpha_d^{(d)} = 1 .$$

From (VIII.8) it follows that there does not exist a non-trivial relation

$$\text{(VIII.10)} \quad \beta_1 X'_{\nu s_1} + \ldots + \beta_k X'_{\nu s_k} = \alpha_{d+1} p_{\nu\, d+1} + \ldots + \alpha_m p_{\nu m} \quad (\nu = 1,\ldots,k,k+d+1,\ldots,n').$$

8.4. Consider now d vectors of order k+m corresponding to (VIII.2),

$$\text{(VIII.11)} \qquad P_\nu = (0,\ldots,0,p_{\nu 1},\ldots,p_{\nu m}) \quad (\nu = k+1,\ldots,k+d) ,$$

where the first k elements of each P_ν consist of zeros. Using these vectors together with the vectors (VIII.3), form the $(k+m)\times n'$-matrix

$$\text{(VIII.12)} \qquad K_x^* = (L_1,\ldots,L_k,P_{k+1},\ldots,P_{k+d},L_{k+d+1},\ldots,L_{n'}) .$$

We consider further the determinants of the order k+m:

$$\text{(VIII.13)} \quad D_{\alpha_1\alpha_2\ldots\alpha_k\alpha_{k+d+1}\ldots\alpha_{k+m}} := \left| L_{\alpha_1} L_{\alpha_2} \ldots L_{\alpha_k} P_{k+1} \ldots P_{k+d} L_{\alpha_{k+1}} \ldots L_{\alpha_{k+m}} \right|' ,$$

where

$$\text{(VIII.14)} \qquad 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k < \alpha_{k+1} < \ldots < \alpha_{k+m} \leq n'$$

and none of the α_χ assumes the values k+1,...,k+d.

On the other hand, we consider vectors of order k+m-d, (VIII.15) $\hat{L}_\nu := (X'_{\nu s_1},\dots,X'_{\nu s_k}, p_{\nu\, d+1},\dots,p_{\nu\, m})$,

obtained from the L_ν by dropping the first d columns of $p_{\nu\mu}$. Correspondingly we define the determinants of order k+m-d,

(VIII.16) $\hat{D}_{\alpha_1,\dots,\alpha_k\alpha_{k+d+1},\dots,\alpha_{k+m}} := \left\| \hat{L}_{\alpha_1}\cdots\hat{L}_{\alpha_k}\hat{L}_{\alpha_{k+d+1}}\cdots\hat{L}_{\alpha_{k+m}}\right\|$,

and the $(k+m-d)\times n$-matrix

(VIII.17) $\hat{K}^*_x := (\hat{L}_1,\dots,\hat{L}_k,\hat{L}_{k+d+1},\dots,\hat{L}_{n'})^t$.

8.5. We are now going to transform in a convenient way the matrix K^*_x without changing its rank. We add to the (k+1)-st column of K^*_x the following columns multiplied subsequently with $\alpha_2^{(1)}$, $\alpha_3^{(1)},\dots,\alpha_m^{(1)}$ and substract then the first k columns multiplied by $\beta_1^{(1)},\dots,\beta_k^{(1)}$. Then we obtain a matrix in which the only elements in the (k+1)-st column not necessarily vanishing are

$$\hat{p}_{k+\delta\; k+1} := \sum_{=1}^{m}\alpha_\nu^{(1)}\, p_{k+\delta\,\nu} .$$

Generally we apply the same transformation to the columns with the index $k+\varepsilon$, $\varepsilon=1,\dots,d$, adding to each such column all other p columns multiplied by $\alpha_1^{(\varepsilon)},\dots,\alpha_{\varepsilon-1}^{(\varepsilon)}, \alpha_{\varepsilon+1}^{(\varepsilon)},\dots,\alpha_m^{(\varepsilon)}$ and then substracting the first k columns multiplied by $\beta_1^{(\varepsilon)},\dots,\beta_k^{(\varepsilon)}$. Then the only elements in the $(k+\varepsilon)$-th column are the expressions

(VIII.18) $$\hat{p}_{k+\delta\; k+\varepsilon} := \sum_{\nu=1}^{d}\alpha_\nu^{(\varepsilon)}\, p_{k+\delta\,\nu} \quad (\varepsilon,\delta=1,\dots,d) .$$

We obtain in this way a matrix of dimensions $n'\times(k+m)$,

$$\text{(VIII.19)} \qquad \bar{K}^*_x := \begin{pmatrix} J_k & O_1 & Q_1 \\ O_2 & \hat{P} & Q_2 \\ J_{n-k} & O_3 & Q_3 \end{pmatrix} .$$

Here the matrices J_k and J_{n-k} are matrices of dimensions $k \times k$ and $(n-k) \times k$ formed with the $X'_{\nu s_\chi}$ for $\nu=1,\dots,k$ and $\nu=k+d+1,\dots,n'$. The matrices O_1, O_2 and O_3 are matrices consisting of zeros, the first of the dimensions $k \times d$, the second of the dimensions $d \times k$ and the third of the dimensions $(n-k) \times d$. Further the matrices Q_1, Q_2 and Q_3 are matrices from the last $(m-d)$ columns of the $p_{\nu\mu}$ with dimensions $k \times (m-d)$, $d \times (m-d)$ and $(n-k) \times (m-d)$. Finally the matrix $\hat{P}$ is the matrix formed with the expressions (VIII.18),

$$\text{(VIII.20)} \qquad \hat{P} := (\hat{p}_{k+\delta\ k+\varepsilon}) \quad (\delta,\varepsilon=1,\dots,d) .$$

Observe that the determinant $|\hat{P}|$ of $\hat{P}$ does not identically vanish in the $p_{d+\delta\,\mu}$, since the coefficients in (VIII.18) do not depend on these $p_{d+\delta\,\mu}$.

8.6. It follows obviously from the decomposition (VIII.19) that the determinants (VIII.13) can be written as

$$\text{(VIII.21)} \quad D_{\alpha_1 \dots \alpha_k \alpha_{k+d+1} \dots \alpha_{k+m}} = |\hat{P}|\, \hat{D}_{\alpha_1 \dots \alpha_k \alpha_{k+d+1} \dots \alpha_{k+m}} .$$

On the other hand the rank of the $n \times (k+m-d)$-matrix $\hat{K}^*_x$ is obviously exactly

$$\text{(VIII.22)} \qquad Rk(\hat{K}^*_x) = k+m-d ,$$

since otherwise we would have a relation of the type (VIII.10). Therefore, by (VIII.21), there exist subdeterminants $\hat{D}_{\alpha_1 \dots \alpha_k \alpha_{k+d+1} \dots \alpha_{k+m}}$

which do not vanish and the rank of (VIII.19) and thence that of K_x^* is exactly k+m,

$$\text{(VIII.23)} \qquad Rk(K_x^*) = k+m \qquad .$$

We can therefore change the order of the X_ν in (II.2b) in such a way that the determinants

$$\text{(VIII.24)} \qquad J := \begin{vmatrix} X'_{1s_1} & \cdots & X'_{1s_k} \\ \vdots & & \vdots \\ X'_{ks_1} & \cdots & X'_{ks_k} \end{vmatrix} ,$$

$$\text{(VIII.25)} \qquad D_{1\cdots k\ k+1\cdots k+d\cdots k+m}$$

and

$$\text{(VIII.26)} \qquad \hat{D}_{1\cdots k\ k+d+1\cdots k+m}$$

do not vanish and we can assume without loss of generality that it is the case from the beginning.

8.7. We now subdivide the sequence k+m+1,...,n' into ℓ consucutive sequences of the lenght d and a last one of the lenght $<$ d which could be also =0. The first ℓ sequences are

$$k+m+1,\ldots,k+m+d;k+m+d+1,\ldots,k+m+2d;\ldots;k+m+(\ell-1)d+1,\ldots,k+m+\ell d$$

where

$$k+m+\ell d \leq n' < k+m+(\ell+1)d$$

and thence

$$\ell \leq \frac{n'-k-m}{d} < \ell+1 \quad ,$$

(VIII.27) $$\ell = \frac{n'-k-m}{d} + \theta_o \quad , \quad 0 \leq \theta_o < 1 \quad .$$

We replace now, for $\lambda=1,2,\ldots,\ell$, in (VIII.5) the rows with the numbers $k+1,\ldots,k+d$ with the rows

$$k+m+(\lambda-1)d+1,\ldots,k+m+\lambda d$$

and denote the determinants obtained in this way by

(VIII.28) $$D_1, D_2, \ldots, D_\ell \quad .$$

All rows of these determinants belong to $\tilde{K}^*_x$ and therefore vanish so that we obtain finally ℓ equations

(VIII.29) $$D_1 = 0 \ , \ \ldots \ , \ D_\ell = 0 \quad .$$

8.9. Observe that each of D_λ contains a rectangle of values of the $p_{\nu\mu}$ which is not contained in any other of the D_λ. Therefore, as $J \neq 0$, the ℓ expressions D_λ are independent as functions of the $p_{\nu\mu}$. But the relations (VIII.29) contain ℓ equations for the k expressions $r_1, r_2, \ldots, r_k$ and we have therefore the inequality

(VIII.30) $$k \leq \ell = \frac{n'-k-m}{d} - \theta_o \quad , \quad 0 \leq \theta_o < 1 \quad .$$

Solving this with respect to k we obtain

(VIII.30a) $$k \leq \frac{n-m}{d+1} + \theta \quad , \quad 0 < \theta \leq \frac{d}{d+1} \quad .$$

8.10. We describe now the method we use for some cases with $1 \leqslant d < m$. We consider the new transformation, introduced in 8.1. and which we call <u>the enlargement</u>, $\hat{T}$, of the original one, T. If we put

$$\text{(VIII.31)} \quad y_\nu = Y_\nu^*(x_\nu, r_\varkappa) \ , \ x_\nu = X_\nu^*(y_\nu, s_\varkappa) \quad (\nu=1,\dots,k,k+d+1,\dots,n')$$

$$\text{(VIII.32)} \qquad x_\nu = y_\nu \qquad (\nu=k+1,\dots,k+d) \qquad ,$$

then T is given by (VIII.31) and $\hat{T}$ by (VIII.31) <u>together</u> with (VIII.32).

We are now going to show that for this enlarged transformation d vanishes, that is to say that no non-trivial relation of the type

$$\text{(VIII.33)} \qquad \sum_{\varkappa=1}^{k} \beta_\varkappa X'_{\nu s_\varkappa} = \sum_{\mu=1}^{n'} \alpha_\mu p_{\nu\mu} \qquad (\nu=1,\dots,n')$$

exists. Indeed, such a relation would be in particular valid for $\nu=1,\dots,k,k+d+1,\dots,n'=n+d$ and therefore be a combination of relations (VIII.6),

$$\beta_\varkappa = \sum_{\delta=1}^{d} u_\delta \beta_\mu^{(\delta)} \quad , \quad \alpha_\mu = \sum_{\delta=1}^{d} u_\delta \alpha_\mu^{(\delta)}$$

$$(\varkappa=1,\dots,k;\mu=1,\dots,m) \qquad .$$

Since the relation (VIII.33) holds also for $\nu=k+1,\dots,k+d$ we would have the relations

$$\sum_{\delta=1}^{d} u_\delta \sum_{\mu=1}^{m} \alpha_\mu^{(\delta)} p_{\nu\mu} = 0 \quad (\nu=k+1,\dots,k+d) \ .$$

Hence the determinant

$$\left| \sum_{\mu=1}^{m} \alpha_{\mu}^{(\delta)} p_{\nu\mu} \right| \quad (\delta, \nu-k=1,\dots,d)$$

would vanish, contrary to the lemma D1 of the Appendix D, as the coefficients $\alpha_{\mu}^{(\delta)}$ do not depend on the $p_{\nu\mu}$ with $k<\nu\leqslant k+d$.

8.11. Therefore the method used in chapter VII can be tried for the enlarged transformation $\hat{T}$ given by (VIII.31), (VIII.32). The expressions of the $r_{\varkappa}$, $s_{\varkappa}$ obtained in this way have to be chosen independent of the $p_{\nu\mu}$ $(k<\nu\leqslant k+d)$ and belong to T. However this is only possible for d=1, as in all other cases (VII.16) is not satisfied.

8.12. We consider now the case d=1. The relations (VIII.6) reduce here to relations which can be written, omitting the superscript 1 and putting n':=n+1, as

$$\text{(VIII.34)} \qquad \sum_{\varkappa=1}^{k} \beta_{\varkappa} x'_{\nu s_{\varkappa}} = \sum_{\mu=1}^{m} \alpha_{\mu} p_{\nu\mu} \quad (\nu=1,\dots,k,k+2,\dots,n') .$$

Here we let ν run through $1,\dots,n'$ omitting k+1. Our enlarged system becomes (VIII.31) together with

$$\text{(VIII.35)} \qquad x_{k+1} = y_{k+1} \qquad .$$

For this enlarged system N=m(n'-k-m)=m(n+1-k-m) is the same as for the original one.

From the formula (VIII.30) it follows for d=1:

$$\text{(VIII.36)} \qquad k+m \leqslant n \leqslant 2k+m-1 \qquad .$$

8.13. We now form in notations of 7.1. for the enlarged system the expressions D and $D_{\mu\varkappa}$.

We have for D:

$$
\text{(VIII.37)}\qquad D = \begin{vmatrix}
X'_{1s_1} & \cdots & X'_{1s_k} & p_{11} & \cdots & p_{1m} \\
\vdots & & \vdots & \vdots & & \vdots \\
X'_{ks_1} & \cdots & X'_{ks_k} & p_{k1} & \cdots & p_{km} \\
0 & \cdots & 0 & p_{k+11} & \cdots & p_{k+1m} \\
X'_{k+2s_1} & \cdots & X'_{k+2s_k} & p_{k+21} & \cdots & p_{k+2m} \\
\vdots & & \vdots & \vdots & & \vdots \\
X'_{k+ms_1} & \cdots & X'_{k+ms_k} & p_{k+m1} & \cdots & p_{k+mm}
\end{vmatrix} ,
$$

while the expressions for $D_{\mu\tau}$ are different for $\mu=k+1$ and $\mu>k+1$:

$$
\text{(VIII.38)}\qquad D_{k+1,\tau} = \begin{vmatrix}
X'_{1s_1} & \cdots & X'_{1s_k} & p_{11} & \cdots & p_{1m} \\
\vdots & & \vdots & \vdots & & \vdots \\
X'_{ks_1} & \cdots & X'_{ks_k} & p_{k1} & \cdots & p_{km} \\
X'_{k+2s_1} & \cdots & X'_{k+2s_k} & p_{k+21} & \cdots & p_{k+2m} \\
\vdots & & \vdots & \vdots & & \vdots \\
X'_{k+ms_1} & \cdots & X'_{k+ms_k} & p_{k+m1} & \cdots & p_{k+mm} \\
X'_{\tau s_1} & \cdots & X'_{\tau s_k} & p_{\tau 1} & \cdots & p_{\tau m}
\end{vmatrix} = 0
$$

$$(\tau=k+m+1,\ldots,n') \qquad ,$$

$$
\text{(VIII.39)} \qquad D_{\mu\tau} = \left|\begin{matrix}
X'_{1s_1} & \cdots & X'_{1s_k} & p_{11} & \cdots & p_{1m} \\
\vdots & & \vdots & \vdots & & \vdots \\
X'_{ks_1} & \cdots & X'_{ks_k} & p_{k1} & \cdots & p_{km} \\
0 & \cdots & 0 & p_{k+11} & \cdots & p_{k+1m} \\
X'_{k+2s_1} & \cdots & X'_{k+2s_k} & p_{k+21} & \cdots & p_{k+2m} \\
\vdots & & \vdots & \vdots & & \vdots \\
X'_{k+ms_1} & \cdots & X'_{k+ms_k} & p_{k+m1} & \cdots & p_{k+mm} \\
X'_{\tau s_1} & \cdots & X'_{\tau s_k} & p_{\tau 1} & \cdots & p_{\tau m}
\end{matrix}\right|^{(\mu)}
$$

$$(\mu=k+2,\ldots,k+m) \qquad ,$$

where the notations $|^{(\mu)}$ signifies that the row corresponding to the index μ is omitted.

8.14. Without loss of generality we can assume that $\alpha_1=1$. Similarly as in 8.5. we add to the (k+1)-st columns in the determinants (VIII.37) and (VIII.39) the columns with the indices k+2,...,m multiplied with the corresponding α_μ and substract the the columns with the indices 1,...,k multiplied with the corresponding β_ν. Then all elements of the (k+1)-st column become 0 save the (k+1)-st element which becomes

$$
\text{(VIII.40)} \qquad p \;:=\; p_{k+11} + \sum_{\mu=2}^{m} \alpha_\mu \, p_{k+1\mu} \qquad .
$$

Then D and $D_{\mu\tau}$ in (VIII.39) become finally

$$\text{(VIII.41)}\qquad D = p\begin{vmatrix} X'_{1s_1} & \cdots & X'_{1s_k} & p_{12} & \cdots & p_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{ks_1} & \cdots & X'_{ks_k} & p_{k2} & \cdots & p_{km} \\ X'_{k+2s_1} & \cdots & X'_{k+2s_k} & p_{k+22} & \cdots & p_{k+2m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{k+ms_1} & \cdots & X'_{k+ms_k} & p_{k+m2} & \cdots & p_{k+mm} \end{vmatrix},$$

$$\text{(VIII.42)}\qquad D_{\mu\tau} = p\begin{vmatrix} X'_{1s_1} & \cdots & X'_{1s_k} & p_{12} & \cdots & p_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{ks_1} & \cdots & X'_{ks_k} & p_{k2} & \cdots & p_{km} \\ X'_{k+2s_1} & \cdots & X'_{k+2s_k} & p_{k+22} & \cdots & p_{k+2m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X'_{k+ms_1} & \cdots & X'_{k+ms_k} & p_{k+m2} & \cdots & p_{k+mm} \\ X'_{\tau s_1} & \cdots & X'_{\tau s_k} & p_{\tau 2} & \cdots & p_{\tau m} \end{vmatrix}^{(\mu)}$$

$$(\mu = k+2,\ldots,k+m) \quad .$$

8.15. From the relations (VIII.33) it follows that the $r^*_\varkappa$ satisfy the equations

$$\text{(VIII.43)}\qquad D_{k+1\,\tau}(x_\nu, r_\varkappa, p_{\nu\mu}) = 0 \quad (\tau = k+m+1,\ldots,n')$$

which do not depend on the $p_{\nu\mu}$ $(k+1 \leqslant \nu \leqslant k+d)$ and their number is

(VIII.44) $$k_1 := n'-k-m = n-k-m+1 \quad .$$

That this is $\leq k$ follows from (VIII.36).

We assume now that

(VIII.45) $$\mathrm{Rk}\left(\frac{\partial(D_{k+1\tau})}{\partial(r_\varkappa)}\right) = k_1 \quad (\tau=k+m+1,\dots,n';\varkappa=1,\dots,k)$$

and that in particular

(VIII.46) $$\mathrm{Rk}\left(\frac{\partial(D_{k+1\tau})}{\partial(r_1,\dots,r_{k_1})}\right) = k_1 \quad (\tau=k+m+1,\dots,k+m+k_1=n') \quad .$$

Now we proved in section 7.5. that the $D_{k+1\tau}/p$ together with the $D_{\mu\tau}$ are independent as functions of the $p_{\nu\mu}$. As their number is N and they do not depend on the $p_{\nu\mu}$ $(k+1\leq\nu\leq k+d)$ they form a complete system of functions with the property U with respect to the original system. Thence the $r^*_\varkappa$ are functions of the $D_{k+1\tau}/p$ and the $D_{\mu\tau}$,

(VIII.47) $$r^*_\varkappa = \Psi_\varkappa(x_\nu,\tfrac{1}{p}D_{k+1\tau},D_{\mu\tau}) \quad (\varkappa=1,\dots,k) \quad .$$

8.16. Since however the $r_\varkappa$ satisfy also (VIII.31) we can replace the $\Psi_\varkappa(x_\nu,\frac{1}{p}D_{k+1\tau},D_{\mu\tau})$ with the $\Psi_\varkappa(x_\nu,0,\dots,0,D_{\mu\tau}) =: \varphi_\varkappa(x_\nu,D_{\mu\tau})$. We assume now that

(VIII.48) $$\mathrm{Rk}\left(\frac{\partial(\varphi_\varkappa)}{\partial(r_\varkappa)}\right) = k-k_1 =: k_2 \quad ,$$

and that in particular

(VIII.49) $$\mathrm{Rk}\ \frac{\partial(\varphi_1,\dots,\varphi_{k_2})}{\partial(r_{k_1+1},\dots,r_k)} = k_2 \quad .$$

Finally we assume that

$$\text{(VIII.50)} \qquad \frac{\partial(D_{k+1\tau}, \varphi_1 - r_{k_1+1}, \ldots, \varphi_{k_2} - r_k)}{\partial(r_\varkappa)} \neq 0 \quad .$$

Then the k expressions $r^*_\varkappa$ can be obtained from the k equations

$$\text{(VIII.51)} \quad D_{k+1\tau} = 0 \ , \ \varphi_\varkappa - r_{k_1+\varkappa} = 0 \quad (\tau = k+m+1, \ldots, n' ; \varkappa = 1, \ldots, k_2)$$

as functions of the original $p_{\nu\mu}$. Further, using (II.14), the expressions $S_\sigma(x_\nu, r_\varkappa)$ can be represented through the y_ν and $q_{\nu\mu}$ and give the representations (II.9) of the $s_\sigma(y_\nu, q_{\nu\mu})$, with which our problem is solved.

APPENDIX A

Lemma A1. Consider the m+k functions of the n+k variables,

$$\text{(A 1)}\quad \beta_\varkappa(x_1,\ldots,x_n;z_1,\ldots,z_k)\ (\varkappa=1,\ldots,k)\ ,\ \alpha_\mu(x_1,\ldots,x_n;z_1,\ldots,z_k)\ (\mu=1,\ldots,m)\ ,$$

all functions being assumed to have continuous first derivatives in convenient domains. Assume that the Jacobian

$$\text{(A 2)}\quad \left|\frac{\partial(\beta_\varkappa)}{\partial(z_\varkappa)}\right| \neq 0$$

and further that the Jacobian matrix of the $\beta_\varkappa$ and α_μ with respect to the $z_\varkappa$ and x_ν

$$\text{(A 3)}\quad \left(\frac{\partial(\beta_\varkappa,\alpha_\mu)}{\partial(z_\varkappa,x_\nu)}\right)$$

with m+k columns has the rank m_o+k, $m_o \leqslant m$.

Consider the k equations

$$\text{(A 4)}\quad \beta_\varkappa(x_\nu,z_\varkappa) = U_\varkappa \quad (\varkappa=1,\ldots,k)$$

solved, for indeterminates $U_1,\ldots,U_k$, with respect to the $z_\varkappa$ and denote the solution

$$\text{(A 5)}\quad \bar{z}_\varkappa(x_1,\ldots,x_n) \quad (\varkappa=1,\ldots,k)\ .$$

Introducing these values of the $z_\varkappa$ into the $\alpha_\mu(x_\nu,z_\varkappa)$ put

$$\text{(A 6)}\quad \alpha_\mu(x_1,\ldots,x_n;\bar{z}_1,\ldots,\bar{z}_k) =: \bar{\alpha}_\mu(x_\nu).$$

Then the rank of the matrix

$$\left(\frac{\partial(\bar{\alpha}_\mu)}{\partial(x_\nu)}\right) \tag{A 7}$$

is $\geq m_o$, that is at most by k less than that of (A 3).

Corollary. If $m_o = m$, then the rank of (A 7) is precisely m.

Proof. The matrix (A 7) has as its ν-th line

$$\alpha'_{1x_\nu} + \sum_{\varkappa=1}^{k} \alpha'_{1z_\varkappa} z'_{\varkappa x_\nu}, \ldots, \alpha'_{mx_\nu} + \sum_{=1}^{k} \alpha'_{mz_\varkappa} z'_{\varkappa x_\nu} , \tag{A 8}$$

where the $z_\varkappa$ are to be replaced, after (A 8) has been written out, by the $\bar{z}_\varkappa$.

In order to prove that the matrix (A 7) has the rank $\geq m_o$, it is sufficient to show that to this matrix k further columns can be added so as to obtain a matrix of the rank $\geq m_o + k$.

But if we add to the general element (A 8) of the ν-th lines the further elements $\bar{z}'_{1x_\nu}, \ldots, \bar{z}'_{kx_\nu}$ we obtain a matrix, whose ν-th line is

$$(\bar{z}'_{1x_\nu}, \ldots, \bar{z}'_{kx_\nu}, \alpha'_{1x_\nu} + \sum_{\varkappa=1}^{k} \alpha'_{1z_\varkappa} \bar{z}'_{\varkappa x_\nu}, \ldots, \alpha'_{mx_\nu} + \sum_{\varkappa=1}^{k} \alpha'_{mz_\varkappa} \bar{z}'_{\varkappa x_\nu}) \tag{A 9}$$

Therefore, subtracting in (A 9) from the (m+1)-th column the first k columns multiplied respectively by $\alpha'_{\mu z_\varkappa}$, the (k+1)-th element of the ν-th line becomes α'_{1x_ν}. Proceeding in the same way with the following columns of (A 9) we obtain the matrix

$$\left(\bar{z}'_{1x_\nu}, \ldots, \bar{z}'_{kx_\nu}, \alpha'_{1x_\nu}, \ldots, \alpha'_{mx_\nu}\right) \quad (\nu = 1, \ldots, n) . \tag{A 10}$$

Multiply this matrix from the left by the square matrix of order k+m:

$$(A\ 11) \qquad \begin{pmatrix} \beta'_{\varkappa z_{\varkappa'}} & 0 \\ 0 & I_n \end{pmatrix}$$

where $\varkappa$ and $\varkappa'$ run from 1 to k and I_m is the unity matrix of order m. We obtain with $\nu=1,\dots,n$:

$$(A\ 12) \qquad \begin{pmatrix} \sum_{\varkappa'} \beta'_{1 z_{\varkappa'}} \bar{z}'_{\varkappa' x_\nu} \\ \vdots \\ \sum_{\varkappa'} \beta'_{k z_{\varkappa'}} \bar{z}'_{\varkappa' x_\nu} \\ \alpha'_{1 x_\nu} \\ \vdots \\ \alpha'_{m x_\nu} \end{pmatrix} .$$

But differentiating totally (A 4) with respect to each ν we obtain

$$\sum_{\varkappa'} \beta'_{\varkappa z_{\varkappa'}} \bar{z}'_{\varkappa' x_\nu} = - \beta'_{\varkappa x_\nu} \quad (\varkappa=1,\dots,k;\nu=1,\dots,n) \quad .$$

Therefore (A 12) becomes

$$\begin{pmatrix} -\beta'_{1x_\nu} \\ \vdots \\ -\beta'_{kx_\nu} \\ \alpha'_{1x_\nu} \\ \vdots \\ \alpha'_{mx_\nu} \end{pmatrix} .$$

And this matrix has, by comparison with (A 3), the exact rank m_o+k. Therefore (A 10) has at least the rank m_o+k and lemma A1 is proved.

Lemma A2. Consider k equations

$$w_\varkappa(r_\varrho, u_\nu) = 0 \quad (\varkappa,\varrho=1,\ldots,k; \nu=1,\ldots,n) \tag{A 13}$$

and assume that the $\frac{\partial w_\varkappa}{\partial r_\varrho}$, $\frac{\partial w_\varkappa}{\partial u_\nu}$ exist and are continuous in convenient domains and that the Jacobian matrix

$$V := \left(\frac{\partial(w_\varkappa)}{\partial(r_\varrho)}\right) \tag{A 14}$$

is non-singular. Assume further that, solving the equations (A 13) with respect to the r_ϱ, we obtain the relations

$$r_\varkappa - M_\varkappa(u_\nu) = 0 \quad (\varkappa=1,\ldots,k) \quad . \tag{A 15}$$

Replacing now the u_ν with continuously differentiable functions of the $r_\varkappa$, put for any continuously differentiable function A of the r_ϱ and u_ν:

$$\frac{dA}{dr_\varrho} := \frac{\partial A}{\partial r_\varrho} + \sum_{\nu=1}^{n} \frac{\partial A}{\partial u_\nu} \frac{\partial u_\nu}{\partial r_\varrho} \quad , \tag{A 16}$$

and consider the matrix

$$\hat{V} := \left(\frac{d(w_\varkappa)}{d(r_\varrho)}\right) \quad . \tag{A 17}$$

Then the relation holds:

$$\left(\frac{d(r_\varkappa - M_\varkappa(u_1,\ldots,u_n))}{dr_\varrho}\right) = V^{-1}\hat{V} \quad . \tag{A 18}$$

<u>If in particular $\hat{V}$ is non-singular</u>, <u>then the matrix</u>

$$\left(\frac{d(r_\varkappa - M_\varkappa)}{d(r_\varrho)}\right) \tag{A 19}$$

<u>is non-singular</u>.

We verify first that, independently of the way in which the u_ν depend on the $r_\varkappa$, we have

$$\Omega := \left(w'_{\varkappa u_\nu}\right) , \tag{A 20}$$

$$\left(M'_{\varkappa u_\nu}\right) = -V^{-1}\Omega . \tag{A 21}$$

Indeed, we have identically

$$w_\varkappa(M_1,\dots,M_k;u_1,\dots,u_n) \equiv 0 ,$$

$$\sum_{\varrho=1}^{k} w'_{\varkappa r_\varrho} d(M_\varrho) + \sum_{\nu=1}^{n} w'_{\varkappa u_\nu} d(u_\nu) \equiv 0 \quad (\varkappa=1,\dots,k) .$$

This can be written, using here the accents to denote the <u>transposed</u>, that is <u>vertical</u> vectors,

$$(d(M_\varkappa))' = -V^{-1}\Omega(d(u_\nu))' = \left(M'_{\varkappa u_\nu}\right)(d(u_\nu))' ,$$

and (A 21) follows since the differentials $d(u_\nu)$ are arbitrary.

Assuming now the u_ν as continuously differentiable functions of the $r_\varkappa$ put

$$\hat{U} := \left(\frac{\partial(u_\nu)}{\partial(r_\varkappa)}\right) . \tag{A 22}$$

The relations (A 16) can then be written applied to the $w_{\varkappa}$ as

$$\left(\frac{d(w_{\varkappa})}{d(r_{\varrho})}\right) = \left(\frac{\partial(w_{\varkappa})}{\partial(r_{\varrho})}\right) + \Omega\left(u'_{1r}\ ,\ldots,u'_{nr}\right)' \qquad ,$$

(A 23) $$\hat{V} = V + \Omega\hat{U} \qquad .$$

On the other hand, by (A 22) and (A 21),

$$\left(\frac{d(r_{\varkappa} - M_{\varkappa})}{d(r_{\varrho})}\right) = I - \left(\frac{d(M_{\varkappa})}{d(r_{\varrho})}\right) = I - \left(M'_{\varkappa u_{\nu}}\right)\hat{U} =$$

$$= I + V^{-1}\Omega\,\hat{U} = V^{-1}\left[V + \Omega\hat{U}\right] = V^{-1}\hat{V} \quad ,$$

and (A 18) is proved.

APPENDIX B

Lemma B1. Consider an $m \times n$-matrix of $n > m$ row vectors $q_\nu = (q_{\nu 1}, \ldots, q_{\nu m})$:

(B 1)
$$Q^* = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} q_{11} & \cdots & q_{1m} \\ \vdots & \cdots & \vdots \\ q_{n1} & \cdots & q_{nm} \end{pmatrix}$$

with arbitrary complex $q_{\nu\mu}$, and a function $Y(q_1, \ldots, q_n)$ of the vectors q_ν, that is of mn variables $q_{\nu\mu}$. Assume that for an arbitrary non-singular $m \times m$-matrix, B, always

(B 2)
$$Y(q_1 B, \ldots, q_n B) = Y(q_1, \ldots, q_n).$$

Then Y is a homogeneous function of dimension 0 of the subdeterminants of order m of the matrix Q^*, more precisely

(B 3)
$$Y = Z(\bar{q}_{m+1}, \ldots, \bar{q}_n) ,$$

(B 4)
$$\bar{q}_\sigma = (\bar{q}_{\sigma 1}, \ldots, \bar{q}_{\sigma m}) , \quad \bar{q}_{\sigma\mu} = \Delta_\mu^{(\sigma)}/\Delta \quad (\sigma = m+1, \ldots, n; \mu = 1, \ldots, m).$$

Here Δ is an arbitrary but fixed subdeterminant of order m from Q^* and $\Delta_\mu^{(\sigma)}$ is another subterminant of order m of Q^*, conveniently chosen, but having $m-1$ rows im common with Δ.

If in particular Δ is the determinant formed with the first m rows of (B 1), then $\Delta_\mu^{(\sigma)}$ is obtained from Δ replacing the μ-th row of Δ by the σ-th row of Q^*.

Proof. Without loss of generality we can assume that Δ is the determinant of the matrix Q formed by the first m rows of Q*,

(B 5) $$Q := (q_1,\dots,q_m)' \ , \quad \Delta := \det Q \quad .$$

If we choose now B in (B 2) as Q^{-1}, the first m of the vectors $q_\sigma Q^{-1}$ reduce to the m unity vectors, $I_1,\dots,I_m$, and we can write

(B 6) $$Y(q_1,\dots,q_n) = Y(q_1Q^{-1},\dots,q_nQ^{-1}) = Y(I_1,\dots,I_m,\bar{q}_{m+1},\dots,\bar{q}_n),$$

(B 7) $$\bar{q}_\sigma = q_\sigma Q^{-1} \quad (\sigma=m+1,\dots,n) \ .$$

Consider the matrix

(B 8) $$A = (A_{\varkappa\mu}) := \Delta Q^{-1} \quad .$$

Then obviously

(B 9) $$\sum_{\varkappa=1}^{m} q_{\mu\varkappa} A_{\varkappa\mu} = \Delta \quad (\mu=1,\dots,m) \ .$$

Observe that the $A_{\varkappa\mu}$ as the algebraic complements of the $q_{\mu\varkappa}$, are for any fixed μ independent of the vector q_μ that is of the m elements $q_{\mu 1},\dots,q_{\mu m}$. Therefore, if we replace in (B 9) q_μ by q_σ ($\sigma > m$) the left side sum is $\Delta_\mu^{(\sigma)}$ defined as the subdeterminant of Q* obtained from Δ replacing there q_μ by q_σ;

(B 10) $$\sum_{\varkappa=1}^{m} q_{\sigma\varkappa} A_{\varkappa\mu} = \Delta_\mu^{(\sigma)} \quad (\sigma > m) \ ,$$

on the other hand the left-hand expression in (B 10) is by (B 8) and (B 7)

(B 11) $$(q_\sigma A)_\mu = \Delta(q_\sigma Q^{-1})_\mu = \Delta\bar{q}_\mu \quad ,$$

where the subscript μ denotes taking the μ-th component of the vector in parentheses. We obtain finally from (B 10) and (B 11) the formula (B 4) and our lemma is proved.

Observe that inversely, if a function $Y(q_1, \ldots, q_m)$ can be written as a function of the quotients of subdeterminants of order m of Q^*, then obviously the formula (B 2) holds.

APPENDIX C

We introduce first some notations useful when dealing with matrices. We denote by $E_{\mu\lambda}$ an m×m-matrix which has 1 as its λ-th element in the μ-th row while all other elements of $E_{\mu\lambda}$ vanish. For the multiplication of such matrices we see at once that, if $\delta_{\lambda\sigma}$ is Kronecker's symbol, then always

$$(C\ 1) \qquad E_{\mu\lambda}\, E_{\sigma\varrho} = \delta_{\lambda\sigma} E_{\mu\varrho} \quad .$$

Then if I denotes the unity matrix of order m, we have

$$(C\ 2) \qquad I = \sum_{\mu=1}^{m} E_{\mu\mu} \quad .$$

<u>Lemma C1</u>. <u>Under the assumptions of lemma B1</u>, <u>necessary and sufficient for the relation (B 2) being satisfied for any arbitrary non-singular</u> m×m-<u>matrix</u> B, <u>is that the Eulerian equations hold</u>:

$$(C\ 3) \qquad \sum_{\nu=1}^{n} q_{\nu\mu} Y'_{q_{\nu\lambda}} = 0 \quad (\mu,\lambda=1,\ldots,m) \quad .$$

<u>Proof</u>. We will have to specialize the matrix B in (B 2) in two particular ways.

$$(C\ 4) \qquad I + (g-1)E_{\lambda\lambda} \quad (\lambda=1,\ldots,m)$$

are m matrices such that

$$Q^*(I + (g-1)E_{\lambda\lambda})$$

is obtained from Q^* multiplying the λ-th column of Q^* with g.

$$\text{(C 5)} \qquad I + gE_{\mu\lambda} \quad (\lambda \neq \mu)$$

are $m(m-1)$ matrices such that generally

$$Q^*(I + gE_{\mu\lambda})$$

is obtained from Q^* if we add to the λ-th column the product of the μ-th column with g.

The matrices of the types (C 4) and (C 5) can in so far be considered as <u>elementary matrices</u>, as any non-singular $m \times m$-matrix B can be written as the product of a final number of such matrices. (This fact was repeatedly used in Kronecker's and Hensel's work on determinants and matrices.)

Our lemma C1 will therefore be proved if we prove that the necessary and sufficient invariancy condition for

$$\text{(C 6)} \qquad B = I + (g-1)E_{\lambda\lambda}$$

is the relation (C 3) for $\mu=\lambda$ and further that the relation (C 3) corresponding to μ and λ is the necessary and sufficient condition of invariancy for

$$\text{(C 7)} \qquad B = I + gE_{\mu\lambda} \quad (\lambda \neq \mu) \quad .$$

As to the relation (C 3) for a $\mu=\lambda$ it is by Euler's theorem equivalent with Y being a homogeneous function of dimension 0 in

$q_{1\lambda}, q_{2\lambda}, \ldots, q_{n\lambda}$ and this is again equivalent with (B 2) being true for

$$B = I + (g-1)E_{\lambda\lambda} \quad .$$

The invariancy with respect to $B = I + gE_{\mu\lambda}$ amounts to the relation, <u>for fixed</u> μ and λ,

$$Y(q_{\nu\mu}, q_{\nu\lambda}+gq_{\nu\mu}) = Y(q_{\nu\mu}, q_{\nu\lambda})$$

where only the variables corresponding to the μ-th and λ-th columnsare written out. This relation is again equivalent to

(C 8) $$\frac{d}{dg} Y(q_{\nu\mu}, q_{\nu\lambda}+gq_{\nu\mu}) = 0 \quad .$$

On the other hand introducing in (B 3) instead of the $q_{\nu\lambda}$ the new variables,

(C 9) $$r_{\nu\lambda} := q_{\nu\lambda} + gq_{\nu\mu}$$

we obtain

$$\sum_{\nu=1}^{n} q_{\nu\mu} Y'_{r_{\nu\lambda}}(q_{\nu\mu}, r_{\varkappa\lambda}) = 0 \quad .$$

But obviously by (C 9)

$$\frac{\partial}{\partial r_{\nu\lambda}} = \frac{\partial}{\partial q_{\nu\lambda}} \quad .$$

We obtain therefore

$$\sum_{\nu=1}^{n} q_{\nu\mu} Y'_{q_{\nu\lambda}}(q_{\nu\mu}, r_{\varkappa\lambda}) = 0$$

and this is identical with (C 8). Our lemma C1 is proved.

We are going now to verify that the system of m^2 equations (C 3) is complete. Indeed, we have

$$\sum_{\nu_1=1}^{n} q_{\nu_1\varrho} \frac{\partial}{\partial q_{\nu_1\sigma}} \sum_{\nu=1}^{n} q_{\nu\mu} Y'_{q_{\nu\lambda}} - \sum_{\nu=1}^{n} q_{\nu\mu} \frac{\partial}{\partial q_{\nu\lambda}} \sum_{\nu_1=1}^{n} q_{\nu_1\varrho} Y'_{q_{\nu_1\sigma}} =$$

$$\left[\sum_{\nu,\nu_1=1}^{n} q_{\nu_1\varrho} q_{\nu\mu} Y''_{q_{\nu\mu} q_{\nu_1\sigma}} - \sum_{\nu,\nu_1=1}^{n} q_{\nu\mu} q_{\nu_1\varrho} Y''_{q_{\nu_1\sigma} q_{\nu\mu}} \right] +$$

$$\sum_{\nu,\nu_1=1}^{n} q_{\nu_1\varrho} \delta_{\nu\nu_1} \delta_{\sigma\mu} Y'_{q_{\nu\mu}} - \sum_{\nu,\nu_1=1}^{n} q_{\nu\mu} \delta_{\nu\nu_1} \delta_{\lambda\varrho} Y'_{q_{\nu_1\sigma}} .$$

But here on the right the expression in the brackets vanishes and we can account for the factor $\delta_{\nu\nu_1}$ taking $\nu_1 = \nu$. We obtain

$$\delta_{\sigma\mu} \sum_{\nu=1}^{n} q_{\nu\varrho} Y'_{q_{\nu\mu}} - \delta_{\lambda\varrho} \sum_{\nu=1}^{n} q_{\nu\mu} Y'_{q_{\nu\sigma}} .$$

We see that combining two of the equations (C 3) by Poisson's parentheses we obtain at the most a linear combination of two of the equations (C 3). The system (C 3) is indeed complete.

This system (C 3) has therefore ∞^{nm-m^2} solutions. But by lemma B1 all solutions of the system (C 3) can be expressed as functions of m-vectors $\overline{q}_{m+1}, \ldots, \overline{q}_n$. It follows that the system of components of these n-m vectors,

$$\frac{\Delta_{\mu}^{(\sigma)}}{\Delta} \qquad (\sigma=m+1,\ldots,n;\mu=1,\ldots,m)$$

is independent.

This independence could be also deduced by lemma B1 from the relation (B 2).

APPENDIX D

Lemma D1. *Consider* d *linear and linearly independent functions* $L_\delta(x_1,\ldots,x_m)$ $(\delta=1,\ldots,d)$ *and* d m-*dimensional vectors* $V_\delta(p_{\delta 1},\ldots,p_{\delta m})$ *with elements* $p_{\delta\mu}$ *as* dm *independent variables*. *Write* $L_\epsilon(V_\delta)$ *for* $L_\epsilon(p_{\delta 1},\ldots,p_{\delta m})$. *Then if* $d \leq m$, *the determinant*

$$\text{(D 1)} \qquad \left| L_\epsilon(V_\delta) \right| \qquad (\epsilon,\delta=1,\ldots,d)$$

does not vanish.

Proof. Put

$$L = \sum_{\mu=1}^{m} \alpha_\mu^{(\delta)} x_\mu \qquad (\delta=1,\ldots,d) \quad .$$

Then, by assumption, the rank of the matrix $(\alpha_\mu^{(\delta)})$ is d. We can therefore, after suitable rearrangement of the indices 1,...,m, assume that the determinant

$$\left| \alpha_\mu^{(\delta)} \right| \qquad (\delta,\mu=1,\ldots,d)$$

is not zero. But then if we replace all $p_{\delta d+1},\ldots,p_{\delta m}$ with zeros, the determinant (D 1) becomes

$$\left| \alpha_\mu^{(\delta)} \right| \; \left| p_{\delta\mu} \right| \qquad (\delta,\mu=1,\ldots,d)$$

and does not therefore vanish.